CONVERGENCE OF STREAMLINE DIFFUSION METHODS FOR THE VLASOV–POISSON–FOKKER–PLANCK SYSTEM

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Abstract. We prove stability estimates and convergence rates for the streamline diffusion and discontinuous Galerkin finite element methods for discretization of the multi–dimensional Vlasov–Poisson–Fokker–Planck system. Our study is an extension of the results derived in [3] and [4]. Qualitative solution properties such as existence, uniqueness and long time behaviour are based on asymptotic studies in [10]. This note is the first part of our studies covering the theoretical aspects and convergence analysis of the proposed methods. In a complementary part, devoted to the numerical results, we shall show the desired accuracy, high resolution and robustness of our approach through simulating some canonical examples.

1. Introduction

In this paper we study the approximate solution for the deterministic, multi-dimensional Vlasov–Poisson–Fokker–Planck (VPFP) system using the streamline diffusion and discontinuous Galerkin finite element methods. We prove stability estimates and derive optimal convergence rates for the regularized VPFP system. This extends the results in [3] and [4] for the multi–dimensional Vlasov–Poisson and the Fokker–Planck equations, respectively.

The VPFP system arising in the kinetic description of a plasma of Coulomb particles under the influence of a self–consistent internal field and an external force can be formulated as follows. Given the initial distribution of particles $f_0(x,v) \geq 0$, in the phase–space variable $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1,2,3$, and the physical parameters $\beta > 0$ and $\sigma > 0$, find the distribution function $f(x,v,t)$ for $t > 0$, satisfying the nonlinear system of evolution equations

$$
\begin{cases}
\partial_t f + v\cdot\nabla_x f + \operatorname{div}_v[(E-\beta v)f] = \sigma\Delta_v f, & \text{in } \mathbb{R}^{2d}\times(0,\infty),\\
f(x,v,0) = f_0(x,v), & \text{for } (x,v)\in\mathbb{R}^{2d},\\
E(x,t) = \dfrac{\theta}{|\mathcal{S}|^{d-1}}\dfrac{x}{|x|^d} *_x \rho(x,t), & \text{for } (x,t)\in\mathbb{R}^d\times(0,\infty),\\
\rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,dv, & E = \theta\tilde{E}, \text{ and } \theta = \pm 1,
\end{cases}
\tag{1.1}
$$

where $x \in \mathbb{R}^d$ is the position, $v \in \mathbb{R}^d$ is the velocity, and $t > 0$ is the time, $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \cdots, \partial/\partial x_d)$, $\nabla_v = (\partial/\partial v_1, \partial/\partial v_2, \cdots, \partial/\partial v_d)$, and $\cdot$ is the

1991 *Mathematics Subject Classification.* 65M12, 65M15, 65M60, 82D10, 35L80.

Key words and phrases. Vlasov–Poisson–Fokker–Planck, streamline diffusion, discontinuous Galerkin, stability, convergence.

[1] Partially supported by the EU contract ERB FMRX CT97 0157.

[2] Partially supported by Polish Academy of Sciences in cooperation with Royal Swedish Academy of Sciences and Polish State Committee for Scientific Research Grant No. 2P03A 007 17.

inner product in $\mathbb{R}^d$. The diffusion parameters β and σ are the viscosity and the thermal diffusivity coefficients, respectively, which are related by $\sigma = \beta\kappa T/m$, with κ being the Boltzmann's constant, T the temperature of the surrounding medium and m the mass of a particle, (the physical parameter σ is very small). $|\mathcal{S}|^{d-1} \sim 1/\omega_d$ is the surface area of the unit disc in $\mathbb{R}^d$. Finally $\rho(x,t)$ is the spatial density and $*_x$ denotes the convolution in x. E and ρ can be interpreted as the electrical field, and charge, respectively.

The macroscopic force field E can also be assumed to be of the form

$$E(x,t) = -\nabla_x\Big(\psi(x) + \phi(x,t)\Big), \tag{1.2}$$

with $\psi(x) \geq 0$ being an external potential force, and $\phi(x,t)$ the internal potential field.

For a gradient field, when E is divergence free and with no viscosity; i.e., for $\beta = 0$, the first equation in (1.1), would become

$$\partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = \sigma\Delta_v f, \tag{1.3}$$

which, with the rest of equations in (1.1), gives rise to the Vlasov–Fokker–Planck system. If in addition $\sigma = 0$, then we obtain the classical Vlasov–Poisson equation corresponding to a zero external force, i.e., $\psi(x) \equiv 0$, and with an internal potential field $\phi(x,t)$ satisfying the Poisson equation

$$\Delta_x\phi(x,t) = -\theta\int_{\mathbb{R}^d} f(x,v,t)dv = -\theta\rho(x,t), \tag{1.4}$$

with the asymptotic boundary condition

$$\begin{cases} \phi(x,t)\to 0, & \text{for} \quad d>2, \quad \text{as} \quad |x|\to\infty, \\ \phi(x,t) = \mathcal{O}(\log|x|), & \text{for} \quad d=2, \quad \text{as} \quad |x|\to\infty. \end{cases} \tag{1.5}$$

We shall concentrate on the following (modified) version of the VPFP equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = \nabla_v\cdot(\beta v f + \sigma\nabla_v f), \tag{1.6}$$

where ϕ is assumed to be the exact solution for the Poisson equation (1.4) given by

$$\phi(x,t) = \theta\int_{\mathbb{R}^{2d}} G(x-y)f(y,v',t)dydv', \tag{1.7}$$

with G being the Green's function associated with the fundamental solution of the Laplace's operator $-\Delta_x$, viz;

$$G(x) = \begin{cases} \dfrac{\omega_d}{|x|^{d-2}}, & \text{for} \quad d>2, \\ \omega_2\log|x|, & \text{for} \quad d=2, \end{cases} \tag{1.8}$$

where $1/\omega_d$, as above, is the area of the unit sphere in $\mathbb{R}^d$. (The case $d=1$, is given as in the original system (1.1).)

Depending on the sign of the parameter θ the VPFP system describes two different physical situations. For $\theta = 1$, the system models a gas of charged particles with an external potential ψ, interacting through a mean electrostatic field $-\nabla_x\phi$ generated by their spatial density ρ. The case $\theta = -1$ corresponds to a VPFP system modelling particles under the effect of the gravitational potential ψ.

In the stochastic approach the solution f for the VPFP system is interpreted as a probability density function for the stochastic process $(X(t),V(t)) \in \mathbb{R}^d\times\mathbb{R}^d$ satisfying the Lagevin equations given by

$$\frac{dX}{dt} = V, \quad \frac{dV}{dt} = E(x,t) - \beta V + \sqrt{(2\sigma)}\kappa(t), \quad \kappa(t) = dW(t), \tag{1.9}$$

where $W(t)$ is a d–dimensional Wiener process, (see [20] for the details), and

$$\int_{\mathbb{R}^{2d}} f(x,v,t)dxdv = \int_{\mathbb{R}^{2d}} f_0(x,v)dxdv. \tag{1.10}$$

The mathematical study of the VPFP system has been considered by several authors in various settings. The deterministic approach is based on controlling the behaviour of the trajectories, i.e., the solutions of the ordinary differential equations underlying the Vlasov–Poisson equation, (see, e.g., [34], [18], and [14]), whereas the stochastic approach is based on the diffusion, stochastic differential equations studied, e.g., in [26], [28], and [30]. Here are some literature concerning the properties of the analytic solution for the VPFP system: Asymptotic behaviour, parabolic limit and stability properties have been carried out in, e.g., [8], [29], and the references therein. Existence of local in time, classical, smooth solution is given in, e.g., [33], and sufficient condition for the global existence of classical solution in three–dimensions can be found in [31]. Existence and uniqueness of smooth global in time solution for large class of initial data are given in [9] and [10] , where the recent paper treats also regularity of the weak solution. Existence of global measure solutions in the one–dimensional case, which uses relationship between VPFP and the two–dimensional Euler equation with vortex sheet as initial data, is studied in [35]. Large time behaviour and steady state are considered in [15]. In a recent work [20], the time–discrete variational formulations are studied by certain Kantorovich type functionals. Propagation of moments, in the Vlasov–Poisson context, is studied in [25].

Finally some basic application aspects can be found in [6] for the diffusive asymptotic limit of the neutron transport equation, in [7] for the study of the radiative transfer model problem, and in [17] where a compactness argument is used to study a semiconductor model.

Compared to the analytical studies the numerical analysis of the VPFP system, both in theory and implementations, is much less developed. In this setting the Monte-Carlo simulations are explained in transport/diffusion context in [24]. In the deterministic approaches the dominant part of numerical studies are using method of characteristics, (similar to the analytic studies of the deterministic problem), and are mostly the well known particle methods developed for the Vlasov–Poisson equation in, e.g., [27], [16], and [13].

