EXTENSIONS OF A REALANALYTIC FUNCTIONAL CALCULUS FOR COMMUTING OPERATORS

MATS ANDERSSON

ABSTRACT. Let $a = (a_1, \ldots, a_n)$ be a tuple of commuting operators on a Banach space. We discuss various conditions equivalent to that the holomorphic (Taylor) functional calculus has an extension to the realanalytic functions or various ultradifferentiable classes. Especially we discuss the possible existence of a functional calculus for smooth functions. We also provide a new simple proof that the existence of such a functional calculus implies the so-called $(\beta)_\varepsilon$ property. One of the equivalent conditions is expressed in terms of possible current (or ultracurrent) extension of the resolvent mapping of a over the spectrum. Our main tool that we introduce in this paper is Fourier transforms of forms and currents.

1. INTRODUCTION

Let $a_1, \ldots, a_n$ be an n-tuple of commuting operators on a Banach space X. For any polynomial or entire function $f(z) = f(z_1, \ldots, z_n)$ one can define an operator $f(a)$ on X, simply by replacing each z_j by a_j in the Taylor expansion for $f(z)$. One then gets an algebra homomorphism $\mathcal{O}(\mathbb{C}^n) \to \mathcal{L}(X)$, usually called a functional calculus, where $\mathcal{L}(X)$ is the space of bounded operators on X. In order to find extensions of this functional calculus, one is led to consider the joint spectrum of the n-tuple a; the relevant definition was found by Taylor, [14] 1970, and can be described as follows. Let T_z be the complex tangent space at the point $z \in \mathbb{C}^n$, and let δ_{z-a} denote contraction with the operator-valued vector field

$$2\pi i \sum_j (z_j - a_j) \frac{\partial}{\partial z}\Big|_z.$$

We then have a complex

$$(1.1) \qquad 0 \leftarrow \Lambda^0 T_z \otimes X \xleftarrow{\delta_{z-a}} \Lambda^1 T_z \otimes X \xleftarrow{\delta_{z-a}} \cdots \xleftarrow{\delta_{z-a}} \Lambda^n T_z \otimes X \leftarrow 0,$$

for each $z \in \mathbb{C}^n$, and the Taylor spectrum $\sigma(a)$ of the n-tuple a is, by definition, the set of all $z \in \mathbb{C}^n$ such that (1.1) is not exact. It turns out that the spectrum is a compact nonempty (unless $X = \{0\}$) subset of $\mathbb{C}^n$. The fundamental result, due to Taylor [15], is

Theorem 1.1 (Taylor). *Let a be an n-tuple of commuting operators on the Banach space X. There is a continuous homomorphism*

$$(1.2) \qquad\qquad \mathcal{O}(\sigma(a)) \to \mathcal{L}(X)$$

that extends the functional calculus $\mathcal{O}(\mathbb{C}^n) \to X$. If $f = (f_1, \ldots, f_n)$ is an analytic mapping, $f_j \in \mathcal{O}(\sigma(a))$ and $f(a) = (f_1(a), \ldots, f_n(a))$, then

$$(1.3) \qquad\qquad \sigma(f(a)) = f(\sigma(a)).$$

The equality (1.3) will be referred to as the spectral mapping property. It was proved in [12] that any two extensions of the functional calculus, which fulfill the properties stated in Theorem 1.1 coincide.

Let $\mathcal{A}$ be an algebra of functions that contains $\mathcal{O}(\sigma(a))$. We say that a admits an $\mathcal{A}$ functional calculus Π if (1.2) has a continuous extension to a homomorphism

$$\Pi \colon \mathcal{A} \to \mathcal{L}(X).$$

A natural attempt to obtain such an extension is by means of the resolvent mapping ω_{z-a}, cf., Section 3. If f is holomorphic in a neighborhood of $\sigma(a)$ and has compact support, then

$$f(a) = -\int \bar{\partial} f(z) \wedge \omega_{z-a} x.$$

However, the same formula may have meaning even for an f that is not necessarily holomorphic in a full neighborhood of $\sigma(a)$, provided that $\bar{\partial} f(z)$ has enough decay when approaching $\sigma(a)$ to balance the growth of (some representative of) the resolvent. In one variable this approach was first exploited by Dynkin, [8]; for several commuting operators a similar approach is used by Droste, [7], and recently by Sandberg, [13]; for the case when $\sigma(a)$ is real, see [4]. Notice that such an approach will always require that $\bar{\partial} f = 0$ on $\sigma(a)$ which is a very strong restriction if $\sigma(a)$ contains some complex structure.

In this paper we will consider algebras $\mathcal{A}$ that contain the algebra $C^\omega(\sigma(a))$ of germs of realanalytic functions on $\sigma(a)$, i.e., (equivalence classes of) functions that are realanalytic in some neighborhood of $\sigma(a)$. The main results are contained in Sections 5,6, and 7. Here we discuss various equivalent conditions for the existence of an extension to various classes of ultradifferentiable functions. Especially we study the case of a functional calculus for smooth functions. We also give a new simple proof for that the existence of a smooth functional calculus implies that the tuple a has the property $(\beta)_\varepsilon$.

However, we begin with discussing some equivalent formulations of the existence of a realanalytic extension in Section 2 and point out that in general such an extence will not be unique.

To be able to relate to the holomorphic calculus we briefly recall its definition in terms of the resolvent mapping. One of our conditions equivalent to the existence of a ultradifferentiable extension of the realanalytic functional calculus is expressed in terms of a possible current (or ultracurrent) extension of the resolvent mapping over the spectrum. Our main tool is Fourier transforms of differential forms and currents, and the necessary definitions and results are introduced in Section 4.

Throughout this paper X is a Banach space and e denotes the identity element in $\mathcal{L}(X)$. If a is a tuple of operators, then (a) denotes the closed subalgebra of $\mathcal{L}(X)$ generated by a, and $(a)'$ denotes the commutant of a, i.e., the algebra of all $b \in \mathcal{L}(X)$ that commute with each a_j.

CONTENTS

2. REALANALYTIC FUNCTIONAL CALCULUS

Let a be a commuting n-tuple with spectrum $\sigma(a)$ and let $C^\omega(\sigma(a))$ denote the space of realanalytic functions defined in some neighborhood. For each $\phi \in C^\omega(\sigma(a))$ we have a function $\tilde{\phi}(z, w)$, holomorphic in a neighborhood of $\{(z, \bar{z}); \ z \in \sigma(a)\}$ in $\mathbb{C}^{2n}$, such that $\phi(z) = \tilde{\phi}(z, \bar{z})$. Sometimes it is natural instead to identify $\tilde{\phi}$ with the realanalytic function (that we also denote $\tilde{\phi}(x, y)$) defined in a neighborhood of $\{(x, y) \in \mathbb{R}^{2n}; \ x + iy \in \sigma(a)\}$ such that $\phi(x + iy) = \tilde{\phi}(x, y)$. he topology of $C^\omega(\sigma(a))$ is defined by the seminorms given by taking supremum of $\tilde{\phi}$ over small neighborhoods in $\mathbb{C}^{2n}$ of $\{(z, \bar{z}); \ z \in \sigma(a)\}$.

Assume that a admits a realanalytic functional calculus $\Pi \colon C^\omega(\sigma(a)) \to \mathcal{L}(X)$, and let a_j^* be the images of $\bar{z}_j$. Then (a, a^*) is a commuting $2n$-tuple of operators, as well as $(\operatorname{Re} a, \operatorname{Im} a)$ if $\operatorname{Re} a = (a + a^*)/2$ and $\operatorname{Im} a = (a - a^*)/2i$. We claim that

$$(2.1) \qquad \sigma(a, a^*) = \{(z, \bar{z}) \in \mathbb{C}^{2n}; \ z \in \sigma(a)\}.$$

By the spectral mapping property for the holomorphic functional calculus, applied to the mapping $(z, w) \mapsto ((z + w)/2, (z - w)/2i)$, (2.1) holds if and only if

$$(2.2) \qquad \sigma(\operatorname{Re} a, \operatorname{Im} a) = \{(x, y); \ x + iy \in \sigma(a)\}.$$

If $\tilde{\phi}(z, w)$ is a holomorphic polynomial, then clearly

$$(2.3) \qquad \Pi(\phi) = \tilde{\phi}(a, a^*),$$

where the right hand side denotes the holomorphic functional calculus of (a, a^*). Moreover, (2.3) also holds for each entire function $\tilde{\phi}(z, w)$, since it can be approximated in neighborhoods of $\sigma(a, a^*)$ by polynomials. In particular, if

$$E_\zeta(z) = e^{2\pi i \operatorname{Re} z \cdot \bar{\zeta}} = e^{\pi i(z \cdot \bar{\zeta} + \bar{z} \cdot \zeta)}, \quad \zeta \in \mathbb{C}^n,$$

then

$$(2.4) \qquad \Pi(E_\zeta) = e^{\pi i(a \cdot \bar{\zeta} + a^* \cdot \zeta)}, \quad \zeta \in \mathbb{C}^n,$$

and therefore, by the continuity of Π, we get the estimate

$$(2.5) \qquad \|e^{\pi i(a \cdot \bar{\zeta} + a^* \cdot \zeta)}\| \leq e^{o(|\zeta|)}, \quad |\zeta| \to \infty.$$

We will need the following simple lemma; for a proof see, e.g., [4].

