On bounded and unbounded idempotents whose sum is a multiple of the identity

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Abstract

We study bounded and unbounded representations of the $*$-algebra $\mathcal{Q}_{n,\lambda}(*)$ generated by n idempotents whose sum equals λe ($\lambda \in \mathbb{C}$, e is the identity).

1 Introduction

Let H be a complex (finite or infinite dimensional) separable Hilbert space and $B(H)$ the algebra of bounded operators on H. An operator $Q \in B(H)$ is said to be idempotent if $Q^2 = Q$. A selfadjoint idempotent P is called orthoprojection. The problem to study families of idempotent is not just an interesting algebraic problem but it also arises in numerous applications to analysis (see, for example, [BGKKRSS] and references therein), theory of operators (see [Wu1] and references therein), mathematical physics (see [Ba, EK] and references therein) etc.

The structure of pairs of idempotents on finite-dimensional space or, equivalently, finite-dimensional representations of the algebra $\mathcal{Q}_2 = \mathbb{C}\langle q_1, q_2, e \mid q_k^2 = q_k, k = 1, 2\rangle$ with the unit e, is non-trivial but well understood (see [Na, GP]). The structure of three and more idempotents or, equivalently, representations of the algebras $\mathcal{Q}_n = \mathbb{C}\langle q_1, \dots, q_n, e \mid q_k^2 = q_k, k = 1, \dots, n\rangle$ for $n \geq 3$ is extremely difficult (the corresponding algebra is wild (see [DF])). Families of idempotents with additional condition that their sum is a multiple of the identity operator or, equivalently, representations of the algebras $\mathcal{Q}_{n,\lambda} = \mathbb{C}\langle q_1, \dots, q_n, e \mid \sum_{k=1}^n q_k = \lambda e, q_k^2 = q_k, k = 1, \dots, n\rangle$, $\lambda \in \mathbb{C}$, are studied less (for some results about finite-dimensional representations and representations by bounded operators on Hilbert space of $\mathcal{Q}_{n,\lambda}$ see the first subsection at each section).

In this paper we consider the problem of describing up to unitary equivalence families of idempotents or, which is equivalent, $*$-representations of $*$-algebras generated by idempotents.

Natural $*$-analogues of the algebras $\mathcal{Q}_n$ and $\mathcal{Q}_{n,\lambda}$, $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are:

(a) the $*$-algebras $\mathcal{Q}_n(*) = \mathbb{C}\langle q_1,\dots,q_n,q_1^*,\dots,q_n^*,e \mid q_k^2 = q_k, k = 1,\dots,n\rangle$ generated by n free idempotents and their adjoints. Denoting $\mathcal{Q}_n^*$ the algebra (not a $*$-algebra) $\mathbb{C}\langle q_1^*,\dots,q_n^* \mid (q_k^*)^2 = q_k^*, k = 1,\dots,n\rangle$, we have $\mathcal{Q}_n(*) = \mathcal{Q}_n \star \mathcal{Q}_n^*$, where $\star$ is the sign of free product of algebras;

(b) the $*$-algebras $\mathcal{Q}_{n,\lambda}(*) = \mathbb{C}\langle q_1,\dots,q_n,q_1^*,\dots,q_n^*,e \mid \sum_{k=1}^n q_k = \lambda e, q_k^2 = q_k, k = 1,\dots,n\rangle$, $\lambda \in \mathbb{C}$, which are generated by n idempotents whose sum is a multiple of the unit, and elements which are adjoint to them. Setting $\mathcal{Q}_{n,\lambda}^* = \mathbb{C}\langle q_1^*,\dots,q_n^*,e \mid \sum_{k=1}^n q_k^* = \lambda e, (q_k^*)^2 = q_k^*, k = 1,\dots,n\rangle$, we have $\mathcal{Q}_{n,\lambda}(*) = \mathcal{Q}_{n,\lambda} \star \mathcal{Q}_{n,\lambda}^*$;

and their factor-$*$-algebras, such as

(c) the $*$-algebras $\mathcal{P}_n = \mathbb{C}\langle p_1,\dots,p_n,e \mid p_k^2 = p_k = p_k^*, k = 1,\dots,n\rangle$ generated by n orthoprojections;

(d) the $*$-algebras $\mathcal{P}_{n,\alpha} = \mathbb{C}\langle p_1,\dots,p_n,e \mid \sum_{k=1}^n p_k = \alpha e, p_k^2 = p_k = p_k^*, k = 1,\dots,n\rangle$, $\alpha \in \mathbb{R}$, generated by n orthoprojections whose sum is a multiple of the unit.

The structure of pairs of orthogonal projections P_1, P_2, or representations of $*$-algebra $\mathcal{P}_2$ is well known (see, for example,[RaS]): irreducible representations of $\mathcal{P}_2$ are one or two-dimensional and any representation is a direct sum (or direct integral) of irreducible ones.

The structure of three and more orthoprojections or representations of the $*$-algebras $\mathcal{P}_n$, $n \geq 3$, is very difficult (the $*$-algebras are $*$-wild). For the definition of $*$-wild algebras we refer the reader to [KS2, OS2].

A number of articles (see [RS, KRS]) are devoted to the structure of families of orthogonal projections whose sum is a multiple of the identity or, equivalently, representations of the $*$-algebras $\mathcal{P}_{n,\alpha}$, $\alpha \in \mathbb{R}$. In particular, there were described the sets of $\alpha \in \mathbb{R}$ such that there exist orthogonal projections $P_1,\dots P_n$ on a Hilbert space H so that $\sum_{k=1}^n P_k = \alpha I$, I is the identity operator. Note that orthoprojections are necessarily bounded operators and $*$-algebras $\mathcal{P}_n$ and $\mathcal{P}_{n,\alpha}$ do not have representations by unbounded operators.

Families of idempotents $Q_1,\dots,Q_n,Q_1^*,\dots,Q_n^*$ or, equivalently, representations of $*$-algebra $\mathcal{Q}_n(*)$ have a simple structure in the case $n = 1$. The situation is similar to the situation for pairs of orthoprojections (see section 2.2). If $n \geq 2$ the problem of unitary classification of all families of idempotents becomes difficult (the $*$-algebra $\mathcal{Q}_n(*)$ is $*$-wild) ([KS2, OS2]).

