NON-HOLOMORPHIC FUNCTIONAL CALCULUS FOR COMMUTING OPERATORS WITH REAL SPECTRUM

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ABSTRACT. We consider n-tuples of commuting operators $a = a_1, \dots, a_n$ on a Banach space with real spectra. The holomorphic functional calculus for a is extended to algebras of ultradifferentiable functions on $\mathbb{R}^n$, depending on the growth of $\|\exp(ia\cdot t)\|$, $t \in \mathbb{R}^n$, when $|t| \to \infty$. In the non-quasi-analytic case we use the usual Fourier transform, whereas for the quasi-analytic case we introduce a variant of the FBI transform, adapted to ultradifferentiable classes.

1. INTRODUCTION

Let X be a Banach space and let $\mathcal{L}(X)$ denote the space of bounded linear operators on X. If $a_1, \dots, a_n \in \mathcal{L}(X)$ are commuting, then $f(a) = f(a_1, \dots, a_n)$ has a definite meaning for any polynomial $f(z) = f(z_1, \dots, z_n)$; in fact for any entire function $f(z)$. Since the polynomials are dense in $\mathcal{O}(\mathbb{C}^n)$ there is a continuous algebra homomorphism

$$\mathcal{O}(\mathbb{C}^n) \to (a) \subset \mathcal{L}(X), \tag{1.1}$$

where (a) denotes the closed subalgebra of $\mathcal{L}(X)$ that is generated by $a_1, \dots, a_n$. To go beyond entire functions one has to consider the joint spectrum of the operators. The appropriate notion of joint spectrum $\sigma(a) = \sigma(a_1, \dots, a_n)$ was introduced by J. Taylor in [17]. Let $\Lambda_z^{p,q}$ denote the space of X-valued (p,q)-forms at $z \in \mathbb{C}^n$, and let $\delta_{z-a}(z)$ denote contraction with the operator-valued $(1,0)$-vector $2\pi i \sum (z_j - a_j)(\partial/\partial z_j|_z)$. Since $\delta_{z-a} \colon \Lambda_z^{p+1,0} \to \Lambda_z^{p,0}$ and $\delta_{z-a} \circ \delta_{z-a} = 0$ (this is equivalent to the commutativity of the a_j), we have a complex

$$0 \leftarrow \Lambda_z^{0,0} \leftarrow \dots \leftarrow \Lambda_z^{n,0} \leftarrow 0. \tag{1.2}$$

Taylor defines the spectrum $\sigma(a)$ as the set of z for which the complex (1.2) is not exact. It turns out that $\sigma(a)$ is a compact subset of $\mathbb{C}^n$ which is nonempty unless $X = \{0\}$.

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Theorem 1.1. *Suppose that $a_1, \dots, a_n$ are commuting operators on a Banach space. There is a continuous algebra homomorphism*

$$\mathcal{O}(\sigma(a)) \to \mathcal{L}(X), \quad f \mapsto f(a),$$

that extends (1.1). *Moreover, if $f = (f_1, \dots, f_m)$ is a mapping, $f_j \in \mathcal{O}(\sigma(a))$, and $f(a) = (f_1(a), \dots, f_m(a))$, then*

$$\sigma(f(a)) = f(\sigma(a)). \tag{1.3}$$

If furthermore $h \in \mathcal{O}(\sigma(f(a))$, then $h \circ f(a) = h(f(a))$.

The basic results about the functional calculus are due to Taylor, [18]. The last statement, the composition rule, was proved by Putinar in [14].

If a is one single operator and $f \in \mathcal{O}(\sigma(a))$, then $f(a)$ is given by the formula

$$f(a)x = \int_{\partial D} f(z)\omega_{z-a}x, \quad x \in X, \tag{1.4}$$

where ω_{z-a} is the resolvent form

$$\omega_{z-a} = \frac{1}{2\pi i}(z-a)^{-1}dz$$

which is holomorphic in $\mathbb{C} \setminus \sigma(a)$. In the several dimensional case, the resolvent $\omega_{z-a}x$, for a fixed element $x \in X$, is a cohomology class in $H^{n-1}(\mathbb{C}^n \setminus \sigma(a), \mathcal{O}_n^X)$ where $\mathcal{O}_n^X$ denotes the sheaf of holomorphic X-valued $(n,0)$-forms. In the original work of Taylor this cohomology class was defined by a Čech co-chain with respect to a certain covering of the complement of the spectrum. In principle, by the Dolbeault isomorphism, one can also represent the resolvent class by a $\bar\partial$-closed differential form of bidegree $(n, n-1)$ in the complement of the spectrum in such a way that the integral representation formula (1.4) still holds. It is then of interest to find a Dolbeault representative as explicitly as possible. This was first done by Vasilescu in [19] for the case when X is a Hilbert space, by an appropriate generalization of the Martinelli-Bochner formula. Vasilescu's construction was later generalized by D Albrecht [3] and [4] and Kordula and Muller [12] to operators on a Banach space satisfying some additional conditions. Finally, the first author gave the construction of the Dolbeault representative in the general case, see [3], and showed how this could be used to develop Taylor's theory in a simpler and more elementary way.

In this paper we will mainly consider the case when the spectrum $\sigma(a)$ is real; then there is always a (a)-valued form $\widetilde{\omega}_{z-a}$ of Cauchy-Fantappie-Leray type such that $\widetilde{\omega}_{z-a}x$ represents the class $\omega_{z-a}x$ for each $x \in X$. In fact, since $\sigma(a)$ is polynomially convex, for a fixed point $z \notin \sigma(a)$, by Cartans's theorem there are $\phi_j \in \mathcal{O}(\sigma(a))$ such that $\sum \phi_j(w)(z_j - w_j) = 1$. Hence, by Taylor's theorem, $\delta_{z-a}s = \sum s_j(z_j - a_j) = e$, if $s = \sum s_j dz_j$, $s_j = \phi_j(a)$, and since the polynomials

are dense in $\mathcal{O}(\sigma(a))$ it follows that s is in (a). It is now easy to obtain a smooth form in $\mathbb{C}^n \setminus \sigma(a)$ such that $\delta_{z-a}s(z) = e$, and one can then take $\widetilde{\omega}_{z-a} = s \wedge (\bar{\partial}s)^{n-1}$, see [3].

If F is a function with compact support that coincides with $f \in \mathcal{O}(\sigma(a))$ is some neighborhood of $\sigma(a)$, then

$$f(a) = -\int \bar{\partial}F(z) \wedge \omega_{z-a}.$$

This formula suggests that, in certain situations, one may have a richer functional calculus. One should look for functions F defined in $\mathbb{C}^n$, with say compact support, such that $|\bar{\partial}F|$ vanishes fast enough on $\sigma(a)$ compared to the growth of some representing form $\widetilde{\omega}_{z-a}$ as above so that the integral

$$F(a) = \int \bar{\partial}F(z) \wedge \widetilde{\omega}_{z-a} \tag{1.5}$$

has a meaning. One easily verifies that the definition only depends on the values of F near $\sigma(a)$. For one single operator this idea was exploited by Dynkin, see [9]. In the higher dimensional case similar ideas have been used by several authors including Waelbrock [20], Nguyen [13], Droste [8], and more recently Sandberg [16]. One difficulty is to prove the multiplicative property $FG(a) = F(a)G(a)$. In [13] this is done in a manner parallel to Taylors method, by considering tensor products. Droste considers the situation when the spectrum lies on a totally real manifold, and in that case he obtains the multiplicative property by approximation with holomorphic functions. Finally Sandberg proves a multidimensional generalization of the so-called resolvent identity, and obtains the multiplicative property from there, following Dynkin's approach. Another difficulty in the several variable case is that there is a variety of possible representatives $\widetilde{\omega}_{z-a}$ of the class ω_{z-a}. Therefore, the growth of a representative of the resolvent class, and much less the growth of the form s, is not an intrinsic property of the n-tuple of operators when $n > 1$, and it would be desirable to have a more easily verified hypothesis on the operators which as closely as possible determines which class of functions that operate on them.

Another natural approach to extend the functional calculus if the spectrum is real (or contained in the torus in $\mathbb{C}^n$) is by the suggestive formula

$$f(a) = \int e^{ia\cdot t}\hat{f}(t)dt, \tag{1.6}$$

where $a \cdot t = \sum a_j t_j$, for $t \in \mathbb{R}^n$, and where f is a function on $\mathbb{R}^n$ and $\hat{f}$ is the usual Fourier transform

$$\hat{f}(t) = \frac{1}{(2\pi)^n}\int e^{-it\cdot x}f(x)dx. \tag{1.7}$$

Clearly this formula gives a meaning to $f(a)$ if $\hat{f}$ has enough decay, which roughly speaking just means that f has enough regularity, compared to the growth of $\|e^{ia\cdot t}\|$. In this case the multiplicativity follows directly and the problem with non-uniqueness of the resolvent representative disappears. It is not clear to us where this idea appeared first, but it is used quite explicitly (in the case of one single operator with spectrum on the circle) already in [21], where the idea is attributed to Beurling. It turns out that a radial growth condition on $\|e^{ia\cdot t}\|$ precisely corresponds to a radial growth condition on some form $\widetilde{\omega}_{z-a}$. In the same way, the regularity of a function on $\mathbb{R}^n$ measured by a radial decay condition on $|\hat{f}(t)|$ more or less corresponds to a radial decay condition on $\bar{\partial}F$ for some extension $F(z)$ of f to $\mathbb{C}^n$. In this case therefore both methods give rise to essentionally the same functional calculus. The purpose of this paper is to consider more general growth conditions, and then no such simple connection between the two methods is possible, because of the so-called edge-of-the-wedge phenomena. It turns out that the Dynkin approach then cannot give as precise statements as the Fourier method; so in this paper we concentrate on the latter. However, for a discussion of the two methods, see Section 6 and [5].

One can consider the mapping $f \mapsto \mu(f) = f(a) \in (a)$ given by Theorem 1.1 as an (a)-valued analytic functional which is carried by $\sigma(a) \subset \mathbb{R}^n$, i.e., an (a)-valued hyperfunction on $\mathbb{R}^n$ with compact support. The regularity of such a hyperfunction is reflected by the growth of its (inverse) Fourier transform $\check{\mu}(t) = \mu(e^{ix\cdot t})$ and is related to how large class of functions that μ operates on. Therefore we should look for optimal such classes of functions. Our starting point is certain Banach algebras $\mathcal{A}_h$ of functions in $\mathbb{R}^n$, first introduced by Beurling [6]. Here h is a nonnegative subadditive function and $f \in \mathcal{A}_h$ if

$$\int |\hat{f}(t)| \exp h(t) dt$$

is finite. Clearly a compactly supported hyperfunction μ is defined on $\mathcal{A}_h$ if $|\check{u}| \leq C \exp h$; in case μ is the holomorphic functional calculus, this action of course is realized by (1.6). However, to find the optimal class of functions on which such a μ will be acting, one has to consider functions only defined in some neighborhood of the support of μ. The core of this paper is to show that the algebras $\mathcal{A}_h$ can be extended to spaces $\mathcal{A}_{h,K}$ of functions defined in some neighborhood of a compact set K, that these spaces essentially are the duals of compactly supported hyperfunctions with the stated regularity, that these spaces actually are algebras, and that the holomorphic functional calculus extends to these spaces, and that the desired spectral mapping property holds, given the appropriate growth condition on $\|e^{ia\cdot t}\|$.

In the case when $\mathcal{A}_h$ contains cutoff functions, the non-quasianalytic case, it turns out that the space $\mathcal{A}_{h,K}$ of functions can be described

simply as the restrictions to K of functions in the space of global functions $\mathcal{A}_h$ on $\mathbb{R}^n$. This is done in Section 2, and the extension of the functional calculus to these algebras is made in Section 3. The work in these two sections to a large extent relies on Gelfand theory. To define the general algebras, and the corresponding functional calculus, we introduce a variant of the so-called FBI transformation that we have adapted to ultradifferentiable classes. This is done in Sections 4 and 5.

