Spectral Synthesis and Operator Synthesis for Compact Groups

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Abstract

Let G be a compact group. We construct an imbedding of the Fourier algebra $\mathrm{A}(G)$ of G into the algebra $\mathrm{V}(G) = \mathcal{C}(G) \otimes^h \mathcal{C}(G)$ and deduce results about parallel spectral synthesis, generalizing a result of Varopoulos [20]. We then characterize which diagonal sets in $G \times G$ which are sets of operator synthesis with respect to the Haar measure, using the definition of operator synthesis is due to Arveson [1]. We apply this result to obtain an analogue of a result of Froelich [8]: a tensor formula for the algebras associated with the pre-orders defined by closed unital subsemigroups of G.

In [20], Varopoulos conducted a study of spectral synthesis for algebras which are the projective tensor products of several commutative semi-simple regular Banach algebras, in particular tensor products of commutative C*-algebras. He used certain relationships between these algebras and the Fourier algebras of compact Abelian groups. One of his remarkable results is that for a compact Abelian group G, there is a natural isometric imbedding of $\mathrm{A}(G)$ into $\mathcal{C}(G)\hat{\otimes}\mathcal{C}(G)$. Moreover, he showed that a closed subset $E \subset G$, is spectral for $\mathrm{A}(G)$ exactly when $E^* = \{(s,t) \in G \times G : s + t \in E\}$ is spectral for $\mathcal{C}(G)\hat{\otimes}\mathcal{C}(G)$. That $\mathrm{A}(G)$ is the closed span of characters is crucial for some of his calculations.

In Section 2, we generalize Varopoulos' imbedding, but use the Haagerup tensor product instead of the projective tensor product. By Grothendieck's inequality, this amounts only to a renorming of Banach algebras. However, this renorming is critical to obtaining an isometric imbedding. It is necessary for us implicitly to use the theory of completely bounded maps of operator algebras, i.e. in the papers of Blecher and Smith ([17], [2]). This is no accident, as it is becoming apparent that the theory of operator spaces is an invaluable tool for the study of the harmonic analysis of non-Abelian groups.

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With our imbedding result in hand, we are then able to generalize Varopoulos' spectral synthesis result, mentioned above, to arbitrary compact groups. We do this in Section 3.

In Section 4, we obtain an analogous result to that in Section 3, but obtain sets of operator synthesis in $G \times G$ with respect to Haar measure, in the sense of Arveson ([1]). We then obtain an analogue of the tensor formula for algebras associated to the pre-order defined by unital semi-groups, due to Froelich [8].

In Section 5 we investigate the analogue of the imbedding from Section 2 for a non-compact group. We show that the most natural analogue of this imbedding fails. Hence the methods of this paper are special to compact groups.

We attempt to give sufficient background references for all of the concepts involved, both at the beginning of each aforementioned section, and in Section 1.

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1 Preliminaries and Notation

For any Banach space $\mathcal{X}$, let $\mathrm{b}_1(\mathcal{X})$ denote the unit ball of $\mathcal{X}$, $\mathcal{X}^*$ denote the space of all bounded linear functionals on $\mathcal{X}$, and $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on $\mathcal{X}$. For a subset $\mathcal{Y}$ of $\mathcal{X}$, let $\mathcal{Y}^\perp$ be the annihilator of $\mathcal{Y}$ in $\mathcal{X}^*$. If $T \in \mathcal{B}(\mathcal{X})$, let $T^* \in \mathcal{B}(\mathcal{X}^*)$ denote the adjoint map. If $\mathcal{M}$ is a von Neumann algebra, let $\mathcal{B}^\sigma(\mathcal{M})$ denote the space of bounded operators which are ultra-weakly continuous on $\mathcal{M}$, and $\mathcal{M}_*$ denote the predual. Then $\mathcal{B}^\sigma(\mathcal{M}) = \{T^* : T \in \mathcal{B}(\mathcal{M}_*)\}$.

Let G be a compact group with normalized Haar integration $\int_G \ldots ds$. Let $\mathrm{L}^p(G)$ denote the space of (almost everywhere equivalence classes) of p-integrable functions, for $p = 1, 2$, with norm $\|\cdot\|_p$. The *left regular representation*, $\lambda : G \to \mathcal{B}(\mathrm{L}^2(G))$ is given by $\lambda(s)g(t) = g(s^{-1}t)$ for all s in G, g in $\mathrm{L}^2(G)$ and almost every t in G. Define $\lambda_1 : \mathrm{L}^1(G) \to \mathcal{B}(\mathrm{L}^2(G))$ by $\lambda_1(f)g = \int_G f(s)\lambda(s)g ds = f*g$ for f in $\mathrm{L}^1(G)$ and g in $\mathrm{L}^2(G)$. Let

$$\mathrm{C}^*(G) = \overline{\{\lambda_1(f) : f \in \mathrm{L}^1(G)\}} \subset \mathcal{B}(\mathrm{L}^2(G))$$

be the (reduced =) enveloping C*-algebra of G and

$$\mathrm{VN}(G) = \overline{\mathrm{span}}^{\mathrm{WOT}}\{\lambda(s) : s \in G\} = \overline{\mathrm{C}^*(G)}^{\mathrm{WOT}} \subset \mathcal{B}(\mathrm{L}^2(G))$$

be the von Neumann algebra of G, where WOT indicates "weak operator topology" closure.

The *Fourier algebra* is the family of functions

$$\mathrm{A}(G) = \{s \mapsto \langle \lambda(s)f|g\rangle : f, g \in \mathrm{L}^2(G)\}$$

as defined by Eymard in [7]. Then $\mathrm{A}(G) \cong \mathrm{VN}(G)_*$ via $\langle\langle \lambda(\cdot)f|g\rangle, T\rangle = \langle Tf|g\rangle$, and is normed thusly, with the norm denoted by $\|\cdot\|_{\mathrm{A}}$. $\mathrm{A}(G)$ is a semi-simple unital Banach algebra of functions on G with spectrum G. If $u \in \mathrm{A}(G)$, then there exist f and g in $\mathrm{L}^2(G)$ such that

(i) $u = \langle \lambda(\cdot)f|g\rangle$, and

(ii) $\|u\|_{\mathrm{A}} = \sup\{|\langle u, T\rangle| : T \in \mathrm{b}_1(\mathrm{VN}(G))\} = \|f\|_2 \|g\|_2$.

(See [7] or [11, 34.15].)

2 Imbedding the Fourier Algebra into the Varopoulos Algebra

As above, we let G denote a *compact* group for this entire section.

We start with a result from [3], which is given there in more general form. We sketch the proof in our case for convenience of the reader.

Proposition 2.1 *The following are equivalent for a complex valued function u on G.*

(i) $u \in \mathrm{b}_1(\mathrm{A}(G))$.

(ii) *There is an operator M_u in $\mathrm{b}_1(\mathcal{B}^\sigma(\mathrm{VN}(G)))$ such that*

$$M_u\lambda(s) = u(s)\lambda(s)$$

for s in G.

(iii) *There is an operator $\tilde{M}_u$ in $\mathrm{b}_1(\mathcal{B}(\mathrm{C}^*(G)))$ such that*

$$\tilde{M}_u\lambda_1(f) = \lambda_1(u{\cdot}f)$$

for f in $\mathrm{L}^1(G)$.

Proof: (i)⇒(ii) Since $\mathrm{A}(G)$ is a Banach algebra, if $u \in \mathrm{b}_1(\mathrm{A}(G))$, then the multiplication operator by u, m_u, is in $\mathrm{b}_1(\mathcal{B}(\mathrm{A}(G)))$. Then $M_u = m_u^* \in \mathrm{b}_1(\mathcal{B}^\sigma(\mathrm{VN}(G)))$.

(ii)⇒(iii) If f in $\mathrm{L}^1(G)$, we can realize $\lambda_1(f) = \int_G f(s)\lambda(s)ds$ as a WOT-converging integral. Then $\tilde{M}_u = M_u|_{\mathrm{C}^*(G)} \in \mathrm{b}_1(\mathcal{B}(\mathrm{C}^*(G)))$.

