A FOURIER SERIES FORMULA FOR ENERGY OF MEASURES WITH APPLICATIONS TO RIESZ PRODUCTS

KATHRYN E. HARE AND MARIA ROGINSKAYA

Abstract. In this paper we derive a formula relating the energy and the Fourier transform of a finite measure on the d-dimensional torus which is similar to the well-known formula for measures on $\mathbb{R}^d$.

We apply the formula to obtain estimates on the Hausdorff dimension of Riesz product measures. These give improvements on the earlier, classical results which were based on completely different techniques.

1. Introduction

Potential theoretic techniques have been quite useful in geometric measure theory to study the behaviour of measures on $\mathbb{R}^d$. There is a well known and important relationship between the t-energy of a measure and its Fourier transform:

$$I_t(\mu) \equiv \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^t} = c_{t,d}\int |x|^{t-d}\left|\widehat{\mu}(x)\right|^2 dx.$$

In this note we derive a similar formula relating the energy and discrete Fourier transform of a measure on the d-dimensional torus:

$$I_t(\mu) \sim \left|\widehat{\mu}(0)\right|^2 + \sum_{z\in\mathbb{Z}^d,\ z\neq 0} |z|^{t-d}\left|\widehat{\mu}(z)\right|^2.$$

The energy of a measure is closely connected with geometric properties of the measure. In particular, if $I_t(\mu) < \infty$, then the Hausdorff dimension of any set of positive μ measure is at least t. We use this fact, together with our formula relating the energy and discrete Fourier transform of a measure on the torus, to study the dimensions of Riesz product measures.

Riesz products are typically singular to Lebesgue measure, so it is of interest to determine their Hausdorff dimension. We prove, for example, that if $\{n_k\}$ is a lacunary sequence of integers satisfying $n_{k+1}/n_k \geq 3$, then the Hausdorff dimension of the Riesz product $\mu = \prod(1+\operatorname{Re} a_j e^{in_j x})$ is at least

$$1 - \limsup_{k\to\infty}\left(\sum_{j=1}^{k-1} \left|a_j\right|^2/(2\log n_k)\right).$$

1991 *Mathematics Subject Classification.* Primary: 28A12, 42A55.

Key words and phrases. energy, Hausdorff dimension, Riesz products.

This research was done while the first author enjoyed the hospitality of the Dept. of Mathematics of Göteborg University and Chalmers Inst. of Tech. It was supported in part by NSERC and the Swedish natural sciences research council.

This improves upon the earlier work of Peyriere [9] and Brown, Moran and Pearce [1] who had shown, using other techniques, that the Hausdorff dimension of a more restricted class of Riesz products was at least $1-\limsup\left(\sum_{j=1}^{k-1}|a_j|/\log n_k\right)$.

Similar results can be obtained for the dimensions of multivariable Riesz products in $\mathbb{T}^d$ and $\mathbb{R}^d$. These are again sharper and apply to a larger class of measures than the earlier results of Peyriere.

2. Energy and the Fourier transform

By a measure we mean a finite, positive Borel measure on a metric space. The *t-energy* of the measure μ, $I_t(\mu)$, is defined as

$$I_t(\mu) \equiv \int\int \frac{d\mu(x)d\mu(y)}{dist(x,y)^t}.$$

Our interest is in the metric spaces $\mathbb{R}^d$ with the usual Euclidean metric $dist(x,y) = |x-y|$, and the d-dimensional torus, $\mathbb{T}^d$, which we view as either $[-1/2,1/2]^d$ or $[-\pi,\pi]^d$ with the usual identification (depending upon which is more convenient). The torus has a group structure and when we write $x-y$ for $x,y\in\mathbb{T}^d$ it should be understood that the binary operation is the group operation. The metric we consider on $\mathbb{T}^d$ is the usual notion of distance on the torus and we denote this metric by $d_{\mathbb{T}}(x,y)$ to distinguish it from the metric on $\mathbb{R}^d$.

The *Hausdorff dimension* of a measure μ is defined as

$$\dim_H\mu \equiv \inf\{\dim_H E : E \text{ is a Borel set with } \mu(E)>0\}.$$

For properties of the Hausdorff dimension of a measure see [3], ch. 10.

