A discussion of some recent results on stationary boundary value problems for equations of Boltzmann type

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1. ON STATIONARY PROBLEMS IN BOUNDED DOMAINS.

This first part will review some recent ideas and results concerning stationary boundary value problems for equations of Boltzmann type. In the second part, those techniques and results will be connected with questions about asymptotics with respect to time and small mean free paths. The report is mainly based on joint research with A. Nouri.

Before turning to our more recent studies of stationary problems under various boundary conditions, I would like to recall some *earlier works* in the field. Concerning the *linearized* Boltzmann equation, existence and uniqueness of stationary solutions in a bounded domain is well presented in Maslova's monograph ([27]), using classical Hilbert space techniques such as the Fredholm alternative. More recently, for the *linear* Boltzmann equation, stationary measure solutions were obtained via measure compactness by Cercignani and Giurin ([18]), and uniqueness of L^1-solutions via a study of the relative entropy by Pettersson and Triolo ([32], [33]). As for the corresponding L^1-existence, we shall return to that in the discussion below.

Concerning the *nonlinear* stationary Boltzmann equation in $I\!R^n$ in the *close to equilibrium* case, the study was started by Grad ([23]) and Guiraud ([24]) in the mid-1960ies and early 1970ies, and was followed by many others, including exterior domain results by Maslova and coworkers ([27]) as well as by Ukai and Asano ([34]) in the early 1980ies. Also some small domain results by Pao ([31])

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in the late 1960ies belong to this group. In such a setting well known perturbation and contraction mapping techniques can be utilized. For large Knudsen numbers Kn (defined below), the unique solvability of stationary problems for the Boltzmann equation was similarly established in Maslova's monograph using Kn^{-1} as a small parameter.

Discrete velocity models, in particular the Broadwell model, have been well researched also far from equilibrium, see e.g.([12], [13], [16], [17], [19]). Inspired by the discrete velocity techniques, with Cercignani and Illner we obtained measure solutions far from equilibrium for continuous velocities and special stationary nonlinear Boltzmann equations in a slab via measure compactness ([3]).

In contrast to all these earlier results, the *recent development* concerning the full nonlinear stationary Boltzmann equation, which is the main topic in this first part, is mainly connected with entropy dissipation control. Consider the stationary *Boltzmann equation in the slab*,

$$\xi \frac{\partial}{\partial x} f(x,v) = Q(f,f)(x,v), \; x \in [-1,1], \; v \in \mathbb{R}^3. \tag{1.1}$$

The nonnegative function $f(x,v)$ represents the density of a rarefied gas at position x and velocity v. The collision operator Q is the classical Boltzmann operator

$$Q(f,f)(x,v) = Q^+(f,f) - Q^-(f,f) =$$
$$\int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \omega)[f'f'^* - ff^*]d\omega dv_*,$$

where $Q^+ - Q^-$ is the splitting into gain and loss terms,

$$f^* = f(x,v_*), \quad f' = f(x,v'), \quad f'^* = f(x,v'_*),$$

and the pre-and post-collisional velocities are connected by

$$v' = v - (v - v_*|\omega)\omega, \quad v'_* = v_* + (v - v_*|\omega)\omega.$$

The velocity component in the x-direction is denoted by ξ, and $(v - v_*|\omega)$ denotes the Euclidean inner product in $\mathbb{R}^3$. Let ω be represented by the polar angle θ (with polar axis along $v - v_*$) and by the azimuthal angle. The function $B(v - v_*, \omega)$ is the collision kernel of the collision operator Q, and is for simplicity taken as $| v - v_* |^\beta b(\theta)$, with

$$-3 < \beta < 2, \quad b \in L^1_+(0,\pi), \quad b(\theta) \geq c > 0, \; a.e..$$

As we shall see below using Greens identity, stationary boundary value problems only control fluxes of quantities which are conserved in the time dependent case.

In a number of stationary cases, energy and entropy dissipation are then the most useful among easily available à priori quantities, whereas equally useful mass and entropy estimates are lacking.

Given a constant $m > 0$ and positive *indata* f_b bounded away from zero on compacts, solutions f to (1.1) are sought such that

$$\int_{-1}^{1}\int_{\mathbb{R}^3}(1+\mid v\mid)^{\beta}f(x,v)dxdv = m, \tag{1.2}$$

$$f(-1,v) = kf_b(-1,v),\ \xi > 0, \quad f(1,v) = kf_b(1,v),\ \xi < 0, \tag{1.3}$$

for some constant $k > 0$. The constant k is determined from the value m of the β-norm (1.2). In this way, the lack of a mass estimate is compensated by forcing a β-norm control m on the solution. If it were not for the problem with small velocities, the condition (1.2) could have been replaced by the condition $k = 1$.

Denote the collision frequency by

$$\nu(x,v) := \int_{\mathbb{R}^3\times S^2} B(v-v_*,\omega)f(x,v_*)dv_*d\omega.$$

Assuming that $\frac{Q^+(f,f)}{1+f} \in L^1_{loc}$, $\frac{Q^-(f,f)}{1+f} \in L^1_{loc}$, the exponential, mild and weak *solution concepts* in the stationary context (1.1-3) can be formulated as follows.