(iii)⇒(i) We have that $\mathrm{A}(G) \cong \mathrm{C}^*(G)^*$ via the dual pairing which gives $\mathrm{A}(G) \cong \mathrm{VN}(G)_*$. Let $m_u = \tilde{M}_u^*$. Then $u = m_u 1 \in \mathrm{A}(G)$ with $\|u\|_{\mathrm{A}} = \|m_u 1\|_{\mathrm{A}} \leq \left\|\tilde{M}_u\right\| \leq 1$. □

Let $\mathcal{C}(G)$ denote the C*-algebra of continuous complex-valued functions on G and

$$\mathrm{V}(G) = \mathcal{C}(G) \otimes^h \mathcal{C}(G)$$

where $\otimes^h$ denotes the Haagerup tensor product. Let us recall the definition of this space: it is the completion of the algebraic tensor product $\mathcal{C}(G) \otimes \mathcal{C}(G)$ with respect to the norm defined by $\|w\|_{\mathrm{V}} = \inf\left\{\|\sum_{i=1}^n |\varphi_i|^2\|_\infty^{1/2} \|\sum_{i=1}^n |\psi_i|^2\|_\infty^{1/2} : w = \sum_{i=1}^n \varphi_i \otimes \psi_i\right\}$. We note that every w in $\mathrm{V}(G)$ can be represented as a norm converging (infinite) sum of elementary tensors, and for such w we have

$$\|w\|_{\mathrm{V}} = \inf\left\{\left\|\sum_{i=1}^\infty |\varphi_i|^2\right\|_\infty^{1/2} \left\|\sum_{i=1}^\infty |\psi_i|^2\right\|_\infty^{1/2} : w = \sum_{i=1}^\infty \varphi_i \otimes \psi_i\right\} \tag{1}$$

where the sums $\sum_{i=1}^{\infty} |\varphi_i|^2$ and $\sum_{i=1}^{\infty} |\psi_i|^2$ are uniformly convergent. Moreover, the infimum is attained. (See [2], for example.) For φ in $\mathcal{C}(G)$, let M_φ denote the multiplication operator on $\mathrm{L}^2(G)$. For $w = \sum_{i=1}^{\infty} \varphi_i \otimes \psi_i$ in $\mathrm{V}(G)$, the operator $T_w : \mathcal{B}(\mathrm{L}^2(G)) \to \mathcal{B}(\mathrm{L}^2(G))$ given by

$$T_w S = \sum_{i=1}^{\infty} M_{\varphi_i} S M_{\psi_i} \tag{2}$$

(with norm convergent sum) is in $\mathcal{B}^\sigma(\mathcal{B}(\mathrm{L}^2(G)))$, with $\|T_w\| = \|w\|_{\mathrm{V}}$ (as in [17]). That $\mathrm{V}(G)$ is a Banach algebra (under pointwise multiplication) follows from that for v and w in $\mathrm{V}(G)$, $T_{vw} = T_v T_w$.

On $\mathcal{C}(G) \otimes \mathcal{C}(G)$, the Haagerup norm is exactly Grothendieck's H-norm, and thus is equivalent to the projective tensor norm by Grothendieck's inequality. Hence $\mathrm{V}(G)$ is the algebra studied by Varopoulos in [20], renormed. We thus call it the *Varopoulos algebra.*

The invariant part of the Varopoulos algebra is given by

$$\mathrm{V}_{\mathrm{inv}}(G) = \{w \in \mathrm{V}(G) : w(sr, tr) = w(s,t) \text{ for } s,t,r \in G\}.$$

It is easily seen to be a closed subalgebra of $\mathrm{V}(G)$.

Theorem 2.2 *The map $N : \mathrm{A}(G) \to \mathrm{V}_{\mathrm{inv}}(G)$ given for s,t in G by*

$$Nu(s,t) = u(st^{-1})$$

is an isometric isomorphism.

Our map N above is denoted M in [20] and [9], a symbol which has already been over used here. Varopoulos defines M only for Abelian compact groups, and shows it is an isometry, but with $\mathrm{V}(G) = \mathcal{C}(G)\hat{\otimes}\mathcal{C}(G)$ (the projective tensor product). Thus for Abelian groups, the Haagerup and projective norms coincide on $\mathrm{V}_{\mathrm{inv}}(G)$. If we could conclude that any continuous function in $\mathrm{V}^\infty(G)$ (defined in Section 4, *infra*) is in $\mathrm{V}(G)$, then this result would follow from [9]. However, the immediately preceding statement is not apparent, and we feel that our proof is illuminating.

Proof of Theorem 2.2: First some notation is needed. Let $\widehat{G}$ denote set of (equivalence classes of) irreducible continuous unitary representations of G. For π in $\widehat{G}$ let $\mathcal{T}_\pi \subset \mathrm{L}^2(G)$ be the space spanned by matrix coefficients of π. By the Weyl-Peter Theorem [11, 27.40], $\mathrm{L}^2(G) = \bigoplus_{\pi \in \widehat{G}} \mathcal{T}_\pi$ (Hilbertian direct sum). Moreover, each $\mathcal{T}_\pi$ is λ-invariant (see [11, 27.20]). Thus for any $F \subset \widehat{G}$, if P_F denotes the orthogonal projection from $\mathrm{L}^2(G)$ onto $\mathcal{T}_F = \bigoplus_{\pi \in F} \mathcal{T}_\pi$, then $\lambda(s) P_F = P_F \lambda(s)$ for any s in G. Let $\mathcal{F}$ denote the set of all finite subsets of $\widehat{G}$, directed by inclusion. For each π in $\widehat{G}$, let $\{e_i^{(\pi)} : i = 1, \ldots, d_\pi^2\}$ ($d_\pi = \dim \pi$) be an ortho-normal basis for $\mathcal{T}_\pi$.

Now let $u \in \mathrm{A}(G)$ and write $u = \langle \lambda(\cdot) f | g \rangle$ where $\|u\|_{\mathrm{A}} = \|f\|_2 \|g\|_2$. Then $Nu(s,t) = \langle \lambda(t)^* f | \lambda(s)^* g \rangle$ for s,t in G. For π in $\widehat{G}$ and $i = 1, \ldots, d_\pi^2$, let

$$\varphi_{\pi,i}(s) = \langle e_i^{(\pi)} | \lambda(s)^* g \rangle \text{ and } \psi_{\pi,i}(s) = \langle \lambda(s)^* f | e_i^{(\pi)} \rangle$$

for s in G. Note that at most countable many $\varphi_{\pi,i}$'s and $\psi_{\pi,i}$'s are non-zero functions, since, for example, $\varphi_{\pi,i} = \langle \lambda(\cdot) e_i^{(\pi)} | g \rangle$, and the are at most countable many π's for which $P_{\{\pi\}} g \neq 0$. It follows from Parseval's formula that

$$Nu(s,t) = \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} \varphi_{\pi,i}(s) \psi_{\pi,i}(t)$$

for all s, t in G. We will have that $Nu \in \mathrm{V}(G)$, and hence in $\mathrm{V}_{\mathrm{inv}}(G)$, if we can show that $\sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} |\varphi_{\pi,i}|^2$ and $\sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} |\psi_{\pi,i}|^2$ converge uniformly in $\mathcal{C}(G)$. For s in G, using Parseval's formula, that each P_F commutes with $\lambda(s)$ and that $P_F \xrightarrow{F \in \mathcal{F}} \mathrm{id}_{\mathrm{L}^2(G)}$ in the WOT, we have

$$\begin{aligned} \sum_{\pi \in F} \sum_{i=1}^{d_\pi^2} |\varphi_{\pi,i}(s)|^2 &= \sum_{\pi \in F} \sum_{i=1}^{d_\pi^2} \langle e_i^{(\pi)} | \lambda(s)^* g \rangle \langle \lambda(s)^* g | e_i^{(\pi)} \rangle \\ &= \langle P_F \lambda(s)^* g | P_F \lambda(s)^* g \rangle = \langle P_F g | g \rangle \\ &\xrightarrow{F \in \mathcal{F}} \langle g | g \rangle = \|g\|_2^2 \end{aligned}$$

for any s in G, and thus convergence is uniform over s in G. Hence we have $\sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} |\varphi_{\pi,i}|^2 = \|g\|_2^2 1$ with uniform convergence. Similarly, $\sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} |\psi_{\pi,i}|^2 = \|f\|_2^2 1$. Thus we have

$$Nu = \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} \varphi_{\pi,i} \otimes \psi_{\pi,i} \in \mathrm{V}_{\mathrm{inv}}(G)$$

with

$$\|Nu\|_{\mathrm{V}} \leq \left\| \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} |\varphi_{\pi,i}|^2 \right\|_\infty \left\| \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi^2} |\psi_{\pi,i}|^2 \right\|_\infty = \|f\|_2 \, \|g\|_2 = \|u\|_{\mathrm{A}}.$$