We will say that t is the *energy dimension* of μ if

$$t=\sup\{s : I_s(\mu)<\infty\}.$$

If $I_t(\mu)<\infty$, then $\dim_H\mu>t$ (c.f. [2], 4.3). Thus the Hausdorff dimension of a measure is always at least the energy dimension. The two dimensions are not always equal, but for many measures they are. For example, if μ is the Hausdorff measure on the classical middle-third Cantor set, then $\dim_H\mu=\log 2/\log 3\equiv s_0$. Since $\mu(B(x,r))\le cr^{s_0}$ for all x, it can be shown that $I_t(\mu)<\infty$ for all $t<s_0$, and hence the energy and Hausdorff dimensions coincide.

There is a very useful relationship between the energy and Fourier transform of a measure on $\mathbb{R}^d$ which can be derived from Parseval's theorem and the well-known fact that if $f(x)=|x|^{-t}$ for $x\in\mathbb{R}^d$, then $\widehat{f}(z)=c_{t,d}|z|^{t-d}$ for some constant $c_{t,d}$, namely,

$$I_t(\mu)=c_{t,d}\int|x|^{t-d}|\widehat{\mu}(x)|^2\,dx. \tag{2.1}$$

A good explanation of the derivation of this formula and some examples of applications are given in [8], ch. 12.

It is natural to ask if there is a similar formula relating the energy and the (discrete) Fourier transform of a measure on the d-dimensional torus. When $d=1$ one can obtain a relationship by using Parseval's theorem and the fact ([11], p.70) that

$$\sum_{n=1}^{\infty} n^{s-1}\cos nx = c_s\,|x|^{-s} + O(1) \text{ for } 0 < s < 1.$$

By finding a suitable kernel on $\mathbb{T}^d$ we will obtain a formula for $d > 1$, as well.

Notation. When we write $f \sim g$ we will mean there are positive constants a, b such that $ag \le f \le bg$.

Lemma 2.1. *Let $0 < t < d$. There is a function F_t defined on $\mathbb{T}^d$ and a C^∞-function ϕ defined on $\mathbb{R}^d$ which have the following properties:*

(1) F_t is positive, integrable, continuous except at the origin and has Fourier coefficients satisfying $\widehat{F_t}(z) \sim |z|^{t-d}$ for $z \in \mathbb{Z}^d$, $z \neq 0$;

(2) ϕ is positive, bounded on $\mathbb{R}^d$ and bounded away from zero on $\mathbb{T}^d$; and

(3) $F_t(x) - \phi(x)\,|x|^{-t}$ is positive and bounded on $\mathbb{T}^d$.

Proof. We will take $\mathbb{T}^d = [-1/2, 1/2]^d$. Let $\widetilde{\psi}$ be any non-negative, $C^\infty(\mathbb{R}^d)$ test function supported on $B(0, 1/4)$ (the ball centred at the origin, with radius $1/4$) satisfying $\widetilde{\psi}\ (0) > 0$. Set $\psi(x) = \widetilde{\psi}(x) + \widetilde{\psi}(-x)$ and let ϕ_1 be the inverse Fourier transform of ψ. Then ϕ_1^2 is non-negative and strictly positive on a neighbourhood of 0. The function $\phi = \phi_1^2 * \phi_1^2$ is a strictly positive, C^∞ function which decays rapidly and whose Fourier transform, $\widehat{\phi} = (\psi * \psi)^2$, is a non-negative function which is supported on $B(0, 1/2)$ and satisfies $\widehat{\phi}(0) > 0$. The boundedness and rapid decay of ϕ ensure that $f_t(x) \equiv \phi(x)\,|x|^{-t}$ is a summable function on $\mathbb{R}^d$ and thus we can form the periodic function

$$F_t(x) \equiv \sum_{a \in \mathbb{Z}^d} f_t(x + a) = \sum_{a \in \mathbb{Z}^d} \phi(x + a)\,|x + a|^{-t}.$$