We end up this section with some additional remarks on tuples of operators with real spectra. From the spectral mapping property in Theorem 1.1 it follows that $\sigma(a)$ is real if and only of $\sigma(a_k)$ is real for each k. Furthermore, if $\sigma(a)$ is real then $\|e^{ia\cdot t}\| = \exp o(|t|)$ when $|t| \to \infty$, and in fact this growth condition characterizes commuting n-tuples with real spectra. More precisely,

Lemma 1.2. *Suppose that $a_1, \dots, a_n$ are commuting operators. Then $\sigma(a)$ is contained in $\mathbb{R}^n$ if and only if there is a (increasing and concave) function $H(s)$ on $[0,\infty)$ such that*

$$\|e^{ia\cdot t}\| \le Ce^{H(|t|)} \quad \textit{and} \quad \lim_{s\to\infty} H(s)/s = 0. \tag{1.8}$$

Proof. Suppose that $\sigma(a) \subset \mathbb{R}^n$. Let D be a neighborhood of $\sigma(a)$ in $\mathbb{R}^n$ and let $D_\eta = \{x+iy;\ x \in D,\ |y| < \eta\}$. If $\widetilde{\omega}_{z-a}$ is a fixed form as above, then, by the formula (1.4), we have that

$$e^{ia\cdot t} = \int_{\partial D_\eta} e^{iz\cdot t}\widetilde{\omega}_{z-a}. \tag{1.9}$$

Let $g(\eta)$ be a convex decreasing function on $(0,\infty)$ such that

$$\sup_{z\in\partial D_\eta} \|\widetilde{\omega}_{z-a}\| \le \exp g(\eta)$$

for small η. If $g^\flat(s) = \inf_{\eta>0}(g(\eta) + s\eta)$, then $g^\flat(s)$ is concave and increasing on $(0,\infty)$ and a simple estimation of (1.9) gives that

$$\|e^{ia\cdot t}\| \le C \exp g^\flat(|t|).$$

For each $\epsilon > 0$ there is a constant C_ϵ such that $g(\eta) \le C_\epsilon$ for $\eta \ge \epsilon$. Therefore $g^\flat(s) \le C_\epsilon + \epsilon s$, i.e., $g^\flat(s) = o(s)$.

Conversely, assume that (1.8) holds. If $w \in \sigma(a)$ it follows by the spectral mapping property (Theorem 1.1) that $e^{iw\cdot t} \in \sigma(e^{ia\cdot t})$, and therefore

$$|e^{iw\cdot t}| \le \|e^{ia\cdot t}\| \le Ce^{h(t)}.$$

Taking $t = -s\operatorname{Im} w$ in this inequality and letting $s \to \infty$ we deduce that $\operatorname{Im} w = 0$ so that $w \in \mathbb{R}^n$. Thus the lemma is proved. □

Remark 1. One can prove the lemma with no explicit reference to the spectral mapping property. In fact, assume that (1.8) holds and let

$w = \alpha - i\beta$ with $\beta \neq 0$. Then $\left\|e^{is(a-w)\cdot\beta}\right\| \lesssim e^{h(s\beta)-|\beta|^2 s} \lesssim e^{-\delta s}$ for some positive δ, and hence

$$c_j = \int_{s=0}^{\infty} \beta_j e^{is(a-w)\cdot\beta} ds$$

makes sense. Furthermore,

$$\sum c_j(w_j - a_j) = \int_0^{\infty} \frac{d}{ds} e^{is(a-w)\cdot\beta} ds = e,$$

which shows that $w \notin \sigma(a)$. □

Contents

2. Algebras of ultra-differentible functions

In view of formula (1.6) and Lemma 1.2 it is natural to consider classes (algebras) of functions whose Fourier transforms have less than exponential decay. Let $h(t)$ be a positive, continuous, and subadditive function in $\mathbb{R}^n$ with $h(0) = 0$. Moreover, assume that $h(t)$ is increasing on rays from the origin and that

$$\lim_{|t|\to\infty} \frac{h(t)}{|t|} = 0. \tag{2.1}$$

A function satisfying these requirements will be referred to as an *admissible weight function*. For instance, if $H(s)$ is concave and nondecreasing on each half axis of $\mathbb{R}$ and $H(0) = 0$, then it is automatically subadditive, and if in addition $H(s) = o(|s|)$ when $s \to \pm\infty$, then functions like $h(t) = H(|t|)$ and $h(t) = H(t \cdot \alpha)$, $\alpha \in \mathbb{R}^n$, are admissible weight functions. If h is admissible, then h^a is admissible for $0 < a < 1$. It is easily verified that the class of admissible weight functions is closed under finite sums and suprema. More generally, if $h_1, \ldots, h_m$ are admissible, then $(h_1^p + \cdots + h_m^p)^{1/p}$ is admissible if $1 \leq p < \infty$. Sometimes we impose the additional assumption that $\exp(-h(t)) = \mathcal{O}(|t|^{-m})$ for all positive m, or equivalently, that

$$\limsup_{|t|\to\infty} \frac{\log(1+|t|)}{h(t)} = 0. \tag{2.2}$$

Let $\mathcal{A}_h$ be the space of tempered distributions f on $\mathbb{R}^n$ such that $\hat{f}$ is a measure and

$$\|f\|_{\mathcal{A}_h} = \int_t |\hat{f}(t)| e^{-h(t)} dt < \infty, \tag{2.3}$$

where the Fourier transform $\hat{f}$ is defined as in (1.7). Then any $f \in \mathcal{A}_h$ is at least continuous, and if (2.2) holds, then $\mathcal{A}_h$ is contained in $C^\infty(\mathbb{R}^n)$. Clearly $\mathcal{A}_h$ is a Banach space of functions that is closed under translations. Moreover, if $\hat{g}$ is bounded, in particular if $g \in L^1(\mathbb{R}^n)$, then $f \mapsto f * g$ is a bounded operator on $\mathcal{A}_h$. Since h is subadditive, $h(u) = h(t + u - t) \le h(t) + h(u - t)$, and therefore

$$e^{h(u)} |\hat{f} * \hat{g}(u)| \le \int_t |\hat{f}(t)| e^{h(t)} |\hat{g}(u - t)| e^{h(u-t)} dt,$$

and integrating this inequality with respect to u and applying Fubini's theorem we get that

$$\|fg\|_{\mathcal{A}_h} \le \|f\|_{\mathcal{A}_h} \|g\|_{\mathcal{A}_h}.$$

Thus $\mathcal{A}_h$ actually is a Banach algebra under pointwise multiplication. We say that the class $\mathcal{A}_h$ is non-quasianalytic if for each compact set E and open neighborhood $U \supset E$ there is a function $\chi \in \mathcal{A}_h$ with support in U which is identically 1 in some neighborhood of E. We recall the following version of the Denjoy–Carleman theorem.

Theorem 2.1. *Let h be an admissible weight function. The class $\mathcal{A}_h$ is non-quasianalytic if and only if*

$$\int_{|t| \ge 1} \frac{h(t) dt}{|t|^{n+1}} < \infty, \tag{2.4}$$

and this holds if and only if there is a concave increasing function $H(s)$ such that $h(t) \le H(|t|)$ and

$$\int_1^\infty \frac{H(s) ds}{s^2} < \infty. \tag{2.5}$$

Remark 2. In [6] this theorem is only stated explicitly for the class $\cap_{c>0} \mathcal{A}_{ch}$ but it holds for each fixed $\mathcal{A}_h$ as well. What is not obvious from Beurling's formulation is that all desired cutoff functions χ can be found in the same space $\mathcal{A}_h$. Notice that the simplest way to obtain functions with small support, by dilation like $f_\delta(x) = \delta^{-n} f(x/\delta)$, does not work. However, the statement is anyway true and for the reader's convenience we supply a direct proof here.

We thus are to prove that $\mathcal{A}_h$ contains all desired kinds of cutoff functions if $h(t) = H(|t|)$, where H is concave and increasing and (2.5) holds. Let us first assume that $n = 1$ and let

$$\tilde{h}(t) = H(|t|) + 2 \log(1 + |t|). \tag{2.6}$$

In view of (2.5) it follows that the Poisson integral $P\tilde{h}$ of $\tilde{h}$ is a positive harmonic function in the upper halfplane. If Φ is a holomorphic function such that $\operatorname{Re}\Phi = P\tilde{h}$, then $g = \exp(-\Phi)$ is a bounded holomorphic function in the upper halfplane and $|g| = \exp(-\tilde{h}(t))$ on the boundary. Therefore g is the Fourier transform of a function $f(x)$ supported on the positive halfaxis and since $|\hat{f}(t)| = \exp(-\tilde{h}(t))$ it follows that $f \in \mathcal{A}_h$. Now $\phi(x) = f(\alpha+x)f(\beta-x)$ is an nonvanishing function in $\mathcal{A}_h$ with support in $(-\epsilon, \epsilon)$ if the real numbers α and β are appropriately chosen. It follows that $f(x) = \phi(x_1)\phi(x_2)\cdots\phi(x_n)$ is in $\mathcal{A}_h$ in $\mathbb{R}^n$ and has support in $\{x;\ \max|x_j| < \epsilon\}$. Since h is a radial function, $\mathcal{A}_h$ is closed under conjugation and if $f \in \mathcal{A}_h$ has compact support, then $|f|^2$ is a non-negative compactly supported function in $\mathcal{A}_h$. One obtains the required function χ by convolution of a function with small support and the characteristic function for a domain which is slighly larger than E. □

Let $\mathcal{A}_{h,0}$ be the subalgebra of $\mathcal{A}_h$ that is generated by the constants and the functions $f \in \mathcal{A}_h$ such that $\hat{f}$ is absolutely continuous, i.e., in $L^1(e^h)$. Each function $f(x)$ in $\mathcal{A}_{h,0}$ is thus continuous and has a limit when $|x| \to \infty$. For this slightly smaller Banach algebra there is a simple description of its maximal ideal space as the one point compactification of $\mathbb{R}^n$.

Lemma 2.2. *The complex homomorphisms $m\colon \mathcal{A}_{h,0} \to \mathbb{C}$ are precisely the point evaluations $x \mapsto f(x)$ for $x \in \mathbb{R}^n \cup \{\infty\}$.*

Proof. If $x \in \mathbb{R}^n \cup \{\infty\}$, then clearly $f \mapsto f(x)$ is a complex homomorphism $\mathcal{A}_{h,0} \to \mathbb{C}$. To see that any homomorphism is of this kind, first notice that any $f \in \mathcal{A}_{h,0}$ can be written uniquely as $f = f_c + \beta$, where $\hat{f}_c$ is in $L^1(e^h)$ and β is a constant. Following [15] Example 11.13 (e) one finds that if $m\colon \mathcal{A}_{h,0} \to \mathbb{C}$ is a homomorphism, then either $m(f_c + \beta) = \beta$ for all $f \in \mathcal{A}_{h,0}$ or

$$m(f) = \int \hat{f}\phi e^h dt$$

for some bounded function ϕ. The multiplicativity property then forces that $\phi(t)e^{h(t)} = e^{it\alpha}$ for some complex number α. In view of the assumption (2.1), α must be real, and thus $m(f) = f(\alpha)$. □

From the lemma and basic Gelfand theory it follows that the ideal generated by $f_1, \ldots, f_m \in \mathcal{A}_{h,0}$ is the whole algebra if and only if the mapping $f = (f_1, \ldots, f_m)$ is nonvanishing on $\mathbb{R}^n \cup \{\infty\}$. In particular, $1/f \in \mathcal{A}_{h,0}$ if $f \in \mathcal{A}_{h,0}$ and $f \neq 0$ on $\mathbb{R}^n \cup \{\infty\}$.