To see that N is surjective, let $w = \sum_{i=1}^\infty \varphi_i \otimes \psi_i \in \mathrm{V}_{\mathrm{inv}}(G)$, with $\|w\|_{\mathrm{V}} = \|\sum_{i=1}^\infty |\varphi_i|^2\|_\infty^{1/2} \cdot \|\sum_{i=1}^\infty |\psi_i|^2\|_\infty^{1/2}$. Let $u(s) = w(s,e)$ for s in G, so $u(st^{-1}) = w(s,t)$ for s, t in G. We will show that $u \in \mathrm{A}(G)$ with $\|u\|_{\mathrm{A}} \leq \|w\|_{\mathrm{V}}$. Clearly $Nu = w$. Let $s \in G$ and $f \in \mathrm{L}^2(G)$. Then for almost all t in G,

$$\begin{aligned} T_w \lambda(s) f(t) &= \sum_{i=1}^\infty M_{\varphi_i} \lambda(s) M_{\psi_i} f(t) \\ &= \sum_{i=1}^\infty \varphi_i(t) \psi_i(s^{-1}t) f(s^{-1}t) \qquad (3) \\ &= w(t, s^{-1}t) f(s^{-1}t) = u(s) \lambda(s) f(t). \end{aligned}$$

Since $T_w \in \mathcal{B}^\sigma(\mathcal{B}(\mathrm{L}^2(G)))$, $T_w \mathrm{VN}(G) \subset \mathrm{VN}(G)$. Thus if we let $M_u = T_w|_{\mathrm{VN}(G)}$, M_u satisfies Proposition 2.1 (ii), with $\|M_u\| \leq \|T_w\| = \|w\|_{\mathrm{V}}$, so $u \in \mathrm{A}(G)$ with $\|u\|_{\mathrm{A}} \leq \|w\|_{\mathrm{V}}$. (This part of the proof should be compared with [15, Theo. 6.4].) $\square$

If $r \in G$, and $w \in \mathrm{V}(G)$, define $r \cdot w$ by

$$r \cdot w(s,t) = w(sr, tr)$$

for s, t in G. Then $(r, w) \mapsto r \cdot w$ is a continuous action of G on $\mathrm{V}(G)$ by isometries. Thus for f in $\mathrm{L}^1(G)$ and w in $\mathrm{V}(G)$, we may define a convolution

$$f \cdot w = \int_G f(r)\, r \cdot w \, dr \tag{4}$$

where the integral is $\mathrm{V}(G)$-valued. Indeed, if $f \in \mathcal{C}(G)$, then $r \mapsto f(r)\, r \cdot w$ is continuous, and hence Bochner integrable. Moreover, $\|f \cdot w\|_{\mathrm{V}} \leq \|f\|_1 \|w\|_{\mathrm{V}}$, for such f. Hence we may unambiguously define $f \cdot w$ for an arbitrary f in $\mathrm{L}^1(G)$ just by taking it as a limit of continuous functions. If $\{e_U\}$ is the bounded approximate identity for $\mathrm{L}^1(G)$ formed by letting $e_U = \frac{1}{m(U)} \chi_U$ (normalized indicator function) for decreasing neighbourhoods U of e, then it is clear that $e_U \cdot w \to w$, as $U \searrow \{e\}$, for all w in $\mathrm{V}(G)$. Then by Cohen's Factorization Theorem [11, 32.22], and [11, 32.33 (a)], $e_\alpha \cdot w \to w$ for any bounded approximate identity $\{e_\alpha\}$ in $\mathrm{L}^1(G)$.

We will now define a map which will be useful in the sequel.

Proposition 2.3 *The map $P : \mathrm{V}(G) \to \mathrm{V}_{\mathrm{inv}}(G)$ given for s, t in g by*

$$Pw(s,t) = \int_G w(sr, tr) dr$$

is a contractive projection and a $\mathrm{V}_{\mathrm{inv}}(G)$-module map.

Proof: Clearly $Pw = 1 \cdot w$, where 1 is the constant function equal to 1, and thus P is a contraction. Its range is in $\mathrm{V}_{\mathrm{inv}}(G)$ by the left invariance of the Haar measure. Since $\mathrm{V}_{\mathrm{inv}}(G) = \{w \in \mathrm{V}(G) : r \cdot w = w \text{ for all } r \text{ in } G\}$, it is clear that $P|_{\mathrm{V}_{\mathrm{inv}}(G)} = \mathrm{id}_{\mathrm{V}_{\mathrm{inv}}(G)}$. That P is a $\mathrm{V}_{\mathrm{inv}}(G)$-module map follows from the same observation. □

3 Spectral Synthesis

Let $\mathcal{A}$ be a unital semi-simple regular commutative Banach algebra with spectrum X, which is thus a compact Hausdorff space. We will identify $\mathcal{A}$ as a subalgebra of $\mathcal{C}(X)$ in our notation. If $E \subset X$ is closed, let

$$\begin{aligned} \mathrm{I}_{\mathcal{A}}(E) &= \{a \in \mathcal{A} : a(x) = 0 \text{ for } x \in E\} \\ \mathrm{I}^0_{\mathcal{A}}(E) &= \{a \in \mathcal{A} : \mathrm{supp}(a) \cap E = \varnothing\} \\ \text{and } \mathrm{J}_{\mathcal{A}}(E) &= \overline{\mathrm{I}^0_{\mathcal{A}}(E)}. \end{aligned}$$

(Note that for φ in $\mathcal{C}(X)$, $\mathrm{supp}(\varphi) = \overline{\{x \in X : \varphi(x) \neq 0\}}$.) We say that E is *spectral* for $\mathcal{A}$ if $\mathrm{I}_{\mathcal{A}}(E) = \mathrm{J}_{\mathcal{A}}(E)$. By [11, 39.18] this is equivalent to the usual notion of spectral synthesis.

We, as usual, let G be a fixed *compact* group. We note that $\mathrm{A}(G)$ and $\mathrm{V}(G)$ are semi-simple regular Banach algebras with spectra G and $G\times G$, respectively. For $\mathrm{A}(G)$ this is shown in [7] and [11, Sec. 34]; for $\mathrm{V}(G)$ this is discussed in [20, 1.5], where we note that $\mathrm{V}(G) = \mathcal{C}(G)\hat{\otimes}\mathcal{C}(G)$.

Recalling the definitions of N and P from the previous section, define $Q : \mathrm{V}(G) \to \mathrm{A}(G)$ by $Q = N^{-1}\circ P$, so for w in $\mathrm{V}(G)$ and s, t in G,

$$Qw(s) = \int_G w(sr, r)dr.$$

Let $\theta : G\times G \to G$ be given by $\theta(s,t) = st^{-1}$. Then θ is an open continuous map. Moreover, $Nu = u\circ\theta$, for u in $\mathrm{A}(G)$.

The following theorem is proved for compact Abelian groups in [20, 8.2]. The proof is motivated by the one given there.

Theorem 3.1 *A closed set $E \subset G$ is spectral for $\mathrm{A}(G)$ if and only if $E^* = \theta^{-1}(E) = \{(s,t) \in G\times G : st^{-1} \in E\}$ is spectral for $\mathrm{V}(G)$.*

Proof: We will write $\mathrm{I}_{\mathrm{A}}(E)$ for $\mathrm{I}_{\mathrm{A}(G)}(E)$, $\mathrm{I}_{\mathrm{V}}(E^*)$ for $\mathrm{I}_{\mathrm{V}(G)}(E^*)$, etc. It is clear that

$$u \in \mathrm{I}_{\mathrm{A}}(E) \Leftrightarrow Nu \in \mathrm{I}_{\mathrm{V}}(E^*). \tag{5}$$

We also have that

$$u \in \mathrm{J}_{\mathrm{A}}(E) \Leftrightarrow Nu \in \mathrm{J}_{\mathrm{V}}(E^*). \tag{6}$$

To see this, first suppose that $u \in \mathrm{J}_{\mathrm{A}}(E)$. Then there is a sequence $\{u_n\} \subset \mathrm{I}^0_{\mathrm{A}}(E)$ such that $u_n \to u$, so $Nu_n \to Nu$. Since for each n, $\mathrm{supp}(u_n) \cap E = \varnothing$, $\mathrm{supp}(Nu_n) \cap E^* = \theta^{-1}(\mathrm{supp}(u_n)\cap E) = \varnothing$, and hence $\{Nu_n\} \subset \mathrm{I}^0_{\mathrm{V}}(E^*)$, showing that $Nu \in \mathrm{J}_{\mathrm{V}}(E^*)$. Conversely, if $Nu \in \mathrm{J}_{\mathrm{V}}(E^*)$, there is a sequence $\{w_n\} \subset \mathrm{I}^0_{\mathrm{V}}(E^*)$ such that $w_n \to Nu$, so $Qw_n \to QNu = u$. For each n it is clear that $\mathrm{supp}(Qw_n) \subset \theta(\mathrm{supp}(w_n))$, and $\theta(\mathrm{supp}(w_n)) \cap E = \theta(\mathrm{supp}(w_n) \cap E^*) = \varnothing$. Thus $\mathrm{supp}(Qw_n) \cap E = \varnothing$ for each n, and we have $\{Qw_n\} \subset \mathrm{I}^0_{\mathrm{A}}(E)$, so $u \in \mathrm{J}_{\mathrm{A}}(E)$.