We consider F_t as a function on the torus; it is clearly positive, integrable and its Fourier coefficients coincide with the Fourier transform of f_t on the integer lattice. As $\widehat{f_t} = \widehat{\phi} * \widehat{|\cdot|^{-t}}$ we have

$$\widehat{F_t}(z) = \widehat{f_t}(z) = \int_{\mathbb{R}^d} \frac{\widehat{\phi}(y)}{|z-y|^{d-t}}dy = \int_{B(0,1/2)} \frac{\widehat{\phi}(y)}{|z-y|^{d-t}}dy \text{ for } z \in \mathbb{Z}^d,$$

with the latter equality arising because $\widehat{\phi}$ is supported on $B(0, 1/2)$. If in addition $z \neq 0$, then $|z - y| \sim |z|$ when $y \in B(0, 1/2)$. As $\widehat{\phi}$ is bounded away from zero in a neighbourhood of the origin it follows that $\widehat{F_t}(z) \sim |z|^{t-d}$.

The function $F_t(x) - \phi(x)\,|x|^{-t}$ is bounded because of the rapid decay of ϕ; F_t is continuous except at 0 for similar reasons. ∎

Theorem 2.2. *Let $0 < t < d$. There are constants $a, b > 0$, depending on t, d such that if μ is any finite, positive Borel measure on $\mathbb{T}^d$, then*

(2.2)
$$a\left(|\widehat{\mu}(0)|^2 + \sum_{z \in \mathbb{Z}^d \setminus 0} |z|^{t-d}\,|\widehat{\mu}(z)|^2\right) \le I_t(\mu) \le b\left(|\widehat{\mu}(0)|^2 + \sum_{z \in \mathbb{Z}^d \setminus 0} |z|^{t-d}\,|\widehat{\mu}(z)|^2\right).$$

Proof. Choose the functions ϕ and $F_t(x)$ found in the lemma and set $g_t(x) = F_t(x) - \phi(x)\,|x|^{-t}$. Since ϕ is bounded away from zero on the torus, $I_t(\mu)$ is comparable to

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}(F_t - g_t)(x-y)d\mu(x)d\mu(y).$$

Let ψ be a non-negative, continuous function supported on $[-1/6,1/6]^d$, satisfying $\int\psi = 1$ and $\widehat{\psi} \geq 0$, and suppose $\{\psi_\varepsilon\}_{\varepsilon>0}$ is an approximate identity where ψ_ε is the function supported on $[-\varepsilon/6,\varepsilon/6]^d$ given by $x \mapsto \varepsilon^{-d}\psi(x/\varepsilon)$.

As $\psi_\varepsilon * F_t$ is a continuous function on the torus, Parseval's theorem gives

$$\int\int(\psi_\varepsilon * F_t)(x-y)d\mu(x)d\mu(y) = \sum_{z\in\mathbb{Z}^d}\widehat{\psi_\varepsilon}(z)\widehat{F}_t(z)\left|\widehat{\mu}(z)\right|^2,$$

and this converges to $\sum_{z\in\mathbb{Z}^d}\widehat{F}_t(z)\left|\widehat{\mu}(z)\right|^2$ as $\varepsilon \to 0$ since $\widehat{\psi_\varepsilon}(z) \to 1$ pointwise from below. By Fatou's lemma

$$\int\int F_t(x-y)d\mu(x)d\mu(y) \leq \liminf_{\varepsilon\to 0}\int\int\psi_\varepsilon * F_t(x-y)d\mu(x)d\mu(y),$$

and thus $I_t(\mu)$ is bounded above by some multiple of

$$\sum_{z\in\mathbb{Z}^d\setminus 0}|z|^{t-d}\left|\widehat{\mu}(z)\right|^2 + \widehat{F}_t(0)\left|\widehat{\mu}(0)\right|^2,$$

verifying the right hand inequality.

To establish the left hand inequality we should first observe that there is a constant C, depending on t, such that for all $\varepsilon > 0$,

$$\psi_\varepsilon * d_{\mathbb{T}}(\cdot,0)^{-t}(z) \leq C d_{\mathbb{T}}(z,0)^{-t} \text{ for all } z \in \mathbb{T}^d.$$

One way to prove this is to decompose $d_{\mathbb{T}}(\cdot,0)^{-t}$ as $d_1 + d_2$ where d_2 equals $d_{\mathbb{T}}(\cdot,0)^{-t}$ restricted to the complement of a small neighbourhood of 0. As d_2 is bounded, $\psi_\varepsilon * d_2$ is bounded. Clearly, $\psi_\varepsilon * d_1(z) = 0$ if z is not sufficiently close to 0, and it is an exercise to check that it is bounded by $C\,|z|^{-t} = C d_{\mathbb{T}}(z,0)^{-t}$ otherwise.