Definition 1. *f is an exponential solution to the stationary Boltzmann problem (1.1-3), if $f \in L^1_{loc}((-1,1)\times\mathbb{R}^3)$, and for almost all v in $\mathbb{R}^3$,*

$$f(1+s\xi,v) = kf_b(-1,v)e^{-\int_{-\frac{2}{\xi}}^{s}\nu(1+\tau\xi,v)d\tau}$$
$$+\int_{-\frac{2}{\xi}}^{s} e^{-\int_\tau^s \nu(1+\sigma\xi,v)d\sigma}Q^+(f,f)(1+\tau\xi,v)d\tau,\ \xi > 0,\ \ s\in(-\frac{2}{\xi},0),$$
$$f(-1+s\xi,v) = kf_b(1,v)e^{-\int_{\frac{2}{\xi}}^{s}\nu(-1+\tau\xi,v)d\tau}$$
$$+\int_{\frac{2}{\xi}}^{s} e^{-\int_\tau^s \nu(-1+\sigma\xi,v)d\sigma}Q^+(f,f)(-1+\tau\xi,v)d\tau,\ \xi < 0,\ \ s\in(\frac{2}{\xi},0).$$

Definition 2. *f is a mild solution to the stationary Boltzmann problem (1.1-3), if $f \in L^1_{loc}((-1,1)\times\mathbb{R}^3)$, and for almost all v in $\mathbb{R}^3$,*

$$f(1+s\xi,v) = kf_b(-1,v) + \int_{-\frac{2}{\xi}}^{s} Q(f,f)(1+\tau\xi,v)d\tau,\ \xi > 0,\ s\in(-\frac{2}{\xi},0),$$
$$f(-1+s\xi,v) = kf_b(1,v) + \int_{\frac{2}{\xi}}^{s} Q(f,f)(-1+\tau\xi,v)d\tau,\ \xi < 0,\ s\in(\frac{2}{\xi},0).$$

Here the integrals for Q^+ and Q^- are assumed to exist separately.

Definition 3. *f is a weak solution of the stationary Boltzmann problem (1.1-3), if* $f \in L^1_{loc}((-1,1)\times I\!R^3)$, *and*

$$\int_{-1}^{1}\int_{I\!R^3}(\xi f\frac{\partial\varphi}{\partial x}+Q(f,f)\varphi)(x,v)dxdv$$

$$= k\int_{v\in I\!R^3;\xi<0}\xi f_b\varphi(1,v)dv - k\int_{v\in I\!R^3;\xi>0}\xi f_b\varphi(-1,v)dv, \qquad (1.4)$$

for every $\varphi \in C^1_c([-1,1]\times I\!R^3)$ *with* $supp\varphi \subset [-1,1]\times\{v\in I\!R^3; \mid \xi \mid \geq \delta\}$ *for some* $\delta > 0$, *and with* φ *vanishing on* $\{(-1,v);\xi<0\}\cup\{(1,v);\xi>0\}$. *In (1.4) the integrals for* Q^+ *and for* Q^- *are assumed to exist separately.*

Remark 1.1. This weak form is somewhat stronger than the mild and exponential ones.

Remark 1.2. It follows from the exponential form that f can be estimated from above and below by the values at ingoing and outgoing boundary,

$$kf_b(-1,v)e^{-\int\nu d\tau} \leq f(x,v) \leq f(1,v)e^{\int\nu d\tau} \qquad (1.5)$$

for $\xi > 0$ and correspondingly for $\xi < 0$. The entropy control, so useful in kinetic weak L^1 compactness arguments, may sometimes be replaced by an entropy dissipation control, when f is distinctly non-maxwellian. Namely, in the integrand of the *entropy dissipation*

$$(ff^* - f'f'^*)log\frac{ff^*}{f'f'^*},$$

at a high concentration with respect to v, i.e. where f is large, one may e.g. use (1.5) to pick a 'large enough' set in (v_*,ω), where f^* is 'sufficiently large' and f' and f'^* 'sufficiently small' for the integrand to behave like $f\log f$. That is a common device for obtaining the results presented here.

Remark 1.3. With an *absorption term* αf added, the time dependent version of (1.1) in weak form becomes

$$\int_0^t\int_{-1}^{1}\int_{I\!R^3}(-\alpha f\varphi+\xi f\frac{\partial\varphi}{\partial x}+Q(f,f)\varphi)(\tau,x,v)d\tau dxdv =$$

$$\int_0^t[k\int_{v\in I\!R^3;\xi<0}\xi f_b\varphi(\tau,1,v)dv - k\int_{v\in I\!R^3;\xi>0}\xi f_b\varphi(\tau,-1,v)dv]d\tau +$$

$$\int_{-1}^{1}\int_{I\!R^3}f\varphi(t,x,v)dxdv - \int_{-1}^{1}\int_{I\!R^3}f\varphi(0,x,v)dxdv. \qquad (1.6)$$

Now

$$\int_{IR^3} Q(f,f)\phi dv = 0, \text{ for } \phi = 1, v, v^2,$$
$$\int_{IR^3} Q(f,f)\phi dv \leq 0 \text{ for } \phi = \log f.$$

From here when $\alpha = 0$, we see that only the fluxes may be controlled in the stationary case for quantities which are themselves controlled in the time dependent case. But for α positive and fixed, mass, energy and entropy are à priori controlled in the stationary case. Therefore the solving of stationary problems can be split into first treating the case of positive α, and then letting α tend to zero. Solutions for α positive can be obtained by variants of the time-dependent solution scheme, so the existence problem is thereby reduced to removing the α-term.

Suppose

$$\int_{\xi>0} [\xi(1+ \mid v \mid^2 + \mid log f_b \mid) + (1+ \mid v \mid)^\beta] f_b(-1,v) dv < \infty,$$
$$\int_{\xi<0} \mid \xi \mid (1+ \mid v \mid^2 + \mid log f_b \mid) + (1+ \mid v \mid)^\beta] f_b(1,v) dv < \infty. \qquad (1.7)$$

In the case of a slab, i.e. for a one dimensional space domain, important velocity integrals, such as $\int \xi^2 f(x,v)dv$, can be controlled à priori in the maximum norm of the space variable x using (1.6). The following theorem can be proved.

Theorem 1.1 ([6]) *Given* $m > 0$, β *with* $0 \leq \beta < 2$ *in the collision kernel, and indata* f_b *satisfying (1.7), there is a weak solution to the stationary problem (1.1-3).*

An analogous result holds for boundary conditions of *diffuse reflection* type,

$$f(-1,v) = M_-(v) \int_{\xi'<0} \mid \xi' \mid f(0,v')dv', \ \xi > 0,$$
$$f(1,v) = M_+(v) \int_{\xi'>0} \xi' f(L,v')dv', \ \xi < 0, \qquad (1.8)$$

where M_- and M_+ are given normalized half-space maxwellians $M_i(v) = \frac{1}{2\pi T_i^2} e^{-\frac{|v|^2}{2Ti}}$, $i \in \{-,+\}$.