In this paper we focus on the deterministic approach and study the stability and convergence of some finite element methods constructed for the problems of fluid dynamics: incompressible Euler and Navier–Stokes equations, as well as conservations laws and convection–diffusion equations see, e.g., [23], [32], [22], [12], [11], and [19]. More precisely this paper extends the results of the first author [3], and [4] in Vlasov–Poisson and Fokker–Planck equations to the VPFP system.

We have considered the h–version of the streamline diffusion and discontinuous Galerkin finite element methods for the VPFP system using piecewise polynomials of degree k, in discretizing the phase–space–time domain $Q_T := \Omega_x \times \Omega_v \times [0,T]$, where $\Omega_x \subset \mathbb{R}^d$, $d = 1,2,3$ is a bounded simply connected spatial domain and the support of $f_0 \subset \Omega_v \subset \mathbb{R}^d$. Assuming a continuous Poisson solver, and sufficient regularity of the exact solution, ($f \in H^{k+1}(\Omega)$, the Sobolev space of L_2 integrable functions having all their partial derivatives of order $k+1$ in L_2, see [1]), we develop stability estimates to guarantee the conservation of qualitative regularity features of the solution and prove sharp error estimates of order $\mathcal{O}(h^{k+1/2})$.

An outline of this paper is as follows. In Section 2 we briefly review the existence, uniqueness and stability of the solution for the continuous VPFP system (1.1). Section 3 is devoted to the study of stability estimates and proof of the convergence rates for the streamline diffusion approximation of the VPFP system and Section 4 is the discontinuous Galerkin counterpart of Section 3.

2. The continuous problem

In this section we summerize some analytic properties such as existence, uniqueness and stability for the solution of the continuous problem. Both local and global in time as well as strong and weak solutions are quoted from the relevant literature. The regularity assumptions on the initial data would guarantee the conservation of mass, momentum and energy. We also present a continuous version of the VPFP system which we have discretized in Sections 3 and 4. For further discussions on these properties as well as entropy relations we refer the reader to [29] and the references therein.

Because of the structural connections we start with the Vlasov–Fokker–Planck (VFP) system and relate the solution properties for the linear Fokker–Planck, linear transport, VFP, Vlasov–Poisson and VPFP system. To this end we present the VFP system

$$\tag{2.1} \begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f - \sigma\Delta_v f = 0, & f(x,v,0) = f_0(x,v), \\ E(x,t) = C_d \displaystyle\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\rho(y,t)\,dy, & \rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,dv. \end{cases}$$

We let σ tend to zero in (2.1) and obtain formally the classical Vlasov–Poisson equations

$$\tag{2.2} \begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = 0, & f(x,v,0) = f_0(x,v), \\ E(x,t) = C_d \displaystyle\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\rho(y,t)\,dy, & \rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,dv. \end{cases}$$

In the sequel we denote by $\|\cdot\|_{m,p}$ the usual norm in the Sobolev space $W^{m,p}(\Omega)$ with $\Omega\subset\mathbb{R}^d$ (see [1]), and by $\|\cdot\|_p$, $1\le p\le\infty$, the L_p norm. $D^m u$ will denote the m-th total derivative of u with respect to (x,v).

Problems (2.1) and (2.2) are studied by, e.g., Arsen'ev [2] and Degond [14]. We summarize the solution properties obtained in [14] in the following two propositions:

Proposition 2.1 (existence, uniqueness and regularity). *Assume that* $f_0\ge 0$,

$$\tag{2.3} f_0\in W^{m,1}(\mathbb{R}^{2d}),\quad (1+|v|^2)^{\gamma/2}\Big(|f_0|+\cdots+|D^m f_0|\Big)\in L^\infty(\mathbb{R}^{2d}),\quad \gamma>d,$$

with $m\ge 1$*. Then the VFP system (2.1) admits a positive, unique classical solution in a time interval* $[0,T)$ *with* $T=\infty$ *for* $d=1,2$ *(global in time solution), and* *finite* $T = T(f_0)$, *if* $d = 3$ *(local in time solution) satisfying*

$$\begin{cases} f \in L^\infty_{loc}\Big([0,T),\, W^{m,1}(\mathbb{R}^{2d})\Big), \qquad E \in L^\infty_{Loc}\Big([0,T),\, W^{m,\infty}(\mathbb{R}^d_x)\Big), \\ (1+|v|^2)^{\gamma/2}\Big(|f| + \cdots + |D^m f|\Big) \in L^\infty_{Loc}\Big([0,T),\, L^\infty(\mathbb{R}^{2d})\Big), \\ \nabla_v(D^m f) \in L^2_{Loc}\Big([0,T),\, L^2(\mathbb{R}^{2d})\Big). \end{cases}$$

Here $m = 1$ *would suffices for existence and uniqueness.*

Now let us denote the solution of the Vlasov–Poisson equation (2.2) by (f, E) and that of the VFP system (2.1) by (f^σ, E^σ), then we have for sufficiently regular initial data f_0, that (f^σ, E^σ) converges to (f, E) as $\sigma \to 0$. More specifically:

Proposition 2.2. *Assume that the initial data* f_0 *is nonnegative and satisfies (2.3), with* $m = 2$, *i.e.,*

$$f_0 \in W^{2,1}(\mathbb{R}^{2d}), \quad (1+|v|^2)^{\gamma/2}\Big(|f_0| + |Df_0| + |D^2 f_0|\Big) \in L^\infty(\mathbb{R}^{2d}),\ \gamma > d. \tag{2.4}$$

Then for any finite time interval $[0, T']$ *(*$T' < T$ *for* $d = 3$*), the solution* (f^σ, E^σ) *of the VFP system converges to the solution* (f, E) *of the Vlasov–Poisson equation in the following sense*

$$\max_{[0,T']}\Big(\|(f^\sigma - f)\|_1 + \|(1+|v|^2)^{\gamma/2}(f^\sigma - f)\|_\infty + \|(E^\sigma - E)\|_\infty\Big) = \mathcal{O}(\sigma). \tag{2.5}$$

The proofs for these results are based on a standard iterative approximation scheme constructed to solve the following linear Fokker–Planck equation: given the electric field $E^n(x,t) \in L^\infty_{Loc}\Big([0,\infty),\, W^{1,\infty}(\mathbb{R}^d_x)\Big)$, find f^{n+1} satisfying

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} + E^n \cdot \nabla_v f^{n+1} - \sigma \Delta_v f^{n+1} = 0, \\ f^{n+1}(x,v,0) = f_0(x,v), \end{cases} \tag{2.6}$$

and then compute the charge density ρ^{n+1} and electrical field E^{n+1} according to

$$\rho^{n+1}(x,t) = \int_{\mathbb{R}^d} f^{n+1}(x,v,t)\,dv, \qquad E^{n+1}(x,t) = C_d \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\rho^{n+1}(y,t)\,dy.$$

Now proposition 2.3 below guarantees the existence of a unique solution f^{n+1} to the equation (2.6) satisfying the stability estimates:

$$f^{n+1} \geq 0, \quad \|\rho^{n+1}(t)\|_1 \leq \|f^{n+1}(t)\|_1 \leq \|f_0\|_1, \quad \|f^{n+1}(t)\|_\infty \leq \|f_0\|_\infty. \tag{2.7}$$

Note that for $\sigma = 0$, equation (2.6) becomes the classical linear transport equation, which can be solved, e.g., by means of characteristics, and the stability properties (2.7) are evident.

To continue we return to the linear Fokker–Planck equation:

$$f_t + v \cdot \nabla_x f + F \cdot \nabla_v f - \sigma \Delta_v f = g, \quad f(x,v,0) = f_0(x,v), \tag{2.8}$$

where

$$F(x,v,t) = \Big(F_i(x,v,t)\Big)_{i=1}^d,$$

is a given vector field and $f_0(x,v)$ and $g(x,v,t)$ are given functions. Existence, uniqueness, stability and regularity properties of the solution for the equation (2.8) are based on one–dimensional classical results by Baquendi and Grisvard [5] for the degenerate type equations. To invoke these results, recalling the definition of the inverse norm H^{-1}; (see [1]), we assume that

$$f_0 \in L_2(\mathbb{R}^{2d}), \qquad g \in L_2\Big([0,T] \times \mathbb{R}^d_x, H^{-1}(\mathbb{R}^d_v)\Big), \tag{2.9}$$

$$F \in L_\infty\Big([0,T] \times \mathbb{R}^{2d}\Big), \qquad \nabla_v \cdot F \in L_\infty\Big([0,T] \times \mathbb{R}^{2d}\Big), \tag{2.10}$$

and define the function space

$$\mathcal{F} = \Big\{ f \in L_2\Big([0,T] \times \mathbb{R}^d_x, H^1(\mathbb{R}^d_v)\Big) \Big| \ f_t + v \cdot \nabla_x f \in L_2\Big([0,T] \times \mathbb{R}^d_x, H^{-1}(\mathbb{R}^d_v)\Big) \Big\}.$$