Since also $\psi_\varepsilon * g_t \leq \|g_t\|_\infty$, it follows that $\psi_\varepsilon * F_t \leq C d_{\mathbb{T}}(z,0)^{-t} + \|g_t\|_\infty$. Thus another application of Parseval's theorem gives

$$\begin{aligned} I_t(\mu) &\geq \frac{1}{C}\left(\liminf\int\int\psi_\varepsilon * F_t(x-y)d\mu(x)d\mu(y) - \|g_t\|_\infty\left|\widehat{\mu}(0)\right|^2\right) \\ &= \frac{1}{C}\left(\sum_{z\in\mathbb{Z}^d}\widehat{F}_t(z)\left|\widehat{\mu}(z)\right|^2 - \|g_t\|_\infty\left|\widehat{\mu}(0)\right|^2\right), \end{aligned}$$

and therefore $I_t(\mu)$ is bounded below by

$$A\sum_{z\in\mathbb{Z}^d\setminus 0}|z|^{t-d}\left|\widehat{\mu}(z)\right|^2 - B\left|\widehat{\mu}(0)\right|^2$$

for suitable positive constants A, B. To complete the proof just note that we trivially have $I_t(\mu) \geq C_1\left|\widehat{\mu}(0)\right|^2$. ∎

It is known that there are singular measures μ on the circle with $\{\widehat{\mu}(n)\} \in l^p$ for $p > 2$. We can give a lower bound on the Hausdorff dimension of such measures.

Corollary 2.3. *If* $\widehat{\mu} \in l^p(\mathbb{Z}^d)$ *for some* $p > 2$, *then* $\dim_H \mu \geq 2d/p$.

Proof. This is an easy consequence of Holder's inequality and the fact that $\sum_{z\in\mathbb{Z}^d\setminus 0}|z|^{(t-d)q}$ is finite when $(t-d)q<-d$. ∎

Example 2.1. *Recall that the energy dimension of the Cantor measure* μ *on the standard middle-third Cantor set is* $\log 2/\log 3$*. Thus for all* $t<\log 2/\log 3$,

$$|\widehat{\mu}(0)|^2+\sum_{n\neq 0}|n|^{t-1}|\widehat{\mu}(n)|^2\sim I_t(\mu)<\infty.$$

3. Riesz Products

A sequence $\{\gamma_k\}\subseteq\mathbb{Z}^d$ is called dissociate if for any positive integer N,

$$\sum_{k=1}^{N}\varepsilon_k\gamma_k=0 \text{ for } \varepsilon_k=0,\pm1,\pm2 \text{ implies } \varepsilon_k=0 \text{ for all } k.$$

A lacunary sequence of positive integers $\{n_k\}$ with $n_{k+1}/n_k\geq 3$ is an example of a dissociate sequence in $\mathbb{Z}$.

Given a dissociate sequence $\{\gamma_k\}$ and sequence of complex numbers $\{a_k\}$ satisfying $\sup_k|a_k|\leq 1$, we define trigonometric polynomials $P_k(x)=\prod_{j=1}^{k}(1+\operatorname{Re}a_je^{i\gamma_j\cdot x})$ for $x\in\mathbb{T}^d$. By a Riesz product measure

$$\mu_{\{a_j\}}\equiv\prod_{j=1}^{\infty}(1+\operatorname{Re}a_je^{i\gamma_j\cdot x})$$

we mean the weak* limit of the measures $P_k(x)dx$ on $\mathbb{T}^d$, which here we identify with $[-\pi,\pi]^d$.