If E^* is spectral for $\mathrm{V}(G)$, then it follows from (5) and (6) that E is spectral for $\mathrm{A}(G)$.

We now suppose that E is spectral for $\mathrm{A}(G)$. For each π in $\widehat{G}$, let $\mathcal{H}_\pi$ be the representation space of π, and $\{e_1, \ldots, e_{d_\pi}\}$ be an ortho-normal basis for $\mathcal{H}_\pi$. For π in $\widehat{G}$ and $i, j = 1, \ldots, d_\pi$,

$$u^{(\pi)}_{ij} = \langle \pi(\cdot)e_j | e_i \rangle.$$

By the Weyl-Peter Theorem, $\mathcal{H}_0 = \mathrm{span}\{u^{(\pi)}_{ij} : \pi \in \widehat{G}, i, j = 1, \ldots, d_\pi\}$ is dense in $\mathrm{L}^2(G)$ and thus dense in $\mathrm{L}^1(G)$. Hence we can find a net $\{u_\alpha\} \subset \mathcal{H}_0$ which forms a bounded approximate identity for $\mathrm{L}^1(G)$.

Now suppose that $w \in \mathrm{I}_{\mathrm{V}}(E^*)$. For each π in $\widehat{G}$ we define "matrix-valued" functions $w^{(\pi)}, \tilde{w}^{(\pi)} : G\times G \to \mathcal{B}(\mathcal{H}_\pi)$ by

$$w^{(\pi)}(s,t) = \int_G w(sr, tr)\pi(r)dr \quad \text{and} \quad \tilde{w}^{(\pi)}(s,t) = \pi(s)w^{(\pi)}(s,t).$$

Then for any s, t, r in G, we have that

$$\tilde{w}^{(\pi)}(sr, tr) = \tilde{w}^{(\pi)}(s, t). \tag{7}$$

Let us fix π for the remainder of this paragraph. For $i, j = 1, \ldots, d_\pi$ we let $w_{ij}^{(\pi)}$ and $\tilde{w}_{ij}^{(\pi)}$ denote the matrix coefficients of $w^{(\pi)}$ and $\tilde{w}^{(\pi)}$ with respect to $\{e_1, \ldots, e_{d_\pi}\}$. Then for each i, j,

$$w_{ij}^{(\pi)} = u_{ij}^{(\pi)} \cdot w. \tag{8}$$

Since $w \in \mathrm{I_V}(E^*)$, $f \cdot w \in \mathrm{I_V}(E^*)$ for each f in $\mathrm{L}^1(G)$, and hence each $w_{ij}^{(\pi)} \in \mathrm{I_V}(E^*)$. Now for each i, j and s, t in G,

$$\tilde{w}_{ij}^{(\pi)}(s, t) = \sum_{k=1}^{d_\pi} u_{ik}^{(\pi)}(s) w_{kj}^{(\pi)}(s, t)$$

$$\text{i.e. } \tilde{w}_{ij}^{(\pi)} = \sum_{k=1}^{d_\pi} u_{ik}^{(\pi)} \otimes 1 \; w_{kj}^{(\pi)} \tag{9}$$

so $\tilde{w}_{ij}^{(\pi)} \in \mathrm{I_V}(E^*)$. By (7), $\tilde{w}_{ij}^{(\pi)} \in \mathrm{V_{inv}}(G)$. It then follows from (5), (6) and the identity $\mathrm{I_A}(E) = \mathrm{J_A}(E)$, that $\tilde{w}_{ij}^{(\pi)} \in \mathrm{J_V}(E^*)$. Using that $w^{(\pi)}(s, t) = \pi(s^{-1})\tilde{w}^{(\pi)}(s, t)$, we have that

$$w_{ij}^{(\pi)} = \sum_{k=1}^{d_\pi} \check{u}_{ik}^{(\pi)} \otimes 1 \; \tilde{w}_{kj}^{(\pi)} \tag{10}$$

(where $\check{u}(s) = u(s^{-1})$), so each $w_{ij}^{(\pi)} \in \mathrm{J_V}(E^*)$.

Finally, we let $\{u_\alpha\}$ be the bounded approximate identity for $\mathrm{L}^1(G)$ chosen from $\mathcal{H}_0$, as above. By (8), for each α, $u_\alpha \cdot w \in \mathrm{span}\{w_{ij}^{(\pi)} : \pi \in \widehat{G}, i, j = 1, \ldots, d_\pi\} \subset \mathrm{J_V}(E^*)$. Since $\mathrm{V}(G)$ is an essential $\mathrm{L}^1(G)$-module, $u_\alpha \cdot w \to w$, and hence $w \in \mathrm{J_V}(E^*)$. □

We remark that amongst the subsets of G, which are spectral for $\mathrm{A}(G)$ are closed subgroups, by a Theorem of Herz [10], and their cosets, the latter because right and left translates are isometric automorphisms of $\mathrm{A}(G)$. Also Ditkin sets, i.e. sets whose boundaries contain no perfect sets, are spectral by [11, 39.26 and 39.31].

By a theorem of Malliavin, of which a proof is given by Varopoulos [20, 9.2.2] (also see [11, 42.18]), every compact Abelian group contains a set which fails spectral synthesis for $\mathrm{A}(G)$. In [21], it is shown that every infinite compact group contains an infinite Abelian subgroup. Hence by [11, 42.27], every infinite compact group contains a set which fails spectral synthesis for $\mathrm{A}(G)$. Thus we obtain the following.

Corollary 3.2 *If G is infinite, then spectral synthesis fails for $\mathrm{V}(G)$.*

4 Operator Synthesis

Let $\mathcal{A}$ and X be as at the beginning of Section 3. For τ in $\mathcal{A}^*$ and a in $\mathcal{A}$, define $a\tau$ in $\mathcal{A}^*$ by $a\tau(b) = \tau(ab)$. Define the support of τ by

$$\mathrm{supp}(\tau) = \{x \in X : a\tau \neq 0 \text{ whenever } a(x) \neq 0\}.$$

Using the regularity of $\mathcal{A}$ and functional calculus, it can be shown that $\mathrm{supp}(\tau)$ consists of all x in X such that for any neighbourhood U of x, there is a in $\mathcal{A}$ for which $\mathrm{supp}(a) \subset U$ and $\tau a \neq 0$. Then it is well known that for a closed set $E \subset X$,

$$\mathrm{J}_{\mathcal{A}}(E)^{\perp} = \{\tau \in \mathcal{A}^* : \mathrm{supp}(\tau) \subset E\} \tag{11}$$

and hence E is spectral for $\mathcal{A}$ if and only if $\mathrm{I}_{\mathcal{A}}(E)^{\perp} = \{\tau \in \mathcal{A}^* : \mathrm{supp}(\tau) \subset E\}$. If $S \in \mathrm{VN}(G)$, then we will denote by $\mathrm{supp}_{\mathrm{VN}}(S)$ the support of S qua linear functional on $\mathrm{A}(G)$. If $u \in \mathrm{A}(G)$, then by Proposition 2.1 we see that $uS = M_u S = T_{Nu} S$.

We now wish to establish a framework in which we may recall the definition of operator synthesis due to Arveson in [1]. To make use of some of the results from [1] and [16] needed in the sequel we will assume that G *is separable*, and hence metrizable by [12, 8.3]. As above, the assumption that G is compact is still in force.