Lemma 2.1. *Suppose that $\alpha_1, \dots, \alpha_N$ are commuting operators. Then $\sigma(\alpha)$ is real if and only if $\|\exp(2\pi i \alpha \cdot \xi)\| \leq C \exp(o(|\xi|))$.*

Notice that $2\pi i(\operatorname{Re} a \cdot \xi + \operatorname{Im} a \cdot \eta) = \pi i(a \cdot \bar{\zeta} + a^* \cdot \zeta)$ if $\zeta = \xi + i\eta$. In view of Lemma 2.1, therefore (2.5) implies that $\sigma(\operatorname{Re} a, \operatorname{Im} a) \subset \mathbb{R}^{2n}$, hence $\sigma(a, a^*) \subset \{w = \bar{z}\}$, and by the spectral mapping property applied to the projection $(z, w) \mapsto z$ we get (2.1). Conversely, (2.1) implies (2.5).

It now follows by approximation that (2.3) holds for all $\tilde{\phi}(z, w)$ that are holomorphic in a neighborhood of $\{(z, \bar{z}); \ z \in \sigma(a)\}$ since this set is polynomially convex. By the holomorphic spectral mapping property,

$$\sigma(\Pi(\phi)) = \sigma(\tilde{\phi}(a, a^*)) = \tilde{\phi}(\sigma(a, a^*)) = \phi(\sigma(a)),$$

and thus the spectral mapping property holds for Π.

If a is a commuting n-tuple and there is a commuting n-tuple $a^* \in (a)'$ such that (2.1) holds, then one can extend the holomorphic functional calculus to an algebra homomorphism $\Pi \colon C^\omega(\sigma(a)) \to \mathcal{L}(X)$ by formula (2.3). Summing up, we have proved

Proposition 2.2. *Assume that a is a commuting n-tuple of operators on X that admits a realanalytic functional calculus $\Pi \colon C^\omega(\sigma(a)) \to \mathcal{L}(X)$, and let $a^* = \Pi(\bar{z})$. Then (2.1) (and (2.2)) and (2.5) hold, and (2.3) holds for all $\tilde{\phi} \in \mathcal{O}(\{(z, \bar{z}); \ z \in \sigma(a)\})$. Moreover, the spectral mapping property $\sigma(\Pi(\phi)) = \phi(\sigma(a))$ holds for all $\phi \in C^\omega(\sigma(a))$.*

Conversely, if there is an n-tuple a^* such that (a, a^*) is commuting, and such that (2.1) or (2.5) hold, then a admits a realanalytic functional calculus defined by (2.3).

In general a possible extension of the holomorphic functional calculus to $C^\omega(\sigma(a))$ is not unique. We say that a tuple q is quasi-nilpotent if $\sigma(q) = \{0\}$. This holds if and only if both q and iq have real spectra, and this in turn holds, in view of Lemma 2.1, if and only if $\| \exp(\pi i q \cdot \zeta)\| \leq \exp o(|\zeta|)$ for all ζ.

Proposition 2.3. *Suppose that $\Pi \colon C^\omega(\sigma(a)) \to \mathcal{L}(X)$ is a $C^\omega(\sigma(a))$ functional calculus and $a^* = \Pi(\bar{z})$. Moreover, assume that $q \in (a, a^*)'$ is a quasi-nilpotent commuting n-tuple. Then there is another $C^\omega(\sigma(a))$ functional calculus Π' such that $a^* + q = \Pi'(\bar{z})$. Conversely, any two $C^\omega(\sigma(a))$ functional calculi Π and Π' such that $\Pi(\bar{z})$ and $\Pi'(\bar{z})$ commute are related in this way for some quasi-nilpotent n-tuple q.*

Proof. If (2.5) holds and q is quasi-nilpotent then also (2.5) holds for $a^* + q$ instead of a^*, and hence $a^* + q$ corresponds to another C^ω functional calculus according to Proposition 2.2. Conversely, if (2.5) holds for both a^* and $a^* + q$ and they are commuting, then $\| \exp(\pi i q \cdot \zeta)\| \leq \exp o(|\zeta|)$ and so q is quasi-nilpotent. □

Notice that thus $\Pi'(f) = \tilde{f}(a, a^* + q)$ whereas $\Pi(f) = \tilde{f}(a, a^*)$.

Example 1. Let a be any nonzero quasi-nilpotent tuple. Then $a^* = a$ and $a^* = 0$ provide two different C^ω extensions. □

Remark 1. Assume that a has real spectrum. Then a admits a natural realanalytic functional calculus corresponding to the choice $a^* = a$ in Proposition 2.2 (notice that then (2.5) holds). There is always a nontrivial extension of this C^ω-functional calculus (depending on the size of $o(|\zeta|)$), see [4]. □

If a admits a C^ω functional calculus, we thus have that

$$(2.6) \qquad\qquad \Pi(\phi) = \tilde{\phi}(\operatorname{Re} a, \operatorname{Im} a),$$

where the right hand side is the natural realanalytic functional calculus ((2.6) is of course equivalent to (2.3)), and hence the question of possible extensions to wider classes of functions is transformed to the question of possible (nonholomorphic) extensions of the natural realanalytic functional calculus of the $2n$-tuple $(\operatorname{Re} a, \operatorname{Im} a)$ with real spectrum. As was noted in Remark 1 above, some nontrivial extension always exists. In subsequent sections we shall consider specific such extensions.

3. THE RESOLVENT MAPPING

In the case of one single operator, the extension of the functional calculus from entire functions can be made by Cauchy's integral formula,

$$(3.1) \quad f(a)x = \int_{\partial D} \omega_{z-a} f x = - \int \bar{\partial}\chi \wedge \omega_{z-a} f x, \quad f \in \mathcal{O}(U), \ x \in X,$$

where $\sigma(a) \subset D \subset\subset U$, and χ is a cutoff function in U which is identically 1 in a neighborhood of $\sigma(a)$, and

$$\omega_{z-a}x = (2\pi i)^{-1}(z-a)^{-1}xdz.$$

In the multidimensional case, for $U \supset \sigma(a)$, ω_{z-a} is a mapping

$$\omega_{z-a} \colon \mathcal{O}(U,X) \to H_{\bar{\partial}}^{n,n-1}(U \setminus \sigma(a), X)$$

that we call the resolvent mapping. If $f \in \mathcal{O}(U)$ and $x \in X$, then $fx \in \mathcal{O}(U,X)$ and $f(a)x$ is given by (3.1) just as in the one-dimensional case. The definition of the resolvent mapping is in short as follows; for more details, see [1] and [2]. Since $\delta_{z-a}\bar{\partial} = -\bar{\partial}\delta_{z-a}$, $\mathcal{L}^{\ell,k}(\mathcal{E},X,V) = \mathcal{E}_{-\ell,k}(V,X)$ is a double complex with the coboundary operators δ_{z-a} and $\bar{\partial}$, and it gives rise to the total complex

$$(3.2) \qquad \xrightarrow{\delta_{z-a}-\bar{\partial}} \mathcal{L}^{m-1}(\mathcal{E},V,X) \xrightarrow{\delta_{z-a}-\bar{\partial}} \mathcal{L}^{m}(\mathcal{E},V,X) \xrightarrow{\delta_{z-a}-\bar{\partial}},$$

where

$$\mathcal{L}^{m}(\mathcal{E},V,X) = \bigoplus_{\ell+k=m} \mathcal{L}^{\ell,k}(V,X) = \bigoplus_{\ell} \mathcal{E}_{\ell,\ell+m}(V,X).$$