3.1. Hausdorff dimension.

Estimates of the Hausdorff dimension of Riesz products on $\mathbb{T}$ were first obtained by Peyriere in [9] using probabilistic ideas. He proved that if $n_{k+1}/n_k\in\mathbb{Z}$ and $n_{k+1}/n_k\geq 3$, then the Hausdorff dimension of the Riesz product measure $\mu_{\{a_j\}}=\prod_{j=1}^{\infty}(1+\operatorname{Re}a_je^{in_jx})$ satisfies

(3.1)
$$1-\liminf_{k\to\infty}\left(\frac{\int\log P_kd\mu_{\{a_j\}}}{\log n_{k+1}}\right)\geq\dim_H\mu_{\{a_j\}}\geq\ 1-\limsup_{k\to\infty}\left(\frac{\int\log P_kd\mu_{\{a_j\}}}{\log n_k}\right)$$

([9], Thm. 2.8). From this formula and (the proof of) [9], Lemma 2.3 one can obtain an upper bound on the Hausdorff dimension of $\mu_{\{a_j\}}$ in terms of the size of the coefficients $\{a_j\}$ and the integers $\{n_j\}$. When the coefficients are small in modulus the Hausdorff dimension of $\mu_{\{a_j\}}$ is approximately bounded above by

$$1-\liminf_{k\to\infty}\left(\frac{1}{4}\sum_{j=1}^{k}|a_j|^2/\log n_k\right). \tag{3.2}$$

In [1], Brown et al extended Peyriere's integral formula (3.1), replacing the divisibility condition by a less restrictive technical condition, and showed that the Hausdorff dimension of these Riesz products was bounded below by

$$1-\limsup_{k\to\infty}\left(\sum_{j=1}^{k}|a_j|/\log n_k\right). \tag{3.3}$$

In our next theorem we use our formula (2.2) relating energy and the Fourier transform to improve this lower bound; the new lower bound (3.5) should be compared with Peyriere's upper bound (3.2).

Theorem 3.1. *Suppose $\{n_j\}$ is a dissociate set of increasing, positive integers and assume there is some $c < 1$ such that $\sum_{j=1}^{k-1} n_j \leq cn_k$ for all k. Let $\mu_{\{a_j\}}$ be the Riesz product $\prod_{j=1}^{\infty}(1 + \operatorname{Re} a_j e^{in_j x})$. Then the energy dimension of $\mu_{\{a_j\}}$ equals $1 - \alpha_0$ where*

$$\alpha_0 = \max\left(\limsup_{k\to\infty}\left(\frac{2\log|a_k| + \sum_{j=1}^{k-1}\log\left(1 + |a_j|^2/2\right)}{\log n_k}\right), 0\right). \tag{3.4}$$

In particular,

$$\dim_H \mu_{\{a_j\}} \geq 1 - \limsup_{k\to\infty}\left(\frac{1}{2}\sum_{j=1}^{k}|a_j|^2/\log n_k\right). \tag{3.5}$$

Before proving this there are several observations we would like to make.

First, note that the technical assumption of our theorem, the condition that $\sum_{j=1}^{k-1} n_j \leq cn_k$ for some $c < 1$, is automatically satisfied when $n_{k+1}/n_k \geq 3$.

In the special case that $a_j = a \geq 0$ for all j, $n_j = q^j$ for some integer $q \geq 3$ and $\mu_a = \prod_{j=1}^{\infty}(1 + a \operatorname{Re} e^{iq^j x})$ there is an exact (theoretical) formula for the Hausdorff dimension given in [4]:

$$\dim_H \mu_a = 1 - \frac{\int \log(1 + \operatorname{Re} a e^{iqx}) d\mu_a}{\log q}.$$

There are approximations for this in terms of a and q (see [5]), however our theorem appears to give some improved estimates.

Corollary 3.2. *If $\mu_a = \prod_{j=1}^{\infty}(1 + \operatorname{Re} a e^{iq^j x})$ then $\dim_H \mu_a \geq 1 - |a^2|/(2\log q)$.*

More generally, if $\log n_k/k \to \log q < \infty$ and $\mu_{\{a_j\}} = \prod_{j=1}^{\infty}(1 + \operatorname{Re} a_j e^{iq^j x})$, then

$$\dim_H \mu_{\{a_j\}} \geq 1 - \frac{\limsup_k \frac{1}{k}\sum_{j=1}^{k}|a_j|^2}{2\log q}.$$

In [9] Peyriere gave examples of Riesz products satisfying $n_{k+1}/n_k \to \infty$ which had Hausdorff dimension one. In fact, another consequence of our theorem is that all such Riesz products have dimension one.