Now suppose that $\mathcal{A}_h$ is non-quasianalytic, let E be a compact subset of $\mathbb{R}^n$, and let

$$I_{h,E} = \{f \in \mathcal{A}_{h,0};\ f = 0 \text{ on } E\}.$$

Since $I_{h,E}$ is a closed ideal in $\mathcal{A}_{h,0}$, the quotient space $\mathcal{A}_{h,0}/I_{h,E}$ is again a Banach algebra which intuitively consists of all restrictions to E of functions in $\mathcal{A}_{h,0}$, with the norm

$$\|\phi\|_{A_{h,0}/I_{h,E}} = \inf\{\|f\|_{\mathcal{A}_{h,0}};\ f \in \mathcal{A}_{h,0} \text{ and f} = \phi \text{ on E}\}.$$

Since we assume that $\mathcal{A}_h$ is non-quasianalytic the definition is unaffected if we replace $\mathcal{A}_{h,0}$ by $\mathcal{A}_h$.

Clearly each point evaluation $f \mapsto f(\alpha)$, $\alpha \in E$, is a homomorphism $\mathcal{A}_{h,0}/I_{h,E} \to \mathbb{C}$. Conversely, any such homomorphism is pulled back to a homomorphism on $\mathcal{A}_{h,0}$ that vanishes on $I_{h,E}$. In view of the previous proposition it is therefore given by a point evaluation $f \mapsto f(\alpha)$ for some $\alpha \in E$. We therefore have

Proposition 2.3. *The maximal ideal space of $A_{h,0}/I_{h,E}$ is precisely E. Hence if $f_1, \dots, f_m \in A_{h,0}/I_{h,E}$, then there are $u_j \in A_{h,0}/I_{h,E}$ such that $\sum u_j f_j = 1$ if and only if $f = (f_1, \dots, f_m)$ is nonvanishing on E.*

Our next objective is to show that one can compose with functions that are holomorphic in some neighborhood of the image.

Proposition 2.4. *Suppose that $f = (f_1, \dots, f_m)$, where $f_j \in A_{h,0}/I_{h,E}$, and suppose that $g \in \mathcal{O}(f(E))$. Then $g \circ f \in A_{h,0}/I_{h,E}$.*

Proof. The spectrum of $f_1, \dots, f_m$ with respect to the Banach algebra $A_{h,0}/I_{h,E}$ is equal to the image of the Gelfand transform, and in view of Proposition 2.3 is is precisely the compact set $f(E)$. Therefore, by the holomorphic functional calculus for Banach algebras, $g(f) = g(f_1, \dots, f_m)$ is an element in $A_{h,0}/I_{h,E}$. (One can consider f_j as commuting operators on the Banach space $X = A_{h,0}/I_{h,E}$, and hence this claim is an instance of Theorem 1.1.) We must check that this object coincides with the pointwise defined function $g \circ f$.

There is a smooth $\mathcal{A}_{h,0}/I_{h,E}$-valued form $s(w) = \sum_1^m s_j(w) dw_j$ in $\mathbb{C}^m \setminus f(E)$ such that $\delta_{w-f} s(w) = \sum_1^m s_j(w)(w_j - f_j) = 1$. If

$$\widetilde{\omega}_{w-f} = \frac{1}{(2\pi i)^m} s \wedge (\bar\partial_w s)^{m-1}$$

then, cf., [3],

$$g(f) = \int_{\partial D} g(w) \widetilde{\omega}_{w-f}. \tag{2.7}$$

For fixed $\alpha \in \mathbb{C}^m$, let $\omega_{w-\alpha}$ denote the cohomology class in $\mathbb{C}^m \setminus \{\alpha\}$ that represents the point evaluation $g \mapsto g(\alpha)$ for holomorphic g. For fixed $x \in E$, $\sum_j s_j(w)(x)(w_j - f_j(x)) = 1$, for $w \in \mathbb{C}^m \setminus f(E)$, hence $\widetilde{\omega}_{w-f}(x)$ represents the class $\omega_{w-f(x)}$, and therefore

$$g(f)(x) = \int_{\partial D} g(w) \omega_{w-f}(x) = \int_{\partial D} g(w) \omega_{w-f(x)} = g(f(x)).$$

□

Later on we will need that if $\mathcal{A}_h$ is non-quasianalytic then $A_{h,0}/I_{h,E}$ contains all functions which are realanalytic in some neighborhood of E. More precisely we have

Lemma 2.5. *Let D be an open set in $\mathbb{R}^n$ and let $D_\eta = D \times \{|y| < \eta\}$ for some $\eta > 0$. If $\chi \in \mathcal{A}_h$ has compact support in D, then $\chi\phi \in \mathcal{A}_h$ for all $\phi \in \mathcal{O}(D_\eta)$ and we have that*

$$\|\phi\chi\|_{\mathcal{A}_h} \leq C_{\chi,h,\eta} \sup_{D_\eta} |\phi|, \quad \phi \in \mathcal{O}(D_\eta). \tag{2.8}$$

The proof is postponed to the end of this section. We are now ready to define our main algebras.

Definition 1. Let h be an admissible weight function which in addition satisfies (2.4). For each compact subset $K \subset \mathbb{R}^n$ we let $\mathcal{A}_{h,K}$ be the inductive limit of the algebras $\mathcal{A}_{h,0}/I_{h,\overline{U}}$, for neighborhoods $U \supset K$.

This means that each function $F \in \mathcal{A}_h$ defines an element in $\mathcal{A}_{h,K}$, and $F, F' \in \mathcal{A}_h$ define the same element if and only if $F = F'$ in some neighborhood of K. The topology is defined by the requirement that any mapping Φ from $\mathcal{A}_{h,K}$ to a topological space Y is continuous if and only if its pullback to $\mathcal{A}_{h,0}/I_{h,\overline{U}}$ is continuous for each $U \supset K$.

Theorem 2.6. *Let h be an admissible weight function which satisfies (2.4) and let $K \subset \mathbb{R}^n$. Then $\mathcal{A}_{h,K}$ is a topological algebra that contains $\mathcal{O}(K)$. Suppose that $f_1, \dots, f_m \in \mathcal{A}_{h,K}$ and $f = (f_1, \dots, f_m)$. Then $f_1, \dots, f_m$ generate the whole algebra if and only if the mapping f is non-vanishing on K. Moreover, if $g \in \mathcal{O}(f(K))$, then $g \circ f \in \mathcal{A}_{h,K}$ as well.*

Proof. If $\phi \in \mathcal{O}(K)$, then $\phi \in \mathcal{O}(\overline{D_\eta})$ for some open $D \supset K$ and $\eta > 0$. If we choose a cutoff function $\chi \in \mathcal{D}(D)$ which is 1 in a neighborhood of K, it follows from Lemma 2.5 that $\phi\chi \in \mathcal{A}_{h,K}$. It is clear that $\mathcal{A}_{h,K}$ is an algebra and $f_1, \dots, f_m \in \mathcal{A}_{h,K}$ generates the whole algebra if and only if $f = (f_1, \dots, f_m) \neq 0$ on K. In fact, if $f \neq 0$ on K, then $f \neq 0$ on $\overline{U}$ for some $U \supset K$, and from Proposition 2.3 it then follows that there are $u_j \in \mathcal{A}_{h,0}/I_{h,\overline{U}}$ such that $\sum f_j u_j = 1$. Moreover, Proposition 2.4 implies that $g \circ f \in \mathcal{A}_{h,K}$ if $f \in \mathcal{A}_{h,K}$ and $g \in \mathcal{O}(f(K))$. □

We shall now briefly discuss the relation between the $\mathcal{A}_h$-classes and the so-called C_M-classes or C^L-classes, cf., [11] Section 8.4. For simplicity we restrict ourselves to the case of global functions. Let $M_0, M_1, \dots$ be a sequence of positive numbers such that

$$M_0 = 1 \quad \text{and} \quad M_k^2 \leq M_{k-1} M_{k+1} \text{ for all } k. \tag{2.9}$$

The latter condition just means that $\log M_k$ is convex. The class C_M consists of all functions on $\mathbb{R}^n$ for which there are constants $C_1, C_2 > 0$ such that

$$|D^\alpha f(x)| \leq C_1 C_2^{|\alpha|} M_{|\alpha|}$$

for all multiindices α. (Some authors instead consider $L_k = M_k^{1/k}$ and call the corresponding class C^L.) It turns out that C_M is an algebra that is non-quasianalytic if and only if $\sum M_k^{-1/k} < \infty$.

Let $H(s)$ be an increasing concave function on $[0, \infty)$ with $H(0) = 0$, and assume in addition that $s \mapsto H(\exp s)$ is a convex function on $\mathbb{R}$. Then its Legendre transform

$$m(x) = \sup_s (xs - H(\exp s)$$

is convex and

$$H(e^s) = \sup_{x \in \mathbb{R}} (xs - m(x)),$$

see, e.g., [11]. Since $m(x) = -\infty$ for $x < 0$, $m(0) = 0$ and $m(x)$ is increasing, it follows that the sequence $M_k = \exp m(k)$ satisfies (2.9), and moreover,

$$\sup_{k \in \mathbb{N}} (ks - \log M_k) \le H(e^s) \le \sup_{k \in \mathbb{N}} (ks - \log M_k) + s. \tag{2.10}$$

There is a close relation between C_M and $\mathcal{A}_h$ if $h(t) = H(|t|)$. In fact we have

Proposition 2.7. *Suppose that $h(t) = H(|t|)$ is an admissible weight function such that $s \mapsto H(e^s)$ is convex and assume that M_k and $h(t) = H(|t|)$ are related as above. If $f \in \mathcal{A}_h$ then $f \in C_M$; more precisely $|D^\alpha f| \le C M_{|\alpha|}$. Conversely, if $f \in C_M$, $|D^\alpha f| \le C C_1^{|\alpha|} M_{|\alpha|}$, and f has compact support, then*

$$|\hat{f}(t)| \le C_2 |t| e^{-h(t/C_1)}, \tag{2.11}$$

for some constant C_2.

Proof. If $f \in \mathcal{A}_h$, then an obvious estimate of the formula

$$D^\alpha f(x) = i^{|\alpha|} \int t^\alpha e^{-h(t)} e^{ix \cdot t} \hat{f}(t) e^{h(t)} dt$$

gives that

$$|D^\alpha f| \le C \sup_t |t|^{|\alpha|} e^{-h(t)} = C e^{\sup_{s>0}(|\alpha| s - H(e^s))} = C M_{|\alpha|}.$$

Conversely, if f has compact support and the condition on $D^\alpha f$ holds, a similar estimate yields that

$$|t^\alpha \hat{f}(t)| \le C C_1^{|\alpha|} M_{|\alpha|}.$$

In view of (2.10) we conclude that

$$\log |\hat{f}(t)| \le C' + \log |t| - H(|t|/C_1).$$

□

Thus, if the hypotheses above on H are fulfilled then, roughly speaking, $\mathcal{A}_h$ is the same class as C^L. There are natural examples of such an H, i.e., $H(0) = 0$, $H(s)$ sub-additive and increasing, and $s \mapsto H(e^s)$ convex. For instance, $H(s) = s^{1/a}$, $a > 1$, corresponding to $M_k = (k+1)^{ka}$, gives rise to the so-called Gevrey class of order a.

If $H(s)$ is concave, increasing and $H(0) = 0$ but $s \mapsto H(e^s)$ is not convex, then there is a slightly larger function $\tilde{H}$ with this extra property.

