Existence and unicity of σ-forms on finite-dimensional modules

Volodymyr Mazorchuk and Lyudmyla Turowska

Chalmers University of Technology and Göteborg University

Abstract

Let $\mathbb{K}$ be an algebraically closed field field and σ its automorphism of order 1 or 2. Let A be an associative algebra over $\mathbb{K}$ with σ-linear involution $*$ and M be an A-module. We investigate the question when M admits a non-degenerate binary form, linear with respect to the first variable and σ-linear with respect to the second one, with respect to which the module M becomes unitarizable, and the uniqueness of this form.

1 Introduction

$*$-Representations of $*$-algebras by linear operators on Hilbert spaces have been extensively studied in modern representation theory since very long and first of all because of their physical significance. On the mathematical side the importance of such representations is obvious from examples in Group Theory (e.g. representations of compact groups) or Lie Theory (e.g. finite-dimensional representations of simple complex Lie algebras). The basic feature of this theory is decomposibility of each $*$-representation into a direct sum or integral of irreducible ones. In particular, the problem of study of unitarizable representations reduces to the problem of study of unitarizable irreducible representations. In its turn the problem of classification of irreducible representations is relatively simple in many cases (for example, in the case of finite-dimensional algebras). We refer the reader to [?] for the history and recent developments in the theory of Hilbert space $*$-representations.

It is known that for many $*$-algebras the set of their representations can not be covered by the set of their $*$-representations (with respect to positively defined inner products). However, a representation which is left might possess an indefinite inner product with respect to which it becomes a $*$-representation. The study of $*$-representations in spaces with indefinite inner products (or indefinite metrics, see [B]) has become rather popular due to various applications to relativistic quantum mechanics, differential equations, C^*-algebras etc. (see [KS] for an overview of this theory). Unlike the Hilbert space case representations in spaces with indefinite metrics can be indecomposable but not irreducible. We note that even for finite-dimensional algebras the problem of classification of all indecomposable representations is usually hopeless (see e.g. [?]).

Surprisingly, much of the work on $*$-representations in spaces with indefinite metrics has been done in purely analytical context and concentrated around theory of operators. The aim of the present paper is to consider the corresponding algebraical questions. Assume that we are given a $*$-algebra, A, over an algebraically closed field, $\mathbb{K}$. Let M be an A-module. The question is if there exists an indefinite inner product on M with respect to which M becomes a $*$-representation. The question naturally generalizes if one also brings an involutive automorphism, σ, of $\mathbb{K}$ into the game (so $*$ becomes σ-linear and one can talk about σ-inner products, or more general σ-forms, i.e. binary forms, linear with respect to one variable and σ-linear with respect to other). In this paper we mostly consider the case of finite-dimensional M and are interested in the following questions: when M admits a non-degenerate σ-symmetric σ-form and how unique this form can be.

The paper is organized as follows. In Section 2 we collect all necessary preliminaries; in Section 3 we answer the question when a given module admits a non-degenerate σ-form; in Section 4 we study σ-forms on simple modules and show the existence and the uniqueness of admissible σ-(skew)symmetric form; in Section 5 we show that, in the case of non-trivial σ, the existence of a non-degenerate admissible σ-form implies the existence of a non-degenerate admissible σ-symmetric σ-form; in Section 6 we prove that the non-degenerate admissible σ-symmetric σ-form is unique (up to naturally equivalence), if any, for every module in the case of trivial σ and for any indecomposable module otherwise; finally, in Section 7 we give several examples which include, in particular, canonical form of an admissible σ-form on indecomposable modules of length two and finish the paper with some remarks and conjectures collected in Section 8.

2 Preliminaries

We fix an algebraically closed field, $\mathbb{K}$, whose characteristics is not 2, and an involutive (possibly trivial) automorphisms, σ, of $\mathbb{K}$. Denote by $\mathbb{F}$ the subfield of elements of $\mathbb{K}$, fixed by σ. If V is a $\mathbb{K}$-vectorspace, by a *σ-form* on V we will mean a form, $(\cdot,\cdot) : V \times V \to \mathbb{K}$, satisfying

$$(\lambda_1 v_1 + \lambda_2 v_2, \lambda_3 v_3 + \lambda_4 v_4) = \\ = \lambda_1\sigma(\lambda_3)(v_1, v_3) + \lambda_1\sigma(\lambda_4)(v_1, v_4) + \lambda_2\sigma(\lambda_3)(v_2, v_3) + \lambda_2\sigma(\lambda_4)(v_2, v_4)$$