Corollary 3.3. *(a) If $k/\log n_k \to 0$ then $\dim_H \mu_{\{a_j\}} = 1$.*

(b) If $n_{k+1}/n_k \to \infty$ then $\dim_H \mu_{\{a_j\}} = 1$.

Proof. (a) Since $\sum_{j=1}^{k}|a_j|^2 \leq k$ the hypothesis implies $\sum_{j=1}^{k}|a_j|^2/\log n_{k+1} \to 0$.

(b) Given $\varepsilon > 0$ choose m such that $\log m > 1/\varepsilon$ and pick $k(m)$ so that for all $k > k(m)$, $n_{k+1}/n_k > m$. If $k = k(m) + j$, then $n_k \geq m^j n_{k(m)}$ and therefore if j (equivalently, k) is sufficiently large, then

$$\frac{k}{\log n_k} \leq \frac{k(m) + j}{j\log m} \leq 2\varepsilon.$$

∎

Proof of Theorem. To calculate the energy dimension of $\mu_{\{a_j\}}$ (we will write μ in what follows) we need to observe that if

$$n \in \Gamma_k \equiv \{\pm n_k + \sum_{j=1}^{k-1} \varepsilon_j n_j : \varepsilon_j = 0, \pm 1\},$$

then

$$|\widehat{\mu}(n)| = \frac{|a_k|}{2} \prod_{j:\varepsilon_j \neq 0} \frac{|a_j|}{2}$$

(where the empty product is one). Furthermore, if $n \in \Gamma_k$ then $|n| \sim n_k$. Thus

$$\sum_{n \in \Gamma_k} |n|^{-\alpha} |\widehat{\mu}(n)|^2 \sim n_k^{-\alpha} \frac{|a_k|^2}{2} \prod_{j=1}^{k-1} \left(1 + \frac{|a_j|^2}{2}\right),$$

and therefore

$$I_{1-\alpha}(\mu) \sim 1 + \sum_{k=1}^{\infty} n_k^{-\alpha} \frac{|a_k|^2}{2} \prod_{j=1}^{k-1} \left(1 + \frac{|a_j|^2}{2}\right). \tag{3.6}$$

Clearly, $I_{1-\alpha}(\mu) = \infty$ if infinitely many of the summands are at least one, and this occurs if

$$-\alpha \log n_k + \log \frac{|a_k|^2}{2} + \sum_{j=1}^{k-1} \log \left(1 + \frac{|a_j|^2}{2}\right) \geq 0$$

for infinitely many k. It is a routine verification to see that if $\alpha < \alpha_0$ then this is indeed the case.

Conversely, $I_{1-\alpha}(\mu) < \infty$ if there is some $A < 1$ such that for all but finitely many k,

$$n_k^{-\alpha} \frac{|a_k|^2}{2} \prod_{j=1}^{k-1} \left(1 + \frac{|a_j|^2}{2}\right) \leq A^k$$

or, equivalently,

$$\alpha \geq \frac{\log |a_k|^2 / 2 + \sum_{j=1}^{k-1} \log \left(1 + |a_j|^2 / 2\right) + k |\log A|}{\log n_k}.$$

It is known that if $\{n_j\}$ is a dissociate sequence of positive integers then

$$\left| \{n_j\} \bigcap [1, 2^k] \right| \leq O(k)$$

([10]) hence there must be some $C > 0$ such that $n_k \geq 2^{Ck}$ for all k. Thus it suffices to show that for all but finitely many k,

$$\alpha \geq \frac{\log |a_k|^2 / 2 + \sum_{j=1}^{k-1} \log \left(1 + |a_j|^2 / 2\right)}{\log n_k} + \frac{|\log A|}{C \log 2}$$

for some $A < 1$. We can certainly achieve this if $\alpha > \alpha_0$ since we can choose A so close to 1 that $|\log A| / C \log 2$ is as small as necessary.