In the present paper we study, up to unitary equivalence, the structure of idempotents whose sum is a multiple of identity, or, equivalently, representations of the $*$-algebras $\mathcal{Q}_{n,\lambda}(*)$, $\lambda \in \mathbb{C}$. In contrast to orthogonal projections there exist unbounded operators Q satisfying $Q^2 = Q$ (unbounded idempotents) (see [P]). We study representations of $\mathcal{Q}_{n,\lambda}(*)$ by bounded operators (i.e., $*$-homomorphisms of $\mathcal{Q}_{n,\lambda}(*)$ to $B(H)$) and the sets $\Lambda_{n,bd} = \{\lambda \in \mathbb{C} \mid \exists Q_1,\dots,Q_n \in B(H), \sum_{k=1}^n Q_k = \lambda I, Q_k^2 = Q_k, k = 1,\dots,n\}$ together with representations by unbounded operators and the corresponding sets $\Lambda_{n,unbd}$ (for exact definitions see subsection 3 at each section).

For $n = 3$ (sect. 3.2) representations of $\mathcal{Q}_{n,\lambda}(*)$ by bounded operators exist only for $\lambda \in \{0, 1, 3/2, 2, 3\}$, the $*$-algebras $\mathcal{Q}_{3,0}(*) \simeq \mathcal{Q}_{3,3}(*)$ are one-dimensional, the $*$-algebras $\mathcal{Q}_{3,1}(*) \simeq \mathcal{Q}_{3,2}(*) \simeq \mathbb{C}^3 \star \mathbb{C}^3$ and $\mathcal{Q}_{3,3/2}(*) \simeq M_2(\mathbb{C}) \star M_2(\mathbb{C})$ are $*$-wild. For $*$-wild algebras we also study additional conditions under which the problem of unitary classification of their representations become transparent. Properties of the representations of $\mathcal{Q}_{n,\lambda}(*)$ by unbounded operators are essentially the same as representations by bounded operators in the case $n = 3$ (sect. 3.3). In particular, $\Lambda_{3,bd} = \Lambda_{3,unbd}$.

For $n = 4$ the situation becomes different. For example, the sum of four bounded idempotents equals zero only if the idempotents are zero operators, which is not the case for unbounded ones (see [BES]). If the sum of bounded idempotents is equal to 1 then the idempotents are mutually orthogonal, but there exist unbounded non-orthogonal idempotents with the sum equal to 1 (see [ERSS]). Moreover, $\Lambda_{4,bd} = \{0, 1, 1 + \frac{k}{k+2} (k \in \mathbb{N}), 2, 3 - \frac{k}{k+2}, 3, 4\}$, while $\Lambda_{4,unbd} = \mathbb{C}$ (Proposition 7). In Section 4 we study the problem of describing of representations of $\mathcal{Q}_{4,\lambda}(*)$ by bounded and unbounded operators and representations of $\mathcal{Q}_{4,\lambda}(*)$ under some additional restrictions. The case $\mathcal{Q}_{4,0}(*)$ is treated in details.

In Section 5, following [RS], we show that $\Lambda_{5,bd} = \mathbb{C}$ and that the problem of unitary classification of already bounded representations of the $*$-algebra $\mathcal{Q}_{n,\lambda}(*)$ for $n \geq 5$ is difficult: the $*$-algebra is not of type I for any $\lambda \in \mathbb{C}$ and it is $*$-wild for some $\lambda \in \mathbb{R}$.

We would like to emphasise one more time that speaking about representation of algebras we mean homomorphisms into algebras of linear operators on a vector space and $*$-homomorphisms into a $*$-algebra of linear bounded or unbounded operators defined on a Hilbert space if we speak about representations or $*$-representations of $*$-algebras. We will also restrict ourself to indecomposable finite-dimensional representations up to similarity when talk about description of representations of algebras. Description of $*$-representations is reducing to the description of irreducible (=indecomposable) representations up to unitary equivalence.

2 Representations of algebras $\mathcal{Q}_{2,\lambda}$ and $*$-algebras $\mathcal{Q}_{2,\lambda}(*)$

2.1 Algebras $\mathcal{Q}_{2,\lambda}$ and their representations

Algebra $Q_{2,\lambda}$ is non-zero only for $\lambda \in \Lambda_2 = \{0, 1, 2\}$. We have $\mathcal{Q}_{2,0} = \mathcal{Q}_{2,2} = \mathbb{C}e$. The algebra $\mathcal{Q}_{2,1}$ is easily seen to be equal to $\mathcal{Q}_1 = \mathbb{C}\langle q, e | q^2 = q\rangle$, its finite-dimensional indecomposable representations are one-dimensional: $\pi(q) = 0$ or $\pi(q) = 1$.

2.2 $*$-Algebras $\mathcal{Q}_{2,\lambda}(*)$ and their representations by bounded operators

$*$-Representations of $\mathcal{Q}_{2,0}(*) = \mathcal{Q}_{2,2}(*) = \mathbb{C}e$ are trivial. The problem of unitary classification of $\mathcal{Q}_{2,1}(*) = \mathcal{Q}_1(*)$ reduces to the problem of describing a single idempotent up to unitary equivalence. Any irreducible representation of $\mathcal{Q}_1(*)$ is one- or two-dimensional and given by

$$(a) \quad \pi(q_1) = 1, \qquad \text{or} \qquad \pi(q_1) = 0; \tag{1}$$

$$(b) \quad \pi(q_1) = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}, \; y \in (0, \infty). \tag{2}$$

(see, for example [D, OS2]). The structure of arbitrary representation of $\mathcal{Q}_1(*)$ by bounded operators is given by the following statement (see [KS2]): for any representation π on a Hilbert space H there exist a unique decomposition $H = H_0 \oplus H_1 \oplus \mathbb{C}^2 \otimes H_2$ and a projection-valued measure $dE(\cdot)$ on H_2 whose support is a bounded subset of $(0, \infty)$ and such that

$$\pi(q_1) = 0 \cdot I_{H_0} \oplus I_{H_1} \oplus \int_0^\infty \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix} \otimes dE(y). \tag{3}$$

2.3 Representations of $\mathcal{Q}_{2,\lambda}(*)$ by unbounded operators

Here we describe $*$-representations of $*$-algebra $\mathcal{Q}_{2,1}(*) = Q_1(*)$ by unbounded operator recalling necessary definitions of the concepts involved.