Proposition 2.3. *(existence and uniqueness) Assume that the conditions (2.9) and (2.10) are fulfilled, then the equation (2.8) has a unique solution in the class $\mathcal{F}$. Further if f is any weak solution of (2.8), belonging to $L_2\Big([0,T] \times \mathbb{R}^{2d}\Big)$, then f also belongs to $\mathcal{F}$ and coincides with the unique classical solution above.*

Proposition 2.4. *(L_∞, maximum principle, and L_1 stabilities) Assume that (2.9) and (2.10) are fulfilled. Then, for the nonnegative data f_0 and g, the solution f of the equation (2.8) provided by proposition 2.3 above is nonnegative and satisfies*

$$f_0 \in L_\infty(\mathbb{R}^{2d}) \ \text{and} \ g \in L_1\Big([0,T], L_\infty(\mathbb{R}^{2d})\Big) \implies f \in L_\infty\Big([0,T] \times \mathbb{R}^{2d}\Big), \tag{2.11}$$

and

$$\|f(t)\|_\infty \le \|f_0\|_\infty + \int_0^t \|g(s)\|_\infty \, ds. \tag{2.12}$$

Further, if in addition to (2.9) and (2.10), F is divergent free; i.e., $\nabla_v \cdot F = 0$, then the solution f of the equation (2.8) satisfies

$$f_0 \in L_1(\mathbb{R}^{2d}) \quad \text{and} \quad g \in L_1\Big([0,T] \times \mathbb{R}^{2d}\Big) \implies f \in L_\infty\Big([0,T], L_1(\mathbb{R}^{2d})\Big) \tag{2.13}$$

and

$$\|f(t)\|_1 \le \|f_0\|_1 + \int_0^t \|g(s)\|_1 \, ds. \tag{2.14}$$

So far we have developed the following:

(i) The solution to VFP system (2.1) converges to that of Vlasov–Poisson equation (2.2) as $\sigma \to 0$.

(ii) The problem of solving VFP system (2.1) is reduced to, iteratively, solving the approximate linear Fokker–Planck equation (2.6), with an initially given field E^n. The corresponding continuous linear Fokker–Planck equation (2.8) has a unique solution satisfying the stability estimates given in proposition 2.4.

(iii) The Fokker–Planck equation (2.8) converges to the linear transport equation as $\sigma \to 0$.

The basic difference between the VFP (2.1) and VPFP system (1.1) is on the difference in the definition of the vector field E, in the two system. Finally we quote an existence and uniqueness result for the VPFP system (1.1) given by Bouchut for the three–dimensional problem [9]:

Proposition 2.5. *For $d = 3$, if $f_0 \ge 0$ satisfies the energy estimates*

$$f_0 \in L_1 \cap L_\infty(\mathbb{R}^6), \quad \exists m > 6 \ \text{with} \ \iint_{\mathbb{R}^6} |v|^m f_0(x,v) \, dxdv < \infty, \tag{2.15}$$

then there exists a unique solution (f, E) to the system (1.1) satisfying

$$f \in \mathcal{C}\Big([0,\infty), L_1(\mathbb{R}^6)\Big), \quad \sup_{0\le t\le T} \|E(t,\cdot)\|_\infty < \infty, \quad \forall T > 0. \tag{2.16}$$

We point out that the Fokker–Planck term, containing a diffusive part, has smoothing effects on the solution of the system (1.1), see Bouchut [10], which for instance can not be maintained for the Vlasov–Poisson equation lacking this diffusive part. In other words: to show that the macroscopic density $\rho(x,t)$ and the force field $E(x,t)$ are smooth functions of x, the particular structure in the Fokker–Planck operator $v\cdot\nabla_v f - \beta\, div_v(vf) - \sigma\Delta_v f$, would play a central role. That is to say that the Fokker–Planck operator, although degenerate, provides a smoothing effect related to its hypoellipticity in the sense that it acts as a convolution operator on the macroscopic quantities, thus it averages in v.

The VPFP system describing a variety of physical situations is subject to certain modifications depending on the considered problem setting. The transport part, $\partial_t f + v\cdot\nabla_x f$, is associated with various field functions E in the Vlasov term $div_v[(E-\beta v)f]$; where for a given field E we obtain the linear VFP problem, whereas the convolution in (1.1) would relate E to the spatial density ρ and lead to the nonlinear VFP equation. In this context the system (1.1) is qualified to be referred as the *Vlasov–Poisson–Fokker–Planck* system only if the function E is replaced by relations of the form (1.2) and (1.4) corresponding to the Poisson equation. The Fokker–Planck term on the right hand side can also been chosen in different ways as well: e.g., with a Maxwellian type initial data f_0, a Fokker–Planck term of the form

$$L(f) = \nabla_v\Big(e^{-|v|^2/2}\nabla_v(e^{|v|^2/2})\Big),$$

is more suitable (see, e.g., [29]).

In our studies we assume that $(x,v)\in\Omega := \Omega_x\times\Omega_v$, where $\Omega_x\subset\mathbb{R}^d$ is a bounded simply connected spatial domain and $\Omega_v := \mathbb{R}^d$ and let $Q_T := \Omega_x\times\Omega_v\times(0,T]$. We also assume that the initial data

$$f_0 \text{ is compactly supported in } \Omega_v = \mathbb{R}^d. \tag{2.17}$$

With these assumptions we consider the VPFP problem of finding (f,ϕ) satisfying a VFP system of the form

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = \nabla_v(\beta v f + \sigma\nabla_v f), & \text{in } Q_T,\\ f(x,v,0) = f_0(x,v), & \text{in } \Omega,\\ f(x,v,t) = 0, \qquad \text{for } (x,v,t)\in\Gamma_v^-\times\mathbb{R}^d\times(0,T], & \end{cases} \tag{2.18}$$

associated with the Poisson equation

$$-\Delta_x\phi(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\,dv, \qquad (x,t)\in\mathbb{R}^d\times(0,T] := \Omega_T, \tag{2.19}$$

where $\nabla_x\phi$ is uniformly bounded and

$$\begin{cases} \nabla_x\phi \to 0, & \text{as } |x|\to\infty,\\ \Gamma_v^- = \{x\in\partial\Omega_x : \mathbf{n}(x)\cdot v < 0\}, & \text{for } v\in\mathbb{R}^d, \end{cases} \tag{2.20}$$

where $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega_x$ at the point $x\in\partial\Omega_x$. We assume that f_0 satisfies (2.17) and all the conditions on propositions 2.1–2.5 are fulfilled. Then as a consequence of our review of the theoretical approaches we have that the system of equations (2.18)–(2.20) admits a sufficiently regular unique solution.

We conclude this section by introducing the notation

$$\nabla f := (\nabla_x f, \nabla_v f) = \Big(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}, \frac{\partial f}{\partial v_1}, \dots, \frac{\partial f}{\partial v_d}\Big), \quad d = 1, 2, 3,$$

$$G(f) := (v, -\nabla_x \phi) = \Big(v_1, \dots, v_d, -\frac{\partial \phi}{\partial x_1}, \dots, -\frac{\partial \phi}{\partial x_d}\Big) = (G_1, \dots, G_{2d}),$$

leading to the following useful observation:

$$\operatorname{div} G(f) = \sum_{i=1}^{d} \frac{\partial G_i}{\partial x_i} + \sum_{i=d+1}^{2d} \frac{\partial G_i}{\partial v_{i-d}} = 0, \quad d = 1, 2, 3. \tag{2.21}$$

3. The streamline diffusion method

The streamline diffusion (SD) method is a finite element method constructed for convection dominated convection–diffusion problems which (i) is higher order accurate and (ii) has good stability properties. The (SD) method was introduced by Hughes and Brooks [21] for the stationary problems. The mathematical analysis for this method in two settings (streamline diffusion and discontinuous Galerkin) are developed for ,e.g., two–dimensional incompressible Euler and Navier–Stokes equations by Johnson and Saranen [23], for multi–dimensional Vlasov–Poisson equation by Asadzadeh [3], for hyperbolic conservations laws by Szepessy [32], and Jaffre', Johnson and Szepessy [22], for the two–dimensional Fermi and Fokker–Planck by Asadzadeh in [4], for advection–diffusion problems by Brezzi, et al in [11], and [12], and also recently, in adaptive setting, by Houston and Süli [19].