for any $v_1, v_2, v_3, v_4 \in V$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{K}$. We call $(\cdot,\cdot)$ *non-degenerate* provided for any $v \in V$ there exist $w_1, w_2 \in V$ such that $(v, w_1) \neq 0$ and $(w_2, v) \neq 0$, and we call $(\cdot,\cdot)$ *σ-symmetric* if $(v, w) = \sigma((w, v))$ for any $v, w \in V$. We remark that if $(\cdot,\cdot)$ is a σ-form then $(\cdot,\cdot)^{\#}$, defined by $(v, w)^{\#} = \sigma((w, v))$, is also a σ-form. We will call $(\cdot,\cdot)^{\#}$ the form *adjoint* to $(\cdot,\cdot)$. Clearly, $(\cdot,\cdot)^{\#} = (\cdot,\cdot)$ if and only if $(\cdot,\cdot)$ is σ-symmetric. Analogously, for a matrix, $M \in Mat_{n\times n}(\mathbb{K})$, we set $M^{\#} = \sigma(M^t)$.

If V comes with a basis, $\{v_i, i \in I\}$, (which means that each element of V decomposes into a unique finite linear combination $\sum_{i\in I} \lambda_i v_i$) then we define its *finitistic σ-dual space*

$V^\#$ as the space of σ-linear functions on V with the basis $\{v_i^\#, i \in I\}$, where $v_i^\#(v_j) = \delta_{i,j}$. It is clear that $(V^\#)^\#$ is canonically isomorphic to V. From now on we assume that all vectorspaces have a fixed basis, so we can always talk about the finitistic dual space. Clearly, in the finite-dimensional case the notions of the finitistic dual and regular dual coincide.

Let V be a $\mathbb{K}$-vectorspace with a basis, $\{v_i, i \in I\}$, and $(\cdot,\cdot)$ be an σ-form on V. We call $(\cdot,\cdot)$ *representable* provided there exists a basis, $\{w_i, i \in I\}$, of V and a subset, $J \subset I$, such that for any $v, w \in V$ with decompositions $v = \sum_{i\in I} b_i v_i$ and $w = \sum_{i\in I} a_i w_i$ we have $(w, v) = \sum_{i\in J} a_i \sigma(b_i)$. The basis $\{w_i, i \in I\}$ will be called *left dual* to $\{v_i, i \in I\}$. The following Lemma is obvious.

Lemma 1. *The following conditions are equivalent:*

1. *$(\cdot,\cdot)$ is representable.*
2. *There is a homomorphism, $\varphi : V \to V^\#$, such that $(w, v) = \varphi(w)(v)$.*

If $(\cdot,\cdot)$ is representable then each v_i is a finite linear combination of w_j, which means that the set $\{j \in I, (v_i, v_j) \neq 0\}$ is finite for any i. Converse is not true. For example, if $I = \mathbb{N}$ and $(\cdot,\cdot)$ is defined by

$$(v_i, v_j) = \begin{cases} 1, & |i-j| \leq 1; \\ 0 & \text{otherwise} \end{cases},$$

then V does not contain any w satisfying $(w, v_i) = \delta_{1,i}$. In what follows we will consider only representable form, so by a σ-form we will always mean a representable σ-form.

Consider $\{v_i, i \in I\}$ as an orthonormal basis with respect to the σ-form $(\cdot,\cdot)$ defined by $(v_i, v_j) = \delta_{i,j}$. Then if $[\cdot,\cdot]$ is a representable σ-form, there exist a unique linear operator, $A_{[\cdot,\cdot]}$, such that $[v, w] = (A_{[\cdot,\cdot]} v, w)$ for any $v, w \in V$ (it sends v_j to $\sum_{i\in I} [v_j, v_i] v_i$). As the example above shows, converse is not true in general.

We remark that in the case of finite-dimensional V any inner product is representable. Another sufficient condition for $(\cdot,\cdot)$ to be representable is given in the following.

Lemma 2. *Assume that V has an orthogonal (with respect to $(\cdot,\cdot)$) decomposition $V = \oplus_{j\in J} V_j$, where all V_j are finite-dimensional. Then $(\cdot,\cdot)$ is representable.*

Proof. Take a basis of V, which is the union of bases of all V_j. As the decomposition of V is orthogonal, the restriction of $(\cdot,\cdot)$ on each V_j is non-degenerate. Hence, we can find a dual basis inside each of V_j and, taking the union of them, we get a dual basis in V. $\square$

Let A be an associative $\mathbb{K}$-algebra equipped with a *σ-involution*, $*$, which is a map, $* : A \to A$, satisfying $(\lambda_1 a_1 + \lambda_2 a_2)^* = \sigma(\lambda_1) a_1 + \sigma(\lambda_2) a_2$ and $(a_1 a_2)^* = a_2^* a_1^*$ for any $a_1, a_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{K}$. If M is an A-module, the space $M^\#$ becomes an A-module via $(a \cdot f)(m) = f(a^* m)$ for any $a \in A$, $f \in M^\#$ and $m \in M$. We will call $M^\#$ the A-module, dual to M. We will say that an A-module, M, admits a *σ-form* (or is *σ-unitarizable*) provided there is a representable σ-form, $(\cdot,\cdot)$, on M such that $(ax, y) = (x, a^* y)$ for any