Let Φ be a dense linear subset of a Hilbert space, H. Let $\mathcal{L}^+(\Phi) = \{X \in \mathcal{L}(\Phi) \mid \Phi \subset \mathcal{D}(X^*), X^*\Phi \subset \Phi\}$, here $\mathcal{D}(X)$ is the domain of the operator X. Then $\mathcal{L}^+(\Phi)$ is an algebra with involution $X^+ = X^*|_\Phi$. By *$*$-representation* of a $*$-algebra $\mathfrak{A}$ by unbounded operators we call a unital $*$-homomorphism $\pi : \mathfrak{A} \to \mathcal{L}^+(\Phi)$ (see, for example, [In, S]). We write also $D(\pi)$ for the domain Φ and call it the domain of the representation π.

Define $\Lambda_{n,unbd}$ to be the set of all $\lambda \in \mathbb{C}$ such that there exists a $*$-representation of $\mathcal{Q}_{n,\lambda}(*)$ by unbounded operators. Since $*$-algebras $\mathcal{Q}_{2,\lambda}(*)$ are non-zero only for $\lambda \in \Lambda_2$, we have $\Lambda_{2,bd} = \Lambda_{2,unbd} = \Lambda_2$.

The class of representations defined above is very large and practically indescribable (see [ST, S, T]). So if one wishes to get structure theorems giving a description of $*$-representations up to unitary equivalence then one should impose some additional conditions on the domain Φ. For example, one can require that Φ consists of bounded (entire, analytical vectors) for some operators of the representation. Recall that a vector $f \in \cap_{k\in\mathbb{N}}\mathcal{D}(X^k) \subset H$ is called *bounded* (*entire*, *analytical*) vector for operator X on H if there is a constant c_f such that $||X^n f|| \le c_f^n ||f||$ for any $n \in \mathbb{N}$ (the function $\sum_{k=0}^{\infty}(||X^k f||/k!)z^k$ is entire or analytical at 0). The set of bounded (entire, analytical) vectors for operator X will be denoted by $H_b(X)$ ($H_e(X)$ and $H_a(X)$ respectively). Imposing this type of conditions we will call these representations *integrable* or *well-behaved* or *good* following the terminology in the theory of representations of Lie algebras ([N, S]). Definitions of equivalent representations, irreducible representations which are necessary for formulating structure theorems, will be given for every particular class of representation considered in the paper.

Let π be a representation of $\mathcal{Q}_1(*)$ defined on a domain $D(\pi)$ of a Hilbert space $H(\pi)$. We say that π is a *well-behaved* representation if $\Delta = \overline{\pi(qq^* + qq^* - (q + q^*))}$ (the closure of the operator $\pi(qq^* + qq^* - (q + q^*))$) is selfadjoint and $D(\pi) = H_b(\Delta)$. Note that $qq^* + qq^* - (q + q^*)$ is a central element of $\mathcal{Q}_1(*)$. Setting $a = q + q^* - e$ and $b = i(q^* - q)$, which are clearly selfadjoint, we obtain $qq^* + qq^* - (q + q^*) = a^2 + b^2 - e$.

Let $H_m(\pi)$ denote the set of all vectors $f \in H_b(\Delta)$ such that $||\Delta^n f|| \le m^n ||f||$. Since $qq^* + qq^* - (q+q^*)$ is central and Δ is selfadjoint, one can show that the subspaces $H_m(\pi)$ is reducing π, i.e., $\pi = \pi_1 \oplus \pi_2$ and $H_m(\pi) = D(\pi_1)$. Moreover, each subrepresentation $\pi|_{H_m(\pi)}$ is bounded. In order to see this, it is enough to show that $\pi(a)$ and $\pi(b)$ are bounded on $H_m(\pi)$. Given $f \in H_m(\pi)$, we have $||\pi(a^2 + b^2)f|| \le (m + 1)||f||$ and $||\pi(a)f||^2 = (\pi(a)^2 f, f) \le (\pi(a^2 + b^2)f, f) \le ||\pi(a^2 + b^2)f|| \cdot ||f|| \le (m+1)||f||^2$, the same holds for $\pi(b)$.

A representation π is called *irreducible* if the only linear subspace reducing π are $\{0\}$ and $D(\pi)$. Since for a well-behaved representation π at least one of $H_m(\pi)$ is non-zero we obtain that any well-behaved irreducible representation is bounded.

We say that two well-behaved representations π_1 and π_2 are *unitarily equivalent* if there exist a unitary operator $U : H(\pi_1) \to H(\pi_2)$ such that $UH_m(\pi_1) = H_m(\pi_2)$ and $U\pi_1(a)f = \pi_2(a)Uf$ for any $f \in H_m(\pi_1)$ and each $m \in \mathbb{N}$.

Now we state a structure theorem. Its proof essentially follows the proof of an analogous statement for bounded representations of $\mathcal{Q}_1(*)$.

Proposition 1. *Any irreducible well-behaved representation of $\mathcal{Q}_1(*)$ is bounded and, up to unitary equivalence, is given by (1) – (2). Any representation π of $\mathcal{Q}_1(*)$ is a direct sum (direct integral) of irreducible ones and given by (3), where the equality holds on vectors $f \in D(\pi)$.*

Remark 1. If a $*$-algebra is defined in terms of generators and commutation relations (the algebraic equalities imposed on the generators), instead of $*$-representations of the $*$-algebra one speaks often about representations of this set of relations, i.e. families of operators satisfying the relations on some invariant dense domain.