We start by introducing a finite element structure on $\Omega_x \times \Omega_v$. Let $T_h^x = \{\tau_x\}$ and $T_h^v = \{\tau_v\}$ denote finite element subdivisions of Ω_x and $\Omega_v = \mathbb{R}^d$, with elements τ_x and τ_v, respectively. Then $T_h = T_h^x \times T_h^v = \{\tau_x \times \tau_v\} = \{\tau\}$ will be a subdivision of $\Omega = \Omega_x \times \Omega_v$ with elements $\tau = \tau_x \times \tau_v$. We also let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of the time interval $I = [0, T]$ into subintervals $I_m = (t_m, t_{m+1}),\ m = 0, 1, \dots, M-1$. Moreover let $\mathcal{C}_h$ be the corresponding subdivision of $Q_T = \Omega \times [0, T]$ into elements $K := \tau \times I_m$, with the mesh parameter $h = \operatorname{diam} K$ and $P_k(K) = P_k(\tau_x) \times P_k(\tau_v) \times P_k(I_m)$ the set of polynomials in x, v and t of degree at most k on K. Furthermore for piecewise polynomials w_i defined on the triangulation $\mathcal{C}'_h = \{K\}$ with $\mathcal{C}'_h \subset \mathcal{C}_h$ and for D_i being some differential operators, we use the notation

$$(D_1 w_1, D_2 w_2)_{Q'} = \sum_{K \in \mathcal{C}'_h} (D_1 w_1, D_2 w_2)_K, \qquad Q' = \bigcup_{K \in \mathcal{C}'_h} K,$$

where $(\cdot, \cdot)_Q$ is the usual $L_2(Q)$ scalar product and $\|\cdot\|_Q$ is the corresponding $L_2(Q)$–norm. To derive our stability and convergence estimates we shall need to assume that the exact solution is in the $L_2(Q)$–based Sobolev space $H^s(Q)$ with the norm $\|\cdot\|_{s,Q}$, where s is a positive integer.

We shall also need the following assumptions:

1. σ is $O(h)$ (or $\sigma = C_\sigma h$ for small constant C_σ) and β is $O(1)$, where $0 < h \ll 1$ is a finite element mesh parameter introduced above.
2. Despite the diffusive nature of the right–hand side of (2.18), since f_0 has compact support in $\Omega_v = \mathbb{R}^d$ we have, for sufficiently large v, that $f(x, v, t) = 0$. Thus the analysis can be carried out on a bounded domain Ω_v^h with all SD–test functions vanishing on $\partial\Omega_v^h$.

In this section we study the SD–method for the VPFP system given by (2.18)–(2.20), with the trial functions being continuous in the x and v variables. We shall use the following notation: for $k = 0, 1, \ldots$, let

$$V_h = \Big\{ g \in \mathcal{H}_0 : g\Big|_K \in P_k(\tau) \times P_k(I_m); \quad \forall K = \tau \times I_m \in \mathcal{C}_n \Big\},$$

be the finite element space where

$$\mathcal{H}_0 = \prod_{m=0}^{M-1} H_0^1(S_m), \quad S_m = \Omega \times I_m, \quad m = 0, 1, \ldots, M-1.$$

and

$$H_0^1 = \Big\{ g \in H^1 : g \equiv 0 \text{ on } \partial\Omega_v^h \Big\}.$$

Further, for convenience, we write

$$(f, g)_m = (f, g)_{S_m}, \qquad \|g\|_m = (g, g)_m^{1/2},$$

and

$$< f, g >_m = (f(\cdot, \cdot, t_m), (g(\cdot, \cdot, t_m))_\Omega, \quad |g|_m = < g, g >_m^{1/2}.$$

Also we present the jump

$$[g] = g_+ - g_-,$$

where

$$\begin{aligned} g_\pm &= \lim_{s \to 0\pm} g(x, v, t+s), && \text{for } (x, v) \in \text{Int}\Omega_x \times \Omega_v^h, \quad t \in I, \\ g_\pm &= \lim_{s \to 0\pm} g(x + sv, v, t+s), && \text{for } (x, v) \in \partial\Omega_x \times \Omega_v^h, \quad t \in I, \end{aligned}$$

and the boundary integrals

$$< f_+, g_+ >_{\Gamma^-} = \int_{\Gamma^-} f_+ g_+ \, (G^h \cdot \mathbf{n}) \, d\nu, \qquad < f_+, g_+ >_{\Gamma_m^-} = \int_{I_m} < f_+, g_+ >_{\Gamma^-} \, d\nu,$$

and

$$< f_+, g_+ >_{\Gamma_I^-} = \int_I < f_+, g_+ >_{\Gamma^-} \, d\nu,$$

with $G^h := G(f^h)$ defined above (see also (3.1)) and

$$\Gamma^- = \Big\{ (x, v) \in \Gamma = \partial(\Omega_x \times \Omega_v^h) : G^h \cdot \mathbf{n} < 0 \Big\},$$

where $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_v)$ with $\mathbf{n}_x$ and $\mathbf{n}_v$ being outward unit normals to $\partial\Omega_x$ and $\partial\Omega_v^h$, respectively. Finally in the sequel $\Omega := \Omega_x \times \Omega_v^h$ and C denotes the general constant independent of the involved parameters on estimations, unless otherwise it is clear from the context or explicitly specified.

Now we are ready to study the stability of the SD-scheme, viz:

3.1. **Stability.** The discrete variational formulation for problem (2.18) reads as follows:

find $f^h \in V_h$ *such that for* $m = 0, 1, \dots, M-1$ *and for all* $g \in V_h$,

$$\begin{aligned}\big(f_t^h + G(f^h)\nabla f^h - \nabla_v(\beta v f^h), g + h(g_t + G(f^h)\nabla g)\big)_m + \sigma\big(\nabla_v f^h, \nabla_v g\big)_m + \\ -h\sigma\big(\Delta_v f^h, g_t + G(f^h)\nabla g\big)_m + \langle f_+^h, g_+\rangle_m - \langle f_+^h, g_+\rangle_{\Gamma_m^-} = \langle f_-^h, g_+\rangle_m.\end{aligned} \tag{3.1}$$

We use the discrete version of (2.21):

$$\text{div } G(f^h) = 0, \tag{3.2}$$

and, for a given appropriate function $\tilde{f}$, define the trilinear form B by

$$\begin{aligned}B\big(G(\tilde{f}); f, g\big) =& \big(f_t + G(\tilde{f})\nabla f, g + h(g_t + G(f^h)\nabla g)\big)_{Q_T} + \\ &+ \sigma\big(\nabla_v f, \nabla_v g\big)_{Q_T} - h\sigma\big(\Delta_v f, g_t + G(f^h)\nabla g\big)_{Q_T} + \\ &+ \sum_{m=1}^{M-1} \langle [f], g_+\rangle_m + \langle f_+, g_+\rangle_0 - \langle f_+, g_+\rangle_{\Gamma_I^-}\end{aligned}$$

and the bilinear form K by

$$K\big(f, g\big) = \big(\nabla_v(\beta v f), g + h(g_t + G(f^h)\nabla g)\big)_{Q_T}.$$

Note that both B and K depend implicitly on f^h (hence also on h) through the term $G(f^h)$. Moreover we define the linear form L as

$$L(g) = \langle f_0, g_+\rangle_0.$$

Using this notation we can formulate the problem (3.1) in the following concise form:

find $f^h \in V_h$ *such that*

$$B\big(G(f^h); f^h, g\big) - K\big(f^h, g\big) = L(g) \qquad \forall g \in V_h. \tag{3.3}$$

We shall derive our stability and convergence estimates for (3.3) in a triple norm defined by

$$\begin{aligned}|||g|||^2 = \frac{1}{2}\Big[2\sigma\|\nabla_v g\|_{Q_T}^2 + |g|_M^2 + |g|_0^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + 2h\|g_t + G(f^h)\nabla g\|_{Q_T}^2 + \\ + \int_{\partial\Omega\times I} g^2\big|G^h\cdot \mathbf{n}\big|\, d\nu\, ds\Big].\end{aligned}$$

Lemma 3.1. *We have that*

$$\forall g \in \mathcal{H}_0 \qquad B\big(G(f^h); g, g\big) \geq \frac{1}{2}|||g|||^2.$$

Proof. Using the definition of B we have the identity

$$
\begin{aligned}
B\big(G(f^h);g,g\big) =& \big(g_t,g\big)_{Q_T} + \sum_{m=1}^{M-1} \langle [g], g_+\rangle_m + \langle g_+, g_+\rangle_0 + \\
&+ \big(G(f^h)\cdot\nabla g, g\big)_{Q_T} - \langle g_+, g_+\rangle_{\Gamma_I^-} + \\
&- h\sigma\big(\Delta_v g, g_t + G(f^h)\nabla g\big)_{Q_T} + \\
&+ \sigma\|\nabla_v g\|^2_{Q_T} + h\|g_t + G(f^h)\nabla g\|^2_{Q_T}.
\end{aligned}
\tag{3.4}
$$

Integration by parts gives that

$$
\big(g_t,g\big)_{Q_T} + \sum_{m=1}^{M-1} \langle [g], g_+\rangle_m + \langle g_+, g_+\rangle_0 = \frac{1}{2}\Bigg[|g|^2_M + |g|^2_0 + \sum_{m=1}^{M-1} |[g]|^2_m\Bigg].
\tag{3.5}
$$