Proposition 2.8. *Suppose that $H(s)$ is concave and increasing on $(0,\infty)$ and that $H(0) = 0$. Furthermore, assume that (2.5) holds. Then there is a concave and increasing function $\tilde{H}(s)$ on $(0,\infty)$ such that $\tilde{H}(0) = 0$, $\tilde{H}(s) \geq H(s)$, $s \mapsto \tilde{H}(e^s)$ is convex, and such that (2.5) holds for $\tilde{H}$ as well.*

Proof. First we observe that $H(e^s)$ is (strictly) convex if and only if $tH'(t)$ is (strictly) increasing. Define $\tilde{H}$ by

$$t\tilde{H}'(t) = \int_0^t (sH'(s))'_+ ds, \quad \tilde{H}(0) = 0,$$

where the $+$ denote the positive part. Then $t\tilde{H}'(t)$ is increasing and $\tilde{H}(0) = 0$. Since

$$\tilde{H}'(t) \geq \frac{1}{t}\int_0^t (sH'(s))' ds = H'(t),$$

it follows that $\tilde{H}(t) \geq H(t)$. It remains to check that $\tilde{H}$ is concave and that (2.4) holds for $\tilde{H}$. Since $H'' \leq 0$ we have that

$$\tilde{H}'(t) \leq \frac{1}{t}\int_0^t H'(s) ds = \frac{H(t)}{t}.$$

Therefore

$$\int_1^\infty \frac{\tilde{H}'(t)}{t} dt \leq \int_1^\infty \frac{H(t)}{t^2} dt < \infty.$$

However, this is equivalent to (2.4) for $\tilde{H}$ since

$$\lim_{t\to\infty} \frac{\tilde{H}(t)}{t} = \lim_{t\to\infty} \tilde{H}'(t) \leq \limsup_{t\to\infty} \frac{H(t)}{t} = 0.$$

Finally $\tilde{H}$ is concave, because

$$(t\tilde{H}'(t))' = (tH'(t))'_+ \leq H'(t) \leq \tilde{H}'(t),$$

which implies that $t\tilde{H}''(t) \leq 0$. □

Proof of Lemma 2.5. First notice that if we can find some cutoff function $\tilde{\chi}$ in $\mathcal{A}_h$ which is identically 1 on the support of χ for which (2.8) holds, then it holds for χ as well since $\|\chi\phi\|_{\mathcal{A}_h} = \|\chi\tilde{\chi}\phi\|_{\mathcal{A}_h} \leq \|\chi\|_{\mathcal{A}_h}\|\tilde{\chi}\phi\|_{\mathcal{A}_h} \leq C\|\tilde{\chi}\phi\|_{\mathcal{A}_h}$.

Given $E \subset\subset D$ it is therefore enough to find some cutoff function $\chi \in \mathcal{A}_h$ with support in D which is identically 1 on E and such that (2.8) holds. In view of Theorem 2.1 and Proposition 2.8 we may assume that $h(t) = H(|t|)$ where H is concave increasing, $H(0) = 0$, and satisfying (2.5). Let us tentatively choose a cutoff function $\chi \in \mathcal{A}_h$ which is identically 1 in a neighborhood of E and has compact support in D. To simplify notation we assume that $n = 1$. Then $|\chi^{(k)}| \leq CM_k$ in view of Proposition 2.7. If $\phi \in \mathcal{O}(D_{2\eta})$ then

$$|\phi^{(k)}| \leq Ck!/\eta^k$$

where C is a constant times $\sup_{D_{2\eta}} |\phi|$. We may also assume that $H(s) \leq s$. It is then easily checked that $m(x) \geq x \log x - x$ and hence $M_k \geq (k/e)^k \sim k!$. (To be precise, at least $\geq k!(1-\epsilon)^k$.) From (2.9) it also follows that $M_{n-k}M_k \leq M_n$. Therefore,

$$|(\phi\chi)^{(n)}| \leq C \sum_0^n \frac{n!}{(n-k)!k!} \frac{1}{\eta^k} k! M_{n-k} \leq \sum_0^n \frac{n!}{(n-k)!k!} \frac{1}{\eta^k} M_n$$
$$\leq C(1+\eta^{-1})^n M_n.$$

Holding in mind that $\chi\phi$ has compact support, it follows from Proposition 2.7 that

$$|\widehat{(\chi\phi)}| \leq Ce^{-h(t/C_1)},$$

where $C_1 = 1 + 1/\eta$. Therefore if we instead choose χ in $\mathcal{A}_{\tilde{h}}$ where $\tilde{h}(t) = h(C_1 t) + (n+1)\log(1+|C_1 t|)$ (and with respect to $\tilde{\eta} = \eta/2$) we get (2.8). □

3. Non-quasianalytic functional calculus

For given commuting operators $a_1, \ldots, a_n$ with real spectrum we shall now consider possible extensions of the holomorphic functional calculus to the non-quasianalytic algebras $\mathcal{A}_{h,K}$ which where introduced in the previous section. Since the Taylor spectrum $\sigma(a)$ is polynomially convex, we know, cf., the introduction, that it is equal to the spectrum with respect to (a). For (a tuple) $b \in (a)$ we let $\hat{b}$ denote the Gelfand transform with respect to the algebra (a), and we recall that $\operatorname{Im} \hat{b} = \sigma_{(a)}(b)$, the spectrum with respect to (a). We begin with a preliminary result where we consider a (not necessarily non-quasianalytic) algebra $\mathcal{A}_h$ of global functions.

Proposition 3.1. *Let h be a non-negative subadditive function, $h(0) = 0$, and suppose that*

$$\|e^{ia\cdot t}\| \leq Ce^{h(t)}. \tag{3.1}$$

Then there is a continuous algebra homomorphism $\Phi\colon \mathcal{A}_h \to (a)$, such that $|\Phi(f)| \leq C\|f\|_{\mathcal{A}_h}$ and $\Phi(f) = f(a)$ for real analytic functions f in

$\mathcal{A}_h$ *satisfying*

$$|\hat{f}(t)| \lesssim e^{-\delta|t|}. \tag{3.2}$$

If $f = (f_1, \dots, f_m) \in \mathcal{A}_h$, *then*

$$f(\sigma(a)) = \sigma(\Phi(f)). \tag{3.3}$$

Moreover,

$$\widehat{\Phi(f)} = f \circ \hat{a} \tag{3.4}$$

and

$$\sigma_{(a)}(f(a)) = \sigma(f(a)). \tag{3.5}$$

If there exists a mapping Φ with the stated properties, then

$$\|e^{ia\cdot t}\| \le C\|e^{ix\cdot t}\|_{\mathcal{A}_h} = Ce^{h(t)},$$

and hence the assumption on the growth of $\|e^{ia\cdot t}\|$ is necessary. It follows that a admits a non-quasianalytic $\mathcal{A}_h$-functional calculus if and only if (3.1) holds for some radial h satisfying (2.5). In particular, a satisfies such a condition if and only if each a_j does. Usually we will write $f(a)$ rather than $\Phi(f)$, but in the proof it is convenient to keep the notational distinction.

Proof. By the assumptions it follows that the definition

$$\Phi(f) = \int e^{ia\cdot t}\hat{f}(t)dt$$

makes sense and that

$$\|\Phi(f)\| \le C\|f\|_{A_h}; \tag{3.6}$$

thus we have a continuous linear mapping $\Phi\colon \mathcal{A}_h \to L(X)$. The function

$$f_R(z) = \int_{|t|<R} e^{iz\cdot t}\hat{f}(t)dt \tag{3.7}$$

is an entire function for each R, so $f_R(a)$ is defined by the holomorphic functional calculus. Moreover,

$$\|f_R\|_{\mathcal{A}_h} \le \|f\|_{\mathcal{A}_h} \quad \text{and} \quad \|f - f_R\|_{\mathcal{A}_h} \to 0 \ R \to \infty. \tag{3.8}$$

We claim that

$$\Phi(f_R) = f_R(a). \tag{3.9}$$

In fact, if D is a neighborhood of $\sigma(a)$ in $\mathbb{R}^n$ and let $D_\eta = D \times \{y;\ |y| < \eta\}$ and ω_{z-a} is the resolvent, then by (1.4),

$$\Phi(f_R) = \int_{|t|<R} e^{ia\cdot t}\hat{f}(t)dt = \int_{|t|<R}\left(\int_{\partial D_\eta} e^{izt}\omega_{z-a}\right)\hat{f}(t)dt =$$
$$\int_{\partial D_\eta}\omega_{z-a}\int_{|t|<R} e^{izt}\hat{f}(t)dt = \int_{\partial D_\eta} f_R(z)\omega_{z-a} = f_R(a).$$

From (3.6), (3.8), and (3.9) it follows that $f_R(a) = \Phi(f_R) \to \Phi(f)$ and since $f_R(a) \in (a)$ hence $\Phi(f) \in (a)$. Moreover, if $f, g \in \mathcal{A}_h$, then $\|fg - f_R g_R\|_{\mathcal{A}_h} \to 0$ and since $(f_R g_R)(a) = f_R(a) g_R(a)$ by the holomorphic functional calculus, we can conclude that $\Phi(fg) = \Phi(f)\Phi(g)$. Moreover, the spectrum depends continuously on commutative perturbations, cf., [16] Proposition 2.6, and since $\sigma(f_R(a)) = f_R(\sigma(a))$ and $f_R \to f$ pointwise, (3.3) follows. If f is entire and satisfies (3.2), then $f_R \to f$ uniformly in a neighborhood of $\mathbb{R}^n$ and therefore $f_R(a) \to f(a)$ by the holomorphic functional calculus. Hence $\Phi(f) = f(a)$ in this case. Since f_R is entire we have that $\widehat{f_R(a)} = f_R \circ \hat{a}$ and then (3.4) follows by continuity. Finally, (3.5) follows from (3.4) and (3.3). □

We will now restrict to the non-quasianalytic case and localize to the spectrum. The first objective is to ensure that $f(a)$ only depends on the values of f in some neighborhood of $\sigma(a)$.

Corollary 3.2. *Let a and h be as in the previous proposition, and assume in addition that $\mathcal{A}_h$ is non-quasianalytic. If $f \in \mathcal{A}_h$ and $f = 0$ in a neighborhood of $\sigma(a)$, then $f(a) = 0$.*

Proof. Since $\mathcal{A}_h$ is non-quasianalytic we can find a function $\phi \in \mathcal{A}_h$ which is nonvanishing on $\sigma(a)$ such that $\phi f \equiv 0$. From the spectral mapping property (3.3) it follows that $\phi(a)$ is invertible, and since moreover $0 = \phi(a) f(a)$, we can conclude that $f(a) = 0$. □

We are now ready for the main result of this section.

Theorem 3.3 (Main theorem in the non-quasianalytic case)**.** *Let h be an admissible weight function that satisfies* (2.4) *and assume that $a_1, \dots, a_n$ are commuting operators in $\mathcal{L}(X)$ such that $\|e^{ia\cdot t}\| \le Ce^{h(t)}$. Then there is a continuous algebra homomorphism*

$$\Phi\colon \mathcal{A}_{h,\sigma(a)} \to (a) \tag{3.10}$$

such that $\Phi(f) = f(a)$ for all $f \in \mathcal{O}(\sigma(a))$. If $f_1, \dots, f_m \in \mathcal{A}_{h,\sigma(a)}$ and $f = (f_1, \dots, f_m)$, then

$$f(\sigma(a)) = \sigma(\Phi(f)), \tag{3.11}$$

and if $g \in \mathcal{O}(f(\sigma(a)))$, then

$$\Phi(g \circ f) = g(\Phi(f)). \tag{3.12}$$

Moreover, for each mapping $f \in \mathcal{A}_{h,\sigma(a)}$ we have that

$$\widehat{\Phi(f)} = f \circ \hat{a} \quad \text{and} \quad \sigma_{(a)}(f(a)) = \sigma(f(a)). \tag{3.13}$$

Proof. In view of Corollary 3.2 the algebra homomorphism Φ from Proposition 3.1 is well-defined on $\mathcal{A}_{h,\sigma(a)}$, and (3.11) as well as (3.13) immediately follows from Proposition 3.1.

If $s(w)$ is as in the proof of Proposition 2.4, then $\delta_{w-\Phi(f)}\Phi(s(w)) = e$. Moreover, by the continuity of Φ,

$$\bar\partial_w \Phi(s(w)) = \Phi(\bar\partial_w s(w))$$

and therefore

$$\Phi(\widetilde\omega_{w-f}) = \widetilde\omega_{w-\Phi(f)}.$$

If $g \in \mathcal{O}(f(\sigma(a)))$ we get from (2.7) that $\Phi(g(f)) = g(\Phi(f))$. But we already know that $g(f) = g \circ f$ and hence (3.12) is proved.