Theorem 1.2 ([7]) *Given* $m > 0$ *and* β *with* $0 \leq \beta < 2$ *in the collision kernel, there is a weak solution to the stationary problem (1.1-2), (1.8).*

In the proofs, one hurdle is mass concentrated at small ξ-velocities. This may be resolved using an à priori control by the mass at large velocities as indicated in Remark 1.2, with the entropy dissipation estimate replacing the usual entropy argument.

A number of *generalizations*, for $v \in I\!R^n$, $n \geq 2$, and $-n < \beta < 2$, together with cases of $b(\theta) > 0$ a.e., or B not in the product form $\mid v - v_* \mid^\beta b(\theta)$, can also be analyzed straightforwardly by the same approach giving similar results, such as mild solutions for $\beta < 0$. The maxwellians in (1.8) can be replaced by other reentry profiles, under suitable conditions on the functions replacing the maxwellians. This type of ideas also lead to corresponding L^1-existence results for the Povzner equation ([8], [30]) and for equations of Enskog type in bounded domains of $I\!R^n$. For the linear Boltzmann equation on the other hand, we are not aware of any useful entropy dissipation control. But here information about small velocity behaviour may instead be directly extracted from the gain term at large velocities, again resulting in a similar L^1-existence picture ([20]).

In the nonlinear cases just discussed, the collision frequency integral along characteristics essentially behaves like a volume integral, which is à priori controlled in the approximation scheme. It is a serious obstacle that this is not so for the stationary nonlinear *Boltzmann equation in* $\Omega \subset I\!R^n$,

$$v \cdot \nabla_x f(x, v) = Q(f, f), \quad x \in \Omega, v \in I\!R^n. \tag{1.9}$$

However, with a new approach this problem can be overcome, at least as long as the other main obstacle to the full result is cancelled, namely the small velocities in the nonlinear collision operator. Consider Q given by

$$\int_{I\!R^n} \int_{S^{n-1}} \chi_\eta(v, v_*, \sigma) B(v - v_*, \sigma)(f(x, v')f(x, v'_*) - f(x, v)f(x, v_*)) dv_* d\sigma$$

with $\eta > 0$, and

$$\chi_\eta(v, v_*, \sigma) = 0 \text{ if } |v| < \eta \text{ or } |v_*| < \eta \text{ or } |v'| < \eta \text{ or } |v'_*| < \eta, \ \chi_\eta(v, v_*, \sigma) = 1 \text{ else.}$$

Take Ω as a strictly convex domain with C^1 boundary. The inward and outward boundaries in phase space are

$$\partial\Omega^+ = \{(x, v) \in \partial\Omega \times I\!R^n; v \cdot n(x) > 0\},$$
$$\partial\Omega^- = \{(x, v) \in \partial\Omega \times I\!R^n; v \cdot n(x) < 0\},$$

where $n(x)$ denotes the inward normal on $\partial\Omega$.

The removal of small velcities through χ_η, allows mass to be estimated by à priori controlled energy, and we may study the equation with given indata instead of the previous β-norm m plus indata profile. So given a function $f_b > 0$ defined on $\partial\Omega^+$, we look for a solution f to (1.9) with

$$f(x,v) = f_b(x,v), \quad (x,v) \in \partial\Omega^+. \tag{1.10}$$

Under these conditions, à priori estimates along characteristics using the exponential solution form, together with new local information from the entropy dissipation control, leads to the following result.

Theorem 1.3 ([10]) *Suppose that $f_b > ae^{-dv^2}$ for some $a, d > 0$ and a.a. $(x,v) \in \partial\Omega^+$, and that*

$$\int_{(x,v)\in\partial\Omega^+} [v \cdot n(x)(1 + v^2 + \ln^+ f_b(x,v)) + 1] f_b(x,v) dx dv < \infty.$$

Then the equation (1.9) has a solution satisfying the boundary condition (1.10).

The technical restrictions on Ω and f_b can be relaxed. In fact we expect the result to hold for the same mathematically and physically natural, non-smooth domains as in the time dependent case ([4]), namely with boundaries having finite Hausdorff measure plus a certain cone condition. However, the removal of the χ_η-truncation probably requires fresh ideas, since only using the ideas of the present proof with no χ_η-truncation seems to permit the alternative that all mass in the limit becomes concentrated at zero velocity.

The proof of Theorem 1.3 As in the previous theorems, the proof starts from solutions to the equation with an extra absorption term αf, which is then removed in the limit $\alpha \to 0$.

$$\alpha f + v \cdot \nabla_x f(x,v) = Q(f,f), \quad x \in \Omega, v \in I\!R^n.$$

Using the relevant, stationary form of Green's formula

$$\int_{\partial\Omega^-} (f(x,v) + \int_{-s^+(x,v)}^{0} \alpha f(x+sv,v) ds) \mid v \cdot n(x) \mid dx dv = \int_{\partial\Omega^+} f_b(x,v) \mid v \cdot n(x) \mid dx dv,$$

we can estimate outgoing mass flow à priori by ingoing mass flow. An exponential estimate of type (1.5) gives uniform estimates of f^α along characteristics outside a small set. So given $\epsilon > 0$, there is C_ϵ independent of α, so that outside a set (depending on α) of characteristics of measure ϵ, it holds that $f^\alpha < C_\epsilon$. This can be arranged so that the weak limit of f^α restricted to the remaining sets, $f_\epsilon = w - \lim f^\alpha_{restr}$ increases with $1/\epsilon$.