It is well known that $\mathcal{L}^{\ell,k}(\mathcal{E},X,V)$ has exact rows if $V \subset \mathbb{C}^n \setminus \sigma(a)$ and therefore (3.2) is exact for such V. If now $U \supset \sigma(a)$ and $f \in \mathcal{O}(U,X)$, then f defines a closed element in $\mathcal{L}^{0}(\mathcal{E}, U \setminus \sigma(a), X)$ and hence there is a solution $u \in \mathcal{L}^{-1}(\mathcal{E}, U \setminus \sigma(a), X)$ to $(\delta_{z-a} - \bar{\partial})u = f$. If u_n denotes the component of u of bidegree $(n, n-1)$ it follows that $\bar{\partial}u_n = 0$ and $\omega_{z-a}f$ is defined as the Dolbeault cohomology class of u_n. In particular, if there is a $v \in \mathcal{L}^{-1}(E, U \setminus \sigma(a), (a)')$ that solves $(\delta_{z-a} - \bar{\partial})v = e$ in $U \setminus \sigma(a)$, then $\omega_{z-a}f$ is defined by the form $v_n f$. However, in general it is not possible to find such a solution v close to $\sigma(a)$, but for large z one can take. e.g.,

$$v = \sum_{\ell=1}^{n} s \wedge (\bar{\partial}s)^{\ell-1},$$

where $\delta_{z-a}s = 1$; a possible choice is

$$s = (2\pi i)^{-1}(|z|^2 - \bar{z}\cdot a)^{-1}\partial|z|^2.$$

All the spaces $\mathcal{O}(V)$, $\mathcal{D}(V)$, $\mathcal{E}(V)$, $\mathcal{S} = \mathcal{S}(\mathbb{C}^n)$, as well as their duals, are nuclear. This implies that one can form topological tensor products like $\mathcal{E}(U,X) = \mathcal{E}(U)\hat{\otimes}X$ in an unambigous way, and that furthermore these tensor product preserve exactness, see, e.g., [9]. For instance, since the Dolbeault complex

$$0 \to \mathcal{O}(V) \to \mathcal{E}_{0,0}(V) \xrightarrow{\bar{\partial}} \mathcal{E}_{0,1}(V) \xrightarrow{\bar{\partial}}$$

is exact if V is pseudoconvex, it follows that the corresponding X-valued Dolbeault complex

$$0 \to \mathcal{O}(V,X) \to \mathcal{E}_{0,0}(V,X) \xrightarrow{\bar{\partial}} \mathcal{E}_{0,1}(V,X) \xrightarrow{\bar{\partial}}$$

is exact as well. We will also consider spaces of X-valued currents, e.g., $\mathcal{D}'_{p,q}(V, X) = \mathcal{D}'_{p,q}(V) \hat{\otimes} X$. For further reference we include

Lemma 3.1. *Suppose that* $V \subset \mathbb{C}^n \setminus \sigma(a)$. *If* $f \in \mathcal{D}'_{n,q}(V, X)$ *and* $\delta_{z-a} f = 0$ *it follows that* $f \equiv 0$ *in* V.

Proof. Let $\psi \in \mathcal{D}_{0,n-q}(V)$; then there is a $\phi \in \mathcal{D}_{1,n-q}(V)$ such that $\delta_{z-a}\phi = \psi$. Thus $f.\psi = f.\delta_{z-a}\phi = \pm \delta_{z-a} f.\phi = 0$ so that $f \equiv 0$. $\qquad\square$

4. Fourier transforms of forms and currents

Our main tool is the Fourier transformation of vectorvalued currents. To begin with we introduce Fourier transforms of forms. Roughly speaking the Fourier transform of a (p, q)-form $f = f_{IJ}(z)dz^J \wedge d\bar{z}^J$ will be $\pm \hat{f}_{IJ}(\zeta)d\zeta^{J'} \wedge d\bar{\zeta}^{I'}$, where $\hat{f}_{IJ}$ is the usual Fourier transform of the coefficient f_{IJ} and I' and J' denote complementary indices. However, to make the formalism run smoothly we give the following definition. Let

$$\omega = \omega(z, \zeta) = 2\pi i \mathrm{Re}\, z \cdot \bar{\zeta} + \mathrm{Re}\, dz \cdot d\bar{\zeta} =$$
$$= \pi i(z \cdot \bar{\zeta} + \bar{z} \cdot \zeta) + (dz \cdot d\bar{\zeta} + d\bar{z} \cdot d\zeta)/2.$$

Since ω has even degree, $\exp(-\omega)$ is welldefined, and for a form $f(z)$ with coefficients in $\mathcal{S}(\mathbb{C}^n)$ we let

$$(4.1) \qquad \mathcal{F}f(\zeta) = \int_z e^{-\omega(z,\zeta)} \wedge f(z).$$

Since we have an even real dimension it is immaterial whether we put all differentials of $d\zeta, d\bar{\zeta}$ to the right or to the left before performing the integration, and thus $\mathcal{F}f(\zeta)$ is a welldefined form with coefficients in $\mathcal{S}$. If $f \in \mathcal{S}_{p,q}$, then

$$\mathcal{F}f(\zeta) = \int_z e^{-2\pi i \mathrm{Re}\, z \cdot \bar{\zeta}} \phi(z) \wedge (-\mathrm{Re}\, dz \cdot d\bar{\zeta})^{2n-p-q}/(2n - p - q)!$$

and hence an $(n - q, n - p)$-form. In what follows we let $\hat{f}$ mean the same as $\mathcal{F}f$.

Proposition 4.1. *We have the following inversion formula*

$$(4.2) \qquad f(z) = (-1)^n \int_\zeta e^{\omega(z,\zeta)} \wedge \hat{f}(\zeta).$$

Of course it can be deduced from the inversion formula for the usual Fourier transform, but we prefer to repeat one of the wellknown argument in the form formalism.

Proof. Take $\psi \in \mathcal{S}_{0,0}$ such that $\psi(0) = 1$. Then

$$\int_\zeta \hat{f}(\zeta) \wedge e^{\omega(w,\zeta)} = \lim_{\epsilon \to 0} \int_\zeta \psi(\epsilon\zeta)\hat{f}(z) \wedge e^{\omega(w,\zeta)} =$$

$$= \lim_{\epsilon \to 0} \int_\zeta \int_z \psi(\epsilon\zeta)\hat{f}(z) \wedge e^{\omega(w-z,\zeta)}.$$

Making the change of variables $\zeta \mapsto \zeta/\epsilon$, $z \mapsto z + w$, the right hand double integrals becomes (the mapping is orientation preserving, so no minus sign appears)

$$\int_\zeta \int_z \psi(\zeta)\hat{f}(z+w) \wedge e^{-\omega(z,\zeta/\epsilon)}$$

and since $\omega(z, \zeta/\epsilon) = \omega(z/\epsilon, \zeta)$ another change of variables $\zeta \mapsto \epsilon\zeta$ gives

$$\int_\zeta \int_z \psi(\zeta)\hat{f}(w+\epsilon z) \wedge e^{-\omega(z,\zeta)}$$

which tends to $c_n f(w)$, where

$$c_n = \int_z \int_\zeta \psi(\zeta)e^{-\omega(z,\zeta)}.$$

Taking for instance $\psi(\zeta) = \exp(-|\zeta|^2)$, a simple computation reveals that $c_n = (-1)^n$.   □

Let δ_{z-a} denote contraction with the vector field $2\pi i(z - a) \cdot \frac{\partial}{\partial z}$ for $a \in \mathbb{C}^n$. Since $\omega = (1/2)(\delta_z - \delta_\zeta - \bar{\partial})(\bar{\zeta} \cdot dz - \bar{z} \cdot d\zeta)$ it follows that

$$(4.3) \qquad (\delta_{z-a} - \delta_\zeta - \bar{\partial})\omega(z,\zeta) = -\pi i a \cdot d\bar{\zeta}.$$

Proposition 4.2. *If* $a \in \mathbb{C}^n$, *then*

$$(4.4) \qquad \mathcal{F}((\delta_{z-a} - \bar{\partial}_z)f(z)) = -(\delta_\zeta + \bar{\partial}_\zeta + A)\mathcal{F}f,$$

where

$$A\phi = \pi i a \cdot d\zeta \wedge \phi$$

for forms $\phi(\zeta)$.

Identifying bidegrees we also get that

$$\mathcal{F}(\delta_{z-a}f) = -(\bar{\partial}_\zeta + A)\mathcal{F}f \quad \text{and} \quad \mathcal{F}(\bar{\partial}f) = \delta_\zeta \mathcal{F}f.$$

Proof. By (4.3) we have that

$$(\delta_{z-a} - \delta_\zeta - \bar{\partial})(e^{-\omega} \wedge f(z)) = A(e^{-\omega} \wedge f) + e^{-\omega} \wedge (\delta_{z-a} - \delta_\zeta - \bar{\partial})f =$$

$$= A(e^{-\omega} \wedge f) + e^{-\omega} \wedge (\delta_{z-a} - \bar{\partial}_z)f.$$

Integrating with respect to z we get (4.4), since $\int \delta_{z-a}g = 0$ and $\int_z \bar{\partial}_z g = 0$ for forms g in $\mathcal{S}$.   □

We now want to extend the Fourier transform to currents in $\mathcal{S}'$, and to this end we first notice that

$$(4.5) \qquad (-1)^n \int_z u(z) \wedge f(z) = \int_\zeta \hat{f}(-\zeta) \wedge \hat{u}(\zeta),$$

for $u, f \in \mathcal{S}$. To see this, just notice that both sides are equal to

$$\int_z \int_\zeta u(z) \wedge \hat{f}(\zeta) \wedge e^{\omega(z,\zeta)} = \int_z \int_\zeta u(z) \wedge \hat{f}(-\zeta) \wedge e^{-\omega(z,\zeta)}.$$

Moreover, one easily checks that if $\check{f}(z) = f(-z)$, then $\mathcal{F}\check{f}(\zeta) = \mathcal{F}f(-\zeta)$. Any $u \in \mathcal{S}$ defines an element in $\mathcal{S}'$ by

$$u.f = \int_z u(z) \wedge f(z), \quad f \in \mathcal{S}.$$

For a general $u \in \mathcal{S}'$ it is therefore natural to define $\hat{u}$ by the formula

$$\hat{u}.\hat{f} = (-1)^n u.\check{f}, \quad f \in \mathcal{S}.$$

It is routine to extend δ_{z-a}, $\bar{\partial}$ etc to $\mathcal{S}'$, and verify that Proposition 4.2 still holds for currents $u \in \mathcal{S}'$.