Let

$$\mathrm{T}(G) = \mathrm{L}^2(G) \hat{\otimes} \mathrm{L}^2(G)$$

where $\hat{\otimes}$ denotes the projective tensor product. A set $K \subset G \times G$ is called *marginally null* (with respect to Haar measure $m \times m$) if there are sets $N_1, N_2 \subset G$ such that $K \subset (N_1 \times G) \cup (G \times N_2)$ and $m(N_1) = m(N_2) = 0$. We note that if $E \subset G$ is closed and non-empty, then E^* is never marginally null. In fact if G is infinite, $\{e\}^*$ has measure 0 but is not marginally null. By [1, 2.2.7], an element ω in $\mathrm{T}(G)$ may be regarded as a function on $G \times G$, defined up to marginally null sets: for marginally almost every (s,t) in $G \times G$, let

$$\omega(s,t) = \sum_{i=1}^{\infty} f_i(t) g_i(s) \text{ where } \omega = \sum_{i=1}^{\infty} f_i \otimes g_i \text{ in } \mathrm{L}^2(G) \hat{\otimes} \mathrm{L}^2(G).$$

Note that the order of s and t above is purposeful, and gives us some technical simplifications in the sequel. For ω in $\mathrm{T}(G)$, define $\mathrm{supp}(\omega) = \{(s,t) \in G \times G : \omega(s,t) \neq 0\}$. Note that $\mathrm{supp}(\omega)$ is defined only up to marginally null sets. If F is a closed subset of $G \times G$, let

$$\begin{aligned} \Phi(F) &= \{\omega \in \mathrm{T}(G) : \mathrm{supp}(\omega) \cap F = \varnothing\} \\ \Phi_0(F) &= \{\omega \in \mathrm{T}(G) : \mathrm{supp}(\omega) \cap U = \varnothing \text{ for some neighbourhood } U \text{ of } F\} \\ \text{and } \Psi(F) &= \overline{\Phi_0(F)}. \end{aligned}$$

By [1, 2.2.8], $\Phi(F)$ is closed in $\mathrm{T}(G)$. A closed subset F of $G \times G$ is a set of *operator synthesis* (or is *synthetic*) with respect to $m \times m$, if $\Phi(F) = \Psi(F)$. We note that there is a more general notion of operator synthesis which does not make explicit reference to the underlying topology on the measure space. See [6] or [16].

It will be convenient for us to make use of a space similar to $\mathrm{V}(G)$. Let $\mathrm{L}^\infty(G)$ denote the usual space of (equivalence classes of) essentially bounded functions on G with norm $\|\cdot\|_\infty$. Then let

$$\mathrm{V}^\infty(G) = \mathrm{L}^\infty(G) \otimes^{w^*h} \mathrm{L}^\infty(G)$$

where $\otimes^{w^*h}$ denotes the weak* Haagerup tensor product of [2]. We note that $\mathrm{V}^\infty(G)$ is the weak* closure of $\mathrm{L}^\infty(G) \otimes^h \mathrm{L}^\infty(G)$ in $(\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G))^*$, and, in fact, comprises all of $(\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G))^*$. The space $\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G)$ is defined differently than $\mathrm{V}(G)$, and requires the maximal operator space structure on $\mathrm{L}^1(G)$. Will will not need to go into detail about this, but refer the interested reader to [5]. In analogy with (1) every w in $\mathrm{V}^\infty(G)$ admits a representation as a weak* converging (infinite) sum of elementary tensors, and for such w we have

$$\|w\|_{\mathrm{V}^\infty} = \inf\left\{ \left\| \sum_{i=1}^\infty |\varphi_i|^2 \right\|_\infty^{1/2} \left\| \sum_{i=1}^\infty |\psi_i|^2 \right\|_\infty^{1/2} : w = \sum_{i=1}^\infty \varphi_i \otimes \psi_i \right\}$$

but with the sums $\sum_{i=1}^\infty |\varphi_i|^2$ and $\sum_{i=1}^\infty |\psi_i|^2$ converging weak* (instead of uniformly). Also, as in (1), the infimum is attained. Note that $\mathrm{V}(G)$ is a norm-closed subspace of $\mathrm{V}^\infty(G)$. For w in $\mathrm{V}^\infty(G)$, we can define T_w just as in (2) (but with the sum $T_w S = \sum_{i=1}^\infty M_{\varphi_i} S M_{\psi_i}$ converging ultra-weakly), and we have $\|w\|_{\mathrm{V}^\infty} = \|T_w\|$. We note that $\mathrm{V}^\infty(G)$ is a Banach algebra. We can identify elements of $\mathrm{V}^\infty(G)$ as functions up to equivalence on marginally null sets. This follows from Proposition 4.1, *infra*.

We say that a complex valued function w on $G \times G$ is a multiplier of $\mathrm{T}(G)$ if for any ω in $\mathrm{T}(G)$, $(s,t) \mapsto w(s,t)\omega(s,t)$ defines an element $m_w\omega$ of $\mathrm{T}(G)$ such that $\|m_w\omega\|_{\mathrm{T}} \leq C \|\omega\|_{\mathrm{T}}$ where C is some constant, so w defines a bounded linear operator m_w on $\mathrm{T}(G)$. It is clear that two multipliers w and w' satisfy $m_w = m_{w'}$ if $w = w'$ marginally almost everywhere. We say that w and w' are equivalent if $m_w = m_{w'}$. We denote the space of equivalence classes of multipliers of $\mathrm{T}(G)$ by $\mathrm{MT}(G)$.

Recall that $\mathcal{B}(\mathrm{L}^2(G)) \cong \mathrm{T}(G)^*$ via $\langle f \otimes g, S \rangle = \langle Sf | \bar{g} \rangle$.

Proposition 4.1 *$\mathrm{V}^\infty(G) = \mathrm{MT}(G)$ with $\|w\|_{\mathrm{V}^\infty} = \|m_w\|$ for w in $\mathrm{V}^\infty(G)$.*

Proof: Let $w \in \mathrm{V}^\infty(G)$ with $w = \sum_{i=1}^\infty \varphi_i \otimes \psi_i$ and $\|w\|_{\mathrm{V}^\infty} = \|\sum_{i=1}^\infty |\varphi_i|^2\|_\infty^{1/2} \|\sum_{i=1}^\infty |\psi_i|^2\|_\infty^{1/2}$. It suffices to show that $m_w f \otimes g \in \mathrm{T}(G)$ with $\|m_w f \otimes g\|_{\mathrm{T}} \leq \|w\|_{\mathrm{V}^\infty} \|f\|_2 \|g\|_2$ for any f and g in $\mathrm{L}^2(G)$, in order to establish that $w \in \mathrm{MT}(G)$ with $\|m_w\| \leq \|w\|_{\mathrm{V}^\infty}$. Using Hölder's inequality we have that

$$\begin{aligned} \|m_w f \otimes g\|_{\mathrm{T}} &= \left\| \sum_{i=1}^\infty M_{\psi_i} f \otimes M_{\varphi_i} g \right\|_{\mathrm{T}} \leq \sum_{i=1}^\infty \|M_{\psi_i} f\|_2 \, \|M_{\varphi_i} g\|_2 \\ &\leq \left(\sum_{i=1}^\infty \|M_{\psi_i} f\|_2^2 \right)^{1/2} \left(\sum_{i=1}^\infty \|M_{\varphi_i} g\|_2^2 \right)^{1/2}. \end{aligned}$$

Then, using Tonelli's Theorem, we have

$$\sum_{i=1}^{\infty} \|M_{\psi_i} f\|_2^2 = \sum_{i=1}^{\infty} \int_G |\psi_i(s) f(s)|^2 ds = \int_G \left(\sum_{i=1}^{\infty} |\psi_i(s) f(s)|^2 \right) ds$$
$$= \int_G \left(\sum_{i=1}^{\infty} |\psi_i(s)|^2 \right) |f(s)|^2 ds \leq \left\| \sum_{i=1}^{\infty} |\psi_i|^2 \right\|_\infty \|f\|_2^2$$

and, similarly $\sum_{i=1}^{\infty} \|M_{\varphi_i} g\|_2^2 \leq \|\sum_{i=1}^{\infty} |\varphi_i|^2\|_\infty \|g\|_2^2$. Hence

$$\|m_w f \otimes g\|_{\mathrm{T}} \leq \left\| \sum_{i=1}^{\infty} |\varphi_i|^2 \right\|_\infty^{1/2} \left\| \sum_{i=1}^{\infty} |\psi_i|^2 \right\|_\infty^{1/2} \|f\|_2 \|g\|_2 = \|w\|_{\mathrm{V}\infty} \|f\|_2 \|g\|_2$$

as required.