Using Green's formula and (3.2) we have also

$$
\begin{aligned}
\big(G(f^h)\cdot\nabla g, g\big)_{\Omega} - \langle g_+, g_+\rangle_{\Gamma^-} =& \\
&= \frac{1}{2}\int_{\partial\Omega} g^2\big(G^h\cdot\mathbf{n}\big)\,d\nu - \int_{\Gamma^-} g^2\big(G^h\cdot\mathbf{n}\big)\,d\nu \\
&= \frac{1}{2}\int_{\partial\Omega} g^2\big|G^h\cdot\mathbf{n}\big|\,d\nu.
\end{aligned}
\tag{3.6}
$$

By the inverse inequality and assumption on σ we get

$$
\begin{aligned}
h\sigma\big|\big(\Delta_v g, g_t + G(f^h)\nabla g\big)_{Q_T}\big| &\le \frac{1}{2}\big(\sigma\|\nabla_v g\|^2_{Q_T} + h\|g_t + G(f^h)\nabla g\|^2_{Q_T}\big) \\
&\le \frac{1}{2}|||g|||^2.
\end{aligned}
\tag{3.7}
$$

Now the proof follows from (3.4)–(3.7). $\square$

Lemma 3.2. *For any constant $C_1 > 0$ we have for any $g \in \mathcal{H}_0$,*

$$
\|g\|^2_{Q_T} \le \Bigg[\frac{1}{C_1}\|g_t + G(f^h)\nabla g\|^2_{Q_T} + \sum_{m=1}^{M} |g_-|^2_m + \int_{\partial\Omega\times I} g^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds\Bigg]\,he^{C_1 h}.
$$

The proof is the same as that of Lemma 3.2 in [3].

3.2. **Error estimates.** Let $\tilde{f}^h \in V_h$ be an interpolant of f with the interpolation error denoted by $\eta = f - \tilde{f}^h$ and set $\xi = f^h - \tilde{f}^h$, so we have

$$
e = f - f^h = \eta - \xi.
$$

The objective in the error estimates is to dominate $|||\xi|||$ by the known interpolation estimates for $|||\eta|||$. Our main result in this section is as follows:

Theorem 3.1. *Assume that $f^h \in V_h$ and $f \in H^{k+1}(Q_T)$, with $k \ge 1$, are the solutions of* (3.3) *and* (2.18)*, respectively, such that*

$$
\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla\eta\|_\infty \le C.
\tag{3.8}
$$

Then there exists a constant C such that

$$
|||f - f^h||| \le Ch^{k+\frac{1}{2}}\|f\|_{k+1,Q_T}.
$$

In the proof of Theorem 3.1 we shall use the following two results estimating the forms B and K.

Lemma 3.3. *Under the assumptions of Theorem 3.1 and with $\tilde{f}^h$, ξ and η defined as above we have that*

$$\begin{aligned}
&\big|B\big(G(f);f,\xi\big)-B\big(G(f^h);\tilde{f}^h,\xi\big)\big| \leq \\
&\qquad \leq \frac{1}{8}|||\xi|||^2 + C\Big[\int_{\partial\Omega\times I}\eta^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds + h^{-1}\|\eta\|^2_{Q_T} + \\
&\qquad\quad + \sum_{m=1}^{M}|\eta_-|^2_m + h(\|\eta_t\|_{Q_T}+\|\nabla\eta\|_{Q_T})^2\Big] + \\
&\qquad\quad + C(\|\xi\|_{Q_T}+\|\eta\|_{Q_T})\|\xi\|_{Q_T} + Ch(\|\xi\|_{Q_T}+\|\eta\|_{Q_T})^2.
\end{aligned}$$

Proof. Using the definition of η we may write

$$\begin{aligned}
B\big(G(f);f,\xi\big) - B\big(G(f^h);\tilde{f}^h,\xi\big) &= \\
&= B\big(G(f^h);\eta,\xi\big) + B\big(G(f);f,\xi\big) - B\big(G(f^h);f,\xi\big) \\
&:= T_1 + T_2 - T_3.
\end{aligned}$$

Now we estimate the terms T_1 and $T_2 - T_3$, separately. For the term T_1 we use the inverse inequality and assumption on σ to obtain

$$\sigma\big|\big(\nabla_v\eta,\nabla_v\xi\big)_{Q_T}\big| \leq \sigma\|\nabla_v\eta\|_{Q_T}\|\nabla_v\xi\|_{Q_T} \leq Ch^{-1}\|\eta\|^2_{Q_T} + \frac{\sigma}{8}\|\nabla_v\xi\|^2_{Q_T} \tag{3.9}$$

and

$$\begin{aligned}
&h\sigma\big|\big(\Delta_v\eta,\xi_t+G(f^h)\nabla\xi\big)_{Q_T}\big| \leq h\sigma\|\Delta_v\eta\|_{Q_T}\|\xi_t+G(f^h)\nabla\xi\|_{Q_T} \leq \\
&\quad \leq C\|\eta\|_{Q_T}\|\xi_t+G(f^h)\nabla\xi\|_{Q_T} \leq Ch^{-1}\|\eta\|^2_{Q_T} + \frac{h}{8}\|\xi_t+G(f^h)\nabla\xi\|^2_{Q_T}.
\end{aligned} \tag{3.10}$$

Then integrating by parts, using (3.2), a similar argument as in the proof of Lemma 3.1 and the fact that Ω_v^h is bounded with zero boundary condition we get

$$\begin{aligned}
&\big(\eta_t + G(f^h)\nabla\eta, \xi + h(\xi_t + G(f^h)\nabla\xi)\big)_{Q_T} + \sum_{m=1}^{M-1}\langle[\eta],\xi\rangle_m \\
&\quad + \langle\eta_+,\xi_+\rangle_0 - \langle\eta_+,\xi_+\rangle_{\Gamma_I^-} = \\
&= -\big(\eta,\xi_t+G(f^h)\nabla\xi\big)_{Q_T} + \langle\eta_-,\xi_-\rangle_M - \sum_{m=1}^{M-1}\langle\eta_-,[\xi]\rangle_m + \\
&\quad + \int_{\partial\Omega\times I}\eta\xi\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds + h\big(\eta_t+G(f^h)\nabla\eta,\xi_t+G(f^h)\nabla\xi\big)_{Q_T}
\end{aligned}$$

which together with (3.9) and (3.10) gives

$$\begin{aligned}
|T_1| \leq \frac{1}{8}|||\xi|||^2 + C\Big[&\int_{\partial\Omega\times I}\eta^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds + h^{-1}\|\eta\|^2_{Q_T} + \\
&+ \sum_{m=1}^{M}|\eta_-|^2_m + h\|\eta_t+G(f^h)\nabla\eta\|^2_{Q_T}\Big].
\end{aligned} \tag{3.11}$$

To bound the last term on the right hand side of (3.11) we use some basic properties of the solution of the Poisson equation together with the definition of G (see [3] for more detailed description) and derive the estimate

$$\|G(f^h) - G(f)\|_{\Omega_T} \le C\|f - f^h\|_{Q_T} \le C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}),$$

which gives

$$\begin{aligned} \|\eta_t + G(f^h)\nabla\eta\|_{Q_T} \le \|\eta_t\|_{Q_T} + \|G(f)\|_\infty \|\nabla\eta\|_{Q_T} + \\ + C\|\nabla\eta\|_\infty(\|\xi\|_{Q_T} + \|\eta\|_{Q_T}). \end{aligned} \tag{3.12}$$

To estimate the term $T_2 - T_3$, we follow a similar argument as in [3] and get

$$\begin{aligned} |T_2 - T_3| \le C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})\|\nabla f\|_\infty \|\xi\|_{Q_T} + \\ + Ch(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})^2\|\nabla f\|_\infty^2 + \frac{1}{8}h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2. \end{aligned} \tag{3.13}$$

Now combining the estimates (3.11)–(3.13), using assumptions of Theorem 3.1 and hiding the term $\frac{1}{8}h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2$ in the triple norm the proof is complete. □

Lemma 3.4. *Under the assumptions of Theorem 3.1 we have*

$$\big|K(f^h, \xi) - K(f, \xi)\big| \le \frac{1}{8}|||\xi|||^2 + C\|\xi\|_{Q_T}^2 + Ch^{-1}\|\eta\|_{Q_T}^2.$$

Proof. Using the definition of ξ and η, we have the identity

$$K(f^h, \xi) - K(f, \xi) = K(\xi, \xi) - K(\eta, \xi) := K_1 - K_2.$$

Below we bound the terms K_1 and K_2, separately. For the first term using the vanishing boundary condition on $\partial\Omega_v^h$ we have

$$\begin{aligned} |K_1| &= \big|\big(\nabla_v(\beta v\xi), \xi + h(\xi_t + G(f^h)\nabla\xi)\big)_{Q_T}\big| = \\ &= \big|\big(d\xi + v\nabla_v\xi, \xi + h(\xi_t + G(f^h)\nabla\xi)\big)_{Q_T}\big| = \\ &= \big|\beta\frac{d}{2}\|\xi\|_{Q_T}^2 + \beta dh\big(\xi, \xi_t + G(f^h)\nabla\xi\big)_{Q_T} + \\ &\quad + \beta h\big(v\nabla_v\xi, \xi_t + G(f^h)\nabla\xi\big)_{Q_T}\big| \le \\ &\le C\beta\|\xi\|_{Q_T}^2 + C\beta h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2 + C_v\beta h\|\nabla_v\xi\|_{Q_T}^2. \end{aligned} \tag{3.14}$$