Remark 2. We have deliberately avoided to introduce the Fourier transform for forms on the "transform side". However, if we define

$$\hat{\phi}(z) = \int_\zeta e^{-\omega(z,\zeta)} \wedge \phi(\zeta),$$

for forms $\phi(\zeta)$ on the "transform" side (notice that this definition is not symmetric in ζ and z) then

$$(4.6) \qquad \int \hat{u}(\zeta) \wedge \phi(\zeta) = \int u(z) \wedge \hat{\phi}(z)$$

since both sides are equal to

$$\int_\zeta \int_z e^{-\omega(z,\zeta)} \wedge u(z) \wedge \phi(\zeta).$$

One can then define the Fourier transform of currents in view of this formula instead. $\qquad \square$

Lemma 4.3. *If $[0]$ denotes the current integration at the point 0, then*

$$(4.7) \qquad \mathcal{F}[0](\zeta) = 1 \quad \text{and} \quad \mathcal{F}1(\zeta) = (-1)^n[0](\zeta).$$

Proof. In fact, for $f \in \mathcal{S}_{0,0}$ we have

$$\widehat{[0]}.\hat{f} = (-1)^n \int_z [0](z) \wedge \check{f}(z) = (-1)^n f(0) = \int_\zeta \hat{f}(\zeta),$$

where the last equality follows from the inversion formula (4.2), holding in mind that $\hat{f}$ is a (n, n)-form. In a similar way we have

$$\hat{1}.\hat{f} = (-1)^n \int_z \check{f}(z) = (-1)^n \int_z f(z) = (-1)^n \hat{f}(0),$$

since in this case f is a (n, n)-form. $\qquad \square$

We say that $u \in \mathcal{L}^{-1}(\mathcal{S}', \mathbb{C}^n)$ is a fundamental solution if

$$(\delta_z - \bar{\partial})u = 1 - [0].$$

From Lemma 4.3 and Proposition 4.2 it follows that u is a fundamental solution if and only if

$$(\delta_\zeta + \bar{\partial})\hat{u}(\zeta) = 1 - (-1)^n[0].$$

For instance, let

$$(4.8) \qquad B(z) = \sum_{\ell=1}^{n} b(z) \wedge (\bar{\partial}b(z))^{\ell-1}, \quad b(z) = \frac{1}{2\pi i}\frac{\partial|z|^2}{|z|^2}.$$

Then $\delta_z b(z) = 1$ and hence $(\delta_z - \bar{\partial}_z)B(z) = 1 - [0]$. Thus $(\delta_\zeta + \bar{\partial}_\zeta)\hat{B}(\zeta) = 1 - (-1)^n[0]$. We claim that actually

$$\hat{B}(\zeta) = \sum_{\ell=1}^{n}(-1)^{\ell-1}b(\zeta) \wedge (\bar{\partial}b(\zeta))^{\ell-1}.$$

In fact, one can verify that $B(z)$ is the only possible fundamental solution that is rotation invariant and 0-homogeneous. Since these properties are preserved by $\mathcal{F}$, the claim follows. Alternatively, one can use the wellknown formulas for Fourier transforms of homogenized $\mathbb{R}^{2n}$-harmonic polynomials. We will refer to $B(z)$ and $\hat{B}(\zeta)$ as the Bochner-Martinelli forms.

It is readily verified that Propositions 4.1 and 4.2 hold for X-valued forms and currents and commuting n-tuples of operators a. This is checked by applying functionals on both sides of each equality.

5. Generalized scalar operators

Assume that $a_1, \dots, a_n$ is a commuting tuple that admits a $\mathcal{E}$-functional calculus, i.e., a continuous algebra homomorphism $\Pi \colon \mathcal{E}(\sigma(a)) \to \mathcal{L}(X)$ that extends (1.2). Such a tuple a is sometimes called a generalized scalar tuple. Here $\mathcal{E}(\sigma(a))$ denotes the algebra of germs of smooth functions on $\sigma(a)$. Thus Π is a $(a)'$-valued distribution in $\mathbb{C}^n$ that is supported on $\sigma(a)$. The continuity assumption implies that there is a nonnegative integer m such that for each $U \supset \sigma(a)$

$$(5.1) \qquad\qquad |\Pi(f)| \leq C_U |f|_{U,m},$$

where $|f|_{U,m} = \sum_{|\alpha| \leq m} \sup_U |D^\alpha f|$.

First we shall relate the existence of a $\mathcal{E}$-functional calculus to the existence of a current extension of the resolvent over the spectrum, cf., Remark 5 below.

Theorem 5.1. *Let a be a commuting n-tuple of operators. Then the following conditions are equivalent:*

(i) a admits a $\mathcal{E}$-functional calculus $\Pi\colon \mathcal{E}(\mathbb{C}^n) \to \mathcal{L}(X)$.

(ii) There is a commuting n-tuple $a^* \in (a)'$ and a number m such that

$$(5.2) \qquad \|e^{\pi i(a\cdot\bar\zeta + a^*\cdot\zeta)}\| \leq C|\zeta|^m, \quad \zeta \in \mathbb{C}^n.$$

(iii) There is a $v \in \mathcal{L}^{-1}(\mathcal{D}', \mathbb{C}^n, (a)')$ such that

$$(5.3) \qquad (\delta_{z-a} - \bar\partial)v = e - [a],$$

where $[a]$ is an $(a)'$-valued (n,n)-current such that

$$(5.4) \qquad \widehat{[a]}(\zeta + \zeta') = \widehat{[a]}(\zeta)\widehat{[a]}(\zeta').$$

In case these statements hold, then $[a]$ is a (n,n)-current supported on $\sigma(a)$, $\Pi(f) = [a].f$ for $f \in \mathcal{S}_{0,0}(\mathbb{C}^n)$, and

$$(5.5) \qquad \widehat{[a]}(\zeta) = e^{-\pi i(a\cdot\bar\zeta + a^*\cdot\zeta)}$$

where $a^* = \Pi(\bar z)$. Moreover,

$$(5.6) \qquad \Pi(f) = (-1)^n \int_\zeta e^{\pi i(a\cdot\bar\zeta + a^*\cdot\zeta)} \hat f(\zeta), \quad f \in \mathcal{S}_{0,0}(\mathbb{C}^n).$$

Proof. If (i) holds and $a_j^* = \Pi(\bar z_j)$, then (a, a^*) is a commuting $2n$-tuple and (2.4) holds, so (5.2) follows from (5.1).

Now assume that (ii) holds, and let $2\operatorname{Re} a \cdot \bar\zeta = a \cdot \bar\zeta + a^* \cdot \zeta$. Then $\exp(-2\pi i\operatorname{Re} a \cdot \bar\zeta)$ is in $\mathcal{S}'(\mathbb{C}^n, (a)')$ so we can define an $(a)'$-valued (n,n)-current $[a]$ by formula (5.5), and then (5.4) will be satisfied since (a, a^*) is commuting. Let $u(z)$ be a fundamental solution in $\mathcal{S}'$, e.g., the Bochner-Martinelli form, cf., Section 4. Then $(\delta_\zeta + \bar\partial_\zeta)\hat u = e - (-1)^n[0]$ and hence

$$(\delta_\zeta + \bar\partial_\zeta + A)(e^{-2\pi i\operatorname{Re} a\cdot\bar\zeta}\hat u) = (1 - (-1)^n[0])e^{-2\pi i\operatorname{Re} a\cdot\bar\zeta} = \widehat{[a]} - (-1)^n[0].$$

By the assumption (5.2), $e^{-2\pi i\operatorname{Re} a\cdot\bar\zeta}\hat u$ is in $\mathcal{S}'$ and by Proposition 4.2 and Lemma 4.3 then (5.3) holds if $\hat v = e^{-2\pi i\operatorname{Re} a\cdot\bar\zeta}\hat u$.