Let Q, Q^* be closed operators such that that there exists a dense domain Φ satisfying the following conditions: (1) Φ is invariant with respect to Q, Q^*; (2) Φ is a core for Q, Q^* (i.e. the closure of operators $Q|_\Phi$ and $Q^*|_\Phi$ are Q and Q^* respectively); (3) $\Phi \subset H_a(QQ^* + Q^*Q - (Q+Q^*))$; (4) $Q^2 f = Qf$ for any $f \in \Phi$. Then Q, Q^* generate a $*$-representation π of $\mathcal{Q}_1(*)$ with the domain Φ by setting $\pi(q)f = Qf$, $\pi(q^*)f = Q^*f$, $f \in \Phi$ and then extending it to the whole algebra. One can show that the representation π^* ($\pi^*(a) = \pi(a^*)^*|_{D(\pi^*)}$, and $D(\pi^*) = \cap_{a\in Q_{2,1}(*)} D(\pi(a)^*)$) is a unique selfadjoint $*$-representation ρ of $\mathcal{Q}_1(*)$ such that $\overline{\rho(q)} = Q$, $\overline{\rho(q^*)} = Q^*$. Recall that ρ is selfadjoint if $\rho = \rho^*$. Note that the domain Φ satisfying $(1) - (4)$ is not uniquely defined. But for every choice of Φ we have the same selfadjoint representation. In particular, the domain Φ can be chosen to be equal $H_b(QQ^* + Q^*Q - (Q+Q^*))$ which implies that the unique selfadjoint representation ρ is π^*, where π is a well-behaved representation. In this case the representation π^* is irreducible iff π is irreducible, two well-behaved representations π_1 and π_2 are unitarily equivalent iff π_1^* and π_2^* are unitarily equivalent in the sense that there exist a unitary operator U of $H(\pi_1^*)$ onto $H(\pi_2^*)$ such that $UD(\pi_1^*) = D(\pi_2^*)$ and $U^{-1}\pi_2^*(a)Uf = \pi_1^*(a)f$, $f \in D(\pi_1^*)$. So the problem to classify all well-behaved representation is equivalent to the problem to classify all selfadjoint representations defined above or all pairs of operators (Q, Q^*). For concepts of the theory of representations by unbounded operators we refer the reader to [S].

Remark 2. There is also a correspondence between well-behaved representations and representations arising from representations of some C^*-algebra. Let $\mathcal{A} = \{f \in C([0,\infty), M_2(\mathbb{C})) \mid f(0)$ is diagonal, $\lim_{t\to\infty} f(t) = 0\}$. $\mathcal{A}$ is a C^*-algebra. Let $q^*(t) = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix}$. One can show that $q^* \in C([0,\infty), M_2(\mathbb{C}))$ is affiliated with $\mathcal{A}$. Moreover, $\mathcal{A}$ is generated by q^* and there exists a dense domain D of $\mathcal{A}$ (for example, $D = \{f \in \mathcal{A} \mid \text{supp } f \text{ is compact}\}$) which is invariant with respect to $q = (q^*)^*$, q^*, D is a core for q and q^* and such that $q^2 a = qa$, $(q^*)^2 a = q^* a$ for any $a \in D$. For the notion of affiliated elements and C^*-algebras generated by unbounded elements we refer the reader to [W1, W2] (see also section 4.3). Let $\mathcal{R}$ denote the set of pairs (Q, Q^*) satisfying the conditions given in Remark 1. Then $\mathcal{R} = \{(\pi(q), \pi(q^*)) \mid \pi$ is a non-degenerate representation of $\mathcal{A}\}$ (recall that π is non-degenerate if $\overline{\pi(\mathcal{A})H} = H$). Here $\pi(q)$, $\pi(q^*)$ is the unique extension of the representation π to affiliated elements. This was essentially proved in [P]. It means that any well-behaved representation arises from a representation of a C^*-algebra and any representation of $\mathcal{A}$ gives rise to a well-behaved representation of $\mathcal{Q}_1(*)$.

3 Representations of algebras $\mathcal{Q}_{3,\lambda}$ and $*$-algebras $\mathcal{Q}_{3,\lambda}(*)$

3.1 Algebras $\mathcal{Q}_{3,\lambda}$ and their representations

All algebras $\mathcal{Q}_{3,\lambda}$ are finite-dimensional. They are non-zero only if $\lambda \in \Lambda_3 = \{0, 1, 3/2, 2, 3\}$. We have $\mathcal{Q}_{3,0} = \mathcal{Q}_{3,3} = \mathbb{C}e$ with trivial representations. The idempotents q_i, $i = 1,2,3$, of $\mathcal{Q}_{3,1} = \mathcal{Q}_{3,2}$ are orthogonal, i.e. $q_i q_j = 0$ $(i \neq j)$. In fact, since $(e - q_3)^2 = e - q_3$, we have $\{q_1, q_2\} = q_1 q_2 + q_2 q_1 = 0$. But the idempotents anti-commute iff $q_1 q_2 = q_2 q_1 = 0$. Therefore $\mathcal{Q}_{3,1} = \mathcal{Q}_{3,2} = \mathbb{C}q_1 \oplus \mathbb{C}q_2 \oplus \mathbb{C}q_3 = \mathbb{C}^3 = \mathcal{Q}_{2,\perp}$ $(= \mathbb{C}\langle q_1, q_2, e \mid q_i^2 = q_i, q_1 q_2 = q_2 q_1 = 0\rangle)$.

The algebra $\mathcal{Q}_{3,3/2}$ is isomorphic to $M_2(\mathbb{C})$. Finite-dimensional representations of $\mathbb{C}^3$ and $M_2(\mathbb{C})$ are easy to describe.

3.2 $*$-Algebras $\mathcal{Q}_{3,\lambda}(*)$ and their representations by bounded operators

As in the algebraic situation we have $\Lambda_{3,bd} = \{0,1,3/2,2,3\}$. In contrast to $\mathcal{Q}_{3,0}(*) = \mathcal{Q}_{3,3}(*) = \mathbb{C}e$ whose representations are trivial, the structure of representations of $\mathcal{Q}_{3,1}(*) = \mathcal{Q}_{3,2}(*)$ and $\mathcal{Q}_{3,3/2}(*)$ is very complicated (the $*$-algebras are $*$-wild). Before proving this we recall some results and constructions concerning wild $*$-algebras. For the general definition and results we refer the reader to [KS2, OS2].