$x, y \in M$ and $a \in A$. In this case $(\cdot,\cdot)$ will be called *admissible* for M. Obviously, if M admits a σ-form, say $(\cdot,\cdot)$, then $(\cdot,\cdot)^{\#}$ is also admissible for M. As a trivial property we recall that if M admits a σ-form and N is a submodule of M then both $N^{\perp} = \{v \in M : (w, v) = 0 \text{ for all } w \in N\}$ and ${}^{\perp}N = \{v \in M : (v, w) = 0 \text{ for all } w \in N\}$ are also submodules of M.

If $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are two different admissible σ-forms on an A-module M we will say that they are *equivalent* if there exists an automorphism, φ, of M and an element, $0 \neq \lambda \in \mathbb{K}$, such that $(v, w) = \lambda[\varphi(v), \varphi(w)]$ for any $v, w \in M$.

We will use the following notation. By $\mathrm{Diag}(d_1, \dots, d_n)$ (resp. $\mathrm{Odiag}(d_1, \dots, d_n)$) we denote the matrix $S = (s_{i,j})_{i,j=1}^{n}$, defined by $s_{i,j} = 0$, $i \neq j$ and $s_{i,i} = d_i$ (resp. $s_{i,j} = 0$, $i+j \neq n+1$ and $s_{i,n+1-i} = d_i$). We will also write $\mathbb{I}$ or $\mathbb{I}_n$ for the unit matrix $\mathrm{Diag}(1, \dots, 1)$.

3 When M admits a σ-form?

The first natural question is when a given module admits a σ-form. The answer below seems to be easy and quite natural.

Theorem 1. *There is a canonical isomorphism between the linear space of representable σ-forms, admissible for M, and the linear space of A-homomorphisms from M to $M^{\#}$.*

Proof. Let $\varphi : M \to M^{\#}$ be an A-module homomorphism. For $v, w \in M$ set $(v, w)_{\varphi} = \varphi(v)(w)$. Then $(v, w)_{\varphi}$ is obviously a σ-form on V, moreover, $(av, w)_{\varphi} = \varphi(av)(w) = \varphi(v)(a^{*}w) = (v, a^{*}w)_{\varphi}$ by the definition of $M^{\#}$. Thus $(v, w)_{\varphi}$ is admissible for M and representable by Lemma 1. Conversely, let $(\cdot,\cdot)$ be an admissible σ-form on M. Then it is representable by definition, which means that there is a linear homomorphism $\varphi : M \to M^{\#}$ such that $(v, w) = \varphi(w)(v)$. Now the admissibility gives $\varphi(v)(aw) = (v, aw) = (a^{*}v, w) = \varphi(a^{*}v)(w)$, which means that φ is an A-module homomorphism. Obviously, the correspondence $\varphi \leftrightarrow (\cdot,\cdot)_{\varphi}$ is a linear bijection. □

Corollary 1. *An A-module, M, admits a non-degenerate σ-form if and only if $M \simeq M^{\#}$.*

Proof. In the proof of Theorem 1 one sees that the form $(\cdot,\cdot)_{\varphi}$ is non-degenerate if and only if the corresponding homomorphism $\varphi : M \to M^{\#}$ is an isomorphism. □

4 σ-forms on simple modules

As we have already mentioned, in the case of positively defined σ-form the theory reduces to the study of simple modules. In our setup this is no longer true but the case of simple modules still remains the easiest one.

Theorem 2. *Assume that $M \simeq M^{\#}$ is simple finite-dimensional. Then there exists unique up to equivalence non-degenerate admissible σ-form on M. Moreover, if σ is non-trivial this form can be always chosen to be σ-symmetric and if σ is trivial this form can be always chosen to be symmetric or skew-symmetric.*

Proof. As any (left or right) kernel of an admissible form is necessarily a submodule of M, the non-degeneracy of the form is trivial provided it is non-zero (this also easily follows from Corollary 1). Let $(\cdot,\cdot)$ be a non-zero admissible form on M, which exists by Theorem 1 and let $[\cdot,\cdot]$ be another non-zero admissible form. As we already know, both $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are non-degenerate which is equivalent to the fact that the corresponding linear operators $A_{(\cdot,\cdot)}$ and $A_{[\cdot,\cdot]}$ are invertible. From $(av,w)=(v,a^*w)$ and $[av,w]=[v,a^*w]$ we also get $aA_{(\cdot,\cdot)}A^{-1}_{[\cdot,\cdot]}=A_{(\cdot,\cdot)}A^{-1}_{[\cdot,\cdot]}a$ for any $a\in A$. Since M is simple and $\mathbb{K}$ is algebraically closed, there exists $\lambda\in\mathbb{K}$ such that $A_{(\cdot,\cdot)}=\lambda A_{[\cdot,\cdot]}$. Hence $(\cdot,\cdot)=\lambda[\cdot,\cdot]$ and thus all non-zero admissible σ-forms on M are equivalent.