Finally, assume that (iii) holds. Then to begin with, $\delta_{z-a}[a] = 0$ so $[a]$ has support on $\sigma(a)$ according to Lemma 3.1. Thus $[a]$ is in $\mathcal{S}'$ although we a priori only assume that v is in $\mathcal{D}'(\mathbb{C}^n, (a)')$, so that (5.4) anyway has meaning. Let f be holomorphic in a neighborhood U of $\sigma(a)$ and let $u \in \mathcal{L}^{-1}(\mathcal{E}, U \setminus \sigma(a), X)$ be a solution to $(\delta_{z-a} - \bar\partial)u = f(z)x$. Then, since v is $(a)'$-valued, $(\delta_{z-a} - \bar\partial)(v \wedge u) = u - vf(z)x$ so that $u_n - v_n fx$ is $\bar\partial$-exact in $U \setminus \sigma(a)$, and hence $v_n fx$ is a current representative of the Dolbeault cohomology class $\omega_{z-a}fx$ in $U \setminus \sigma(a)$. Now, let $\Pi(f) = [a].f$ and let χ be a cutoff function that is 1 in a neighborhood of $\sigma(a)$. Since $[a]$ is supported on $\sigma(a)$ and $\bar\partial v_n = [a]$ we have that

$$\Pi(f)x = [a].\chi fx = v_n fx.\bar\partial\chi = -\int \bar\partial\chi \wedge \omega_{z-a}fx = f(a)x$$

according to (3.1). Thus Π is an extension of the holomorphic functional calculus and (5.4) ensures that it is multiplicative. Thus (iii) implies (i).

The first stated relations between Π, $[a]$, and a^* follow from the proof above, whereas (5.6) follows since

$$\Pi(f) = [a].f = (-1)^n \int_\zeta \widehat{[a]}(-\zeta).\hat{f}(\zeta) = (-1)^n \int e^{-2\pi i \operatorname{Re} a \cdot \bar{\zeta}} \hat{f}(\zeta).$$

$\square$

Remark 3. The proof of $(ii) \to (iii)$ above is based on the possibility to define "translates" $f_a(z) = f(z - a)$ of currents $f \in \mathcal{S}'(\mathbb{C}^n, (a)')$, by multiplying with $\exp(-2\pi i \operatorname{Re} a \cdot \bar{\zeta})$ on the Fourier transform side, provided that (5.2) holds. Given a fundamental solution u, and taking $v = u_a$, we have that $(\delta_{z-a} - \bar{\partial})u_a = 1 - [a]$, since $[a]$ is the a-translate of $[0]$. Of course we can think of $v(z)$ as the "convolution"

$$v(z) = \int_w [a](z - w) \wedge u(w).$$

If $u(z)$ is chosen to be smooth outside 0, like the Bochner-Martinelli form, it follows that v is smooth outside $\sigma(a)$. Hence there is a smooth solution to $(\delta_{z-a} - \bar{\partial})v = e$ there. In particular, $\delta_{z-a}v_{1,0}(z) = e$, which implies that $\sigma(a)$ coincides with the spectrum with respect to the commutative subalgebra (a, a^*) of $\mathcal{L}(X)$, and with the splitting spectrum, see e.g., [2]. $\square$

Remark 4. We can write more suggestively

$$f(a) = \Pi(f) = \int_z f(z)[a](z),$$

and so it is natural to think of $[a]$ as a generalized spectral measure (or rather a spectral current).

Let X be a Hilbert space and a a commuting tuple of normal operators; this means that (a, a^*) is a commuting tuple, where a_j^* is the Hilbert space adjoints of a_j. Then $(\operatorname{Re} a, \operatorname{Im} a)$ is a selfadjoint commuting tuple and hence $\| \exp(\pi i (a \cdot \bar{\zeta} + a^* \cdot \zeta)) \| \leq 1$. From Theorem 5.1 it follows that a admits a $\mathcal{E}$ functional calculus. However, as is well-known, the spectral current $[a]$ in this case actually is a measure (a (n, n)-current of order zero) so there is even a $C(\sigma(a))$ functional calculus. $\square$

It a admits a $\mathcal{E}$-functional calculus we know from Section 2 that $\sigma(\operatorname{Re} a, \operatorname{Im} a) = \{(x, y);\ x + iy \in \sigma(a)\}$. Moreover, the mapping $\phi(z) \mapsto \tilde{\phi}(x, y) = \phi(x + iy)$ extends to an isomorphism

$$\mathcal{E}(\sigma(a)) \simeq \mathcal{E}(\sigma(\operatorname{Re} a, \operatorname{Im} a)),$$

where $\sigma(\operatorname{Re} a, \operatorname{Im} a)$ is considered as a subset of $\mathbb{R}^{2n}$, and since the realanalytic functions are dense the equality (2.6) extends to these

spaces. Since the spectral mapping property holds for the natural C^ω functional calculus for the commuting $2n$-tuple $(\operatorname{Re} a, \operatorname{Im} a)$, see [4], we have

Proposition 5.2. *If a admits a $\mathcal{E}$ functional calculus $\Pi\colon \mathcal{E}(\mathbb{C}^n) \to \mathcal{L}(X)$, then $\sigma(\Pi(f)) = f(\sigma(a))$ for all $f = (f_1, \ldots, f_m)$, where $f_k \in \mathcal{E}(\sigma(a))$.*

Remark 5. Assume that a admits a $\mathcal{E}$ functional calculus $[a]$, let $\Pi\colon \mathcal{E}(\sigma(a)) \to (a)'$ be a linear continuous mapping, and let π be the $(a)'$-valued (n, n)-current supported on $\sigma(a)$ such that $\Pi(f) = \pi.f$ for smooth f. Then the following conditions are equivalent:

(i) Π is a linear extension of the holomorphic functional calculus such that $\Pi(z_j f) = a_j \Pi(f)$ for all a_j.

(ii) There are $c_\alpha \in (a)'$ such that

$$(5.7) \qquad \pi(z) = [a] + \sum_{0 < |\alpha| \le m} c_\alpha \frac{\partial^\alpha}{\partial \bar{z}^\alpha}[a].$$

(iii) There is a $(a)'$-valued current solution to $(\delta_{z-a} - \bar{\partial})v = e - \pi$ in $\mathbb{C}^n$.

Suppose that (i) holds. Then $\delta_{z-a}\pi(z) = 0$, and thus, cf., Remark 3, $\delta_z \pi(z + a) = 0$, which means that

$$\pi(z + a) = \sum_{|\alpha| \le m} c_\alpha \frac{\partial^\alpha}{\partial \bar{z}^\alpha}[0].$$

Moreover, since $\pi.1 = e$ and thus $\pi(\cdot + a).1 = e$, it follows that $c_0 = e$. By "translating" back we get (5.7). On the other hand, if (5.7) holds, it is easy to modify v_n in a solution v to (5.3) so that the equation in (iii) is satisfied. Finally, if (iii) holds, then as before it follows that π is supported on $\sigma(a)$ and that $\delta_{z-a}\pi = 0$. As in the proof of Theorem 5.1 this implies (i).

Thus any choice of coefficients $c_\alpha \in (a)'$ gives a current extension of the representative u_n of the Dolbeault cohomology class resolvent ω_{z-a} over the spectrum $\sigma(a)$, and a linear extension Π of the holomorphic functional calculus such that (i) holds, but in general the extension will not be multiplicative. In fact, it is if and only if $\hat{\pi}(\zeta + \zeta') = \hat{\pi}(\zeta)\hat{\pi}(\zeta')$. $\qquad\square$

However, in general there are several possible multiplicative extensions of the holomorphic functional calculus. The following result is a multivariable version of a classical theorem, see [6].

Theorem 5.3. *Suppose that Π and Π' are two $\mathcal{E}$-functional calculi such that $\Pi(\bar{z})$ and $\Pi'(\bar{z})$ are commuting. Then $q = \Pi(\bar{z}) - \Pi'(\bar{z})$*

is a nilpotent (commuting) tuple, and

$$(5.8) \qquad \Pi'(f) = \sum_{|\alpha| \le m+m'} \frac{q^\alpha}{\alpha!} \Pi\Big(\frac{\partial^\alpha f}{\partial \bar{w}^\alpha}\Big),$$

where m and m' are the orders of Π and Π' respectively. Conversely, if Π is a $\mathcal{E}$-functional calculus, $a^* = \Pi(\bar{z})$, and $q \in (a, a^*)'$ is a commuting nilpotent tuple, then (5.8) defines another $\mathcal{E}$-functional calculus.