It remains to verify that $\Phi(f)$ coincides with $f(a)$ in case that f is realanalytic in a neighborhood of $\sigma(a)$. First notice that $\Phi(e^{ix\cdot t}) = e^{i\Phi(x)\cdot t}$ by the previous part of the proof. However, $\Phi(e^{ix\cdot t}) = e^{ia\cdot t}$ for all $t \in \mathbb{R}^n$, and therefore $\Phi(x_j) = a_j$. It follows that $\Phi(p) = p(a)$ for all polynomials $p(x)$. Now, let D_η be a set as in Lemma 2.5 such that f is holomorphic in some neighborhood of its closure. Let $\chi \in \mathcal{A}_h$ such that $\chi = 1$ in a neighborhood of $\sigma(a)$ and has compact support in D. If p_k are polynomials such that $p_k \to f$ uniformly on D_η, then by the lemma $\Phi(p_k) \to \Phi(f)$. On the other hand $\Phi(p_k) = p_k(a) \to f(a)$ by the holomorphic functional calculus, and the proof is complete. □

Example 1. Let a be a commuting tuple with $\sigma(a) = \{0\}$. By the spectral mapping theorem this is equivalent to that $\sigma(a_k) = \{0\} \subset \mathbb{C}$ for all k. If $f(z) = \sum_{\alpha,\beta} \partial^\alpha \bar\partial^\beta f(0) z^\alpha \bar z^\beta / \alpha!\beta!$ is the germ of a realanalytic function at the origin, with the usual multiindex notation, then

$$f(a) = \sum_{\alpha,\beta} \frac{\partial^\alpha \bar\partial^\beta f(0)}{\alpha!\beta!} a^{\alpha+\beta}. \tag{3.14}$$

If all a_k are nilpotent, then the sum is finite and hence μ is a distribution and (3.14) provides the extension to smooth functions. For instance if X is finite dimensional, say dim $X = N$, and $\sigma(a) = \{0\}$, then the spectrum of the operator $w \cdot a$ is $\{0\}$ for any $w \in \mathbb{C}^n$ by the spectral mapping theorem, and hence $w\cdot a$ is nilpotent. Therefore $(w\cdot a)^{N+1} = 0$ for all $w \in \mathbb{C}^n$ which implies that $a^\alpha = 0$ if $|\alpha| \geq N+1$. It follows that (3.14) only involves derivatives of f up to order at most N, so $f \mapsto f(a)$ is a distribution of order N. □

Example 2. Let h be an admissible weight function such that (2.4) holds, let K be any compact subset of $\mathbb{R}^n$, and consider the Banach algebra $X = \mathcal{A}_{h,0}/I_{h,K}$. In view of Lemma 2.5 $\phi \mapsto x_j\phi$ defines a tuple of bounded commuting operators a_j, and from Proposition 2.4 we conclude that $\sigma(a) = K$. If $g \in \mathcal{O}(K)$ then $g(a) = g$, see the proof of Proposition 2.4, and therefore the holomorphic functional calculus for a has a natural extension to the algebra X. It is clear that it cannot be extended further in any reasonable way. Recall that $\|\phi\|_X = \inf\{\|\Phi\|_{\mathcal{A}_{h,0}};\ \Phi = \phi \text{ on } K\}$. Since X is a Banach algebra, the operator norm $\|e^{ia\cdot t}\|$ is less than or equal to $\|e^{ix\cdot t}\|_X$, and $\|e^{ix\cdot t}\|_X \leq \|e^{ix\cdot t}\|_{\mathcal{A}_{h,0}} = e^{h(t)}$. Hence $\|e^{ia\cdot t}\| \leq e^{h(t)}$, and thus Theorem 3.3 gives us an extension

of the holomorphic functional calculus to the algebra $\mathcal{A}_{h,K} \to \mathcal{L}(X)$, which is just slightly smaller than the optimal one. This means that in general Theorem 3.3 is close to the best possible. □

Example 3. Assume that a admits a $\mathcal{A}_h$ functional calculus, where $\mathcal{A}_h$ is non-quasianalytic. Then $\mathcal{A}_h$ admits partitions of unity, and therefore, cf., [10], Theorem 6.1.13, a has a spectral capacity. This in particular means that X has a rich structure of a-invariant subspaces (if $\sigma(a)$ is not too small). □

Example 4. If μ is the ultradifferentable operator-valued function $f \mapsto \mu(f) = f(a)$ it is natural to write

$$f(a) = \int_{\sigma(a)} f(z)d\mu(z), \quad f \in \mathcal{O}(\sigma(a)). \tag{3.15}$$

and think of μ as a generalized spectral measure. In case μ is a measure (3.15) provides an extension of the functional calculus to any bounded Borel function ϕ. This, for instance, is the case if X is a Hilbert space and a is an n-tuple of commuting self-adjoint operators. □

Example 5. Without introducing the technical machinery in Section 4 in order to define the $\mathcal{A}_h$-norms locally for arbitrary admissible weight functions h, one can make an elementary extension of the results in this section that allow us to include all polynomials in the functional calculus. Let $E_m(z) = \exp(-mz^2)$, $z^2 = \sum z_j^2$, and let $\mathcal{A}_{h,m} = \{f;\ E_m f \in \mathcal{A}_h\}$ with the norm $\|f\|_m = \|E_m f\|_{\mathcal{A}_h}$. For $m' < m$ we have continuous inclusions $\mathcal{A}_{h,m'} \to \mathcal{A}_{h,m}$, and we let $\mathcal{A}_{h,\infty}$ be the inductive limit. Clearly $\mathcal{A}_{h,\infty}$ is an algebra that contains all polynomials. Assume that $\|e^{ia\cdot t}\| \lesssim e^{h(t)}$ for some admissible $h(t)$. We can extend the mapping $\Phi\colon \mathcal{A}_h \to (a)$ from Proposition 3.1 to a mapping

$$\Phi\colon \mathcal{A}_{h,\infty} \to (a) \tag{3.16}$$

by letting $\Phi(f) = \Phi(E_m f))E_{-m}(a)$ if $f \in \mathcal{A}_{h,m}$. It is readily checked that this definition is non-ambiguous, and that $\mathcal{A}_{h,\infty} \to (a)$ so defined is a continuous algebra homomorphism. One just use the multiplicativity of $\mathcal{A}_h \to (a)$ and the fact that the functions $x_j E_m(x)$ satisfy (3.2). Moreover, the spectral mapping property holds for $f \in \mathcal{A}_{h,\infty}$ as well, since $E_{-m}(a)\Phi((E_m f)_R) \to \Phi(f)$ in operator norm when $R \to \infty$. □

4. Algebras of locally defined ultradifferentiable functions

We shall now consider algebras of locally defined ultra-differentiable functions which are not necessarily non-quasianalytic. Again our starting point is the algebras $\mathcal{A}_h$, but since we no longer have access to cutoff functions the localization to neighborhoods U of a compact subset K of $\mathbb{R}^n$ is a more delicate matter. If $h(t)$ is radial, one can use the relation to spaces C_M (or C^L), which are naturally defined in each open set U.

The condition $f \in C^L$ in U can be reformulated as the existence of a sequence of compactly supported distributions f_N tending to f in U such that $|\hat{f}_N(t)| \leq C(CL_N/|t|)^N$. One can also obtain a microlocal varaint by restriction the condition on t to a cone, see [11] Section 8.4. To define the realanalytic singular support of a hyperfunction on $\mathbb{R}^n$ one can use the so-called FBI transform, see, e.g., [11] Section 9.6. We will use a variant of the FBI transform to define the $\mathcal{A}_h$ condition locally in $\mathbb{R}^n$. Moreover, we obtain a generalization of Fourier's inversion formula which makes it possible to define $u.f$ if u is a hyperfunction with compact support in U whose inverse Fourier transform is bounded by $e^{h(t)}$. More precisely we will determine the dual space of space of compactly supported hyperfunctions whose Fourier transforms satisfy a growth condition like $e^{h(t)}$. Of course, primarily we have compactly supported operator-valued hyperfunctions like $\exp(ia \cdot t)$ in mind.

As before we assume that h is an admissible weight function. Note that the condition (2.1) means that for any $\epsilon > 0$ there is a constant C_ϵ such that

$$h(t) \leq \epsilon|t| + C_\epsilon. \tag{4.1}$$

We will furthermore require that (2.2) holds, that h is C^1 outside the origin, and that

$$|\nabla h(t)| \leq C_\epsilon e^{\epsilon h(t)} \tag{4.2}$$

for each $\epsilon > 0$. For each admissible weight function h of this kind and compact set E in $\mathbb{R}^n$ we shall define an algebra $\widetilde{\mathcal{A}}_{h,E}$ which intuitively consists of all functions f which locally (in some neighborhood of E) belong to $\mathcal{A}_{ch}$ for some $c > 1$.

To present our version of the FBI transform, we start with a smooth function f on $\mathbb{R}^n$ with compact support; then

$$\begin{aligned} f(\xi) = \frac{1}{(2\pi)^n} \int_\tau e^{i\tau\cdot\xi} \int_x e^{-i\tau\cdot x} f(x)dxd\tau \\ = \lim_{R\to\infty} \frac{1}{(2\pi)^n} \int_{|\tau|<R} \int_x e^{i\tau\cdot(\xi-x)} f(x)dxd\tau. \end{aligned} \tag{4.3}$$

In the last expression all the integrals are absolutely convergent. Let us now formally change the path of integration with respect to τ in the last integral to the cycle $\gamma\colon t \mapsto t + i\lambda(t)(\xi - x)$. Since then

$$d\tau = \big(1 + \nabla\lambda(t) \cdot (\xi - x)\big)dt$$

we get the formula

$$f(\xi) = \int_t e^{it\cdot\xi} \hat{f}_{\lambda(t),\xi}(t)dt, \tag{4.4}$$

where

$$(4.5)\quad \hat{f}_{\lambda(t),\xi}(t) = \frac{1}{(2\pi)^n}\int_x e^{-it\cdot x-\lambda(t)(\xi-x)^2}\big(1+\nabla\lambda(t)\cdot(\xi-x)\big)f(x)dx.$$

We claim that (4.4) actually is true if $0 \le \lambda(t) \le C|t|$. To see this we estimate the difference I_R of the integrals in (4.3) and (4.4), where the integration in t is restricted to $\{|t| < R\}$. With no loss of generality we may assume that $\xi = 0$. Since $e^{-i\tau\cdot x}d\tau_1 \wedge \ldots \wedge d\tau_n$ is a closed form, Stokes' theorem implies that I_R is equal to the corresponding integral over the cycle $\tau = t - iux$, $|t| = R$, $0 \le u \le \lambda(t)$. Since $d\tau_1 \wedge \ldots \wedge d\tau_n = i\sum(-1)^k x_k \widehat{dt_k} \wedge du$ on this cycle we have that

$$I_R = \int_x\int_{|t|=R}\int_{u=0}^{\lambda(t)} e^{-it\cdot x-ux^2}f(x)\sum(-1)^{k+1}x_k\widehat{dt_k}\wedge du\wedge dx =$$

$$= \int_x\int_{|t|=R} e^{-it\cdot x}\sqrt{\lambda}f(x)\sum g_k(\sqrt{\lambda}x)\widehat{dt_k}\wedge dx,$$

where $g_k(x) = (-1)^{k+1}x_k(1-e^{-x^2})/x^2$. Since g_k as well as all their derivatives are bounded and f has compact support we get that

$$\left|\Delta_x^m\left(\sqrt{\lambda}f(x)g_k(\sqrt{\lambda}x)\right)\right| \le C_m\lambda^{m+1/2},$$

and hence

$$R^{2m}|I_R| \le C\int_{|t|=R}\lambda^{m+1/2}d\sigma(t).$$

Since $\lambda \le CR$ if $|t| = R$, we get that $|I_R| \le C_m R^{-2m+2n-1+m+1/2}$, and choosing m large enough we see that $I_R \to 0$.