Since $\log(1 + x) \leq x$ for $x > 0$ inequality (3.5) follows directly. ∎

Remark 3.1. *Similar statements can obviously be made for Riesz products* $\prod_{j=1}^{\infty}(1+\mathrm{Re}\, a_j e^{i\gamma_j \cdot x})$ *in* $\mathbb{T}^d$ *where* $\{\gamma_j\} \subseteq \mathbb{Z}^d$ *is dissociate and satisfies* $\sum_{j=1}^{k-1} |\gamma_j| \leq c\,|\gamma_k|$ *for some* $c < 1$.

3.2. **Random Riesz products.** One can similarly define a random Riesz product as the weak* limit of the sequence of measures

$$\prod_{j=1}^{k}(1+\mathrm{Re}\, a_j e^{i\gamma_j\cdot(x+\omega_j)})dx$$

where $\omega = \{\omega_j\}$ is a sequence of independent and identically distributed, random variables on $\mathbb{T}^d$.

The results of Peyriere and Brown et al were extended by Fan in [4] to random Riesz products. He showed, for example, that if $n_{k+1}/n_k \geq 3$, $\lim(k/\log n_k) = \xi$ exists and $a_k = a \in \mathbb{R}$, then the Hausdorff dimension of the random Riesz product $\mu_{a,\omega} = \prod_{j=1}^{\infty}(1+a\,\mathrm{Re}\, e^{i\gamma_j\cdot(x+\omega_j)})$ equals

$$1-\ \xi\left(1-\sqrt{1-a^2}+\log\left((1+\sqrt{1-a^2})/2\right)\right)$$

In contrast, we can show that the energy dimension is generally smaller.

Proposition 3.4. *If* $n_{k+1}/n_k \geq 3$, $\lim_{k\to\infty}(k/\log n_k) = \xi \neq 0$ *and* $a_k = a \in \mathbb{R}$, *then almost surely the Hausdorff dimension of* $\mu_{a,\omega}$ *exceeds the energy dimension.*

Proof. As $\left|\widehat{\mu_{a,\omega}}(n)\right| = \left|\widehat{\mu_a}(n)\right|$, formula (3.6) gives that

$$I_{1-\alpha}(\mu_{a,\omega}) \sim 1+\frac{a^2}{2}\sum_{k=1}^{\infty} n_k^{-\alpha}\left(1+\frac{a^2}{2}\right)^{k-1}.$$

Thus the $(1-\alpha)$-energy is infinite if, for infinitely many k,

$$\alpha \leq \frac{(k-1)\log(1+a^2/2)}{\log n_k} \to \xi\log(1+a^2/2)$$

Since a simple calculus argument shows

$$\log(1+a^2/2) > 1-\sqrt{1-a^2}+\log\left((1+\sqrt{1-a^2})/2\right) \text{ if } a\neq 0$$

the result is immediate. ∎

3.3. **Riesz products on** $\mathbb{R}^d$**.** Any measure μ on $\mathbb{T}^d$ (which we now identify with $[-1/2,1/2]^d$) can be extended periodically to μ_e on $\mathbb{R}^d$ (the extension μ_e being a measure which is infinite except in the trivial case when $\mu = 0$). Let ϕ be any strictly positive, $C^\infty(\mathbb{R}^d)$ function, whose Fourier transform is a positive function supported in $B(0,1/2)$ satisfying $\widehat{\phi}(0) > 0$. Assume also that ϕ decays sufficiently rapidly to ensure that $\phi\mu_e$ is a finite measure.

Following Peyriere, we will say that ν is a Riesz product measure on $\mathbb{R}^d$ if $\nu = \phi\mu_e$ where μ is a Riesz product on $\mathbb{T}^d$ and ϕ is as above.

Proposition 3.5. *Suppose* μ *is a measure on* $\mathbb{T}^d$ *and* $\phi\mu_e$ *is a finite measure on* $\mathbb{R}^d$ *where* μ_e *is the periodic extension of* μ *to* $\mathbb{R}^d$ *and* ϕ *is as described above. Then* $I_t(\mu)$ *is comparable to* $I_t(\phi\mu_e)$.