Proof. If $[a]$ and $[a]'$ are the corresponding (n,n)-currents, then

$$(5.9) \qquad \widehat{[a]}'(\zeta) = e^{-\pi i (a\cdot\zeta + a^*\cdot\xi + q\cdot\zeta)} = \widehat{[a]}(\zeta) e^{-\pi i q\cdot\zeta},$$

and since

$$\|\widehat{[a]}(\zeta)\| \lesssim |\zeta|^m \quad \text{and} \quad \|\widehat{[a]}'(\zeta)\| \lesssim |\zeta|^{m'},$$

it follows that

$$\|e^{\pi i q\cdot\zeta}\| \lesssim |\zeta|^{m+m'},$$

which implies that $q^\alpha = 0$ for $|\alpha| > m + m'$. Observe that

$$\pi i q \cdot \zeta \widehat{[a]}(\zeta) = \mathcal{F}(q \cdot \frac{\partial}{\partial \bar{z}}[a]),$$

so that

$$e^{-\pi i q\cdot\zeta}\widehat{[a]} = \mathcal{F} \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} q^\alpha \frac{\partial^\alpha}{\partial\bar{z}^\alpha}[a].$$

In view of (5.9) this means that (5.8) holds. The converse is obtained by arguing backwards. $\qquad\square$

Remark 6. If $f \in C^\omega(\sigma(a))$ and $f(z) = \tilde{f}(z,\bar{z})$ as in Section 2, then then $\Pi(a) = \tilde{f}(a,a^*)$, and $\Pi'(a) = \tilde{f}(a, a^* + q)$. Thus one can think of (5.8) as the Taylor expansion of $\tilde{f}(a, a^*+q)$ at the "point" (a, a^*). $\quad\square$

6. THE PROPERTY $(\beta)_{\mathcal{E}}$ FOR GENERALIZED SCALAR OPERATORS

Let a be an n-tuple of commuting operators. It has the property $(\beta)_{\mathcal{E}}$ if the complex

$$(6.1) \quad 0 \to \mathcal{S}_{n,0}(\mathbb{C}^n, X) \xrightarrow{\delta_{z-a}} \ldots \xrightarrow{\delta_{z-a}} \mathcal{S}_{1,0}(\mathbb{C}^n, X) \xrightarrow{\delta_{z-a}} \mathcal{S}_{0,0}(\mathbb{C}^n, X) \to 0$$

is exact at $\mathcal{S}_{k,0}(\mathbb{C}^n, X)$ for $k > 0$ and the range is closed at $\mathcal{S}_{0,0}(\mathbb{C}^n, X)$. For a background to the this notion, see [4]. For large $|z|$ the (local) exactness of the complex follows by means of the homotopy operator $Sf = s \wedge f$, where, e.g., $s = \sum \partial |z|^2/(|z|^2 - a\cdot\bar{z})$. Therefore, one can just as well replace $\mathcal{S}(\mathbb{C}^n, X)$ by $\mathcal{E}(\mathbb{C}^n, X)$ or $\mathcal{D}(\mathbb{C}^n, X)$ in the definition of $(\beta)_{\mathcal{E}}$.

The main result in this section is a new and simple proof of the following theorem due to Eschmeier and Putinar, [9].

Theorem 6.1. *If the tuple a admits a $\mathcal{E}$ functional calculus, then it has the property $(\beta)_{\mathcal{E}}$.*

If a admits a $\mathcal{E}$ functional calculus, then it induces a mapping $\mathcal{E}(\mathbb{C}^n, X) \to X$ intuitively defined as $f \mapsto f(a)$. If f is in $\mathcal{S}$ it can be defined by the current $[a]$ acting on f or, equivalently, by the formula

$$f(a) = (-1)^n \int_{\zeta} e^{2\pi i \operatorname{Re} a \cdot \bar{\zeta}} \hat{f}(\zeta).$$

This mapping is $\mathcal{S}(\mathbb{C}^n)$-linear (this follows from (5.4)) and it commutes with each a_j (since $[a]$ is $(a)'$-valued). We define the closed subspace $\mathcal{S}_{0,0}^a(\mathbb{C}^n, X)$ of $\mathcal{S}_{0,0}(\mathbb{C}^n, X)$ as

$$\mathcal{S}_{0,0}^a = \left\{ f \in \mathcal{S}_{0,0}(\mathbb{C}^n, X); \ \frac{\partial^{\alpha} f}{\partial \bar{z}^{\alpha}}(a) = 0 \ \forall \alpha \right\}.$$

It immediately follows that $\delta_{z-a} u(z)|_{z=a} = 0$, and therefore $f \in \mathcal{S}_{0,0}^a$ if $f(z) = \delta_{z-a} u(z)$ for some $u \in \mathcal{S}_{1,0}(\mathbb{C}^n, X)$. However also the converse is true.

Theorem 6.2. *Assume that a is a generalized scalar, and let $f \in \mathcal{S}(\mathbb{C}^n, X)$. Then the range of*

$$\delta_{z-a} \colon \mathcal{S}_{1,0}(\mathbb{C}^n, X) \to \mathcal{S}_{0,0}(\mathbb{C}^n, X)$$

is precisely $\mathcal{S}_{0,0}^a(\mathbb{C}^n, X)$, hence it is closed, and the sequence

$$(6.2) \quad 0 \to \mathcal{S}_{n,0}(\mathbb{C}^n, X) \xrightarrow{\delta_{z-a}} \ldots$$

$$\xrightarrow{\delta_{z-a}} \mathcal{S}_{1,0}(\mathbb{C}^n, X) \xrightarrow{\delta_{z-a}} \mathcal{S}_{0,0}(\mathbb{C}^n, X) \to \mathcal{S}_{0,0}(\mathbb{C}^n, X)/\mathcal{S}_{0,0}^a(\mathbb{C}^n, X) \to 0.$$

is exact.

In the case when $X = 0$ and a is just a point in $\mathbb{C}^n$, e.g., $a = 0$, then this theorem is a wellknown nontrivial result of Malgrange, see [9].

Clearly, Theorem 6.2 implies Theorem 6.1.

Proof of Theorem 6.2. Since

$$\mathcal{F}\left(\frac{\partial^{\alpha} f}{\partial \bar{z}^{\alpha}}\right) = (\pi i \zeta)^{\alpha} \mathcal{F} f(\zeta)$$

and

$$\frac{\partial^{\alpha} f}{\partial \bar{z}^{\alpha}}(a) = (-1)^n \int_{\zeta} e^{2\pi i \operatorname{Re} a \cdot \bar{\zeta}} (\pi i \zeta)^{\alpha} \hat{f}(\zeta),$$

it follows that $\mathcal{S}_{0,0}^a(\mathbb{C}^n, X)$ via the Fourier transform corresponds to

$$\mathcal{S}_{n,n}^a(\mathbb{C}^n, X) = e^{-2\pi i \operatorname{Re} z \cdot \bar{\zeta}} \mathcal{S}_{n,n}^0(\mathbb{C}^n, X),$$

where

$$\mathcal{S}_{n,n}^0(\mathbb{C}^n, X) = \left\{ \phi \in \mathcal{S}_{n,n}(\mathbb{C}^n, X); \ \int_{\zeta} \zeta^{\alpha} \phi(\zeta) = 0 \ \ \forall \alpha \right\}.$$

Therefore, (6.2) is exact if and only if

$$(6.3) \quad 0 \to \mathcal{S}_{n,0}(\mathbb{C}^n, X) \xrightarrow{\bar{\partial}_\zeta + A} \mathcal{S}_{n,1}(\mathbb{C}^n, X) \xrightarrow{\bar{\partial}_\zeta + A} \dots$$

$$\xrightarrow{\bar{\partial}_\zeta + A} \mathcal{S}_{n,n-1}(\mathbb{C}^n, X) \xrightarrow{\bar{\partial}_\zeta + A} \mathcal{S}_{n,n}(\mathbb{C}^n, X) \to \frac{\mathcal{S}_{n,n}(\mathbb{C}^n, X)}{e^{-2\pi i \operatorname{Re} a \cdot \bar{\zeta}} \mathcal{S}_{n,n}^0(\mathbb{C}^n, X)} \to 0$$

is exact. On the other hand, since multiplication with $e^{2\pi i \operatorname{Re} a \cdot \bar{\zeta}}$ is a continuous isomorphism on $\mathcal{S}(\mathbb{C}^n, X)$, (6.3) is exact if and only if

$$(6.4) \quad 0 \to \mathcal{S}_{n,0}(\mathbb{C}^n, X) \xrightarrow{\bar{\partial}_\zeta} \mathcal{S}_{n,1}(\mathbb{C}^n, X) \xrightarrow{\bar{\partial}_\zeta} \dots$$

$$\xrightarrow{\bar{\partial}_\zeta} \mathcal{S}_{n,n-1}(\mathbb{C}^n, X) \xrightarrow{\bar{\partial}_\zeta} \mathcal{S}_{n,n}(\mathbb{C}^n, X) \to \frac{\mathcal{S}_{n,n}(\mathbb{C}^n, X)}{\mathcal{S}_{n,n}^0(\mathbb{C}^n, X)} \to 0$$

is. However, Malgrange's theorem means that (6.4) is exact when $X = \mathbb{C}$, and since the spaces $\mathcal{S}_{n,q}$ are nuclear, see, e.g., [9], the exactness is preserved when applying $\hat{\otimes} X$. Thus the theorem is proved. $\square$