The integral in (4.5) defines in fact an entire function of ξ. Let

$$\|f\|_{\alpha,h,\xi} = \sup_t e^{h(t)}|\hat{f}_{\alpha h(t),\xi}(t)|$$

for $\xi \in \mathbb{C}^n$.

Definition 2. Let V be a compact subset of $\mathbb{R}^n$. A function $f \in C_0^\infty(\mathbb{R}^n)$ is in $\widetilde{\mathcal{A}}_{h,K}$ if there are $c > 1$ and $\alpha_0 > 0$ such that

$$(4.6)\qquad \|\phi\|_{\alpha,ch,\xi} \le C$$

uniformly for ξ in some neighborhood open neighborhood of K in $\mathbb{C}^n$.

Proposition 4.1. *If $f \in C_0^\infty(\mathbb{R}^n)$ vanishes identically in a neighborhood of K then $f \in \widetilde{\mathcal{A}}_{h,K}$.*

Thus it is meaningful to say that a function defined only in some neighborhood of K is in $\widetilde{\mathcal{A}}_{h,K}$.

Proof. Suppose that $f = 0$ in a 2δ-neighborhood of K in $\mathbb{R}^n$, let $\xi = a + ib$, and suppose that a is in a δ-neighborhood of K and $|b| < \delta/\sqrt{2}$. By (4.2) we then have that

$$|\hat{f}_{\alpha h(t),\xi}(t)| \leq \int_{|x-a|>\delta} e^{-\alpha h(t)((x-a)^2-b^2)}|f(x)|(1+e^{\epsilon h(t)})dx \lesssim e^{-(\alpha\delta^2/2-\epsilon)h(t)},$$

so it is enough to choose α such that $\alpha\delta^2/2 - \epsilon > 1$. □

Remark 3. As discussed before, we are only interested here in weight functions h that satisfy (2.1). However, the definition of the spaces $\widetilde{\mathcal{A}}_{h,K}$ of locally defined functions works for any h that is subadditive (and then automatically satisfies the estimate $h(t) \leq C|t|$) provided that one weaken the definition of $\widetilde{\mathcal{A}}_{h,K}$ by replacing $c > 1$ by $c > 0$ in the definition. Thus one can have classes of functions which are microlocally realanalytic. □

Let $e_s(z) = e^{is\cdot z}$ for $s \in \mathbb{R}^n$.

Proposition 4.2. *Suppose that $\phi \in \mathcal{O}(V)$, V open and $V \supset K$. Then $e_s\phi$ is in $\widetilde{\mathcal{A}}_{h,K}$. More precisely, there is an α_0 (only depending on V) such that if $1 < c' < c''$ and $\alpha' \geq \alpha_0$, then*

$$\|e_s\phi\|_{\alpha,c'h,\xi} \leq C_{\alpha'}e^{c''h(s)}\sup_V|\phi|, \tag{4.7}$$

where $C_{\alpha'}$ is uniform in $s \in \mathbb{R}^n$ and ξ in some neighborhood (depending on α') of K in $\mathbb{C}^n$.

Proof. Let

$$Tf_{\lambda,\xi}(t) = \frac{1}{(2\pi)^n}\int e^{-it\cdot x-\lambda(x-\xi)^2}f(x)dx. \tag{4.8}$$

We first prove that if $f(x) = \phi(x)e_s(x)$, then

$$e^{(1-\epsilon)h(t)}|Tf_{\alpha h(t),\xi}(t)| \leq C_{\alpha,\epsilon}e^{h(s)}\sup_V|\phi|. \tag{4.9}$$

We then replace h by $(c'/(1-\epsilon))h$ and choose ϵ so that $c'/(1-\epsilon)+\epsilon c' \leq c''$. We obtain the desired estimate for $\hat{f}_{\alpha h(t),\xi}(t)$ by applying (4.9) to $f(x) = x\phi(x)e_s(x)$ as well, and using (4.2).

To prove (4.9) we assume that $f = e_s\phi$ and that $|\phi| \leq 1$ in V. To begin with, we furthermore assume that $\xi = 0$, and that the ball $\{|x| \leq \delta\}$ is contained in $V \cap \mathbb{R}^n$. The number α_0 will depend on this δ. The integral in (4.8) over the set $|x| > \delta$ is estimated as in the proof of Proposition 4.1. The integrand in (4.8) is a holomorphic $(n,0)$-form, hence a closed form, and therefore we can change the integration to the cycle $x \mapsto z = x + i\eta$, where $\eta\cdot(t-s) = |\eta||t-s|$ over the ball $|x| < \delta$. Let us call this integral A. We then also obtain an integral over $\{z =$

$x + ir\eta;\ |x| = \delta,\ 0 \le r \le 1\}$ as well. In this case $dz = \sum \eta_k \widehat{dx_k} \wedge dr$. Let us call this integral B. Since $h(t) - h(s) \le h(t-s)$ we have that

$$e^{h(t)-h(s)}|A| \le \int_{|x|<\delta} e^{h(t-s)-|\eta||t-s|+\alpha h(t)|\eta|^2}.$$

Now fix an $\epsilon > 0$. Take η so that $|\eta|^2 \le \epsilon/\alpha$. Then by (4.1) we get that

$$e^{h(t)-h(s)}|A| \le C_\alpha e^{\epsilon h(t)}.$$

If $\xi = ib$ for small real b we see that instead of $\lambda|\eta|$ we get $\lambda(|\eta| + |b|)^2$ which can be absorbed as well, with a slightly smaller choice of $|\eta|$.

Now we consider the term B. In this case the integrand admits the estimate

$$e^{-r\eta\cdot(t-s)} e^{-\alpha h(t)(\delta^2 - |b|^2)}.$$

Thus, if α_0 is slightly larger than $1/\delta^2$, then we get the estimate $\lesssim e^{-h(t)}$. Thus the proof is complete. □

Lemma 4.3. *Assume that $f \in \widetilde{\mathcal{A}}_{h,K}$, that (4.6) holds uniformly in a neighborhood V of K in $\mathbb{C}^n$, and that $\phi \in \mathcal{O}(V)$. There is $\alpha_0 > 0$ such that if $c' < c'' < c$ and $\alpha \ge \alpha_0$, then*

$$\|\phi f e_s\|_{\alpha',c'h,\xi} \lesssim \sup_V |\phi| e^{c''h(s)} \sup_{z\in V} \|f\|_{\alpha,ch,z} \tag{4.10}$$

uniformly for ξ in a neighborhood of K in $\mathbb{C}^n$.

Proof. By the inversion formula (4.4) we have that

$$f(x) = \int_t e^{it\cdot x} \hat{f}_{\alpha h(t),x}(t)dt \tag{4.11}$$

and thus

$$\phi(x)f(x)e_s(x) = \int \phi(x) e^{i(s+t)\cdot x} \hat{f}_{\alpha h(t),x}(t)dt. \tag{4.12}$$

By Proposition 4.2, for large α' and $c' < c''$, we have that

$$\|\phi(x)e^{i(s+t)\cdot x}\hat{f}_{\alpha h(t),x}(t)\|_{\alpha',c'h,\xi} \le \sup_V |\phi| e^{c''h(s+t)} \sup_{z\in V} |\hat{f}_{\alpha h(t),z}(t)| \lesssim$$
$$\sup_V |\phi| \sup_{z\in V} \|f\|_{\alpha,ch,z} e^{-ch(t)+c''h(s+t)}.$$

Now (4.10) follows by applying Minkowski's inequality to (4.12), keeping in mind that h is subadditive and that $c'' < c$. □

Letting $\phi e_s = 1$ we get

Corollary 4.4. *If $f \in \widetilde{\mathcal{A}}_{h,K}$ then there is $\alpha_0 > 0$ and $c > 1$ such that for any $\alpha \ge \alpha_0$ (4.6) holds for ξ in in some neighborhood of K in $\mathbb{C}^n$.*

By compactness it follows that $f \in \widetilde{\mathcal{A}}_{h,K}$ if and only if $f \in \widetilde{\mathcal{A}}_{h,\xi_0}$ for each $\xi_0 \in K$. Moreover, give two functions in $\widetilde{\mathcal{A}}_{h,K}$ we may always assume that (4.6) holds for both of them with the same c and α.

Clearly $\widetilde{\mathcal{A}}_{h,K}$ is a vector space, and $f_j \to 0$ in $\widetilde{\mathcal{A}}_{h,K}$ if and only if there are $c > 1$ and $\alpha > 0$ such that $\|f_j\|_{\alpha,ch,\xi} \to 0$ uniformly for ξ in some neighborhood in $\mathbb{C}^n$ of K.

Proposition 4.5. *$\mathcal{O}(K)$ is a dense subspace of $\widetilde{\mathcal{A}}_{h,K}$.*

Proof. Assume f is as in Lemma 4.3 and let

$$f_R(z) = \int_{|t|<R} e^{iz\cdot t} \hat{f}_{\alpha h(t),x}(t)dt.$$

Then $f_R(z)$ is entire and $f_R \to f$ pointwise in $\mathbb{R}^n$. On the other hand, cf., the proof of Lemma 4.3,

$$\|f - f_R\|_{\alpha',c'h,\xi} \lesssim \int_{|t|\geq R} e^{(c'-c)h(t)}dt,$$

which tends to 0 when $R \to \infty$. □

Theorem 4.6. *The space $\widetilde{\mathcal{A}}_{h,K}$ is an algebra. More precisely, for some $c > 1$ and large enough α' we have that*

$$\|f\phi e_s\|_{\alpha',ch,\xi} \leq Ce^{h(s)} \sup_{z\in V} \|f\|_{\alpha,ch,z} \sup_{z\in V} \|\phi\|_{\alpha,ch,z} \tag{4.13}$$

uniformly for ξ in some neighborhood of K.

Proof. Assume that $f, \phi \in \widetilde{\mathcal{A}}_{h,K}$. If we apply Minkowski's inequality to the representation (4.12) and use Lemma 4.3, we get the estimate

$$\|f\phi e_s\|_{\alpha',c'h,\xi} \lesssim \int_t \sup_{z\in V} \|\phi\|_{\alpha,ch,z} e^{c''h(s+t)} \sup_{z\in V} |\hat{f}_{\alpha h(t),z}(t)|dt \tag{4.14}$$

and the right hand side is readily estimated by the right hand side of (4.13) as before. □

Corollary 4.7. *If f is in $\mathcal{A}_{ch,\infty}$ for some $c > 1$, then $f \in \widetilde{\mathcal{A}}_{h,K}$.*

Proof. First assume that $f \in \mathcal{A}_h$. Then

$$f(x) = \int_t e^{ix\cdot t} \hat{f}(t)dt,$$

and in the same way as above we get that $\|f\|_{\alpha,c'h,\xi}$ is bounded in a neighborhood of $\mathbb{R}^n$ if $c' < c$. The general case, i.e. when $E_m f \in \mathcal{A}_h$, now follows since $\widetilde{\mathcal{A}}_{h,K}$ is an algebra. □

We also have some partial converses.