The only thing left is to choose a σ-symmetric representative. The adjoint form $(\cdot,\cdot)^\#$ is also admissible for M and hence $(\cdot,\cdot)^\#=\lambda(\cdot,\cdot)$. From $(\cdot,\cdot)^{\#\#}=(\cdot,\cdot)$ we get $\lambda\sigma(\lambda)=1$. Choosing μ satisfying $\mu^2=\lambda$ ($\mathbb{K}$ is algebraically closed) we get $\mu\sigma(\mu)=\pm1$ and $(\mu(\cdot,\cdot))^\#=\sigma(\mu)\lambda(\cdot,\cdot)=\pm\mu(\cdot,\cdot)$. In case of $+$ sign the form $\mu(\cdot,\cdot)$ is already σ-symmetric. Otherwise, in the case of non-trivial σ, the form $\gamma\mu(v,w)$, where $\sigma(\gamma)=-\gamma$, will be σ-symmetric. □

5 When M admits a non-degenerate σ-symmetric σ-form?

Certainly, Corollary 1 does not guarantee that the corresponding form will be σ-symmetric. Nevertheless, in the case of non-trivial σ, this is true.

Theorem 3. *Assume that σ is non-trivial. Then M admits a non-degenerate σ-symmetric σ-form if and only if $M\simeq M^\#$.*

Proof. Let $(\cdot,\cdot)$ be a non-degenerate σ-form on M given by Corollary 1. Since σ is not trivial, there exists $0\neq s\in\mathbb{K}$ such that $\sigma(s)=-s$. Set $[\cdot,\cdot]_1=(\cdot,\cdot)+(\cdot,\cdot)^\#$ and $[\cdot,\cdot]_2=s\big((\cdot,\cdot)-(\cdot,\cdot)^\#\big)$. Clearly, both these σ-forms are σ-symmetric. Then $(\cdot,\cdot)=([\cdot,\cdot]_1+s^{-1}[\cdot,\cdot]_2)/2$ is non-degenerate ($\mathrm{char}(\mathbb{K})\neq2$). Consider the polynomial $f(x)=\det([\cdot,\cdot]_1+x[\cdot,\cdot]_2)$, which is known to be non-zero as $f(s^{-1})\neq0$. Hence, as the filed $\mathbb{F}$ is infinite, there exist infinitely many $t\in\mathbb{F}$ such that $f(t)\neq0$. So, if t is any of them, the σ-form $[\cdot,\cdot]_1+t[\cdot,\cdot]_2$ is non-degenerate, σ-symmetric, and admissible for M. □

In the case of trivial σ this is no longer true. An example of a simple module, which does not have any admissible symmetric form can be found in Section 7. However, the following holds.

Corollary 2. *If σ is trivial and $M\simeq M^\#$ has a simple socle then M admits either a non-degenerate symmetric or non-degenerate skew-symmetric form.*

Proof. Let $(\cdot,\cdot)$ be a non-degenerate σ-form admissible for M, $[\cdot,\cdot]_1=(\cdot,\cdot)+(\cdot,\cdot)^\#$ and $[\cdot,\cdot]_2=(\cdot,\cdot)-(\cdot,\cdot)^\#$. We claim that at least one of $[\cdot,\cdot]_1$ or $[\cdot,\cdot]_2$ is non-degenerate. Indeed, otherwise their kernels, being submodules of M, should intersect non-trivially as M has a simple socle. But the last would mean that this socle belongs to the kernel of $(\cdot,\cdot)$. □

Conjecture 1. *If σ is trivial and $M \simeq M^{\#}$ then M admits either a non-degenerate symmetric or non-degenerate skew symmetric form.*

6 Unicity of the non-degenerate σ-symmetric admissible σ-form

The main question we are interested in here is how unique a non-degenerate σ-symmetric σ-form can be. It happens that the answer again depends on σ. In this section we assume M to be finite-dimensional.

