A NOTE ON THE BOREL-CANTELLI LEMMA

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ABSTRACT A generalization of the Erdös-Rényl formulation of the Borel-Cantelli lemma is obtained.

1 Introduction

The following Borel-Cantelli lemma plays an execeptionally important role in probability theory: If $A_1, A_2, \dots$ is a sequence of events on a common probability space (Ω, F, P) and if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup A_n) = 0$; if $A_1, A_2, \dots$ is a sequence of independent events and if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\limsup A_n) = 1$. Here $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$.

Many investigations were devoted to the second part of the Borel-Cantelli lemma in attempts to weaken the independence condition that means mutual independence of events $A_1, \dots, A_n$ for every n. A short history of the problem can be found, for example, in [6], Section 6.7.

Erdös and Rényi [2] discovered that the independence condition in the second part of the Borel-Cantelli lemma can be replaced by the weaker condition of pairwise independence of events $A_1, A_2, \dots$. More general results have been proved independently by Kochen and Stone [3] and Spitzer [7]. The Erdös-Rényi proof is contained in [2] and [6]. Erdös and Rényi also found that the condition of pairwise independence of events $A_1, A_2, \dots$ can be replaced by the weaker condition $P(A_k A_j) \le P(A_k)P(A_j)$ for every k and j such that $k \ne j$.

Lamperti [4] formulated the following proposition. If $A_1, A_2, \dots$ is a sequence of events such that

$$\sum_{n=1}^{\infty} P(A_n) = \infty \quad \text{and} \quad P(A_k A_j) \le C P(A_k) P(A_j)$$

for all $k, j > N$ and some constants C and N, then $P(\limsup A_n) > 0$.

We present a more general and more precise theorem of this type. The above mentioned results of Erdös and Rényi follow from our theorem.

[0] *Key words and phrases: Borel-Cantelli lemma, generalizations of the Borel-Cantelli lemma, pairwise independent events*

2 Theorems

THEOREM **2.1.** *Let $A_1, A_2, \ldots$ be a sequence of events satisfying conditions*

$$\sum_{n=1}^{\infty} P(A_n) = \infty \tag{1}$$

and

$$P(A_k A_j) \leq C P(A_k) P(A_j) \tag{2}$$

for all $k, j > L$ such that $k \neq j$ and for some constants $C \leq 1$ and L. Then

$$P(\limsup A_n) \geq \frac{1}{C} \tag{3}$$

Note that under condition (1) the probability $P(\limsup A_n)$ can take on any value in the closed interval $[0, 1]$. Martikainen and Petrov [5] found conditions that are necessary and sufficient for the equality $P(\limsup A_n) = \alpha$ where $0 \leq \alpha \leq 1$ (see also [6], Section 6.1).

The following proposition (due to Erdös and Rényi [2]) is an immediate corollary of Theorem 2.1.

THEOREM **2.2.** *Let $A_1, A_2, \ldots$ be a sequence of events satisfying conditions (1) and*

$$P(A_k A_j) \leq P(A_k) P(A_j) \tag{4}$$

for all sufficiently large k and $j, k \neq j$. Then

$$P(\limsup A_n) = 1. \tag{5}$$

If $A_1, A_2, \ldots$ is a sequence of pairwise independent events condition (4) is satisfied with the sign of equality.

3 Proof of Theorem 2.1

It is possible to suggest several different proofs of THEOREM 2.1. We shall present a proof based on the following inequality of Chung and Erdös [1]:

$$P\Big(\cup_{k=1}^{n} A_k \Big) \geq \Big(\sum_{k=1}^{n} P(A_k) \Big)^2 \Big/ \sum_{k,j=1}^{n} P(A_k A_j)$$

where $A_1, \ldots, A_n$ are arbitrary events (see also [6], Section 6.1, where this inequality is proved by means of a strengthening of Lyapunov's inequality for moments of a random variable).

Let $A_1, A_2, \ldots$ be a sequence of events satisfying the conditions of THEOREM 2.1. By the Chung-Erdös inequality we have

$$P(\Big(\cup_{k=n}^{N} A_k \Big) \geq \Big(\sum_{k=n}^{N} P(A_k) \Big)^2 \Big/ \sum_{k,j=n}^{N} P(A_k A_j) \tag{6}$$

Assuming that $n < N$. It follows from condition (2) that

$$\sum_{k,j=n}^{N} P(A_k A_j) \leq C T_1 + T_2 \tag{7}$$

if $n > L$, where

$$T_1 = \sum_{\substack{k,j=n \\ k\neq j}}^{N} P(A_k) P(A_j),\ T_2 = \sum_{k=n}^{N} P(A_k). \tag{8}$$

Since $C \geq 1$, (7) implies the inequality

$$\sum_{k,j=n}^{N} P(A_k A_j) \leq C(T_1 + T_2). \tag{9}$$

Obviously,

$$T_1 = \Big(\sum_{k=n}^{N} P(A_k)\Big)^2 - \sum_{k=n}^{N} (P(A_k))^2.$$

Therefore

$$T_1 + T_2 \leq \Big(\sum_{k=n}^{N} P(A_k)\Big)^2 + \sum_{k=n}^{N} P(A_k). \tag{10}$$

Taking into account inequalities (6)-(10), we obtain

$$P\Big(\cup_{k=n}^{N} A_k\Big) \geq C^{-1}\Big(\sum_{k=n}^{N} P(A_k)\Big)^2 \Big\{\Big(\sum_{k=n}^{N} P(A_k)\Big)^2 + \sum_{k=n}^{N} P(A_j)\Big\}^{-1}$$
$$\geq C^{-1}\Big\{1 + (\sum_{k=n}^{N} P(A_k)\Big)^{-1}\Big\}^{-1}.$$

When $N \to \infty$ and n is fixed. We get $1 + (\sum_{k=n}^{N} P(A_k))^{-1} \to 1$ by condition (1). Thus,

$$\liminf_{N\to\infty} P\Big(\cup_{k=n}^{N} A_k\Big) \geq C^{-1}$$

and

$$P\Big(\cup_{k=n}^{\infty} A_k\Big) \geq C^{-1}. \tag{11}$$

Putting $B_n = \cup_{k=n}^{\infty} A_k$ we observe that $B_1 \supset B_2 \supset \dots$ and $\cap_{n=1}^{\infty} B_n = \limsup A_n$. Hence the limit exists

$$\lim P = P(B_n) = P(\cap_{n=1}^{\infty} B_n) = P(\limsup A_n),$$

and (11) implies the inequality $P(\limsup A_n) \geq C^{-1}$. $\square$

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