If we try to use this partial limit of the approximate solutions $f = s - \lim f_\epsilon$ as a candidate for a true solution, it remains to prove that the limit satisfies the desired problem. We use the *iterated integral form* of the equation, which makes it easy to remove the solution all along characteristics by putting the test function to zero along them,

$$\int_{\partial\Omega^+} (f_b\varphi)(x,v) \mid v \cdot n(x) \mid dxdv$$

$$+\int_{\partial\Omega^-} (\int_{-s^+(x,v)}^{0} [-\alpha f\varphi + Q(f,f)\varphi + fv\cdot\nabla_x\varphi](x+\sigma v, v)d\sigma) \mid v\cdot n(x) \mid dxdv = 0.$$

This form of the problem (1.9), (1.10), is equivalent to the mild and exponential forms, in which case the iterated collision integral is well defined even when Q is not integrable. The truncation of test functions is possible, since our test functions are in L^∞ and are only required to be differentiable along characteristics. The argument $s^+(x,v)$ is the time it takes to reach the boundary from x along the line with velocity $-v$.

One difficulty with the removal procedure just described, is the following. Consider the loss term $f^\alpha \int d\omega \int Bf^{\alpha *}dv_*$. It may happen at a point x along a retained characteristic for f^α, that other characteristics through the same space point x are not retained. This may decrease the collision frequency at x, which is an integral in v_*. For an approximation of the present type to deliver the correct equation in the final limit, the effect of that decrease should disappear in the limit. The main step in proving this, is the following lemma, quantifying in what sense the contribution of the large f^α-values is small from those space points that support a non-negligible amount of "good" characteristics. (The influence from the other relevant space points is then also shown to be negligible in the limit.)

To present this *key lemma*, let $\gamma = \frac{v}{|v|}$, and let ζ_{xv} be a characteristic through $x \in \Omega$ in direction v. Denote by $\mathbf{X}_{\mathbf{n}}^{\alpha}(\gamma)$ the subset of Ω consisting of those characteristics in direction γ for which f^{α} is 'reasonably bounded and nontangential, and with the collision frequency integral along ζ_{xv} also reasonably bounded' for 'most $|v|$'. For the precise quantification of 'reasonably bounded' a relevant version of (1.5) is used. The reader is referred to [10] for details. In velocity space restrict to those v with $\eta \leq |v| \leq V$ where $V >> \eta$. Set

$$\mathbf{f}_{\lambda}^{\alpha} := f^{\alpha} \text{ if } f^{\alpha} \geq \lambda, \quad f_{\lambda}^{\alpha} := 0 \text{ else,}$$
$$\mathbf{a}_0(z) = \max\{1, \log z\} \text{ and inductively } \mathbf{a}_{i+1}(z) = \max\{1, \log a_i(z)\}.$$

Define

$$\mathbf{O}_{\alpha,\lambda} := \{x \in \Omega; \int_{\eta \leq |v| \leq V} f_{\lambda}^{\alpha}(x,v)dv > 0\},$$

and

$$\mathbf{O}_{\alpha,i,n,\lambda} := \{x \in O_{\alpha,\lambda}; \text{meas}\{\mu \in S^2; x \in X_n^{\alpha}(\mu)\} > \frac{4\pi}{i}\}.$$

Lemma 1.4 *Let V, i, n be given in $\mathbb{N}$ and sufficiently large. For λ large enough with respect to V, i, n, it holds that*

$$\int_{O_{\alpha,i,n,\lambda}} \int_{\eta \leq |v| \leq V} f_{\lambda}^{\alpha}(x,v)dvdx \leq g_1(i,n,\lambda),$$

where the function g_1 does not depend on α,

$$g_1(i,n,\lambda) := \frac{ci^8 n^2 a_{i^3}(\lambda)}{a_{i^3-1}(\lambda)},$$

with c not depending on V, i, n, λ, α.

The lemma holds for $n \geq e^{e^{e^i}}$, and $\lambda = e^{e^{\cdot^{\cdot^{e^n}}}}$ with i^4 exponentials, which are the values used in the proof of Theorem 1.3. The proof of Lemma 1.4 is split into a number of geometrically different situations, which each after appropriate analysis is resolved using the entropy dissipation control.

2. ON LONG TIME BEHAVIOUR AND BOUNDARY LAYERS

This second part will focus on long time behaviour of initial boundary value problems for the Boltzmann equation, and on boundary layer asymptotics, two problem areas where the stationary results of the first part play a role.

Besides their intrinsic interest, stationary solutions to the Boltzmann equation come up as natural candidates for *time asymptotics* of corresponding evolutionary problems after the transients have died down. Rigorous convergence results in various topologies for the limit of infinite time are known, when the boundary conditions are periodic, or specular reflections as well as diffuse reflections with temperature and pressure constant around the boundary. An important case is *strong* convergence in L^1 to relevant global maxwellians. This was first discovered by NSA techniques ([1]), but once the strong L^1-convergence was properly understood, it did not take long to find also a standard proof ([26]).

I will next discuss a particularly interesting case of the strong L^1-convergence ([5], [4]). The key to convergence with time is here, as usual, global in time control of the entropy dissipation integral. Consider the time dependent equation

$$(\partial_t + v \cdot \nabla_x) f = Q(f,f), \quad t \in I\!R^3, \quad x \in \Omega, \quad v \in I\!R^3, \tag{2.1}$$

where Ω is bounded, strictly convex and smooth, and Q as before denotes the Boltzmann collision operator, together with an initial condition

$$f(0,x,v) = f_0(x,v), \quad x \in \Omega, \quad v \in I\!R^3. \tag{2.2}$$

Here f_0 has finite mass, energy, and entropy. Let the maxwellian diffuse reflection on the boundary be

$$f(t,x,v) = M(\xi) \int_{v' \cdot n(x) < 0} | v' \cdot n(x) | \, f(t,x,v') dv', \tag{2.3}$$
$$t \in I\!R^+, \quad x \in \partial\Omega, \quad v \cdot n(x) > 0.$$

where

$$M(v) = c_0 exp(-.5\theta \mid v \mid^2),$$

is a normalized maxwellian with c_0 a normalization constant and $\frac{1}{\theta} > 0$ is a constant temperature. The relevant equilibrium solution is $f_s = cM$ with

$$c = \frac{\int_{\Omega \times I\!R^3} f_0(x,v) dx dv}{\int_{\Omega \times I\!R^3} M(v) dx dv}.$$

Recall the following existence result ([5], [28]).