Proof. Formally, the periodic extension μ_e of μ is the convolution of the compactly supported measure μ defined on $[-1/2,1/2]^d \subseteq \mathbb{R}^d$, with the distribution $P = \sum_{a\in\mathbb{Z}^d}\delta_a$ where δ_a is the point mass measure at $a \in \mathbb{R}^d$. Hormander's theorem ([6], 7.2.1) gives $\widehat{P} = P$, thus $\widehat{\mu_e}$ is the distribution $\widehat{\mu * P} = \sum_{a\in\mathbb{Z}^d}\widehat{\mu}(\cdot)\delta_a$.

Let $\nu = \phi\mu_e$ where ϕ is as above. Since ν is a finite measure on $\mathbb{R}^d$ the classical formula can be applied:

$$I_t(\nu) = c\int |x|^{t-d}\,|\widehat{\nu}(x)|^2\,dx.$$

Now $\widehat{\nu}(x) = \widehat{\mu_e} * \widehat{\phi}(x) = \sum_{a\in\mathbb{Z}^d}\widehat{\mu}(a)\widehat{\phi}(x-a)$, and as $\operatorname{supp}\widehat{\phi} \subseteq B(0,1/2)$ there can be at most one non-zero term in the sum for any given x. Thus we can decompose the integral above as

$$\begin{aligned} I_t(\nu) &= \sum_{a\in\mathbb{Z}^d}|\widehat{\mu}(a)|^2\int_{B(0,1/2)}\left|\widehat{\phi}(x)\right|^2|x+a|^{t-d}\,dx \\ &\sim |\widehat{\mu}(0)|^2 + \sum_{a\in\mathbb{Z}^d\setminus 0}|\widehat{\mu}(a)|^2\,|a|^{t-d} \end{aligned}$$

since $\widehat{\phi}$ is bounded away from zero in a neighbourhood of the origin. But this is comparable to $I_t(\mu)$ according to our formula (2.2). ∎

Corollary 3.6. *If $\nu = \phi\mu_e$ is a Riesz product measure on $\mathbb{R}^d$ corresponding to the Riesz product μ on $\mathbb{T}^d$, then the energy dimensions of ν and μ are equal.*

Remark 3.2. *The Hausdorff dimensions of ν and μ are also equal since $\phi\mu_e$ and μ_e have the same null sets.*

References

[1] Brown, G., Moran, W. and Pearce, C., *Riesz products, Hausdorff dimension and normal numbers*, Math. Proc. Camb. Phil. Soc. **101**(1987), 529-540.

[2] Falconer, K., *Fractal geometry; mathematical foundations and applications*, Wiley and Sons, Chichester (1990).

[3] Falconer, K., *Techniques in fractal geometry*, Wiley and Sons, Chichester (1997).

[4] Fan, Ai Hua, *Quelqes proprietes des produits de Riesz*, Bull. Sci. Math. **117**(1993), 421-439.

[5] Fan, Ai Hua, *Une formule approximative de dimension pour certains produits de Riesz*, Monatsh. Math. **118**(1994), 83-89.

[6] Hormander, L., *The analysis of partial differential operators I*, Springer-Verlag, Berlin, 1983.

[7] Katznelson, Y., *An introduction to harmonic analysis*, Dover Press, New York, 1976.

[8] Mattila, P., *Geometry of sets and measures in Euclidean spaces*, Cambridge studies in Advanced Mathematics **44**, Cambridge Univ. Press, Cambridge, 1995.

[9] Peyriere, J., *Etude de quelques proprietes des produits de Riesz*, Ann. Inst. Fourier, Grenoble **25**(1975), 127-169.

[10] Rudin, W., *Trigonometric series with gaps*, J.Math. Mech. **9**(1960), 203-227.

[11] Zygmund, A., *Trigonometric series* I, Cambridge Univ. Press, Cambridge, 1959.

Dept. of Pure Mathematics, University of Waterloo, Waterloo, Ont., N2L 3G1, Canada

E-mail address: `kehare@uwaterloo.ca`

Dept. of Mathematics, Chalmers TH and Goteborg University, Eklandagatan 86, SE-41296, Sweden

E-mail address: `maria@math.chalmers.se`