The term K_2 is estimated using the integration by parts, inverse inequality and boundedness of Ω_v^h, according to

$$\begin{aligned} |K_2| &= \big|\big(\nabla_v(\beta v\eta), \xi + h(\xi_t + G(f^h)\nabla\xi)\big)_{Q_T}\big| = \\ &= \beta\big|\big(\nabla_v(v\eta), \xi\big)_{Q_T} + h\big(\nabla_v(v\eta), \xi_t + G(f^h)\nabla\xi\big)_{Q_T}\big| \le \\ &\le \beta\big|\big(v\eta, \nabla_v\xi\big)_{Q_T}\big| + \beta h(\|\nabla_v(v\eta)\|_{Q_T}^2 + \|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2) \le \\ &\le \beta((h^{-\frac{1}{2}}\|v\eta\|_{Q_T})(h^{\frac{1}{2}}\|\nabla_v\xi\|_{Q_T}) + \\ &\quad + C_v h^{-1}\|\eta\|_{Q_T}^2 + h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2) \le \\ &\le \beta(h\|\nabla_v\xi\|_{Q_T}^2 + C_v h^{-1}\|\eta\|_{Q_T}^2 + h\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2). \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), recalling the assumption on β and hiding the terms of the form $Ch\|\xi_t + G(f^h)\nabla\xi\|_{Q_T}^2$ and $Ch\|\nabla_v\xi\|_{Q_T}^2$ in $|||\xi|||^2$ the proof is complete. □

Now using Lemmas 3.3 and 3.4 the proof of Theorem 3.1 is straightforward.

Proof of Theorem 3.1. The exact solution f satisfies

$$B\big(G(f);f,g\big)-K\big(f,g\big)=L(g)\qquad \forall g\in V_h,$$

so that by Lemma 3.1 and some algebraic labour we get

$$\begin{aligned}\frac{1}{2}|||g|||^2 &\leq B\big(G(f^h);f^h-\tilde{f}^h,\xi\big)=\\ &=L(\xi)+K\big(f^h,\xi\big)-B\big(G(f^h);\tilde{f}^h,\xi\big)=\\ &=B\big(G(f);f,\xi\big)-B\big(G(f^h);\tilde{f}^h,\xi\big)+K\big(f^h,\xi\big)-K\big(f,\xi\big)\\ &:=\Delta B+\Delta K.\end{aligned}\tag{3.16}$$

Now we use Lemmas 3.3 and 3.4 to bound the terms ΔB and ΔK, respectively. Further estimating $\|\xi\|^2_{Q_T}$ and $\|\eta\|^2_{Q_T}$ by Lemma 3.2 with sufficiently large C_1, and also using (3.16) we obtain

$$\begin{aligned}|||\xi|||^2 \leq C\bigg[\int_{\partial\Omega\times I}\eta^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds+h^{-1}\|\eta\|^2_{Q_T}+\\ +\sum_{m=1}^{M}|\eta_-|^2_m+h\|\eta\|^2_{1,Q_T}+\sum_{m=1}^{M}|\xi_-|^2_m h\bigg].\end{aligned}\tag{3.17}$$

Finally, by a Grönwalls type estimate, proceeding as in [3] the proof is complete. □

4. The discontinuous Galerkin method

4.1. **Stability.** In this section we use trial functions which are polynomials of degree $k\geq 1$ on each element K of the triangulation and may be discontinuous across inter–element boundaries in time, space and velocity variables.

To define a finite element method based on discontinuous trial functions we introduce the following notation: if $\zeta=(\zeta_1,\zeta_2,\dots,\zeta_{2d})$, $d=1,2,3$, is a given smooth vector field on Ω we define for $K\in\mathcal{C}_h$ the inflow (outflow) boundary with respect to ζ as

$$\partial K_{-(+)}(\zeta)=\Big\{(x,v,t)\in\partial K:\mathbf{n}_t(x,v,t)+\mathbf{n}(x,v,t)\cdot\zeta(x,v,t)<0(>0)\Big\},\tag{4.1}$$

where $(\mathbf{n},\mathbf{n}_t):=(\mathbf{n}_x,\mathbf{n}_v,\mathbf{n}_t)$ denotes the outward unit normal to $\partial K\subset Q_T$. Further, for $k\geq 0$ we define the function spaces

$$\begin{aligned}W_h&=\big\{g\in L_2(Q_T):g|_K\in P_k(K)\quad\forall K\in\mathcal{C}_h\big\},\\ \mathbf{W}_h^d&=\big\{\mathbf{w}\in[L_2(Q_T)]^d:\mathbf{w}|_K\in[P_k(K)]^d\quad\forall K\in\mathcal{C}_h\big\},\end{aligned}$$

where $d=1,2,3$ is the dimension of the velocity domain.

To derive a variational formulation, for the diffusive part of (1.1), based on discontinuous trial functions we introduce the operator $R\colon W_h\to\mathbf{W}_h^d$ defined in [11], and [12]. More precisely, given $g\in W_h$ we define R by the following relation

$$\big(R(g),\mathbf{w}\big)_{Q_T}=-\sum_{\tau_x\times I_m}\int_{\tau_x\times I_m}\sum_{e\in\mathcal{E}_v}\int_e[\![g]\!]\mathbf{n}_v\cdot(\mathbf{w})^0\,d\nu\qquad\forall\,\mathbf{w}\in\mathbf{W}_h^d,$$

where we denote by $\mathcal{E}_v$ the set of all interior edges of the triangulation T_h^v of the discrete velocity domain Ω_v^h. Moreover for an appropriately chosen function χ, we define

$$(\chi)^0 = \frac{\chi + \chi^{ext}}{2},$$
$$[\![\chi]\!] = \chi - \chi^{ext},$$

where χ^{ext} denotes the value of χ in the element τ_v^{ext} having $e \in \mathcal{E}_v$ as the common edge with τ_v. Hence, roughly speaking, $[\![\chi]\!]$ corresponds to the jump and $(\chi)^0$ is the average value of χ in the velocity variable.

Next for $e \in \mathcal{E}_v$ we define the operator $r_e\colon W_h \to \mathbf{W}_h^d$ to be the restriction of R to the elements sharing the edge $e \in \mathcal{E}_v$, i.e.,

$$\big(r_e(g), \mathbf{w}\big)_{Q_T} = -\sum_{\tau_x \times I_m} \int_{\tau_x \times I_m} \int_e [\![g]\!] \mathbf{n}_v \cdot (\mathbf{w})^0 \, d\nu, \qquad \forall\, \mathbf{w} \in \mathbf{W}_h^d.$$

One can easily verify that

$$\sum_{e \subset \partial\tau_v \cap \mathcal{E}_v} r_e = R \quad \text{on } \tau_v, \tag{4.2}$$

for any element τ_v of the triangulation of Ω_v^h. As a consequence of this we have the following estimate

$$\|R(g)\|_K^2 \le \gamma \sum_{e \subset \partial\tau_v \cap \mathcal{E}_v} \|r_e(g)\|_K^2, \tag{4.3}$$

where τ_v corresponds to the element K and $\gamma > 0$ is a constant depending on d.

Now, since the support of each r_e is the union of elements sharing the edge e, we can evidently deduce

$$\sum_{e \in \mathcal{E}_v} \|r_e(g)\|_{Q_T}^2 = \sum_{K \in \mathcal{C}_h} \sum_{e \subset \partial\tau_v \cap \mathcal{E}_v} \|r_e(g)\|_K^2. \tag{4.4}$$

Using these notations we are now ready to formulate the variational formulation for the discontinuous Galerkin approximation of (2.18) as:

find $f^h \in W_h$ such that for $m = 0, 1, \dots, M-1$ and for all $g \in W_h$

$$\begin{aligned}
&\big(f_t^h + G(f^h)\nabla f^h - \nabla_v(\beta v f^h), g + h(g_t + G(f^h)\nabla g)\big)_{Q_T} + \\
&+ \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(G)} [f^h] g_+ |\mathbf{n}_t + G^h \cdot \mathbf{n}| \, d\nu + \sigma\big(\nabla_v f^h, \nabla_v g\big)_{Q_T} + \\
&+ \sigma\big(\nabla_v f^h, R(g)\big)_{Q_T} + \sigma\big(R(f^h), \nabla_v g\big)_{Q_T} + \lambda\sigma \sum_{e \in \mathcal{E}_v} \big(r_e(f^h), r_e(g)\big)_{Q_T} + \\
&- h\sigma\big(\Delta_v f^h, g_t + G(f^h)\nabla g\big)_{Q_T} = 0,
\end{aligned} \tag{4.5}$$

where $[u] = u_+ - u_-$, with $u_\pm = \lim_{s \to 0\pm} u((x,v) + G(f^h)s, t+s)$, $\lambda > 0$ is a given constant, $f_-^h(x,v,0) = f_0(x,v)$, and in $\partial K_-(G)$; $G := G(f^h)$.