Theorem 4. 1. *Assume that σ is trivial. Then any $M \simeq M^{\#}$ admits only one, up to equivalence, non-degenerate σ-symmetric σ-form.*

2. *Assume that σ is not trivial. Then any indecomposable $M \simeq M^{\#}$ admits only one, up to equivalence, non-degenerate σ-symmetric σ-form.*

Proof. We start, in the general case, with fixing a module, M, and two non-degenerate, admissible σ-symmetric σ-forms, $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$, which are given in some (fixed) basis by matrices, F_1 and F_2 correspondingly. For an element, $a \in A$, we denote its matrix written in the same basis also by a. Then the admissibility of F_1 and F_2 reads $a^tF_1 = F_1\sigma(a)$ and $a^tF_1 = F_1\sigma(a)$. As F_2 is non-degenerate, we have $a^tF_1F_2^{-1} = F_1F_2^{-1}a^t$. Denoting $C = (F_1F_2^{-1})^t$ we have $aC = Ca$ for any $a \in A$ and $F_1 = C^tF_2$, which is equivalent to $(\cdot,\cdot)_1 = (C\cdot\cdot,\cdot)_2$. Multiplying C (and F_2) with a scalar, we can always assume that 1 is an eigenvalue of C. Hence, if M is indecomposable, we automatically get that it is the unique eigenvalue by Fittings lemma.

First we claim that C is the matrix of a linear operator which is self-adjoint with respect to the form F_2. Indeed, we know that both F_1 and F_2 are σ-symmetric. Hence, for any $x, y \in M$ we have

$$(C\cdot x, y)_2 = (x,y)_1 = \sigma((y,x)_1) = \sigma((C\cdot y, x)_2) = (x, C\cdot y)_2 = (C^{\#}\cdot x, y)_2,$$

and thus $C^tF_2 = F_2\sigma(C)$.

Now we want to find a very particular square root of C. For this we will distinguish the cases of trivial and non-trivial σ. Since M is finite-dimensional, there always exists $f_C(x) \in \mathbb{K}[x]$ such that $(f_C(C))^2 = C$. Set $D = f_C(C)$. If σ is trivial we automatically get $aD = Da$ for any $a \in A$ and $D^tF_2 = F_2D$ $(D = \sigma(D))$. Now we want to find some D with the same properties for non-trivial σ under the assumption of indecomposability of M. We know already that in this case the unique eigenvalue of C is 1 and hence the matrix $C - 1$ is nilpotent. For $k \in \mathbb{N}$ we define

$$d_k = \frac{\prod_{i=1}^{k}\left(\frac{1}{2} - i + 1\right)}{k!} \quad \text{and} \quad f(x) = 1 + \sum_{i=1}^{\dim(M)} d_kx^k$$

(this is the dim(M)-th polynomial approximation of the Tailor expansion for $(1+x)^{1/2}$). We claim that $f(x) \in \mathbb{K}[x]$. For this we recall that, by our assumptions, $\mathrm{char}(\mathbb{K}) \neq 2$, and thus it is enough to show that $\mathbb{Q} \ni d_k = l/2^m$ for $l, m \in \mathbb{Z}$. We have

$$d_k = (-1)^k \frac{\prod_{i=1}^{k}(2i-1)}{2^k k!}.$$

If p is an odd prime, it occures

$$\sum_{i=1}^{\infty} \left[\frac{2k-1}{p^i}\right] - \sum_{i=1}^{\infty} \left[\frac{k-1}{p^i}\right]$$

times in the numerator (here $[s]$ denotes the integral part of s) while in the denominator it occures

$$\sum_{i=1}^{\infty} \left[\frac{k}{p^i}\right]$$

times. The necessary inequality (that the first number of occurrences is greater) now follows directly from the evident inequality $[s+t] \geq [s] + [t]$ for non-negative real s, t.

The arguments above show more. It follows immediately that the coefficients of $f(x)$ belong to the prime subfield of $\mathbb{K}$ and hence, belong to $\mathbb{F}$. Thus $f(x) \in \mathbb{F}[x]$, which implies $f(\sigma(X)) = \sigma(f(X))$ for any matrix X. In particular, $f(C^{\#}) = f(C)^{\#}$. Set $D = f(C)$. We again obviously have $D^2 = C$ and $aD = Da$ for any $a \in A$. From $f(C^{\#}) = f(C)^{\#}$ it follows also that $D^t F_2 = F_2 \sigma(D)$.

Finally, we claim that D provides the necessary equivalence between F_1 and F_2. Indeed, $F_1 = C^t F_2 = D^t D^t F_2 = D^t F_2 \sigma(D)$ and thus $(x,y)_1 = (D \cdot x, D \cdot y)_2$ for any $x, y \in M$ (we remark that we possibly have already multiplied F_2 with some $\lambda \in \mathbb{K}$, see arguments above). As $aD = Da$ for any $a \in A$ we get the equivalence of F_1 and F_2 as claimed. □

Below one can find, in the case of non-trivial σ, an example of a decomposable module which admits more than one, up to equivalence, non-degenerate σ-symmetric σ-form.