Now suppose that $w \in \mathrm{MT}(G)$. For φ, ψ in $\mathrm{L}^\infty(G)$, the adjoint of $M_\psi \otimes M_\varphi$, qua operator on $\mathrm{T}(G)$, is $T_{\varphi \otimes \psi}$. Indeed, if $f, g \in \mathrm{L}^2(G)$ and $S \in \mathcal{B}(\mathrm{L}^2(G))$, then

$$\langle M_\psi \otimes M_\varphi (f \otimes g), S \rangle = \langle M_\psi f \otimes M_\varphi g, S \rangle = \langle S M_\psi f | M_{\bar{\varphi}} \bar{g} \rangle$$
$$= \langle M_\varphi S M_\psi f | \bar{g} \rangle = \langle f \otimes g, T_{\varphi \otimes \psi} S \rangle.$$

Since $m_w M_\psi \otimes M_\varphi = M_\psi \otimes M_\varphi m_w$ for all φ, ψ in $\mathrm{L}^\infty(G)$, $T_{\varphi \otimes \psi} m_w^* = m_w^* T_{\varphi \otimes \psi}$. Hence m_w^* is a $\mathrm{L}^\infty(G)$-bimodule map on $\mathcal{B}(\mathrm{L}^2(G))$, and thus, by [17, 3.1], $m_w^* = T_{w'}$ for some w' in $\mathrm{V}^\infty(G)$. It follows that $w' = w$ and hence $\|m_w\| = \|T_w\| = \|w\|_{\mathrm{V}\infty}$. □

We remark that the proof above works for a non-compact group as well. However, the following result is particular to the compact case. It is noted in [16].

Corollary 4.2 *The map $J : \mathrm{V}^\infty(G) \to \mathrm{T}(G)$ given by $Jw = m_w 1 \otimes 1$ is a contractive injection.*

Proof: That J is contractive is evident. If $w \in \ker J$, then $w = 0$ marginally almost everywhere, and hence almost everywhere. Since $\mathrm{V}^\infty(G)$ imbeds contractively into $\mathrm{L}^\infty(G \times G)$ by [2, Cor. 3.8], $w = 0$. □

Note that J may be considered to be the "identity" map on elementary tensors.

If $S \in \mathcal{B}(\mathrm{L}^2(G))$, define

$$\mathrm{supp}_{\mathcal{B}}(S) = \{(s,t) \in G \times G : \text{for any nbhds. } U \text{ of } s, V \text{ of } t, \text{ there are } f, g \text{ in } \mathrm{L}^2(G)$$
$$\text{such that } \mathrm{supp}(f) \subset V, \mathrm{supp}(g) \subset U \text{ and } \langle Sf | g \rangle \neq 0\}.$$

(We note that for $f \in \mathrm{L}^2(G)$, $\mathrm{supp}(f) = \{s \in G : f(s) \neq 0\}$. This notion is distinct from the one we used for continuous functions.) Then $\mathrm{supp}_{\mathcal{B}}(S)$ is closed in $G \times G$. See [6] or [16] for a notion of the support of an operator which does not make explicit use of the underlying topology.

A couple of results in [16] tie together the contents of the preceding few paragraphs. We amalgamate them and quote them here in a form that will be useful to us.

Proposition 4.3 *If $F \subset G\times G$ is closed, then F is synthetic for $G\times G$ if and only if $T_w S = 0$ whenever* $\mathrm{supp}_{\mathcal{B}}(S) \subset F$ *and $w \in \mathrm{V}^\infty(G)$ with $Jw \in \Phi(F)$.*

We note that we only use that for synthetic F, $T_w S = 0$ whenever $\mathrm{supp}_{\mathcal{B}}(S) \subset F$ and $w \in \mathrm{I}_{\mathrm{V}(G)}(F)$, i.e. $w \in \mathrm{V}(G)$ with $Jw \in \Phi(F)$.

We can imbed $\mathrm{A}(G)$ into $\mathrm{T}(G)$, just as we imbedded it into $\mathrm{V}(G)$. The map $\tilde{Q}$ is defined in [9], where it is denoted by P.

Proposition 4.4 *Define $\tilde{N} : \mathrm{A}(G) \to \mathrm{T}(G)$ by*

$$\tilde{N}u(s,t) = u(st^{-1})$$

for (marginally almost) all (s,t) in $G\times G$. Define $\tilde{Q} : \mathrm{T}(G) \to \mathrm{A}(G)$ by

$$\tilde{Q}\omega(s) = \int_G \omega(sr, r)dr$$

for s in G. Then $\tilde{N}$ and $\tilde{Q}$ are contractions satisfying $\tilde{Q}\circ\tilde{N} = \mathrm{id}_{\mathrm{A}(G)}$. Hence $\tilde{N}$ is an isometry.

Proof: Let N and J be the maps from Theorem 2.2 and Corollary 4.2 respectively. Then $\tilde{N} = J\circ N$, and hence is a contraction. To see that $\tilde{Q}$ is a contraction, it suffices to see that $\|\tilde{Q}f \otimes g\|_{\mathrm{A}} \leq \|f\|_2 \|g\|_2$ for any elementary tensor $f \otimes g$ in $\mathrm{T}(G)$. Indeed, for s in G we have

$$\tilde{Q}f \otimes g(s) = \int_G f(r)g(sr)dr = \int_G f(s^{-1}r)g(r)dr = \langle \lambda(s)f|\bar{g}\rangle$$

and the desired inequality is clear. Also, that $\tilde{Q}\circ\tilde{N} = \mathrm{id}_{\mathrm{A}(G)}$, is obvious. □

If for r in G and ω in $\mathrm{T}(G)$ we define

$$r\cdot\omega(s,t) = \omega(sr, tr)$$

for marginally almost all (s,t) in $G\times G$, then $(r,\omega) \mapsto r\cdot\omega$ is a continuous action by isometries. Hence it induces an $\mathrm{L}^1(G)$-module action, given for f in $\mathrm{L}^1(G)$ by

$$f\cdot\omega = \int_G f(r)r\cdot\omega dr$$

making $\mathrm{T}(G)$ an essential Banach $\mathrm{L}^1(G)$-module. Let

$$\mathrm{T}_{\mathrm{inv}}(G) = \{\omega \in \mathrm{T}(G) : r\cdot\omega = \omega \text{ for all } r \text{ in } G\}.$$

Proposition 4.5 $\mathrm{T}_{\mathrm{inv}}(G) = \tilde{N}\mathrm{A}(G)$.

Proof: That $\tilde{N}\mathrm{A}(G) \subset \mathrm{T}_{\mathrm{inv}}(G)$ is clear. Now, in analogy to the map P in Proposition 2.3, let $\tilde{P} : \mathrm{T}(G) \to \mathrm{T}(G)$ be given by $\tilde{P}\omega = 1\cdot\omega$. Then $\tilde{P}$ is a contractive projection onto $\mathrm{T}_{\mathrm{inv}}(G)$, and $\tilde{P}\circ J = J\circ P$ on $\mathrm{V}(G)$. Suppose $\omega \in \mathrm{T}_{\mathrm{inv}}(G)$. Then if $\{w_n\}$ is a sequence in $\mathrm{V}(G)$ such that $Jw_n \to \omega$, we have $J\circ Pw_n \to \tilde{P}\omega = \omega$. Since each $J\circ Pw_n \in \tilde{N}\mathrm{A}(G)$ and $\tilde{N}\mathrm{A}(G)$ is closed, $\omega \in \tilde{N}\mathrm{A}(G)$. □

We can now state the main result of this section. This was proved for locally compact Abelian groups in [8].

Theorem 4.6 *Let G be a separable compact group and E be a closed set in G. Then E is spectral for $\mathrm{A}(G)$ if and only if E^* is synthetic for $G\times G$ with respect to $m\times m$.*

Proof: Let us suppose that E is spectral for $\mathrm{A}(G)$. First, we will see that if $\omega \in \Phi(E^*) \cap \mathrm{T}_{\mathrm{inv}}(G)$, then $\omega \in \Psi(E^*)$. From Proposition 4.5, there is u in $\mathrm{A}(G)$ such that $\omega = \tilde{N}u$. Since $\mathrm{supp}(\omega) = \{(s,t) \text{ in } G\times G : u(st^{-1}) = \omega(s,t) \neq 0\}$, $u \in \mathrm{I}_{\mathrm{A}}(E)$. Since $\mathrm{I}_{\mathrm{A}}(E) = \mathrm{J}_{\mathrm{A}}(E)$, there is a sequence $\{u_n\}$ in $\mathrm{I}^0_{\mathrm{A}}(E)$ such that $u_n \to u$. Then, for each n, $\mathrm{supp}(\tilde{N}u_n) \subset \mathrm{supp}(u_n)^*$, and since $\mathrm{supp}(u_n)^*$ is closed, its compliment U_n is a neighbourhood of E^*, so $\mathrm{supp}(\tilde{N}u_n)\cap U_n = \varnothing$ and hence $\tilde{N}u_n \in \Phi_0(E^*)$. Then, since $\tilde{N}u_n \to \tilde{N}u = \omega$, $\omega \in \Psi(E^*)$.