The proof above may be rephrased in the following way. First notice that there is a a continous mapping $f(z) \mapsto f_a(z) = f(z - a)$ on $\mathcal{S}(\mathbb{C}^n, X)$ if a is a generalized scalar, obtained by multiplying by with $\exp(-2\pi i \operatorname{Re} a \cdot \bar{\zeta})$ on the Fourier transform side, cf., Remark 3. By Malgrange's theorem (the X-valued version, obtained from the usual one by some reference to nuclearity of $\mathcal{S}$), (6.2) is exact if $a = 0$. We then obtain the exactness in general by making the "translation" by a. More concretely, if we have got an X-valued $f(z)$ such that $\delta_{z-a} f(z) = 0$, then $\delta_z f(z + a) = 0$ and by the exactness hence we can solve $\delta_z v(z) = f(z + a)$. Thus $\delta_{z-a} u = f$ if $u(z) = v(z - a)$.

Remark 7. In [9] Theorem 6.1 is first proved for $n = 1$, whereas general case is then obtained by an ingenious induction argument. The one-variable case of Theorem 6.2 and $a = 0$ is via the Fourier transformation equivalent to the following statement:

Suppose $n = 1$ and $\phi \in \mathcal{S}_{0,1}(\mathbb{C}, X)$. Then there is a solution to $\bar{\partial} u = \phi$ in $\mathcal{S}(\mathbb{C}, X)$ if and only if

$$\int_\zeta \zeta^k \phi(\zeta) \wedge d\zeta = 0, \quad k = 0, 1, 2, \dots .$$

To prove this statement, let $u(z)$ be the unique solution that vanishes at ∞. Because of the moment conditions,

$$u(\zeta) = \int_\xi \left(\frac{\xi}{\zeta} \right)^m \frac{f(\xi) \wedge d\xi}{\zeta - \xi},$$

for any nonnegative integer m, and hence $u \in \mathcal{S}_{0,0}(\mathbb{C}, X)$. $\square$

We conclude with a result which is somehow dual to Theorem 6.2, but which seems to be more elementary.

Proposition 6.3. *Suppose that a is a generalized scalar tuple, with $\mathcal{E}$-functional calculus $[a]$. Then the sequence*

$$0 \to \mathcal{D}'_{n,0}(\mathbb{C}^n, X) \xrightarrow{\delta_{z-a}} \mathcal{D}'_{n-1,0}(\mathbb{C}^n, X) \xrightarrow{\delta_{z-a}} \ldots \xrightarrow{\delta_{z-a}} \mathcal{D}'_{0,0}(\mathbb{C}^n, X) \to 0$$

is exact except at $k = n$, where the kernel is

$$\{p(\partial/\partial\bar{z})[a]; \ p(z) \ \ polynomial\}.$$

The same holds for $\mathcal{S}'$ or $\mathcal{E}'$ instead of $\mathcal{D}'$. In particular, Proposition 6.3 implies that

$$0 \to \mathcal{D}'_{n,0}(V, X) \xrightarrow{\delta_{z-a}} \mathcal{D}'_{n-1,0}(V, X) \xrightarrow{\delta_{z-a}} \ldots \xrightarrow{\delta_{z-a}} \mathcal{D}'_{0,0}(V, X) \to 0$$

is exact if $V \subset \mathbb{C}^n \setminus \sigma(a)$.

Sketch of proof. For large z there is an $(a)'$-valued smooth form s such that $\delta_{z-a}s = e$ so if $f \in \mathcal{D}'_{p,0}$, $p < n$, then $\delta_{z-a}f(z) = 0$ then $u = s \wedge f$ is a solution to $\delta_{z-a}u = f$ for large z, and hence we may assume that f is in $\mathcal{S}'$. Via the Fourier transformation and multiplication with $\exp(2\pi i \mathrm{Re}\, a \cdot \bar{\zeta})$ the problem then is reduced to solving the $\bar{\partial}$ equation for X-valued currents in $\mathcal{S}'$, and this can be done, e.g., by weighted integral formulas. If $f \in \mathcal{D}'_{n,0}$, and $\delta_{z-a}f = 0$, we know from Lemma 3.1 that $f = 0$ outside the spectrum; in particular f is in $\mathcal{S}'(\mathbb{C}^n, X)$, and $(\bar{\partial}_\zeta + A)\hat{f}(\zeta) = 0$; thus

$$\bar{\partial}_\zeta(e^{-2\pi i \mathrm{Re}\, a \cdot \bar{\zeta}} \hat{f}(\zeta)) = 0,$$

so that $\hat{f}(\zeta) = p(\zeta)e^{-2\pi i \mathrm{Re}\, a \cdot \bar{\zeta}}$ for some holomorphic polynomial p.   □

7. Ultradifferentiable functional calculus

Most of the results from Section 5 hold for algebras of ultradifferentiable functions instead of $\mathcal{E}$. For simplicity we restrict to the nonquasianalytic case; however, for instance all Gevrey classes will be included.

Let $h(\zeta)$ be a nonnegative, continuous, subadditive function on $\mathbb{C}^n$ with $h(0) = 1$, and let $\mathcal{A}_h$ be the space of all tempered distributions f such that $\hat{f}$ is a measure and

$$\|f\|_{\mathcal{A}_h} = \int_\zeta |\hat{f}(\zeta)| e^{h(\zeta)} < \infty.$$

Then $\mathcal{A}_h$ is an algebra, and if $\exp(-h(t)) = \mathcal{O}(|t|^{-m})$ for all m, then $\mathcal{A}_h \subset \mathcal{E}$. We will also assume that

$$(7.1) \qquad \int \frac{h(\zeta)}{1 + |\zeta|^{2n+1}} < \infty,$$

which ensures that $\mathcal{A}_h$ is nonquasianalytic, i.e., it contains cutoff functions with arbitrary small supports, see [4] and [5]. Typically is $h(\zeta) = |\zeta|^\alpha$, $0 < \alpha < 1$, which gives the Gevrey classes, but also nonradial h are

allowed. These algebras were introduced by Beurling in [5]. Since we have access to cutoff functions we can easily localize the $\mathcal{A}_h$-condition; more precisely, for an open set V we can define $\mathcal{A}_h(V)$ as the algebra of functions f in V such that $\chi f \in \mathcal{A}_h$ for all cutoff functions $\chi \in \mathcal{A}_h$ with support in V. It turns out that $C^\omega(V)$ is continuously embedded in $\mathcal{A}_h(V)$ and dense; for a proof see [4]. For a compact set K we define $\mathcal{A}_h(K)$ as the inductive limit of the spaces $\mathcal{A}_h(V)$, $V \supset K$. It is not hard to see that the dual space $\mathcal{A}_h(V)$ consists of all ultradistributions u with compact support in V such that $|\hat{u}(\zeta)| \leq C \exp h(-\zeta)$. We say that a ultradistribution u is in $\mathcal{A}_h'$ if $\chi u \in \mathcal{A}_h'(\mathbb{C}^n)$ for each cutoff function $\chi \in \mathcal{A}_h$. These definitions can easily be extended to vector-valued ultracurrents.

Remark 8. In the quasianalytic case one can localize the $\mathcal{A}_h$-condition by means of a variant of the FBI transform and define spaces $\widetilde{\mathcal{A}_h}(V)$ that consist of all functions f such that roughly speaking f, locally in V, belong to $\mathcal{A}_{ch}$ for some $c > 1$. The dual space then consists of hyperfunctions u such that $|\hat{u}(\zeta)| \leq C_c \exp ch(-\zeta)$ for all $c > 1$, see [4]. It is possible to prove an analogue to Theorem 7.1 for these spaces as well. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We say that a admits a $\mathcal{A}_h$ functional calculus if there is a continuous mapping
$$\Pi\colon \mathcal{A}_h(\sigma(a)) \to \mathcal{L}(X)$$
that extends the holomorphic functional calculus. We have the following analogue to Theorem 5.1. However, since we have no analogue to Lemma 3.1 for $\mathcal{A}_h$ we have to formulate condition *(iii)* somewhat differently.

Theorem 7.1. *Let a be a commuting n-tuple of operators. Then the following are equivalent:*

(i) a admits a $\mathcal{A}_h$-functional calculus $\Pi\colon \mathcal{A}_h(\sigma(a)) \to \mathcal{L}(X)$.