Let $\mathfrak{S}_2$ denote the unital $*$-algebra generated by free selfadjoint elements, a, b, and let $C^*(\mathcal{F}_2)$ be the group C^*-algebra of the free group $\mathcal{F}_2$ with two generators. For a unital $*$-algebra A we denote by Rep A the category of $*$-representations of A whose objects are unital $*$-representations of A considered up to unitary equivalence and its morphisms are intertwining operators. If A, B are $*$-algebras, $A \otimes B$ denote the $*$-algebra that consists of all finite sums of the form $a \otimes b$, $a \in A$, $b \in B$.

Assume that for a unital $*$-algebra, $\mathfrak{A}$, there exists a unital $*$-homomorphism $\psi : \mathfrak{A} \to M_n(\mathbb{C}) \otimes \mathfrak{S}_2$ (or to $M_n(\mathbb{C}) \otimes C^*(\mathcal{F}_2)$) for some $n \in \mathbb{N}$. Then ψ generates a functor F_ψ : Rep $\mathfrak{S}_2 \to$ Rep $\mathfrak{A}$ (or F_ψ : Rep $C^*(\mathcal{F}_2) \to$ Rep $\mathfrak{A}$) defined as follows:

- $F_\psi(\pi) = id \otimes \pi$ for any $\pi \in$ Rep $\mathfrak{S}_2$ (or $\pi \in$ Rep $C^*(\mathcal{F}_2)$), where id is the identity representation of $M_n(\mathbb{C})$;
- $F_\psi(C) = I_n \otimes C$ if C intertwines representations π_1, $\pi_2 \in$ Rep $\mathfrak{S}_2$ (or π_1, $\pi_2 \in$ Rep $C^*(\mathcal{F}_2)$), here I_n is the identity operator on $\mathbb{C}^n$.

If the functor F_ψ is full then the $*$-algebra $\mathfrak{A}$ is $*$-wild. To see that F_ψ is full one has to check that any operator intertwining two representations $F_\psi(\pi_1)$, $F_\psi(\pi_2)$ with π_1, $\pi_2 \in$ Rep $\mathfrak{S}_2$ (or π_1, $\pi_2 \in$ Rep $C^*(\mathcal{F}_2)$) is equal to $F_\psi(C)$, where C is an operator which intertwines π_1 and π_2. In particular, we have that $F_\psi(\pi)$, is irreducible iff π is irreducible, two representations $F_\psi(\pi_1)$, $F_\psi(\pi_2)$ are unitarily equivalent iff π_1 and π_2 are unitarily equivalent. In this case we say that the problem of unitary classification of all representations of $\mathfrak{A}$ contains as a subproblem the problem of unitary classification of representations of $\mathfrak{S}_2$ (or $C^*(\mathcal{F}_2)$).

Proposition 2. *(a) The $*$-algebra $\mathcal{Q}_{3,1}(*) = \mathcal{Q}_{3,2}(*) = \mathcal{Q}_{2,\perp}(*)$ is $*$-wild.*
(b) The $*$-algebra $\mathcal{Q}_{3,3/2}(*)$ is $*$-wild.

Proof. (a) Following [KS2, Theorem 6] or [OS2, Theorem 59], define a $*$-homomorphism $\psi : \mathcal{Q}_{2,\perp}(*) \to M_3(\mathbb{C}) \otimes \mathfrak{S}_2$ by

$$\psi(q_1) = \begin{pmatrix} e & e & a+ib \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \psi(q_2) = \begin{pmatrix} 0 & -e & -e \\ 0 & e & e \\ 0 & 0 & 0 \end{pmatrix}.$$

One can easily check that the generated functor F_ψ is full, i.e., the $*$-algebra $\mathcal{Q}_{2,\perp}(*) = \mathcal{Q}_{3,1}(*)$ is $*$-wild.

(b) To see that $\mathcal{Q}_{3,3/2}(*) \simeq M_2(\mathbb{C}) \star M_2(\mathbb{C})^*$ is $*$-wild we define a $*$-homomorphism $\psi : M_2(\mathbb{C}) \star M_2(\mathbb{C})^* \to M_2(\mathbb{C}) \otimes \mathfrak{S}_2$ by

$$\psi(e_{11}) = \begin{pmatrix} e & -a-ib) \\ 0 & 0 \end{pmatrix}, \ \psi(e_{12}) = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}, \psi(e_{21}) = \begin{pmatrix} a+ib & -(a+ib)^2 \\ e & -a-ib) \end{pmatrix},$$

where e_{ij} are the matrix units in $M_2(\mathbb{C})$. We leave it to the reader to check that the functor F_ψ is full, i.e., the $*$-algebra $\mathcal{Q}_{3,3/2}(*) \simeq M_2(\mathbb{C}) \star M_2(\mathbb{C})^*$ is $*$-wild. $\square$

Note that by Proposition 2 the $*$-algebras $Q_{2,\perp}(*)$ and $Q_{3,3/2}(*)$ have infinite-dimensional irreducible representations.

Imposing additional conditions on representations, the problem of describing them up to unitary equivalence might become easier. For example, representations of $\mathcal{P}_{3,1} = \mathcal{P}_{3,2}$ and $\mathcal{P}_{3,3/2}$ or, equivalently, representations of $\mathcal{Q}_{3,1}(*) = \mathcal{Q}_{3,2}(*) = \mathcal{Q}_{2,\perp}(*)$ and $\mathcal{Q}_{3,3/2}(*)$ with the condition that the images of q_i, $i = 1,2,3$ are selfadjoint are very simple. There exist three non-unitarily equivalent irreducible representations of $\mathcal{P}_{3,1}$: $\pi_i(p_i) = 1$, $\pi_i(p_j) = 0$, $i \neq j$, $i = 1,2,3$, on $H = \mathbb{C}^1$. There exists a unique irreducible representation of $\mathcal{P}_{3,3/2}$ on $H = \mathbb{C}^2$:

$$\pi(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \pi(p_2) = \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix}, \ \pi(p_3) = \begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix}.$$