Proposition 4.8. *If $f \in \widetilde{\mathcal{A}}_{h,\mathbb{R}^n}$ in the sense that $\|f\|_{\alpha,ch,\xi}$ is uniformly bounded in some neighborhood $\{z;\ |y| < 2\delta\}$, then $Ef \in \mathcal{A}_h$; in particular $f \in \mathcal{A}_{h,\infty}$.*

Proof. If $\phi(z)$ is bounded and holomorhic i a 2δ-neighborhood V of $\mathbb{R}^n$, then $|\widehat{E\phi}(t)| \leq C \sup_V |\phi| \exp(-\delta|t|)$. Thus $\|E\phi\|_{\mathcal{A}_h} \leq C \sup_V |\phi|$. We will now use the representation

$$e^{-x^2} f(x) = \int \hat{f}_{\alpha h(t),x}(t) e^{-x^2} e^{ix\cdot t} dt. \tag{4.15}$$

Since

$$\|\hat{f}_{\alpha h(t),x}(t) e^{-x^2} e^{ix\cdot t}\|_{\mathcal{A}_h} \leq \|\hat{f}_{\alpha h(t),x}(t) e^{-x^2}\|_{\mathcal{A}_h} \|e^{ix\cdot t}\|_{\mathcal{A}_h} \lesssim \sup_{z\in V} |\hat{f}_{\alpha h(t),z}(t)| e^{h(x)} \lesssim e^{-(c-1)h(t)},$$

an estimate of (4.15) gives that $E\phi \in \mathcal{A}_h$. □

In the non-quasianalytic case we have

Proposition 4.9. *Suppose that f is a smooth function with compact support in $\mathbb{R}^n$, and suppose that (2.4) holds. Then $f \in \tilde{\mathcal{A}}_{h,K}$ if and only if $f \in \mathcal{A}_{ch,K}$ for some $c > 1$.*

Proof. If $f \in \mathcal{A}_{ch,K}$ then by definition it is realized by a compactly supported function f in $\mathcal{A}_{ch}$, and therefore it is in $\tilde{A}_{h,K}$ according to Corollary 4.7.

If $f \in \tilde{\mathcal{A}}_{h,K}$, then actually $f \in \tilde{\mathcal{A}}_{h,\overline{U}}$ for some neighborhood $U \supset K$ in $\mathbb{R}^n$. Take a cutoff function $\chi \in \mathcal{A}_{ch}$, for some fixed $c > 1$, which is supported in U and identically 1 on a neighborhood of K. Then, by Theorem 4.6, χEf is in $\tilde{\mathcal{A}}_{h,\overline{U}}$ and has compact support in U. It now follows that, for some $c' > 1$ and $\alpha' > 0$, $\|\chi E^{-1} f\|_{\alpha',c'h,\xi}$ is uniformly bounded in a δ-neighborhood neighborhood of $\mathbb{R}^n$. In fact, since $\chi H^{-1} f \in \tilde{\mathcal{A}}_{h,\overline{U}}$ such an estimate holds in some complex neighborhood of K, and since the function vanishes in a neighborhood of $\mathbb{R}^n \setminus U$, the same estimate holds in a complex neighborhood of this set, cf., the proof of Proposition 4.1. From Proposition 4.8 we deduce that $f = \chi f = E\chi E^{-1} f$ is in $\mathcal{A}_{c'h}$. □

Remark 4. If h and h' are admissible functions such that $h'(t)/h(t) = o(1)$ when $|t| \to \infty$, then $\widetilde{\mathcal{A}}_{h',K} \subset_{\neq} \widetilde{\mathcal{A}}_{h,K}$ since clearly $\mathcal{A}_h \setminus \mathcal{A}_{h'}$ is nonempty.

It follows that the inequality $\|e_s\|_{A_{h,K}} \leq \exp h(s)$ essentially is an equality, because an estimate like $\leq \exp h'$ combined with the inversion formula roughly speaking implies that $\mathcal{A}_{h,K}$ is contained in $\mathcal{A}_{h',K}$. □

It is natural to say that $f \in \tilde{\mathcal{A}}_{h,V}$, for an open $V \subset \mathbb{R}^n$, if $f \in \widetilde{\mathcal{A}}_{h,K}$ for all compacts $K \subset V$. Suppose that u is a hyperfunction with support in V which has a continuous extension to $\tilde{\mathcal{A}}_{h,V}$ (recall that the entire functions are dense in $\tilde{\mathcal{A}}_{h,V}$). For each $c > 1$ we then must have that

$$|u.e_s| \lesssim \|e_s\|_{\alpha,ch,\xi} \lesssim e^{c'' h(s)},$$

for $c'' > c$, according to Proposition 4.2. Thus we have proved one half of

Proposition 4.10. *The dual space of $\tilde{\mathcal{A}}_{h,V}$ consists of all hyperfunctions u with support in V such that*

$$|\check{u}(t)| \leq C_c e^{ch(t)}$$

for each $c > 1$.

Of course $\check{u}(t) = u.e_t$ here. We begin with a lemma.

Lemma 4.11. *Suppose that the hyperfunction u has support in K and that*

$$|\check{u}(t)| \leq Ae^{h(t)}.$$

For each $U \supset K$, U open in $\mathbb{C}^n$, we have

$$|u.e_s\phi| \leq Ae^{h(s)}C_U \sup_U |\phi|, \quad \phi \in \mathcal{O}(U).$$

Proof. Since $h(t) \leq H(|t|)$ for some $H(s)$ that is $o(s)$, we can find a representing form ω for u in $\mathbb{C}^n \setminus K$ such that the size of $|\omega|$ only depends on A. We then get that

$$|u.\phi| \leq AC_U \sup_U |\phi|, \quad \phi \in \mathcal{O}(U),$$

where C_U is independent of A. Now, $u.e_s\phi = e_s u.\phi$ and since $e_s u$ is supported on K as well, the general statement follows from the estimate

$$|(e_s u)\check{}(t)| = |\check{u}(t+s)| \leq Ae^{h(s+t)} \leq Ae^{h(s)}e^{h(t)}.$$

□

Proof of Proposition 4.10. Suppose that $f \in \widetilde{\mathcal{A}}_{h,V}$ and that (4.6) holds uniformly for $\xi \in U$, where U is a complex neighborhood of the support K of u. Since $f_R \to f$ in $\widetilde{\mathcal{A}}_{h,K}$, cf., the proof of Proposition 4.5, we have that

$$u.f = \int_t u.\big(\hat{f}_{\alpha h(t),x}(t)e^{ix\cdot t}\big)dt,$$

and hence by the lemma above and the assumption on u

$$|u.f| \lesssim \int_t e^{c'h(t)}e^{-ch(t)}dt$$

which is finite if $c' < c$. □

5. Ultradifferentiable functional calculus

We are now in position to extend Theorem 3.3 to the algebras $\tilde{\mathcal{A}}_{h,\sigma(a)}$.

Theorem 5.1 (Main Theorem)**.** *Let h be an admissible weight function that is C^1 outside the origin that satisfies* (4.2) *and* (2.2)*. Assume that a_k are commuting operators such that $\|e^{ia\cdot t}\| \leq e^{h(t)}$. Then there exists a continuous homomorphism*

$$\widetilde{\mathcal{A}}_{h,\sigma(a)} \to (a) \tag{5.1}$$

which coincides with the holomorphic functional calculus in case that f is realanalytic. Moreover, $\sigma(f(a)) = f(\sigma(a))$ if $f = (f_1, \dots, f_m)$.

Proof. Suppose that $f \in \widetilde{\mathcal{A}}_{h,K}$ and that

$$|\hat{f}_{\alpha h(t),\xi}| \le Ce^{-ch(t)}, \tag{5.2}$$

holds uniformly in the complex neighborhood V of $K = \sigma(a)$. Since $\xi \mapsto \hat{f}_{\lambda,\xi}(t)$ is holomorphic in V we can define $\hat{f}_{\alpha h(t),a}(t)$ by the holomorphic functional calculus and (5.2) implies that

$$\|\hat{f}_{\alpha h(t),a}(t)\| \le Ce^{-ch(t)}.$$

Thus

$$f^\alpha(a) = \int e^{ia\cdot t}\hat{f}_{\alpha h(t),a}(t) \tag{5.3}$$

has meaning and defines an element in (a). We may assume that this definition is independent of $\alpha \ge \alpha_0$ if α_0 and c are as in Corollary 4.7. Let

$$f_R^\alpha(\xi) = \int_{|t|<R} e^{i\xi\cdot t}\hat{f}_{\alpha h(t),\xi}(t)dt.$$

Then $f_R^\alpha(\xi)$ are entire functions, and we want to prove that $f_R^{\alpha_1}(a) - f_R^{\alpha_2}(a) \to 0$ if $\alpha_0 \le \alpha_1 < \alpha_2$. To this end, consider the entire function $g_R(\xi) = f_R^{\alpha_1}(\xi) - f_R^{\alpha_2}(\xi)$. By Stokes' theorem

$$g_R(\xi) = f_R^{\alpha_1}(\xi) - f_R^{\alpha_2}(\xi) =$$
$$\int_s\int_{|t|=R}\int_{u=\alpha_1}^{\alpha_2} e^{-it(\xi-s)-uh(t)(s-\xi)^2}f(s)h(t)\sum\widehat{dt_k}(s_k-\xi_k)dsdu =$$
$$\int_{|t|=R}\int_{u=\alpha_1}^{\alpha_2} e^{-i\xi\cdot t}\sum_k S_k(t)f_{uh(t),\xi}h(t)\widehat{dt_k}du$$

where

$$S_k f_{\lambda,\xi}(t) = \int_s e^{it\cdot s-\lambda(s-\xi)^2}f(s)(s_k-\xi_k)ds.$$

As before we know that

$$|S_k f_{uh(t),\xi}(t)| \le Ce^{-ch(t)}$$

uniformly in some neighborhood of K in $\mathbb{C}^n$ if $\alpha_1 \le u \le \alpha_2$. (A dissection of the proof of Proposition 4.2, on which Corollary 4.7 is based, reveals that the neighborhood can be chosen uniformly for α running over a compact set.) Hence we get the estimate

$$|g_R(a)| \lesssim \int_{|t|=R} e^{-ch(t)}\|e^{-ia\cdot t}\|h(t)d\sigma(t)$$

and the right hand side tends to 0 since $h(t) \to \infty$ when $|t| \to \infty$, cf., (2.2), and $\exp(\epsilon h) \ge h^2$ when h is large.

Thus we can define $f(a)$ as $f^\alpha(a)$ for approriate α and clearly

$$|f(a)| \leq C \sup_{\xi \in V} \|f\|_{\alpha,ch,\xi}$$

so the mapping (5.1) is indeed continuous. Since $|f_R^\alpha|_{\alpha',ch,\xi}$ is bounded uniformly in R, it follows from Theorem 4.6 that $f_R^\alpha g_R^\alpha \to fg$ in $\widetilde{\mathcal{A}}_{h,K}$. Hence $(f_R^\alpha g_R^\alpha)(a) \to (fg)(a)$. On the other hand, by the multiplicativity for the holomorphic functional calculus and the continuity we have that

$$(f_R^\alpha g_R^\alpha)(a) = f_R^\alpha(a) g_R^\alpha(a) \to f(a)g(a)$$

and thus the mapping (5.1) is a continuous algebra homomorphism. The spectral mapping property follows as before. □

Remark 5. We do not know if some composition rule holds in this case. Actually we do not even know if $1/f$ belongs to $\widetilde{\mathcal{A}}_{h,K}$ when $f \in \widetilde{\mathcal{A}}_{h,K}$ and $f \neq 0$. □

6. Relation to estimates of the resolvent

For a nonnegative function k on $\mathbb{R} \setminus \{0\}$ we let

$$k^\sharp(y) = \sup_{\{t;\ yt>0\}} (k(t) - ty).$$

and

$$k^\flat(t) = \inf_{\{y;\ yt>0\}} (k(y) + ty).$$

If h is admissible, then $h^\sharp(\eta)$ tends to ∞ when $\eta \to \pm 0$. Notice also that $h^\sharp$ is always convex on each semi-axis, and $g^\flat$ is always concave on each semi-axis. If k is any function in $\mathbb{R}^n \setminus \{0\}$ and $a \in S^{n-1}$, i.e., $a \in \mathbb{R}^n$ and $|a| = 1$, then we let

$$k^a(\eta) = k(\eta a), \quad \eta > 0$$

be the restriction of k to the ray from 0 determined by a. We let $k^{a\sharp}$ and $k^{a\flat}$ denote the functions $(k^a)^\sharp$ and $(k^a)^\flat$, respectively, and we extend the definition to all $\eta \in \mathbb{R}$ by letting $h^{a\sharp}(\eta) = \infty$ for $\eta < 0$ and $g^{a\flat}(s) = -\infty$ for $s < 0$.