Suppose $\omega \in \Phi(E^*)$. Let $\{u^{(\pi)}_{ij} : \pi \in \widehat{G}, i,j = 1,\ldots,d_\pi\}$ and $\{u_\alpha\}$ be as in the proof of Theorem 3.1. If for π in $\widehat{G}$ and $i,j = 1,\ldots,d_\pi$, we let

$$\omega^{(\pi)}_{ij} = u^{(\pi)}_{ij}\cdot\omega$$

then $\omega^{(\pi)}_{ij} \in \Phi(E^*)$. Note that both $\Phi(E^*)$ and $\Psi(E^*)$ are $\mathrm{MT}(G)$-submodules of $\mathrm{T}(G)$. Analogously to (9), let

$$\tilde{\omega}^{(\pi)}_{ij} = \sum_{k=1}^{d_\pi} m_{u^{(\pi)}_{ik}\otimes 1}\omega^{(\pi)}_{kj}.$$

Then each $\tilde{\omega}^{(\pi)}_{ij} \in \Phi(E^*) \cap \mathrm{T}_{\mathrm{inv}}(G)$, so $\tilde{\omega}^{(\pi)}_{ij} \in \Psi(E^*)$. Finally, in analogy with (10) we have

$$\omega^{(\pi)}_{ij} = \sum_{k=1}^{d_\pi} m_{\check{u}^{(\pi)}_{ik}\otimes 1}\,\tilde{\omega}^{(\pi)}_{kj}$$

and hence $\omega \in \Psi(E^*)$.

Now suppose that E^* is synthetic for $G\times G$. First, we will show that for S in $\mathrm{VN}(G)$, $\mathrm{supp}_{\mathcal{B}}(S) \subset \mathrm{supp}_{\mathrm{VN}}(S)^*$. If $(s,t) \in \mathrm{supp}_{\mathcal{B}}(S)$ but $st^{-1} \notin \mathrm{supp}_{\mathrm{VN}}(S)$, find neighbourhoods U of s and V of t such that $UV^{-1} \cap W = \varnothing$, where W is a neighbourhood of $\mathrm{supp}_{\mathrm{VN}}(S)$, and then take f and g in $\mathrm{L}^2(G)$ such that $\mathrm{supp}(f) \subset V$, $\mathrm{supp}(g) \subset U$ and $0 \neq \langle Sf|g\rangle$. Now $\langle Sf|g\rangle = \langle u, S\rangle$ where $u = \langle\lambda(\cdot)f|g\rangle$. However, $\mathrm{supp}(u) \subset \mathrm{supp}(g)\mathrm{supp}(f)^{-1} \subset \overline{UV^{-1}}$, so $u \in \mathrm{I}^0_{\mathrm{A}}(E)$ and hence $\langle u, S\rangle = 0$ by (11), contradicting that $\langle Sf|g\rangle \neq 0$.

Thus if $\mathrm{supp}_{\mathrm{VN}}(S) \subset E$, then $\mathrm{supp}_{\mathcal{B}}(S) \subset E^*$. If $u \in \mathrm{I}_{\mathrm{A}}(E)$, then $Nu \in \mathrm{I}_{\mathrm{V}}(E^*)$ in $\mathrm{V}(G)$. By Proposition 4.3, $uS = T_{Nu}S = 0$, so $0 = \langle 1, uS\rangle = \langle u, S\rangle$. Then E is spectral for $\mathrm{A}(G)$ by (11). □

We remark that, by [16] if a closed set $F \subset G \times G$ is a set of operator synthesis with respect to any finite Borel measures μ and ν on G then F is spectral for $V(G)$. For special sets E^*, where $E \subset G$, our theorem gives an even stronger result that it is enough to require that E^* is synthetic just with respect to the Haar measure. It is an interesting open question if synthesis of E^* in the Varopoulos algebra and hence operator synthesis with respect to

the Haar measure implies operator synthesis with respect to any pair (μ, ν) of finite Borel measures.

Some examples of sets of spectral sets for $\mathrm{V}(G)$, namely E^*, where E is a spectral set for $\mathrm{A}(G)$, were already given in Section 3. Another class of sets are sets of finite width, i.e. $E = \{(x, y) \in G \times G : \varphi_i(x) \leq \psi_i(y), i = 1, \ldots, n\}$, where $\varphi_i : G \to Z$, $\psi_i : G \to Z$ are continuous functions to a compact ordered metric space Z. That sets of finite width are spectral for $\mathrm{V}(G)$ was proved in [16] using the notion of operator synthesis and its connection with synthesis in the algebra $\mathrm{V}(G)$. We note that sets of finite width are of operator synthesis with respect to any pair (μ, ν) of finite Borel measures is shown in [19] as well as in [16]. Thus we have two classes of examples that which show that there is an important relationship between spectral sets for $\mathrm{V}(G)$ and sets of operator synthesis.

Let Σ be a closed subsemigroup containing the identity. Define a pre-order on G by $x \leq y$ if $xy^{-1} \in \Sigma$. Let $\mathrm{Alg}(\Sigma)$ be an operator algebra associated to this pre-order (see [1]). By [1], $\mathrm{Alg}(\Sigma)$ consists of all operators on $\mathrm{L}^2(G)$ whose support is in Σ^*. We denote by $\mathrm{Alg}_{min}(\Sigma)$ the ultra-weak closure of sets of pseudo-integral operators whose support is in Σ^*. We recall that a pseudo-integral operator supported in a closed set E arises from a measure whose support is in E. By [1], a closed set Σ^* is synthetic if and only if $\mathrm{Alg}_{min}(\Sigma) = \mathrm{Alg}(\Sigma)$, which is the same as having that $\mathrm{Alg}(\Sigma)$ is reflexive.

The following was shown by Froelich for separable Abelian groups in [8]. Our proof is similar to his.

Theorem 4.7 *Let G be a separable compact group and Σ_1, Σ_2 be closed subsemigroups with the identity which are spectral for $\mathrm{A}(G)$. Then $\mathrm{Alg}(\Sigma_1)\bar{\otimes}\mathrm{Alg}(\Sigma_2) = \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$ if and only if $\Sigma_1 \times \Sigma_2$ is spectral for $\mathrm{A}(G\times G)$. Here $\bar{\otimes}$ denotes the ultra-weak closure of the linear span of all elementary tensors.*

Proof: Let $T \in \mathrm{Alg}(\Sigma_1)$, $S \in \mathrm{Alg}(\Sigma_2)$. Then $T \otimes S \in \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$. Since Σ_1 and Σ_2 are spectral for $\mathrm{A}(G)$, Σ_1^* and Σ_2^* are synthetic by Theorem by Theorem 4.6. Hence there are nets $\{T_\alpha\}$, $\{S_\beta\}$ of pseudo-integral operators supported in Σ_1^* and Σ_2^*, respectively, such that $T_\alpha \to T$ and $S_\beta \to S$ ultra-weakly. By [8, 2.2.2], $T_\alpha \otimes S_\beta$ is a pseudo-integral operator on $\mathrm{L}^2(G\times G)$ supported in $(\Sigma_1 \times \Sigma_2)^*$, and since $T_\alpha \otimes \mathrm{id} \to T \otimes \mathrm{id}$ ultra-weakly we have first that $T \otimes S_\beta \in \mathrm{Alg}_{min}(\Sigma_1 \times \Sigma_2)$ for any β and by the same argument we obtain $T \otimes S \in \mathrm{Alg}_{min}(\Sigma_1 \times \Sigma_2)$. From the tensor product formula we have $\mathrm{Alg}_{min}(\Sigma_1 \times \Sigma_2) = \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$ which implies that $(\Sigma_1 \times \Sigma_2)^*$ is a set of operator synthesis. Hence, by Theorem 4.6, $\Sigma_1 \times \Sigma_2$ must be a set of spectral synthesis.