Requiring that images of $q_1, q_2 \in \mathcal{Q}_{2,\perp}(*)$ satisfy the conditions $\pi(q_1)\pi(q_2^*) = 0$ and $\pi(q_2^*)\pi(q_1) = 0$ we obtain that irreducible representations are one- or two-dimensional and, up to unitary equivalence, given by

$$\begin{gathered}\text{(a) } \pi(q_1) = \varepsilon_1, \ \pi(q_2) = \varepsilon_2, \ \varepsilon_i \in \{0,1\}, \varepsilon_1\varepsilon_2 = 0; \\ \text{(b) } \pi(q_1) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}, \ \pi(q_2) = 0 \text{ and } \pi(q_1) = 0, \ \pi(q_2) = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}, \text{ where } \alpha > 0.\end{gathered} \tag{4}$$

3.3 Representations of $\mathcal{Q}_{3,\lambda}(*)$ by unbounded operators

Since the $*$-algebra $\mathcal{Q}_{3,\lambda}(*)$ is non-zero only for $\lambda \in \Lambda_3 = \Lambda_{3,bd}$, we have $\Lambda_{3,unbd} = \Lambda_3 = \{0, 1, 3/2, 2, 3\}$.

Consider unbounded representations of a factor-$*$-algebra of $\mathcal{Q}_{3,1}(*) = \mathcal{Q}_{2,\perp}(*)$, namely, $\mathcal{Q}_{2,\perp}(*)/J$, where J is the tw-sided $*$-ideal generated by $q_1 q_2^*$ and $q_2^* q_1$. Let π be a representation of $\mathcal{Q}_{2,\perp}(*)/J$ on a domain $D(\pi) \subset H(\pi)$ and let

$$\Delta_1 = \overline{\pi(q_1 q_1^* + q_1^* q_1 - (q_1 + q_1^*))}, \ \Delta_2 = \overline{\pi(q_2 q_2^* + q_2^* q_2 - (q_2 + q_2^*))}.$$

We say that π is *well-behaved* if Δ_1 and Δ_2 are selfadjoint, Δ_1, Δ_2 strongly commute (i.e., spectral projections of Δ_1, Δ_2 mutually commute) and $D(\pi) = H_b(\Delta_1, \Delta_2)$, the set of bounded vectors for both Δ_1 and Δ_2. For each $m \in \mathbb{N}$ denote by $H_m(\pi)$ the set $E_{\Delta_1}((-m,m))E_{\Delta_2}((-m,m))H(\pi)$, where $E_{\Delta_i}(\cdot)$ is a resolution of the identity for the selfadjoint operator Δ_i. We have $D(\pi) = \cup_{m\in\mathbb{N}} H_m(\pi)$. It is easy to see that $H_m(\pi)$ is reducing π and the subrepresentation $\pi|_{H_m(\pi)}$ is bounded. With the same definition of irreducibility and unitary equivalence as in section 2.3 we have

Proposition 3. *Any irreducible well-behaved representation π of $\mathcal{Q}_{2,\perp}(*)/J$ is bounded and given by (4).*

For any well-behaved representation π on a Hilbert space $H(\pi)$ there exist a unique decomposition $H(\pi) = H_{00} \oplus H_{01} \oplus H_{10} \oplus \mathbb{C}^2 \otimes H_1 \oplus \mathbb{C}^2 \otimes H_2$ and projection-valued measures

$dE_1(\cdot)$, $dE_2(\cdot)$ on H_1 and H_2 respectively such that

$$\pi(q_1) = 0 \cdot I_{H_{00}} \oplus 0 \cdot I_{H_{01}} \oplus I_{H_{10}} \oplus \int_0^\infty \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix} \otimes dE_1(y) \oplus 0 \cdot I_{H_2},$$

$$\pi(q_2) = 0 \cdot I_{H_{00}} \oplus I_{H_{01}} \oplus 0 \cdot I_{H_{10}} \oplus 0 \cdot I_{H_1} \oplus \int_0^\infty \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix} \otimes dE_2(y),$$

where the equalities hold on $D(\pi)$.

Proof. The first statement follows from the fact that any irreducible representation is bounded. The second one follows from Proposition 1. In fact, since

$$\Delta_2 \pi(q_i) f = \pi(q_i) \Delta_2 f, \; \Delta_2 \pi(q_i^*) f = \pi(q_i^*) \Delta_2 f$$

for any $f \in D(\pi) = H_b(\Delta_1, \Delta_2)$, $i = 1, 2$, it follows that

$$E_{\Delta_2}(\delta) \pi(q_i) f = \pi(q_i) E_{\Delta_2}(\delta) f, \; E_{\Delta_2}(\delta) \pi(q_i^*) f = \pi(q_i^*) E_{\Delta_2}(\delta) f, \; i = 1, 2,$$

for any Borel δ and any $f \in D(\pi)$ (see [OS1][Theorem 1]). This implies that $E_{\Delta_2}(\{0\})H(\pi)$ is reducing π: $H(\pi) = H^1 \oplus H^2$, where $H^1 = E_{\Delta_2}(\{0\})H(\pi)$, $H^2 = H_1^\perp$, and $\pi = \pi_1 \oplus \pi_2$, where $\pi_i = \pi|_{D(\pi) \cap H^i}$, $i = 1, 2$.