Theorem 2.1 *There exists a mild solution*

$$f \in C(\mathbb{R}^+, L^1(\Omega \times \mathbb{R}^3)), \quad f \geq 0,$$

to the initial boundary value problem (2.1-3).

Formally the equation gives

$$(\partial_t + v \cdot \nabla_x)(f log \frac{f}{M}) = Q(f,f) log \frac{f}{M} + Q(f,f).$$

Integrating this over $[0,t] \times \Omega \times \mathbb{R}^3$ and using the initial and boundary values, implies

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^3} (f log \frac{f}{M})(t,x,v) dx dv - \int_0^t \int_{\partial\Omega \times \mathbb{R}^3} v \cdot n(x)(f log \frac{f}{M}) d\tau dx dv \\ + \int_0^t \int_{\Omega \times \mathbb{R}^3} e(f) d\tau dx dv \\ \leq \int_{\Omega \times \mathbb{R}^3} f_0 log \frac{f_0}{M}(x,v) dx dv, \qquad (2.4) \end{aligned}$$

where

$$e(f) = \frac{1}{4} \int_{\mathbb{R}^3} \int_{B^+} B(\mid v - v_* \mid, u)(f' f'_* - f f_*) log \frac{f' f'_*}{f f_*} dv_* du.$$

The previous argument can easily be made rigorous. Since $e(f) \geq 0$ and the boundary integral is non-positive by Darrozes & Guiraud's inequality (for a discussion of this inequality, see e.g. [15]), it follows that

$$\int_{\Omega \times \mathbb{R}^3} f log \frac{f}{M}(t,x,v) dx dv < c,$$

and

$$0 \leq \int_0^{+\infty} \int_{\Omega \times \mathbb{R}^3} e(f)(t,x,v) dt dx dv < c.$$

The density f is a maxwellian, when the integrand in e is zero a.e.. And the desired convergence to a maxwellian is obtained by an analysis of how f is close to a maxwellian, when the integral of e for large times is close to zero. Once the limit is proved to be a maxwellian, the limit boundary condition selects (via Green's identity or directly) the precise limit maxwellian. This leads to the following convergence result.

Theorem 2.2 ([5]) *Let f be a solution of the initial boundary value problem (2.1-3) with nowhere vanishing collision kernel. When t tends to infinity, $f(t,.,.)$ converges strongly in $L^1(\Omega \times I\!R^3)$ to the global maxwellian cM.*

In the theorem M comes from the boundary condition (2.3), and c gives the conservation of mass ($c = \frac{\int f_0}{\int M}$).

Remark 2.1. Here the maxwellian is *uniquely determined* by the initial value and the boundary condition. No uniqueness has so far been found in the cases of periodic, specular, or direct reflection boundary conditions.
Specific for the kinetic case, and not generally correct in fluid dynamics situations, the natural restrictions on the domain are few, only that the boundary has finite (n-1)-dimensional Hausdorff measure - for reasonable traces to exist - and obeys a certain cone condition - to ensure that a molecule which falls on the surface has a strictly positive probability to be reflected to some body angle of size (uniformly over the surface) bounded from below. Theorem 2.2 can be generalized to that *natural* type of *boundary* [4].

For the *linear* Boltzmann equation, this convergence result has also been generalized to the case of a *space dependent* diffuse reflection boundary condition [32]. An entropy inequality of type (2.4) holds with analogous convergence consequences. Namely, let $f(t,x,v)$ denote a timedependent and F(x,v) a stationary solution to the linear Boltzmann equation with diffuse reflection boundary condition e.g. (2.3). Let φ denote a convex real-valued C^1-function defined on the positive real numbers. Then a relative entropy H_F^{φ} can be defined,

$$H_F^{\varphi}(f)(t) = \int_{\Omega}\int_{I\!R^3} \varphi\Big(\frac{f(t.,x,v)}{F(x,v)}\Big)F(x,v)dxdv.$$

The linear Boltzmann equation can be written as

$$\frac{d}{dt}[F(x+tv,v).\varphi(\frac{f(t,x+tv,v)}{F(x+tv,v)})] = [\varphi'(\frac{f}{F})(Qf) + (\varphi(\frac{f}{F}) - \frac{f}{F}\varphi'(\frac{f}{F}))(Qf)](t,x+tv,v).$$

Integrating this, using Green's identity and a change of variables, gives

$$H_F^{\varphi}(f)(t) - \int_0^t \int_{\partial\Omega\times I\!R^3} v\cdot n(x)F\varphi(\frac{f}{F})d\tau dxdv + \int_0^t P_F^{\varphi}(f)(\tau)d\tau \leq H_F^{\varphi}(f_0),$$

with

$$P_F^{\varphi}(f)(t) = \int_{\partial\Omega\times I\!R^3\times I\!R^3\times S^2} dxdvdv_*d\omega Y(x,v'_*)F(x,v').[\varphi(\frac{f(t,x,v')}{F(x,v')}) - \varphi(\frac{f(t,x,v)}{F(x,v)}) - (\frac{f(t,x,v')}{F(x,v')} - \frac{f(t,x,v)}{F(x,v)})\varphi'(\frac{f(t,x,v)}{F(x,v)})] \tag{2.5}$$

Here $P_F^\varphi(f)(t) \geq 0$, since by convexity

$$\varphi(b) - \varphi(a) \geq (b-a)\varphi'(a).$$

The boundary integral in the left hand side is negative by the Darrozes & Guiraud inequality.
However, we are not aware of any suitable generalization of (2.5) to the case of x-dependent reflexion on the boundary for the *nonlinear* Boltzmann equation. In fact, it is an important unresolved question, whether a time dependent solution really converges to a stationary one with time, when the temperature and pressure are varying over the boundary. Such convergence is strongly suggested from close to global equilibrium situations, from the linear case, and from a wealth of numerical evidence by the Kyoto school and others. On the other hand, the stationary solutions are sometimes not unique in the nonlinear case, and unstable periodic solutions to (2.1-3) may exist.