7 Examples

7.1 A simple module, which does not admit any non-zero symmetric form

Let $\mathbb{K} = \mathbb{C}$ and σ be trivial. Denote by A a free $*$-algebra with two generators, a and b, which means that A, as a $\mathbb{K}$-algebra, is generated by a, a^*, b, b^*. Then the two-dimensional A-module M, defined by

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^* = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b^* = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

admits the following skew-symmetric form:

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As it follows from Section 4, the above means that M does not admit any symmetric non-degenerate form.

7.2 A module, σ-unitarizable with respect to at least two essentially different σ-symmetric σ-forms

Assume that $\mathbb{K} = \mathbb{C}$, σ is the complex conjugation, $(A, *)$ is a $\mathbb{C}$-algebra with involution and M is a simple A-module, unitarizable with respect to a positively defined form, say F). In particular, $M \simeq M^{\#}$. Then the module $M \oplus M$ is unitarizable with respect to $F \oplus F$, the last inner product being clearly positively defined again. At the same time $M \oplus M^{\#} \simeq M \oplus M$ is σ-unitarizable with respect to the inner product

$$\begin{pmatrix} 0 & F \\ F^{\#} & 0 \end{pmatrix}.$$

The last form is evidently indefinite, hence is not equivalent to the previous one. As a more precise example one can take A to be the group algebra over $\mathbb{C}$ of a finite group and M to be the trivial A-module.

7.3 Canonical form for the σ-form on a Jordan cell

Let $A \simeq \mathbb{K}[x]$ with trivial involution and σ be trivial as well. We are going to show that any finite-dimensional indecomposable A-module is σ-unitarizable and that the corresponding form is unique up to equivalence by producing the canonical form for any admissible σ-form. It is well-known that finite-dimensional indecomposable A-module are given by Jordan cells, i.e. they are parametrized by pairs $(\lambda, n) \in \mathbb{K} \times \mathbb{N}$ and the matrix, corresponding to x, for fixed (λ, n) is the Jordan cell $J_n(\lambda)$. The condition of σ-unitarizability of $J_n(\lambda)$ with respect to a form, $F = (f_{i,j})_{i,j=1}^n$, reads as follows: $(J_n(\lambda))^t F = F J_n(\lambda)$. From this one immediately has $f_{i,j} = 0$, $i + j \leq n$ and $f_{i,j} = f_{k,l}$, $i + j = k + l$, in particular, F is symmetric. Finally, one can easily find a transformation matrix, $S = (S_{i,j})_{i,j=1}^n$, satisfying $s_{i,j} = 0$, $i > j$ and $s_{i,j} = s_{i+1,j+1}$ for all i, j (this guarantees $SJ_n(\lambda)S^{-1} = J_n(\lambda)$), and such that $S^t F S = \mathrm{Odiag}(1, 1, \dots, 1)$. From this it follows that the canonical form of F for $J_n(\lambda)$ is $\mathrm{Odiag}(1, 1, \dots, 1)$ and the form itself is unique.

Using analogous arguments one can see that in the case of non-trivial σ the only σ-unitarizable modules will be those, which correspond to $J_n(\lambda)$, $\lambda \in \mathbb{F}$, and they all admit only one, up to equivalence, non-degenerate σ-form, whose canonical form is also $\mathrm{Odiag}(1, 1, \dots, 1)$.

7.4 An indecomposable module, which admits both σ-symmetric and non-symmetric non-degenerate σ-forms

After the example with the Jordan cell from Subsection 7.3 one can ask if the σ-form on an indecomposable module is unique up to equivalence. The answer is certainly no as any module admits a zero form, which can not be equivalent to any other. Moreover, using Theorem 1 it is easy to construct a module which admits both non-degenerate and degenerate σ-forms, which certainly also inequivalent. So, we have a natural assumption left, what will happen for non-degenerate forms. But even under this additional assumption the answer remains negative. We will show this by giving an example of an indecomposable module, which admits both σ-symmetric and non-symmetric non-degenerate σ-forms (which clearly can not be equivalent to each other). For this we take the complex algebra A, generated by $1, e, a, b, c$ with relations $e^2 = e$, $ea = a(1-e) = a$, $eb = b(1-e) = b$, $ec = c(1-e) = c$, $ab = ba = ac = ca = cb = bc = 0$ and $*$ defined by $e^* = 1-e$, $a^* = b$ and $c^* = c$. If we consider the 4-dimensional A-module given by

$$e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

it will admit the following σ-forms:

$$F_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

The second form is σ-symmetric (for any σ!) and the first one is not. One can easily check that this module is indecomposable.