To proceed we define the discontinuous Galerkin trilinear form B_{DG} by

$$\begin{aligned}B_{DG}\big(G(\tilde f);f,g\big) =& \big(f_t+G(\tilde f)\nabla f, g+h(g_t+G(f^h)\nabla g)\big)_{Q_T}+\\&+\sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)'}[f]g_+|n_t+G^h\cdot\mathbf{n}|\,d\nu+\\&+\langle f_+,g_+\rangle_0-h\sigma\big(\Delta_v f,g_t+G(f^h)\nabla g\big)_{Q_T}+\\&+\sigma\big(\nabla_v f,\nabla_v g\big)_{Q_T}+\lambda\sigma\sum_{e\in\mathcal{E}_v}\big(r_e(f),r_e(g)\big)_{Q_T}+\\&+\sigma\big(\nabla_v f,R(g)\big)_{Q_T}+\sigma\big(R(f),\nabla_v g\big)_{Q_T}\end{aligned}$$

and the bilinear form K as in the streamline diffusion method, i.e.,

$$K\big(f,g\big)=\big(\nabla_v(\beta v f),g+h(g_t+G(f^h)\nabla g)\big)_{Q_T}.$$

Note that again both B_{DG} and K depend implicitly on f^h (hence also on h) through the term $G(f^h)$. Moreover we define the linear form L as before

$$L(g)=\langle f_0,g_+\rangle_0.$$

Now we can formulate the problem (4.5) in the following concise form:

find $f^h\in W_h$ *such that*

$$B_{DG}\big(G(f^h);f^h,g\big)-K\big(f^h,g\big)=L(g)\qquad\forall g\in W_h. \tag{4.6}$$

We shall refer to (4.5) or (4.6) as the *DG-scheme.*

We derive our stability estimate and prove convergence rates for the DG-scheme (4.6) in the triple

$$\begin{aligned}|||g|||^2=\frac{1}{2}\Big[&2\sigma\|\nabla_v g\|^2_{Q_T}+2\sigma\sum_{e\in\mathcal{E}_v}\|r_e(g)\|^2_{Q_T}+2h\|g_t+G(f^h)\nabla g\|^2_{Q_T}+|g|^2_M+\\&+|g|_0^2+\sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)'}[g]^2|\mathbf{n}_t+G^h\cdot\mathbf{n}|\,d\nu+\int_{\partial\Omega_+\times I}g^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds\Big],\end{aligned}$$

where $\partial K_-(G)'=\partial K_-(G)\setminus\Omega\times\{0\}$.

Lemma 4.1. *There exists a constant* $\alpha>0$ *independent of* h *such that*

$$\forall g\in W_h\qquad B_{DG}\big(G(f^h);g,g\big)\geq\alpha|||g|||^2.$$

Proof. Using the definition of B_{DG} and (4.4) we have that

$$\begin{aligned} B_{DG}\big(G(f^h);g,g\big) =& |g|_0^2 + \sum_{K\in\mathcal{C}_h}\Big[\big(g_t,g\big)_K + \big(G(f^h)\nabla g,g\big)_K + \\ &+ \int_{\partial K_-(G)'}[g]g_+|\mathbf{n}_t + G^h\cdot\mathbf{n}|\,d\nu + \\ &+ h\|g_t + G(f^h)\nabla g\|_K^2 - h\sigma\big(\Delta_v g, g_t + G(f^h)\nabla g\big)_K + \\ &+ \sigma\|\nabla_v g\|_K^2 + 2\sigma\big(\nabla_v g, R(g)\big)_K + \lambda\sigma\sum_{e\subset\partial\tau_v\cap\mathcal{E}_v}\|r_e(g)\|_K^2\Big] \\ :=& \sum_{i=1}^{9} T_i. \end{aligned}$$

Now we estimate the terms $T_1,\dots,T_9$, separately. Integrating by parts we get

$$\begin{aligned} T_1+T_2+T_3+T_4 =& \frac{1}{2}\Big[|g|_M^2 + |g|_0^2 + \\ &+ \sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)'}[g]^2|\mathbf{n}_t + G^h\cdot\mathbf{n}|\,d\nu + \int_{\partial\Omega_+\times I} g^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\Big]. \end{aligned} \tag{4.7}$$

Using (4.2) and (4.3) we deduce for some $\varepsilon>0$, that

$$\begin{aligned} T_7+T_8+T_9 \geq& \\ \geq& \sigma\sum_{K\in\mathcal{C}_h}\Big[(1-\varepsilon)\|\nabla_v g\|_K^2 - \frac{1}{\varepsilon}\|R(g)\|_K^2 + \lambda\sum_{e\subset\partial\tau_v\cap\mathcal{E}_v}\|r_e(g)\|_K^2\Big] \geq \\ \geq& \sigma\sum_{K\in\mathcal{C}_h}\Big[(1-\varepsilon)\|\nabla_v g\|_K^2 + \big(\lambda - \frac{\gamma}{\varepsilon}\big)\sum_{e\subset\partial\tau_v\cap\mathcal{E}_v}\|r_e(g)\|_K^2\Big]. \end{aligned} \tag{4.8}$$

As for the term T_6 we use an estimate similar to (3.7) to obtain

$$h\sigma\big|\big(\Delta_v g, g_t + G(f^h)\nabla g\big)_{Q_T}\big| \leq \delta|||g|||^2, \tag{4.9}$$

where $0<\delta<1-\varepsilon$, and all the constants depending on $\sigma;(C_\sigma)$, as well as σ itself are assumed to be sufficiently small. Finally combining (4.7)–(4.9), including the term T_5, and taking $\alpha=\min(1-\varepsilon-\delta,\lambda-\frac{\gamma}{\varepsilon})$, (which is positive for $\frac{\gamma}{\lambda}<\varepsilon<1$ and $0<\delta<1-\varepsilon$), the proof is complete. □

Lemma 4.2. *For any constant $C_1>0$ we have for $g\in W_h$*

$$\begin{aligned} \|g\|_{Q_T}^2 \leq& \Big[\frac{1}{C_1}\|g_t + G(f^h)\nabla g\|_{Q_T}^2 + \sum_{m=1}^{M}|g_-|_m^2 + \\ &+ \sum_{K\in\mathcal{C}_h}\int_{\partial K_-(G)''}[g]^2\big|G^h\cdot\mathbf{n}\big|\,d\nu + \int_{\partial\Omega_+\times I} g^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds\Big]he^{C_1 h}, \end{aligned}$$

where

$$\partial K_-(G)'' = \{(x,v,t)\in\partial K_-(G') : \mathbf{n}_t(x,v,t)=0\}.$$

The proof is similar to that of Lemma 4.2 in [3], and therefore is omitted.

4.2. **Error estimates.** We use the same notation as in the SD–method with $\tilde f^h \in H_0^1(Q_T)$ denoting the interpolant of the exact solution f.

The main result of this section is the following error estimate:

Theorem 4.1. *Assume $f^h \in W_h$ and $f \in H^{k+1}(Q_T)\cap W^{k+1,\infty}(Q_T)$, with $k \geq 1$, are the solutions of* (4.6) *and* (2.18)*, respectively, such that*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla\eta\|_\infty \leq C. \tag{4.10}$$

Then there exists a constant C such that

$$|||f - f^h||| \leq Ch^{k+\frac{1}{2}}.$$

To prove this convergence rate we shall need the following results:

Lemma 4.3. *Let $u \in L^2\big(\Omega_x \times I, H^1(\Omega_v)\big)$ with $\Delta_v u \in L^2(Q_T)$, and let $w \in W_h$. Then*

$$\sum_{K\in\mathcal{C}_h}\int_{\tau_x\times I_m}\int_{\partial\tau_v} w\frac{\partial u}{\partial \mathbf{n}_v} = \sum_{\tau_x\times I_m}\int_{\tau_x\times I_m}\sum_{e\in\mathcal{E}_v}\int_e [\![w]\!]\mathbf{n}_v\cdot(\nabla_v u)^0.$$