Stationary solutions are also of importance in *rarefied gas dynamics*, which deals with gas phenomena where Navier Stokes type equations are not valid in some significant region of the flow field. Here one tool is the Chapman Enskog theory relating the Navier Stokes equation to the Boltzmann equation of kinetic theory via small parameter asymptotics. A typical parameter is the Knudsen number Kn, the ratio of the molecular mean free path (in ordinary air 10^{-5} cm) to a typical length scale for the flow. This length scale could be based on the gradients occuring in the flows. Often the regions are very thin, where deviation from the Navier Stokes behaviour is expected, and the non Navier Stokes terms become important. The broad picture that emerges from such formal arguments and from related experiments, is one of normal regions where the gas flow follows the macroscopic fluid equations, plus thin shock layers, boundary layers, and initial layers, where matching conditions are sought between fluid regions on each side of the shock, or between outside initial or boundary control and interior fluid behaviour.

From that picture, I will focus on *boundary layers*. In the form of half-space problems they come up on the microscopic scale when the macroscopic hydrodynamic limits cannot match all boundary conditions on the kinetic side which are naturally present. Such boundary layer problems have been extensively studied by functional analytic tools and energy methods in the case of *linear asymptotics*; from Grad ([22]) in the 1960ies, via Guiraud ([25]) in the 1970ies, to a.o. Bardos-Caflish-Nicolaenko ([11]), Golse-Poupaud ([21]), and Cercignani ([14]) in the 1980ies.
When it comes to *fully nonlinear asymptotics*, the work by the Kyoto group is prominent on asymptotic analysis and numerical studies of a wide range of half space problems. As a complement to their work, we shall here discuss some rigorous mathematical results for the fully nonlinear Kn-asymptotics in a particular

case of the half-space Milne problem ([9]).

An integrable cylindrically symmetric maxwellian

$$M(v) := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} e^{-\frac{(\xi-u)^2+\eta^2+\zeta^2}{2T}}, \quad v=(\xi,\eta,\zeta)\in I\!R^3,$$

(with $\rho \geq 0$ and $T>0$), is uniquely determined by its three moments

$$\rho = \int M(v)dv, \quad \rho u = \int \xi M(v)dv, \quad \rho(u^2+T) = \int v^2 M(v)dv.$$

However, it is well known that for nonzero bulk velocity, there can be zero, one or two maxwellians with given fluxes

$$\int \xi M(v)dv, \quad \int \xi^2 M(v)dv, \quad \int \xi v^2 M(v)dv,$$

as stated in the following lemma.

Lemma 2.3 *Let* $(c_i)_{1\leq i\leq 3}$*, with* $c_1 \neq 0$*, be given.*
(i) If $c_2 \leq 0$ *or* $c_1c_3 \leq 0$ *or* $c_1c_3 > \frac{25}{16}c_2^2$*, there is no maxwellian with fluxes* $(c_i)_{1\leq i\leq 3}$*.*
(ii) If $c_1c_3 = \frac{25}{16}c_2^2$*, there is a unique maxwellian with fluxes* $(c_i)_{1\leq i\leq 3}$*.*
(iii) If $0 < c_2^2 < c_1c_3 < \frac{25}{16}c_2^2$*, there are two maxwellians with fluxes* $(c_i)_{1\leq i\leq 3}$*.*
(iv) If $c_2^2 \geq c_1c_3 > 0$*, there is a unique maxwellian with fluxes* $(c_i)_{1\leq i\leq 3}$*.*

For convenience, we recall a short proof.
Proof of Lemma 2.3 The unknown ρ, u, T defining an integrable maxwellian M, are solutions to the system

$$\rho \geq 0, \; T>0, \; \rho u = c_1, \; \rho(u^2+T) = c_2, \; \rho u(u^2+5T) = c_3.$$

Since $c_1 \neq 0$, there are no positive solutions ρ and T when $c_2 \leq 0$ or $c_1c_3 \leq 0$. Since $c_1 \neq 0$,

$$\rho = \frac{c_1}{u}, \quad T = \frac{c_2}{c_1}u - u^2,$$

where u is a solution to

$$4c_1u^2 - 5c_2u + c_3 = 0, \tag{2.6}$$

$$c_1u > 0, \quad 0 < |u| < \frac{c_2}{|c_1|}. \tag{2.7}$$

For $c_1c_3 > \frac{25}{16}c_2^2$, there is no real solution u to equation (2.6). For $c_1c_3 = \frac{25}{16}c_2^2$, the solution $\frac{5c_2}{8c_1}$ to equation (2.6) satisfies (2.7). For $0 < c_2^2 < c_1c_3 < \frac{25}{16}c_2^2$, both solutions to equation (2.6),

$$u_\epsilon = \frac{5c_2 + \epsilon\sqrt{25c_2^2 - 16c_1c_3}}{8c_1}, \quad \epsilon \in \{-,+\}, \tag{2.8}$$

satisfy (2.7). For $c_2^2 \geq c_1 c_3 > 0$, only $u = \frac{5c_2 - \sqrt{25c_2^2 - 16c_1c_3}}{8c_1}$ satisfies (2.7).

Remark 2.2. We note for $c_1 > 0$, that $0 \leq u_- \leq u_+$, $T_+ \leq \frac{3\rho^2}{5} \leq T_-$. The Mach number is defined by $\mathbf{M}_\epsilon^2 = \frac{3u_\epsilon^2}{5T_\epsilon}$. Then

$$u_\epsilon^2 = \frac{c_3 \mathbf{M}_\epsilon^2}{c_1(3 + \mathbf{M}_\epsilon^2)}, \; T_\epsilon = \frac{3c_3}{5c_1(3 + \mathbf{M}_\epsilon^2)}.$$

With

$$\sin^2\theta = \frac{16c_1c_3}{25c_2^2}, \;\; 0 \leq \theta \leq \frac{\pi}{2},$$

we get

$$\mathbf{M}_-^2(\theta) = \frac{3}{4\mathrm{ctg}^2\frac{\theta}{2} - 1}, \;\; \mathbf{M}_+^2(\theta) = \frac{3}{4\mathrm{tg}^2\frac{\theta}{2} - 1},$$

where $\mathbf{M}_-(\theta)$ is subsonic and $\mathbf{M}_+(\theta)$ is supersonic.