Since $\Delta_2|_{H_1} = 0$, we get that $\pi_1(q_2)$, $\pi_1(q_2^*)$ are bounded. Therefore, restricting π_1 to the $*$-subalgebra $\mathbb{C}\langle q_2, q_2^*, e \mid q_2^2 = q_2 \rangle = \mathcal{Q}_1(*)$, we obtain a bounded representation of $\mathcal{Q}_1(*)$ such that the image of $q_2 q_2^* + q_2^* q_2 - (q_2 + q_2^*)$ is zero. It follows from structure theorem for bounded representations of $\mathcal{Q}_1(*)$ (section 2.2) that $\pi_1(q_2)$ is an orthoprojection. Using the same arguments as before and the fact that $\pi(q_1)\pi(q_2) = \pi(q_2)\pi(q_1) = \pi(q_1^*)\pi(q_2) = \pi(q_2)\pi(q_1^*) = 0$, one obtains that $H_1^1 = \ker \pi(q_2)$ is reducing π_1, $\pi_1^1 = \pi_1|_{H_1^1}$ restricted to the $*$-subalgebra $\mathbb{C}\langle q_1, q_1^*, e \mid q_1^2 = q_1 \rangle = \mathcal{Q}_1(*)$ is a well-behaved representation of $\mathcal{Q}_1(*)$ in the sense given in section 2.3, $\pi_1(q_1)|_{(H_1^1)^\perp} = 0$, $\pi_1(q_2)|_{(H_1^1)^\perp} = I$.

Since the kernel of $\Delta_2|_{H^2}$ is $\{0\}$ and $\pi(q_i)\pi(q_j) = \pi(q_i)\pi(q_j^*) = \pi(q_j^*)\pi(q_i^*) = 0$, $i \neq j$, we obtain $\pi_2(q_1) = \pi_2(q_1^*) = 0$ and π_2 restricted to $*$-subalgebra $\mathbb{C}\langle q_2, q_2^*, e \mid q_2^2 = q_2 \rangle = \mathcal{Q}_1(*)$ is a well-behaved representations of $\mathcal{Q}_1(*)$. This completes the proof. □

Clearly, if the support of the measures $dE_1(\cdot)$ and $dE_2(\cdot)$ are unbounded, the representation π from the proposition is unbounded. Any well-behaved representation of $\mathcal{Q}_{2,\perp}(*)/J$ is a representation of $Q_{2,\perp}(*)$.

Remark 3. As for $\mathcal{Q}_1(*)$ there exists a correspondence between well-behaved representations of $\mathcal{Q}_{2,\perp}(*)/J$ and representations of some C^*-algebra, namely the C^*-algebra $\mathcal{A} \oplus \mathcal{A}$, where $\mathcal{A} = \{f \in C([0, \infty), M_2(\mathbb{C})) \mid f(0) \text{ is diagonal }, \lim_{t \to \infty} f(t) = 0\}$. Setting $q_1(t) = q(t) \oplus 0$, $q_2(t) = 0 \oplus q(t)$, where $q \; \eta \; \mathcal{A}$ is the element which was defined in Remark 2, we obtain that q_1, q_2 generate $\mathcal{A} \oplus \mathcal{A}$ as affiliated elements, there exists a dense domain $D' \subset \mathcal{A} \oplus \mathcal{A}$ (for example, $D' = D \oplus D$, where $D = \{f \in \mathcal{A} \mid \text{supp } f \text{ is compcat}\}$) such that D' is invariant with respect to q_i, q_i^*, $i = 1, 2$, D is a core for q_i, q_i^* and the relations $q_i^2 = q_i$, $q_1 q_2 = q_2 q_1 = q_1^* q_2 = q_2 q_1^* = 0$ hold on D'. Moreover, if we let $\mathcal{R}$ denote the set of pairs $(\overline{\pi(q_1)}, \overline{\pi(q_2)})$, where π is a well-behaved representation of $\mathcal{Q}_{2,\perp}(*)/J$ then $\mathcal{R} = \{\rho(q_1), \rho(q_2) \mid \rho \text{ is a non-degenerate representation of } \mathcal{A} \oplus \mathcal{A}\}$. Here $\rho(q_i)$ is the unique extension of the representation ρ to affiliated elements.

Another example of unbounded representations of $\mathcal{Q}_{2,\perp}(*)$ can be derived from Proposition 2. Namely, let $\alpha_n \in \mathbb{R}$ be unbounded sequence of numbers and let

$$Q_1 = \oplus_{n=1}^{\infty} \begin{pmatrix} 1 & 1 & \alpha_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Q_2 = \oplus_{n=1}^{\infty} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be operators on the Hilbert space $H = \oplus_{n=1}^{\infty} H_n$, $H_n = \mathbb{C}^3$. Q_1, Q_2 determine a $*$-representation of $Q_{2,\perp}(*)$ defined on the set of, for example, finite vectors, i.e., vectors which are finite linear combinations of vectors from H_n.

The similar example of representation can be constructed for the algebra $Q_{3,3/2}(*)$.

We are not going to describe unbounded representations of $Q_{2,\perp}(*)$, $\mathcal{Q}_{3,3/2}(*)$, because already the problem of describing all their bounded $*$-representations is very complicated.

4 Representations of algebras $\mathcal{Q}_{4,\lambda}$ and $*$-algebras $\mathcal{Q}_{4,\lambda}(*)$

4.1 Algebras $\mathcal{Q}_{4,\lambda}$ and their representations

For each $\lambda \in \mathbb{C}$ the algebra $\mathcal{Q}_{4,\lambda}$ is non-zero. To see this we give a concrete example of idempotents q_1, q_2, q_3, q_4 as operators defined on a linear space X. The construction is a generalization of an example given in [BES].

Let X be a linear space of complex-valued functions defined on $\mathbb{C}$. Consider the operators q_1, q_2, q_3, $q_4 \in L(X)$ defined by

$$\begin{aligned}
(q_1 f)(z) &= z(f(z) + f(1-z)), \\
(q_2 f)(z) &= z(f(z) - f(1-z)), \\
(q_3 f)(z) &= (\lambda/2 - z)f(z) + (1 - \lambda/2 + z)f(\lambda - 1 - z), \\
(q_4 f)(z) &= (\lambda/2 - z)f(z) - (1 - \lambda/2 + z)f(\lambda - 1 - z).
\end{aligned}$$

Simple computation shows that $q_1 + q_2 + q_3 + q_4 = \lambda I$ and $q_i^2 = q_i$, $i = 1, 2, 3, 4$. This representations is infinite dimensional.