Proof. The regularity of u implies that $\nabla_v u\cdot\mathbf{n}_v$ is continuous across the inter–element boundaries. Hence we have

$$\begin{aligned}\sum_{K\in\mathcal{C}_h}\int_{\tau_x\times I_m}\int_{\partial\tau_v} w\frac{\partial u}{\partial \mathbf{n}_v} &= \sum_{\tau_x\times I_m}\int_{\tau_x\times I_m}\sum_{e\in\mathcal{E}_v}\int_e \Big[w^+\mathbf{n}_v^+\cdot\nabla_v u^+ + w^-\mathbf{n}_v^-\cdot\nabla_v u^-\Big] = \\ &= \sum_{\tau_x\times I_m}\int_{\tau_x\times I_m}\sum_{e\in\mathcal{E}_v}\int_e \Big[w^+\mathbf{n}_v^+\cdot\frac{\nabla_v u^+ + \nabla_v u^-}{2} + w^-\mathbf{n}_v^-\cdot\frac{\nabla_v u^+ + \nabla_v u^-}{2}\Big] = \\ &= \sum_{\tau_x\times I_m}\int_{\tau_x\times I_m}\sum_{e\in\mathcal{E}_v}\int_e [\![w]\!]\mathbf{n}_v\cdot(\nabla_v u)^0,\end{aligned}$$

and the proof is completes. ☐

Lemma 4.4. *Under the assumptions of Theorem 4.1 we have that*

$$\begin{aligned}\big|B_{DG}\big(G(f);f,\xi\big) - B_{DG}\big(G(f^h);\tilde f^h,\xi\big)\big| &\leq \tilde C|||\xi|||^2 + Ch^{2k+1} + \\ &+ C\Big[\int_{\partial\Omega_+\times I}\eta^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds + h^{-1}\|\eta\|_{Q_T}^2 + \sum_{m=0}^{M}|\eta|_m^2\Big] + \\ &+ C(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})\|\xi\|_{Q_T} + Ch(\|\xi\|_{Q_T} + \|\eta\|_{Q_T})^2,\end{aligned}$$

where the constant $\tilde C < 1$.

Proof. Once again by the definition of the interpolation error η we may write

$$\begin{aligned}B_{DG}\big(G(f);f,\xi\big) - B_{DG}\big(G(f^h);\tilde f^h,\xi\big) &= \\ &= B_{DG}\big(G(f^h);\eta,\xi\big) + B_{DG}\big(G(f);f,\xi\big) - B_{DG}\big(G(f^h);f,\xi\big) \\ &:= T_1 + T_2 - T_3.\end{aligned}$$

To estimate the term $T_2 - T_3$ we proceed as in the proof of Lemma 3.3 (cf. (3.13)). For the term T_1 we have

$$
\begin{aligned}
T_1 = \langle \eta_+, \xi_+\rangle_0 + \sum_{K\in\mathcal{C}_h} \Big[& \big(\eta_t + G(f^h)\nabla\eta, \xi + h(\xi_t + G(f^h)\nabla\xi)\big)_K + \\
& + \int_{\partial K_-(G)'} [\eta]\xi_+ |\mathbf{n}_t + G^h\cdot\mathbf{n}|\, d\nu - h\sigma\big(\Delta_v\eta, \xi_t + G(f^h)\nabla\xi\big)_K + \\
& + \sigma\big(\nabla_v\eta, \nabla_v\xi\big)_K + \lambda\sigma \sum_{e\in\mathcal{E}_v} \big(r_e(\eta), r_e(\xi)\big)_K + \\
& + \sigma\big(R(\eta), \nabla_v\xi\big)_K + \sigma\big(\nabla_v\eta, R(\xi)\big)_K \Big] := \sum_{i=1}^{8} S_i.
\end{aligned} \tag{4.11}
$$

Thus we need to estimate S_i, $1\le i\le 8$. For the term S_1 we have

$$|S_1| \le |\eta_+|_0^2 + |\xi_+|_0^2. \tag{4.12}$$

Integration by parts leads to an estimate for S_2+S_3,

$$
\begin{aligned}
|S_2+S_3| \le & \,|\sum_{K\in\mathcal{C}_h} \int_{\partial K_-(G)'} \eta_-[\xi]|\mathbf{n}_t + G^h\cdot\mathbf{n}|\, d\nu| + \\
& + |\int_{\partial\Omega_+\times I} \eta_-\xi_- \big|G^h\cdot\mathbf{n}\big|\, d\nu\, ds|.
\end{aligned} \tag{4.13}
$$

To bound the first term on the right hand side of (4.13), the crucial part is to estimate a term of the form

$$T = \sum_{K\in\mathcal{C}_h} \int_{\partial K_-(G)''} \eta_-[\xi]|G^h\cdot\mathbf{n}|\, d\nu.$$

To this approach using Cauchy–Schwartz inequality we have for $\delta>0$ that

$$|T| \le \frac{C}{\delta} \sum_{K\in\mathcal{C}_h} \int_{\partial K_-(G)''} |\eta_-|^2 |G^h\cdot\mathbf{n}|\, d\nu + C\delta \sum_{K\in\mathcal{C}_h} \int_{\partial K_-(G)''} [\xi]^2 |G^h\cdot\mathbf{n}|\, d\nu, \tag{4.14}$$

where the last sum can be hidden in $|||\xi|||^2$, and the first sum is estimated below

$$
\begin{aligned}
& \sum_{K\in\mathcal{C}_h} \int_{\partial K_-(G)''} |\eta_-|^2 |G^h\cdot\mathbf{n}|\, d\nu \le \\
& \qquad \le \|\eta\|_\infty^2 \sum_{K\in\mathcal{C}_h} \Big[\int_{\partial K_-(G)''} |G^h\cdot\mathbf{n}|^2\, d\nu + \int_{\partial K_-(G)''} d\nu \Big] \le \\
& \qquad \le C\|\eta\|_\infty^2 \sum_{K\in\mathcal{C}_h} \big[Ch^{-1}\|G(f^h)\|_K^2 + Ch^{2d}\big],
\end{aligned} \tag{4.15}
$$

where $d = 1,2,3$. Further, the interpolation error η satisfies

$$\|\eta\|_\infty \le Ch^{k+1}\|f\|_{k+1,\infty}. \tag{4.16}$$

Hence by (4.14)–(4.16) and by assumptions of the lemma we obtain

$$|T| \le Ch^{2k+1} + \frac{1}{C_1}|||\xi|||^2, \tag{4.17}$$

where C_1 is a sufficiently large constant. So that using once again the Cauchy–Schwartz inequality we obtain for $S_2 + S_3$ the estimate

$$
\begin{aligned}
|S_2 + S_3| \le Ch^{2k+1} + \frac{1}{C_1} |||\xi|||^2 + \\
+ C \int_{\partial\Omega_+ \times I} \eta^2 \big|G^h \cdot \mathbf{n}\big| \, d\nu \, ds + C \sum_{m=1}^{M} |\eta|_m^2 .
\end{aligned} \tag{4.18}
$$

The terms S_4 and S_5 are estimated as in Lemma 3.3. Moreover, from the definition of operators R and r_e and from the fact that η is a continuous function we can easily deduce that $S_6 = 0$ and $S_7 = 0$. Thus it remains to estimate the term S_8. To this end we use (4.3), (4.4), the inverse inequality and assumption on σ to obtain

$$
\begin{aligned}
|S_8| &\le \sum_{K \in \mathcal{C}_h} \sigma \|\nabla_v \eta\|_K \|R(\xi)\|_K \le \sum_{K \in \mathcal{C}_h} \left(C\sigma \|\nabla_v \eta\|_K^2 + \frac{\sigma}{C_1} \|R(\xi)\|_K^2 \right) \le \\
&\le Ch^{-1} \|\eta\|_{Q_T}^2 + C_2 \sigma \sum_{e \in \mathcal{E}_v} \|r_e(\xi)\|_{Q_T}^2 ,
\end{aligned} \tag{4.19}
$$

where, as above, C_1 is taken to be large enough. Finally combining the estimates for the terms T_1 and $T_2 - T_3$ we obtain the desired result. □

Now we are ready to prove our error estimate.

Proof of Theorem 4.1. From the definition of B_{DG} and Lemma 4.3 we deduce that the exact solution f satisfies the variational formulation

$$
B_{DG}\big(G(f); f, g\big) - K\big(f, g\big) = L(g) \qquad \forall g \in W_h .
$$

So using Lemma 4.1 and some algebraic labour we get

$$
\begin{aligned}
\alpha |||g|||^2 &\le B_{DG}\big(G(f^h); f^h - \tilde{f}^h, \xi\big) = \\
&= L(\xi) + K\big(f^h, \xi\big) - B_{DG}\big(G(f^h); \tilde{f}^h, \xi\big) = \\
&= B_{DG}\big(G(f); f, \xi\big) - B_{DG}\big(G(f^h); \tilde{f}^h, \xi\big) + K\big(f^h, \xi\big) - K\big(f, \xi\big) \\
&:= \Delta B + \Delta K .
\end{aligned} \tag{4.20}
$$

Here the term ΔK is similar to the one given in the SD–method and therefore is estimated in an analogous way as in the proof of Lemma 3.4. Furthermore a bound for the term ΔB is given by Lemma 4.4. Now we complete the proof combining Lemma 4.2 by a similar argument used in the proof of Theorem 3.1. □

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