Define for $0 < \mu < \lambda$

$$V_\lambda := \{v \in I\!R^3; \mid v \mid \leq \lambda\}, \quad V_\lambda' = \{v \in V_\lambda; \mu \leq \mid \xi \mid\}.$$

By a perturbative argument there are $\lambda_0 < \infty$ and $0 < \mu_0$, so that for $\lambda \geq \lambda_0, 0 < \mu < \mu_0$, (iii-iv) of Lemma 2.3 hold for the maxwellian fluxes, also when the integrals are truncated with respect to V_λ'. (More precisely the following holds.

Lemma 2.4 *Let $(c_i)_{1 \leq i \leq 3}$, with $0 < c_1c_3 < \frac{25}{16}c_2^2$ and $c_1c_3 \neq c_2^2$ be given. There are $\lambda_0 < \infty$ and $\mu_0 > 0$, such that for $\lambda \geq \lambda_0$, $0 < \mu < \mu_0$, (iii-iv) of Lemma 2.3 hold for the truncated maxwellian fluxes*

$$(c_1, c_2, c_3) = (\int_{V_\lambda'} \xi M(v) dv, \int_{V_\lambda'} \xi^2 M(v) dv, \int_{V_\lambda'} \xi v^2 M(v) dv). \qquad (2.9)$$

In the case $c_1c_3 = c_2^2$, let (ρ_-, u_-, T_-) be the values of (ρ, u, T) for $\lambda = \infty, \mu = 0$ when $\epsilon = -$ in (2.8), and correspondingly (ρ_+, u_+, T_+) with $T_+ = 0$ for $\epsilon = +$. Given any neighbourhoods $\mathbf{O}_-$ and $\mathbf{O}_+$ of (ρ_-, u_-, T_-) and (ρ_+, u_+, T_+) respectively, then $(\rho(\lambda,\mu), u(\lambda,\mu), T(\lambda,\mu))$ is either in $\mathbf{O}_-$ or in $\mathbf{O}_+$ for λ, μ^{-1} large enough. Moreover, $(\rho(\lambda,\mu), u(\lambda,\mu), T(\lambda,\mu))$ is uniquely determined in the $\mathbf{O}_-$-case.)

By Theorem 1.1 there are solutions to the stationary Boltzmann equation in a slab with given indata on the boundary. For the sequel we shall also need to relate the distance of density functions from the set of maxwellians, to the magnitude of the collision integrand.

Lemma 2.5 *Consider a set of non-negative functions f that is weakly compact in $L^1(\mathbb{R}^3)$. Given $\epsilon, \eta > 0$, there is $\delta > 0$, such that if*

$$\mid ff_* - f'f'_* \mid < \delta$$

in $V'_\lambda \times V'_\lambda \times S^2$ outside of some subset of measure smaller than δ, then for some maxwellian M_f (depending on f),

$$\int_{V'_{\lambda-\eta}} \mid f - M_f \mid dv < \epsilon.$$

Lemma 2.5 was proved in the $\mathbb{R}^3$ case in [2] and [29]. Those proofs also imply the present local version.

Denote by (ξ, η, ζ) the three components of $v \in \mathbb{R}^3$ and set $\sigma = \sqrt{\eta^2 + \zeta^2}$. The hydrodynamic limit is considered for subsequences of f^ϵ, solutions to

$$\xi \frac{\partial f^\epsilon}{\partial x} = \frac{1}{\epsilon} Q(f^\epsilon, f^\epsilon), \quad x \in]-1, 1[, \quad v \in \mathbb{R}^3, \tag{2.10}$$

$$f^\epsilon(-1, v) = M_l(v), \ \xi > 0, \quad f^\epsilon(1, v) = M_r(v), \ \xi < 0, \tag{2.11}$$

when the mean free path ϵ tends to zero. Here

$$M_l(v) := \frac{\rho_l}{(2\pi T_l)^{\frac{3}{2}}} e^{-\frac{v^2}{2T_l}}, \ M_r(v) := \frac{\rho_r}{(2\pi T_r)^{\frac{3}{2}}} e^{-\frac{v^2}{2T_r}},$$

and

$$Q(f,f)(x,v) := \int_{\mathbb{R}^3 \times S^2} b(\theta) \chi(v, v_*, \omega) \mid v - v_* \mid^\beta (f'f'_* - ff_*) dv_* d\omega.$$

Moreover,

$$\chi(v, v_*, \omega) = 0 \quad \text{if} \quad \mid v \mid \geq \lambda, \text{or} \mid v_* \mid \geq \lambda, \text{or} \mid v' \mid \geq \lambda, \text{or} \mid v'_* \mid \geq \lambda,$$

$$\text{or} \quad \mid \xi \mid \leq \mu, \text{or} \mid \xi_* \mid \leq \mu, \text{or} \mid \xi' \mid \leq \mu, \text{or} \mid \xi'_* \mid \leq \mu,$$

$$\chi(v, v_*, \omega) = 1 \quad \text{else}, \quad \beta \in [0, 2[, \quad b \in L^1_+(0, \pi), \quad b(\theta) \geq c > 0, \quad a.e..$$