Definition 3. If h is a nonnegative function in $\mathbb{R}^n$, then

$$h^\sharp(y) = \sup_{a \in S^{n-1}} g^{a\flat}(a \cdot t).$$

If g is a nonnegative function defined in $\mathbb{R}^n \setminus \{0\}$, then

$$g^\flat(t) = \inf_{a \in S^{n-1}} h^{a\sharp}(a \cdot y).$$

We say that a function k in $\mathbb{R}^n \setminus \{0\}$ is convex (concave) on rays if k^a is convex (concave) for all $a \in S^{n-1}$. If h is radial, say $h(t) = H(|t|)$, then $h^\sharp(y)$ is radial, more precisely $h^\sharp(y) = g(y) = G(|y|)$, where $G(-\xi)$ is just minus the Legendre transform of $H(s)$. Similarily for $g^\flat$. The following propostion was proved in [5]

Proposition 6.1. *If h (g) is concave (convex) on rays then $h^\sharp$ ($g^\flat$) has convex level sets, and if h (g) has convex level sets, then $h^\sharp$ ($g^\flat$) is convex (concave) on rays.*

If g is convex on rays, then $g^{\flat\sharp} \geq g$ and we have equality if and only if all the level sets $\{g \geq A\}$ are convex. Similarily, if h is concave on rays then $h^{\sharp\flat} \leq h$ with equality if and only if all the level sets $\{h \leq A\}$ are convex.

For $c > 0$ let $g_c(y) = cg(y/c)$, and notice that $g_c^\flat = cg^\flat$. For an open set U in $\mathbb{R}^n$ consider smooth functions such that there exists a smooth extension $F(x+iy)$ to some neighborhood $U_\delta = U \times \{|y| < \delta\}$ such that

$$\sup_{U_\delta} |F| + \sup_{z \in U_\delta} |\bar\partial F(z)| e^{g(-y)} < \infty. \tag{6.1}$$

Let $\mathcal{M}_{g,U}$ consist of all smooth functions in U such that (6.1) holds for g_c for some $c > 1$. For a compact set E, let $\mathcal{M}_{g,E}$ consist of all functions on E that belong to some $\mathcal{M}_{g_c,U}$, where $U \supset E$ and $c > 1$.

Theorem 6.2. *For an open or compact set E we have the inclusions*

$$\mathcal{M}_{g,E} \subset \widetilde{\mathcal{A}}_{g^\flat,E} \tag{6.2}$$

and if h is concave on rays, also

$$\widetilde{\mathcal{A}}_{h,E} \subset \mathcal{M}_{h^\sharp,E}. \tag{6.3}$$

In case, h is admissible, concave on rays and has convex level sets (and is C^1 and satisfies the extra condition (4.2)), e.g., in the radial case, we thus have equality. However, the function $h(t) = \sqrt{t_1} + \sqrt{t_2}$ is admissible, but no admissible h' with convex level sets is equivalent to h, in the sense that it give rise to the same algebra.

Proof. We may assume that $E = V$ is open. To prove (6.2) we first assume that (6.1) holds, and let $a = (1, 0, \dots) \in S^{n-1}$; we may also assume that $B(0, 2\delta) \subset V$, and that F has its support in $|y| < \eta << \delta$; it is enough to prove that $|f|_{\alpha',(1-\epsilon)h,\xi}$ is bounded for ξ in some complex neighborhood of 0. In the expression (4.8) for $Tf_{\alpha h(t)}$, we first consider the integral for $|x| < \delta$. By Stokes' theorem this integral is ($z_1 = x_1 + iy_1$)

$$I = \int_{|x|<1, -\eta<y_1<0} \frac{\partial F}{\partial \bar z_1}(z_1, x') e^{-t_1 z_1 - \lambda(z-\xi)^2} dx dy_1,$$

and by the standard estimates, for $\lambda = \alpha h(t)$, $h = g^{a\flat}$, and $|\xi| \leq \eta$, we get

$$|I| \lesssim \int_{|x|<\delta, -\eta<y_1<0} e^{-g(-y_1,0)+t_1 y_1} e^{3\alpha\eta h(t)} dx dy_1 \lesssim e^{-g^{a\flat}(t)} e^{3\alpha\eta h(t)}$$

if $|b|$ is small enough. Taking infimum over all a we get the estimate

$$|I| \lesssim e^{-(1-2\alpha\eta)h(t)}.$$

The integral over $|x| \geq \delta$ is estimated as before, and finally, as in previous proofs, we get the estimate

$$|\hat{f}_{\alpha h(t),\xi}(t)| \leq Ce^{-(1-3\alpha\eta)h(t)}.$$

If we start with g_c rather than g and η is small enough we get (6.2).

For the converse, first assume that

$$e^{ch(t)}|\hat{f}_{\alpha h(t),\xi}(h(t))| \lesssim C$$

uniformly locally in complex neighborhoods of E for some $c > 1$, and write

$$f(x) = \int_{|a|=1} f_{a,x}(a \cdot x) d\sigma(a),$$

where

$$f_{a,x}(s) = \int_0^\infty \hat{f}_{\alpha h(ra),x}(ra) e^{irs} r^{n-1} dr$$

is a one variable function of s that depends holomorphically on x. It is now enough to extend each $f_{a,z}(s)$ to $(\zeta = s + i\eta)$ $F_{a,z}(\zeta)$ with control of $\partial F_{a,z}/\partial\bar{\zeta}$, and let

$$F(z) = \int_{|a|=1} F_{a,z}(a \cdot z) d\sigma(a).$$

Take $c' > 1$ and a cutoff function $\chi(s)$ on $\mathbb{R}$ that is 0 for $s > 1$ and 0 for $s < 1/c'$, let $\phi^a(r) = (h^a)'(r) = a \cdot \nabla h(a \cdot r)$, and let

$$F_{a,z}(\zeta) = \int_0^\infty \hat{f}_{\alpha h(ra),z}(ra) e^{ir\zeta} \chi(\eta/\phi_a(r)) r^{n-1} dr.$$

We now have that $(\eta > 0)$

$$|\partial F_{a,z}(s - i\eta)/\partial\bar{\zeta}| \leq \int_r e^{-h_a(r)+\eta r} \chi'(\eta/\phi_a(r)) e^{-\epsilon r} r^{n-1} dr/\phi_a(r).$$

However, $h^a(r) - \phi^a(r)r \geq h^{a\sharp}(\phi^a(r))$ and the integration only takes place where $\eta \leq \phi^a(r) \leq c\eta$, so we get the estimate

$$\lesssim \frac{1}{\eta} e^{-h^{a\sharp}(c\eta)} \lesssim e^{-h^{a\sharp}(c'\eta)},$$

where the last inequality uses that fact that $h^{a\sharp}(\eta)/\log(1/\eta) \to \infty$ when $\eta \to 0$; this is a consequence of (2.2). (A similar estimate combined with the dominated convergence theorem shows that $F_{a,z}(\zeta)$ really is an extension of $f_{a,z}(s)$). Summing up we have that

$$|\bar{\partial} F(\bar{z})| \lesssim \int_{|a|=1} e^{-h^{a\sharp}(c'\eta)} d\sigma(a) \leq e^{-h^\sharp(c'y)}.$$

If we instead start with ch, $c' > 1$, instead of h we get $\exp(-ch^\sharp(c'y/c)) \leq \exp(-c''g(y/c'')) = \exp(-g_{c''}(y))$ if $1 < c' < c$ and $c'' = c/c'$, and thus $f \in \mathcal{M}_{g,E}$. □

It is clear that $\mathcal{M}_{g,E}$ is an algebra and that $\psi \circ f$ belongs to $\mathcal{M}_{g,E}$ if $f \in \mathcal{M}_{g,E}$ and $\psi \in \mathcal{O}(f(E))$. Assume now that h is such that $\widetilde{\mathcal{A}}_h, \sigma(a) = \mathcal{M}_{h,\sigma(a)}$. Then

$$|\psi \circ f|_{\widetilde{\mathcal{A}}_h,\sigma(a)} \lesssim \sup_W |\psi||f|_{\widetilde{\mathcal{A}}_h,\sigma(a)}$$

where the norm signs stand for appropriate seminorms, and W is some complex neigborhood of $\sigma(a)$. Therefore, if $f_R \to f$ as before, and we write $\psi(z) - \psi(w) = \sum(z_j - w_j)\psi_j(z,w)$, which is possible since $\sigma(a)$ is real and hence a Stein compact, we get that

$$g \circ f_R - g \circ f = ((f_j)_R - f_j)\psi_j(f, f_R) \to 0$$

in $\widetilde{\mathcal{A}}_h, \sigma(a)$. Hence,

$$\Phi(\psi \circ f) \leftarrow \psi \circ f_R(a) = \psi(f_R(a)) \to \psi(\Phi(f)),$$

or more simply stated, $\psi \circ f(a) = \psi(f(a))$.

References

[1] D.W. Albrecht: *Explicit formulae for Taylor's functional calculus*, Multivariable operator theory (Seattle, WA, 1993), 1-5, Contemp. Math. 185, AMS. Providence, RI (1995).

[2] D.W. Albrecht: *Integral formulae for special cases of Taylor's functional calculus*, Studia Math. bf 105 (1993), 51-68.

[3] M. Andersson: *Taylor's functional calculus with Cauchy–Fantappie–Leray formulas*, International Math. Research Notes **6** (1997), 247-258.

[4] M. Andersson: *Correction to "Taylor's functional calculus with Cauchy–Fantappie–Leray formulas"*, International Math. Research Notes **2** (1998), 123-124.

[5] M. Andersson & B. Berndtsson: *Almost holomorphic extensions of ultradifferentiable functions*, preprint.2001

[6] A. Beurling: *On quasianalyticity and general distributions*, Lecture notes, Stanford (1961).

[7] B. Droste: *Holomorphic approximation of ultradifferentiable functions*, Math. Ann. **257**, 293-316 (1981).

[8] B. Droste: *Extension of analytic functional calculus mappings and duality by $\bar{\partial}$-closed forms with growth*, Math. Ann. **261**, 185-200 (1982).

[9] E. M. Dynkin: *An operator calculus based on the Cauchy-Green formula. (Russian)*, Investigations on linear operators and the theory of functions, III. Zap. Nauv cn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **30** (1972), 33–39..

[10] J. Eschmeier & M. Putinar: *Spectral Decompositions and Analytic Sheaves*, Clarendon Press, Oxford 1996.

[11] L. Hörmander: *The Analysis of Linear Partial Differential Operators I*, Sec. Ed. Springer-Verlag (1990).

[12] V. Kordula & V. Müller: *Vasilescu-Martinelli formula for operators in Banach spaces*, Studia Math. **113** (1995) 127-139.

[13] t. H. Nguyen: *Calcul fonctionnel dependant de la croissance des coefficients spectraux*, Ann. Inst. Fourier **27** (1977), 169-199.

[14] M. Putinar: *The superposition property for Taylor's functional calculus*, J. Operator Theory **7** (1982), 149-155.

[15] W. Rudin: *Functional Analysis*, McGraw-Hill (1973).

[16] S. SANDBERG: *On non-holomorphic functional calculus for commuting operators*, Preprint (1999).

[17] J. L. TAYLOR: *A joint spectrum for several commuting operators*, J. Funct. Anal. **6** (1970), 172-191.

[18] J. L. TAYLOR: *The analytic-functional calculus for several commuting operators*, Acta Math. **125** (1970), 1-38.

[19] F.H. VASILESCU: *Analytic functional calculus and spectral decompositions*, Mathematics and its Applications (East European Series), D. Reidel Publ. Co., Dordrecht-Boston, Mass. (1982).

[20] L. WAELBROECK: *Calcul symbolique lié à la croissance de la résolvant*, Rend. Sem. Mat. e fis. di Milano, XXXIV (1964).

[21] J. WERMER: *The existence of invariant subspaces*, Duke Math. J. **19** (1952), 615-622.

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