To see the converse, suppose that $\Sigma_1 \times \Sigma_2$ is a set of spectral synthesis. Then, by Theorem 4.6, $(\Sigma_1 \times \Sigma_2)^* = \{((t_1, t_2), (s_1, s_2)) : t_1 s_1^{-1} \in \Sigma_1, t_2 s_2^{-1} \in \Sigma_2\}$ is synthetic, and hence $\mathrm{Alg}_{min}(\Sigma_1 \times \Sigma_2) = \mathrm{Alg}(\Sigma_1 \times \Sigma_2)$. Let T_μ be a pseudo-integral operator on $\mathrm{L}^2(G\times G)$ whose support is in $(\Sigma_1 \times \Sigma_2)^*$, where μ is an associated measure. Since μ is supported in $(\Sigma_1 \times \Sigma_2)^*$, it is w^*-limit of a net of measures where each of them is a linear combination of tensor products of measures whose supports are in Σ_1^* and Σ_2^*, respectively. This implies $T_\mu \in \mathrm{Alg}_{min}(\Sigma_1)\bar{\otimes}\mathrm{Alg}_{min}(\Sigma_2) = \mathrm{Alg}(\Sigma_1)\bar{\otimes}\mathrm{Alg}(\Sigma_2)$ and hence $\mathrm{Alg}_{min}(\Sigma_1 \times \Sigma_2) = \mathrm{Alg}(\Sigma_1)\bar{\otimes}\mathrm{Alg}(\Sigma_2)$. Thus $\mathrm{Alg}(\Sigma_1 \times \Sigma_2) = \mathrm{Alg}(\Sigma_1)\bar{\otimes}\mathrm{Alg}(\Sigma_2)$. □

Since any closed subgroup of a compact group is synthetic, we have $\mathrm{Alg}(H_1)\bar{\otimes}\mathrm{Alg}(H_2) = \mathrm{Alg}(H_1 \times H_2)$, where H_1 and H_2 are closed subgroups of G.

5 The Case of a Non-Compact Group

The goal of this section is to begin, for non-compact groups, the study of the natural analogue of Theorem 2.2. We will now suppose the G is a non-compact locally compact separable group. We let $\mathrm{V}^\infty(G) = \mathrm{L}^\infty(G) \otimes^{w^*h} \mathrm{L}^\infty(G)$, just as for a compact group. If G is discrete, then $\mathrm{V}^\infty(G) = \ell^\infty(G) \otimes^{w^*h} \ell^\infty(G)$ is the algebra of Schur multipliers on $\mathcal{B}(\ell^2(G))$. Hence we may call elements of $\mathrm{V}^\infty(G)$ *measurable Schur multipliers* on $\mathcal{B}(\mathrm{L}^2(G))$.

We say that G is *amenable* if there is a state (mean) m on $\mathrm{L}^\infty(G)$ which is invariant for left translations by G. See [14] for information on amenable groups.

We can define the left regular representation λ, the group von Neumann algebra $\mathrm{VN}(G)$, and the Fourier algebra $\mathrm{A}(G)$, just as we did for compact groups in Section 1. This is all done in [7]. Then $\mathrm{A}(G)$ is a commutative semi-simple regular Banach algebra with spectrum G. It is a subset of the space $\mathcal{C}_0(G)$ of continuous functions vanishing at infinity.

Proposition 5.1 *The map $N : \mathrm{A}(G) \to \mathrm{V}^\infty(G)$ given by*

$$Nu(s,t) = u(st^{-1})$$

is a contraction. It is an isometry if G is amenable.

Proof: By [7], if $u \in \mathrm{A}(G)$, we may write $u = \langle \lambda(\cdot) f | g \rangle$ for some $f, g \in \mathrm{L}^2(G)$ with $\|u\|_\mathrm{A} = \|f\|_2 \|g\|_2$. If $\{e_n\}$ is an ortho-normal basis for $\mathrm{L}^2(G)$, let

$$\varphi_n = \langle \lambda(\cdot) e_n | g \rangle \quad \text{and} \quad \psi_n \langle f | \lambda(\cdot) e_n \rangle.$$

Then each $\varphi_n, \psi_n \in \mathcal{C}_0(G) \subset \mathrm{L}^\infty(G)$ with $\sum_{n=1}^\infty |\varphi_n|^2 = \|g\|_2^2 1$ and $\sum_{n=1}^\infty |\psi_n|^2 = \|f\|_2^2 1$ (weak* converging sums). Since $Nu = \sum_{n=1}^\infty \varphi_n \otimes \psi_n$, we have $\|Nu\|_{\mathrm{V}^\infty} \leq \|f\|_2 \|g\|_2 = \|u\|_\mathrm{A}$.

If G is amenable, it has a bounded approximate identity composed of norm 1 elements (see [13] or [14]), so for u in $\mathrm{A}(G)$, $\|u\|_\mathrm{A} = \sup\{\|uv\|_\mathrm{A} : v \in \mathrm{b}_1(\mathrm{A}(G))\}$. For $w = Nu$, repeat the computation (3) to see that $\|u\|_\mathrm{A} \leq \|Nu\|_{\mathrm{V}^\infty}$. Note that for amenable groups, an analogue of Proposition 2.1 holds. □

Define an action of G on $\mathrm{V}^\infty(G)$ by $r \cdot w(s,t) = w(sr, tr)$ for r in G, w in $\mathrm{V}^\infty(G)$ and marginally almost all (s,t) in $G \times G$. The action is one of isometries on $\mathrm{V}^\infty(G)$. However, for a general w in $\mathrm{V}^\infty(G)$, $r \mapsto r \cdot w$ is continuous only if $\mathrm{V}^\infty(G)$ is given the weak* topology. Recall that $\mathrm{V}^\infty(G) \cong (\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G))^*$. Let $\mathrm{V}^\infty_{\mathrm{inv}}(G) = \{w \text{ in } \mathrm{V}^\infty(G) : r \cdot w = w \text{ for all } r \text{ in } G\}$. We note that it can be shown that $\mathrm{V}^\infty_{\mathrm{inv}}(G)$ is isometrically isomorphic to the completely bounded multipliers (i.e. the Herz-Schur multipliers) of $\mathrm{A}(G)$. See [18].

Let G be amenable for the remainder of the section. It will be notationally convenient to regard the mean m as a finitely additive measure on G, absolutely continuous with respect to the left Haar measure m.

Proposition 5.2 *The map $P : \mathrm{V}^\infty(G) \to \mathrm{V}^\infty_{\mathrm{inv}}(G)$ given by*

$$Pw = \int_G r \cdot w \, d\mathrm{m}(r)$$

(i.e. $\langle x, Pw \rangle = \int_G \langle x, r \cdot w \rangle d\mathrm{m}(r)$ for x in $\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G)$) is a contractive projection and a $\mathrm{V}^\infty_{\mathrm{inv}}(G)$-module map.

The proof is very similar to that of Proposition 2.3. Let $\mathrm{V}_0(G) = \mathcal{C}_0(G) \otimes^h \mathcal{C}_0(G)$.

Proposition 5.3 $\mathrm{V}_0(G) \subset \ker P$.

Proof: It suffices to show that $\varphi \otimes \psi \in \ker P$ for any φ, ψ in $\mathcal{C}_0(G)$ having compact supports. Then for f, g in $\mathrm{L}^1(G)$, the function $r \mapsto \langle f \otimes g, r\cdot(\varphi \otimes \psi)\rangle = \int_G f(s)\varphi(sr)ds \int_G g(t)\psi(tr)dt$ is in $\mathcal{C}_0(G)$. Hence $\langle f \otimes g, P\varphi \otimes \psi\rangle = \int_G \langle f \otimes g, r\cdot(\varphi \otimes \psi)\rangle d\mathrm{m}(r) = 0$. It follows that $\langle x, P\varphi \otimes \psi\rangle = 0$ for all x in $\mathrm{L}^1(G) \otimes^h \mathrm{L}^1(G)$, and hence $P\varphi \otimes \psi = 0$. □

Collecting the previous two results we obtain the following.

Corollary 5.4 $N\mathrm{A}(G) \cap \mathrm{V}_0(G) = \{0\}$.

Hence we will not be able to recover information about spectral synthesis for $\mathrm{V}_0(G)$ via that for $\mathrm{A}(G)$ in the direct way that we are able to for compact groups.

It is unclear at the present if $N\mathrm{A}(G)$ lies within $\mathcal{C}_0(G) \otimes^{eh} \mathcal{C}_0(G)$, where $\otimes^{eh}$ denotes the extended Haagerup tensor product of [4]. It is shown in [16] that $\mathcal{C}_0(G) \otimes^{eh} \mathcal{C}_0(G)$ does not consist entirely of functions which are continuous on $G\times G$. Hence $G\times G$ does not even continuously imbed in the spectrum of $\mathcal{C}_0(G) \otimes^{eh} \mathcal{C}_0(G)$. Any questions regarding spectral synthesis for $\mathcal{C}_0(G) \otimes^{eh} \mathcal{C}_0(G)$ should prove to be challenging.

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