For λ finite, the factor $|v - v_*|^\beta$ only introduces minor changes in the arguments, so we shall only discuss the case $\beta = 0$. Under the boundary conditions (2.11), there are cylindrically symmetric (with respect to the variables (ξ, σ))

functions f^ϵ solutions to (2.10-11). Only such solutions are considered in the following. In particular,

$$\int \xi\eta f^\epsilon(x,v)dv = \int \xi\zeta f^\epsilon(x,v)dv = 0$$

under the cylindrical symmetry. By Green's formula the fluxes

$$(c_i^\epsilon)_{1\le i\le 3} = (\int_{|\xi|\ge\mu,|v|\le\lambda} \xi(1,\xi,v^2)f^\epsilon(x,v)dv)$$

are constant in x with ϵ-independent bounds determined by M_l and M_r. Denote by $(c_i^{\epsilon_j})_{1\le i\le 3}$ a converging subsequence with limit $(c_i(\lambda,\mu))_{1\le i\le 3}$, when $\epsilon_j \to 0$. Either $c_1(\lambda,\mu) = 0$ or $c_1(\lambda,\mu) \neq 0$. We only discuss such sequences of solutions with $c_1(\lambda,\mu) \neq 0$, and then - possibly after a change of x-direction - take $c_1(\lambda,\mu) > 0$, also requiring $c_1^{\epsilon_j} > 0$ for all j. Such systems can be considered to model an evaporation-condensation situation with evaporation at $x = -1$ and condensation at $x = 1$. We shall further assume (for a subfamily in λ,μ) the existence of $\lim_{\lambda,\mu^{-1}\to\infty} c_i(\lambda,\mu) = c_i$, for $i = 1,2,3$, with $c_1 > 0$. The quantities λ_0 and μ_0 as defined in Lemma 2.4, may be taken locally constant with respect to (c_1,c_2,c_3) satisfying the conditions of the lemma, and with λ_0, μ_0^{-1} so large that negative T's are excluded. From here on we only consider such $\lambda \ge \lambda_0, 0 < \mu \le \mu_0$, and $0 < c_1c_3 < \frac{25}{16}c_2^2$.

For the fully non-linear Boltzmann equation in the present model setup, the following results have been proved about the half-space problem.

Theorem 2.6 ([9]) *Denote by*

$$g^\epsilon(\frac{x+1}{\epsilon}, v) := f^\epsilon(x,v), \quad a.a.\ x \in]-1,1[, \quad v \in I\!R^3.$$

Then there is a sequence (ϵ_j) *with* $\lim_{j\to\infty}\epsilon_j = 0$*, such that* (g^{ϵ_j}) *converges weakly in* $L^1([-1,1]\times I\!R^3)$ *to a function* g*, which is a weak solution to the half-space problem,*

$$\xi\frac{\partial g}{\partial x} = Q(g,g), \quad x \ge 0, \quad v \in I\!R^3,$$
$$g(0,v) = M_l(v), \quad \xi > 0,$$

in the sense that for any $x_0 > 0$*, for any test function* φ *with support in* $[0,x_0[\times V'_\lambda$

$$\int_{\xi>0} \xi M_l(v)\varphi(0,v)dv + \int_0^{x_0}\int_{I\!R^3}(\xi g\frac{\partial\varphi}{\partial x} + Q(g,g)\varphi)dxdv = 0.$$

Remark 2.3. Test functions are here L^∞-functions, differentiable in the x-variable for a.e. $v \in V'_\lambda$ with $\varphi(0,v) = 0$ for $\xi < 0$.

An analogous result holds for $h^\epsilon(\frac{1-x}{\epsilon}, v) := f_\epsilon(x,v)$ and M_r.

Theorem 2.7 ([9]) *Denote by* S_δ *the union of* $\{v \in V'_\lambda; \mu \leq |\xi| \leq \mu+\delta, \sigma \leq 4\mu+\delta\}$ *and* $\{v \in V'_\lambda; \mu \leq |\xi| \leq 3\mu+\delta, \sigma \leq \delta\}$*. If* $c_1c_3 = c_2^2$*, then include in* S_δ *also a* δ*-neighbourhood in* V'_λ *of* $(\frac{c_2}{c_1}, 0, 0)$*. Either for all* $\delta > 0$

$$\lim_{x\to\infty} \int_{V'_\lambda \setminus S_\delta} g(x,v)dv = 0,$$

or

$$\lim_{x\to\infty} \int |\, g(x,v) - M_-(v) \,|\, dv = 0,$$

or

$$\lim_{x\to\infty} \int |\, g(x,v) - M_+(v) \,|\, dv = 0$$

in the case $c_1c_3 \neq c_2^2$*, respectively*

$$\liminf_{x\to\infty} \int |g(x,v) - M_{\lambda,\mu}(v)|dv = 0$$

in the case $c_1c_3 = c_2^2$*. Here* M_-, M_+ *are those defined in Lemma 2.4, and in the notations of that lemma the infimum is taken over* $M_{\lambda,\mu}$ *corresponding to* $\mathbf{O}_+$ *and satisfying* (2.9).

Remark 2.4. The solution g of the half-space problem in Theorem 2.6 satisfies the Milne problem in the sense of Theorem 2.7. The M_+-alternative is only possible in the case (iii) of Lemma 2.3.

Theorem 2.8 ([9]) *Suppose* $c_1c_3 \neq c_2^2$*. There is a sequence* (ϵ_j) *such that* $\lim_{j\to\infty} \epsilon_j = 0$*, and* (f^{ϵ_j}) *converges in weak* measure sense to a non-negative element* f *of* $L^1((-1,1); \mathcal{M}(V'_\lambda))$ *that satisfies*

$$\int_{V'_\lambda} \xi(1, \xi, v^2) f(x,v)dv = (c_1(\lambda,\mu), c_2(\lambda,\mu), c_3(\lambda,\mu)).$$

Moreover, there are measurable non-negative functions $\theta_-(x), \theta_+(x)$ *with* $0 \leq \theta_-(x) + \theta_+(x) \leq 1, -1 \leq x \leq 1$*, such that for test functions* ϕ *with support in* $V'_\lambda \setminus S_\delta$ *for some* $\delta > 0$*,*

$$\int \phi f(x,v)dv = \int (\theta_- M_- + \theta_+ M_+)\phi dv.$$

Here we have written $f(x,v)dv$ for the measure in the $v-$variable defined by $f(x,.)$.

Remark 2.5. There is a (more involved) version in the case $c_1 c_3 = c_2^2$ for this very weak hydrodynamic type of behaviour in the interior of the slab.

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