(ii) There is a commuting n-tuple $a^* \in (a)'$ such that

$$(7.2) \qquad\qquad \|e^{\pi i(a \cdot \bar{\zeta} + a^* \cdot \zeta)}\| \leq C e^{h(\zeta)}.$$

(iii) There is an $(a)'$-valued ultracurrent v in $\mathcal{L}^{-1}(\mathbb{C}^n, \mathcal{A}_h', (a)')$ that is smooth outside $\sigma(a)$, such that

$$(7.3) \qquad\qquad (\delta_{z-a} - \bar{\partial})v = e - [a],$$

where $[a]$ is an $(a)'$-valued (n,n)-ultracurrent in $\mathcal{A}_h'$ such that

$$(7.4) \qquad\qquad \widehat{[a]}(\zeta + \zeta') = \widehat{[a]}(\zeta)\widehat{[a]}(\zeta').$$

In case these statements hold, then $[a]$ is a (n,n)-ultracurrent supported on $\sigma(a)$ such that $\Pi(f) = [a].f$ for $f \in \mathcal{S}_{0,0}(\mathbb{C}^n)$, and

$$(7.5) \qquad\qquad \widehat{[a]}(\zeta) = e^{-\pi i(a \cdot \bar{\zeta} + a^* \cdot \zeta)}$$

where $a^* = \Pi(\bar{z})$. Moreover,

$$(7.6) \qquad \Pi(f) = (-1)^n \int_\zeta e^{\pi i (a \cdot \bar{\zeta} + a^* \cdot \zeta)} \hat{f}(\zeta).$$

for $f \in \mathcal{A}_h$.

Proof. Assume that (i) holds and define, as usual, $a^* = \Pi(\bar{z})$. As before we have that $\Pi(E_\xi) = \exp(2\pi i \mathrm{Re}\, a \cdot \bar{\xi})$. Since $\hat{f}(-\xi) = E_{-\xi}.f = (-1)^n \hat{E}_\xi.\hat{f}$ we have that $\hat{E}_\xi(\zeta) = (-1)^n[-\xi]$ and hence

$$\|E_\xi\|_{\mathcal{A}_h} = \int_\zeta [-\xi](\zeta) e^{h(\zeta)} = e^{h(-\zeta)},$$

By the continuity of Π we get the estimate (7.2).

Now assume that (7.2) holds. Then, as was proved in Section 2, $\sigma(\mathrm{Re}\, a, \mathrm{Im}\, a)$ is real and equal to $\{(x, y); \ x + iy \in \sigma(a)\}$, and from [4] it follows that the formula (7.6) defines a $\mathcal{A}_h(\sigma(a))$ functional calculus; in particular, a $(a)'$-valued ultradistribution $[a]$ in $\mathcal{A}_h'$ with support on $\sigma(a)$. Moreover, $\Pi(f) = \tilde{f}(a, a^*)$ for entire functions so $\widehat{[a]}(\zeta) = \exp(-2\pi i \mathrm{Re}\, a \cdot \bar{\zeta})$, and so (7.4) will be satisfied since (a, a^*) is commuting. Let $u(z)$ be the Bochner-Martinelli form, cf., Section 4, and let $\hat{v} = e^{-2\pi i \mathrm{Re}\, a \cdot \bar{\zeta}} \hat{u}$. Since $\hat{u}$ is bounded it follows that $|\hat{v}(\zeta)| \leq C \exp h(-\zeta)$ and hence v is in $\mathcal{A}_h'$, and as before (7.3) holds. Since v is a convolution with $[a]$ and u it is smooth outside the support of $[a]$, i.e., $\sigma(a)$.

Finally, assume that (iii) holds. By the extra assumption on v it follows from Lemma 3.1 that $[a]$ is supported on $\sigma(a)$, and then (i) follows in precisely same way as in the proof of Theorem 5.1. $\qquad \square$

From the isomorphism

$$\mathcal{A}_h(\sigma(a)) \simeq \mathcal{A}_h(\sigma(\mathrm{Re}\, a, \mathrm{Im}\, a)), \quad \phi \mapsto \tilde{\phi},$$

and the spectral mapping property for the $\mathcal{A}_h$ functional calculus for tuples with real spectrum, see [4], we get

Proposition 7.2. *Suppose that a is a commuting tuple that admits a $\mathcal{A}_h(\sigma(a))$ functional calculus $\Pi \colon \mathcal{A}_h(\sigma(a)) \to \mathcal{L}(X)$. Then $\sigma(\Pi(f)) = f(\sigma(a))$.*

As one can expect we also have an $\mathcal{A}_h$-analogue to Proposition 2.3 and Theorem 5.3.

Theorem 7.3. *Suppose that Π and Π' are two $\mathcal{A}_h(\sigma(a))$-functional calculi such that $\Pi(\bar{z})$ and $\Pi'(\bar{z})$ are commuting. Then $q = \Pi(\bar{z}) - \Pi'(\bar{z})$ is a quasi-nilpotent (commuting) tuple satisfying*

$$(7.7) \qquad \|e^{\pi i q \cdot \bar{\zeta}}\| \leq C e^{h(\zeta) + h(-\zeta)}.$$

Conversely, if f Π is a $\mathcal{A}_{h_1}$-functional calculus, $a^* = \Pi(\bar{z})$, and $q \in (a, a^*)'$ is a commuting nilpotent tuple such that $\|\exp(\pi i q \cdot \bar{\zeta})\| \leq$

$C \exp h_2(\zeta)$, *then, if* $h = h_1 + h_2$, *there is a* $\mathcal{A}_h$ *functional calculus* Π' *such that* $\Pi'(\bar{z}) = a^* + q$.

Proof. If $a^* = \Pi(\bar{z})$ and $a^* + q = \Pi'(\bar{z})$, then from Theorem 7.1 we have that $\| \exp(i\pi(a \cdot \zeta + a^* \cdot \zeta)\| \leq C \exp h(\zeta)$ and $\| \exp(i\pi(a \cdot \zeta + (a^* + q) \cdot \zeta)\| \leq C \exp h(\zeta)$, and from this we immediately get (7.7). The second statement is concluded in a similar way, again using Theorem 7.1. $\square$

As in the $\mathcal{E}$-case, and for the same reason, Π and Π' are related by the formula

$$(7.8) \qquad \Pi'(f) = \sum_{|\alpha| \geq 0} \frac{q^\alpha}{\alpha!} \Pi\Big(\frac{\partial^\alpha f}{\partial \bar{w}^\alpha}\Big),$$

for all say realanalytic f.

REFERENCES

[1] M. ANDERSSON: *Taylor's functional calculus for commuting operators with Cauchy-Fantappie-Leray formulas*, International Math. Research Notices **6** (1997), 247-258.
[2] M. ANDERSSON: *Taylor's functional calculus and the resolvent mapping*, Preprint Göteborg (2000).
[3] M. ANDERSSON: *A new approach to integral represention with weights*, Preprint Göteborg (2000).
[4] M. ANDERSSON & B. BERNDTSSON: *Non-holomorphic functional calculus for commuting operators with real spectrum*, In preparation.
[5] A. BEURLING: *On quasianalyticity and general distributions*, Lecture notes, Stanford (1961).
[6] I. COLOJOARA & C. FOIAS: *Theory of generalized spectral operators*, Gordon and Breach (1968).
[7] B. DROSTE: *Extension of analytic functional calculus mappings and duality by $\bar{\partial}$-closed forms with growth*, Math. Ann. **261**, 185-200 (1982).
[8] E. M. DYNKIN: *An operator calculus based on the Cauchy-Green formula. (Russian)*, Investigations on linear operators and the theory of functions, III. Zap. Nauv cn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **30** (1972), 33–39..
[9] J. ESCHMEIER & M. PUTINAR: *Spectral Decompositions and Analytic Sheaves*, Clarendon Press, Oxford 1996.
[10] V. MÜLLER: *The splitting spectrum differs from the Taylor spectrum*, Studia Math. **123** (1997) 291–294. .
[11] M. PUTINAR: *Uniqueness of Taylor's functional calculus*, Proc. Amer. Math. Soc. **89)** (1983), 647-650.
[12] S. SANDBERG: *On non-holomorphic functional calculus for commuting operators*, Preprint Göteborg (1999).
[13] J.L. TAYLOR: *A joint spectrum for several commuting operators*, J. Funct. Anal. **6** (1970), 172-191.
[14] J.L. TAYLOR: *The analytic-functional calculus for several commuting operators*, Acta Math. **125** (1970), 1-38.

DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY AND THE UNIVERSITY OF GÖTEBORG, S-412 96 GÖTEBORG, SWEDEN
E-mail address: matsa@math.chalmers.se