7.5 σ-symmetric σ-form for an indecomposable finite-dimensional module of length two and its canonical form

We want to finish this Section with more abstract example. Although Theorem 4 guarantees the unicity of an admissible σ-symmetric σ-form on an indecomposable finite-dimensional module, it says nothing about the canonical matrix of this form. In this subsection we are going to describe it in the simplest situation.

So, let M be an indecomposable finite-dimensional module of length two over some $*$-algebra A. Let N be its unique proper simple submodule. Then the quotient $L = M/N$ is also simple. We know that M admits a non-degenerate σ-form, say $F = (\cdot,\cdot)$. In particular, $M \simeq M^{\#}$, which immediately implies $L \simeq N^{\#}$. Consider $N^{\perp}$. It is a proper submodule of M, hence coincides with N. This means that the restriction of $(\cdot,\cdot)$ on N is zero. As $M \simeq M^{\#}$, the module $L^{\#}$, which is the unique proper submodule of $M^{\#}$, is isomorphic to N. Choose some basis, $\{v_i : i \in I\}$, in N and extend it to the basis $\{v_i, v_i^{\#} : i \in I\}$ of M. Denote by $\{w_i, w_i^{\#} : i \in I\}$ the corresponding dual basis (since the form F is σ-symmetric, the left and right dual bases coincide) and consider the set $\{v_i, w_i : i \in I\}$. From the non-degeneracy of F it follows that these elements are linearly independent, hence form a basis. In this basis the matrix of F has the following form: $F = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & S \end{pmatrix}$, where $S^{\#} = S$, and the element $a \in A$ acts via $\xi(a) = \begin{pmatrix} \pi(a) & \tau(a) \\ 0 & \pi(a^*)^{\#} \end{pmatrix}$.

Lemma 3. *There is a basis of the form $\{v_i, w_i + \sum_j x_{i,j} v_j : i \in I\}$ such that the matrix of F, written in this basis, is $\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & \hat{S} \end{pmatrix}$, where $\hat{S}^{\#} = \hat{S}$, and each $\tau(a)$ satisfies $\tau(a) = \tau(a^*)^{\#}$.*

Proof. The first statement is trivial because the above transformation affects only the S-part of F. To prove the second statement we will consider two different cases.

Case 1. Assume that the A-modules N and L are not isomorphic. As the annihilator of M is a $*$-ideal in A, passing to the image of A in $\mathcal{L}(M)$ we can assume that A is finite-dimensional. Let e and f denote the idempotents from some orthogonal decomposition of $1 \in A$, which act as $\mathbb{I}$ on N and L respectively. As L and N are not isomorphic we have $ef = fe = 0$, hence

$$\xi(e) = \begin{pmatrix} \mathbb{I} & X \\ 0 & 0 \end{pmatrix}, \quad \xi(f) = \begin{pmatrix} 0 & -X \\ 0 & \mathbb{I} \end{pmatrix}.$$

Now there is a basis of the form $\{v_i, w_i + \sum_j x_{i,j} v_j : i \in I\}$ in which $X = 0$. Take any $a \in A$ and consider eaf. One has $\xi(eaf) = \begin{pmatrix} 0 & \tau(a) \\ 0 & 0 \end{pmatrix}$. Now from $\xi(eaf)^t F = F\sigma(\xi((eaf)^*))$ we derive $\tau(a)^{\#} = \tau(a^*)$ as desired.

Case 2. Now we assume that the A-modules N and L are isomorphic. Thus, passing to the image in $\mathcal{L}(M)$, we can assume that A is a primary algebra, and hence is isomorphic to $Mat_{n\times n}(B)$, where B is a local $\mathbb{K}$-algebra (see for example [DK, Chapter 3]). As $\mathbb{K}$ is algebraically closed, $B/\operatorname{rad}(B) \simeq \mathbb{K}$ and B has the canonical subalgebra, $\mathbb{K}1$, which is mapped isomorphically to $\mathbb{K}$ under the above epimorphism. Hence A has a canonical subalgebra, $C \simeq Mat_{n\times n}(\mathbb{K})$, which is mapped isomorphically to $Mat_{n\times n}(\mathbb{K})$ under the canonical epimorhism $A/\operatorname{rad}(A) \simeq Mat_{n\times n}(\mathbb{K})$. As $Mat_{n\times n}(\mathbb{K})$ is simple, the module $M|_C$ is completely reducible and therefore isomorphic to $N|_C \oplus L|_C$ (moreover $N|_C \simeq L|_C$). Hence there is a basis of the form $\{v_i, w_i + \sum_j x_{i,j} v_j : i \in I\}$, such that for any $c \in C$ its matrix, written in this basis, has the property $\tau(c) = 0$. As any $a \in A$ can be uniquely written as $a = c + r$, $c \in C$ and $r \in \operatorname{rad}(A)$ and for all $r \in \operatorname{rad}(A)$ we have $\pi(r) = \pi(r^*)^{\#} = 0$. We complete the proof by the same arguments as in Case 1, applied to this r instead of eaf. $\square$

Because of Lemma 3 we may assume that the basis is chosen such that $\tau(a) = \tau(a^*)^{\#}$ and $F = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & S \end{pmatrix}$, where $S^{\#} = S$. Using $\tau(a) = \tau(a^*)^{\#}$, the unitarizability of M

gives $\pi(a)S = S\pi(a^*)^\#$ for all $a \in A$. Hence the matrix $T = \begin{pmatrix} \mathbb{I} & -\frac{1}{2}S \\ 0 & \mathbb{I} \end{pmatrix}$ satisfies $T\xi(a) = \xi(a)T$ for all $a \in A$ and finally we get $T^tFT = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$, obtaining the necessary canonical form.

Finally, we remark that the arguments above work also for infinite-dimensional module M as well provided N and L have scalar automorphisms only.

8 Conclusive remarks

All the results above have an immediate generalization to the classical version of infinite-dimensional weight modules, i.e. assume that A contains a commutative subalgebra, B, stable under $*$, and there is a set, C, of generators of B such that $b^* = \lambda_b b$, $\lambda_b \in \mathbb{K}$, for any $b \in C$. An A-module, M, is called a *generalized B-weight* module, provided M, as a B-module, decomposes into a direct sum of finite-dimensional modules and the multiplicity of each simple B-module in M is finite. In this situation it is straightforward that with respect to any admissible σ-form M decomposes into an orthogonal direct sum of finite-dimensional subspaces (B-weight spaces). It is clear that for such M all the arguments above remain valid. This applies, for example, to simple weight modules with finite-dimensional weight spaces or to Gelfand-Zetlin modules over simple complex Lie algebras (see [D, DFO]). We will discuss this applications in the subsequent papers.

Using the arguments analogous to that presented in Subsection 7.4 one can show that the canonical form of an admissible non-degenerate σ-symmetric σ-form on an indecomposable finite-dimensional module of length 3 is

$$\begin{pmatrix} 0 & 0 & \mathbb{I} \\ 0 & S & 0 \\ \mathbb{I} & 0 & 0 \end{pmatrix},$$

where $S = S^\#$ is an admissible non-degenerate σ-symmetric σ-form on a simple module. Based on this we want to formulate the following conjecture:

Conjecture 2. *The canonical form of an admissible non-degenerate σ-symmetric σ-form on an indecomposable finite-dimensional module is*

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \mathbb{I} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \mathbb{I} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \mathbb{I} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & S & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \mathbb{I} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbb{I} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbb{I} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where S is a block-diagonal matrix, whose diagonal blocks are admissible non-degenerate σ-symmetric σ-form on simple modules.

Acknowledgements

We would like to thank Prof. T.Brüstle, Prof. S.König, Prof. V.Ostrovskyi and Prof. V.Shulman for helpful discussions.

References

[B] *J.Bognár*, Indefinite inner product spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 78. Springer-Verlag, New York-Heidelberg, 1974.

[D] *J.Dixmier*, Enveloping algebras. Revised reprint of the 1977 translation. Graduate Studies in Mathematics, 11. American Mathematical Society, Providence, RI, 1996.

[Dr] *Yu.Drozd*, Tame and wild matrix problems. (Russian) Matrix problems (Russian), pp. 104–114. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977.

[DK] *Yu. Drozd, V.Kirichenko*, Finite-dimensional algebras. Translated from the 1980 Russian original and with an appendix by Vlastimil Dlab. Springer-Verlag, Berlin, 1994.

[DFO] *Yu.A.Drozd, V.M.Futorny, S.A.Ovsienko*, Harish-Chandra subalgebras and Gelfand-Zetlin modules. Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 79–93, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 424, Kluwer Acad. Publ., Dordrecht, 1994.

[KS] *E.Kissin, V.Shulman*, Representations on Krein spaces and derivations of C^*-algebras. Pitman Monographs and Surveys in Pure and Applied Mathematics, 89. Longman, Harlow, 1997.

[OS] *V.Ostrovskyi and Yu.Samoilenko*, Introduction to the theory of representations of finitely presented $*$-algebras I, Representations by bounded operators, Rev. Math. & Math. Phys., vol. 11, 1999.

Department of Mathematics, Chalmers University of Technology and Göteborg University, Göteborg, SE-412 96, Sweden;
e-mails: V.M.: `volody@math.chalmers.se`; L.T.: `turowska@math.chalmers.se`