Each algebra $\mathcal{Q}_{4,\lambda}$ is infinite dimensional. A linear basis for $\mathcal{Q}_{4,\lambda}$ is constructed in [RSS]. It was also proved that for $\lambda \neq 2$ the algebras are not algebras with the standard polynomial identities (PI-algebras), but for any $x_1, \dots, x_4 \in \mathcal{Q}_{4,2}$ the following equality holds

$$\sum_{\sigma \in S_4} (-1)^{p(\sigma)} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0$$

where $p(\sigma)$ is the parity of permutation $\sigma \in S_4$.

Let $\Lambda_{n,fd}$ be the set of all $\lambda \in \mathbb{C}$ for which there exists a finite-dimensional representation of $\mathcal{Q}_{n,\lambda}$. The set $\Lambda_{4,fd}$ does not cover the whole complex plane, because $TrQ \in \mathbb{N}$ for any idempotent Q in finite-dimensional space and therefore $\Lambda_{n,fd} \subset \mathbb{Q}$ for any $n \in \mathbb{N}$. We have the following

Proposition 4. $\Lambda_{4,fd} = \Lambda_{4,bd} = \{2 \pm \frac{2}{k}, 2 \mid k \in \mathbb{N}\}$.

Proof. Direct computation shows that

$$\begin{aligned}
\mathcal{Q}_{4,\lambda} &= \mathbb{C}\langle r_1, r_2, r_3, r_4, e \mid r_k^2 = e; \textstyle\sum_{k=1}^4 r_k = (2-\lambda)e\rangle = \\
&= \mathbb{C}\langle x_1, x_2, x_3 \mid \{x_1, x_2\} = x_3, \{x_1, x_3\} = x_2, \{x_2, x_3\} = x_1, \\
&\qquad (\lambda - 2)^2 (x_1^2 + x_2^2 + x_3^2 + 1/4e) = 1\rangle,
\end{aligned}$$

where

$$\begin{aligned} r_1 &= 2q_1 - e = (2-\lambda)(-x_1 + x_2 + x_3 + 1/2e),\\ r_2 &= 2q_2 - e = (2-\lambda)(x_1 - x_2 + x_3 + 1/2e),\\ r_3 &= 2q_3 - e = (2-\lambda)(x_1 + x_2 - x_3 + 1/2e),\\ r_4 &= 2q_4 - e = (2-\lambda)(-x_1 - x_2 - x_3 + 1/2e). \end{aligned} \tag{5}$$

If there exists a representation, π, of $\mathbb{C}\langle x_1, x_2, x_3, e \mid \{x_1, x_2\} = x_3, \{x_1, x_3\} = x_2, \{x_2, x_3\} = x_1, (\lambda-2)^2(x_1^2 + x_2^2 + x_3^2 + 1/4e) = 1\rangle$ by bounded operators in a Hilbert space H, then the operators $\pi(x_i) \otimes \sigma_i$, $i = 1, 2, 3$, with $\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ define a representation of the universal enveloping algebra $U(sl(2,\mathbb{C}))$ of the Lie algebra $sl(2,\mathbb{C})$ with an extra condition on the Casimir operator. Namely if we take a basis X_1, X_2, X_3 in $sl(2,\mathbb{C})$ with the Lie bracket defined as

$$[X_1, X_2] = X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2. \tag{6}$$

and denote by Δ the Casimir operator $X_1^2 + X_2^2 + X_3^2$ in $U(sl(2,\mathbb{C}))$ then $\rho(X_i) = \pi(x_i) \otimes \sigma_i$ is a representation of $U(sl(2,\mathbb{C}))$ so that $(\lambda-2)^2\rho(\Delta) = (\lambda-2)^2/4 - I$. In the Hilbert space $H \otimes \mathbb{C}^2$ one can choose an equivalent scalar product so that the operators $\rho(X_i)$ are skew-selfadjoint and ρ is a $*$-representation of the corresponding $*$-algebra defined by the condition $X_i^* = -X_i$, $i = 1, 2, 3$ (a $*$-representation of the Lie algebra $su(2)$) It is known that $*$-representations of the $*$-algebra such that the image of the Casimir operator equals $(1/4 - 1/(\lambda-2)^2)$ exist if and only if $\lambda \in \{2 \pm \frac{2}{k}, 2 \mid k \in \mathbb{N}\}$. This implies that $\Lambda_{n,fd} \subset \Lambda_{n,bd} \subset \{2 \pm \frac{2}{k}, 2 \mid k \in \mathbb{N}\}$.

To see the other inclusion, note that for any $\lambda \in \{2 \pm \frac{2}{k}, 2 \mid k \in \mathbb{N}\}$ there exists a finite-dimensional representation, π, of $U(sl(2,\mathbb{C}))$. Then $\rho(x_i) = \pi(X_i) \otimes \sigma_i$, $i = 1, 2, 3$, define a finite-dimensional representation of $\mathbb{C}\langle x_1, x_2, x_3, e \mid \{x_1, x_2\} = x_3, \{x_1, x_3\} = x_2, \{x_2, x_3\} = x_1, (\lambda-2)^2(x_1^2 + x_2^2 + x_3^2 + 1/4e) = 1\rangle$ and therefore $\mathcal{Q}_{4,\lambda}$. The proof is complete. □

Proposition 5. *Any finite-dimensional representation of* $\mathcal{Q}_{4,\lambda}$, $\lambda \in \Lambda_{4,bd} \setminus \{2\}$ *is equivalent to a* $*$*-representation of* $\mathcal{P}_{4,\lambda}$.

Proof. Let π be a finite-dimensional representation of $\mathcal{Q}_{4,\lambda}$, $\lambda \in \Lambda_{4,bd} \setminus \{2\}$, on a vector space V_π. The procedure is to find a scalar product, $(\cdot,\cdot)_\pi$ (a non-degenerate positive-definite hermitian form) such that

$$(\pi(q_i)\varphi, \psi)_\pi = (\varphi, \pi(q_i)\psi)_\pi, \quad \forall\, \varphi, \psi \in V_\pi,\ i = 1, 2, 3, 4,$$

or, equivalently,

$$(\pi(x_i)\varphi, \psi)_\pi = (\varphi, \pi(x_i)\psi)_\pi, \quad \forall\, \varphi, \psi \in V_\pi,\ i = 1, 2, 3,$$

for the generators x_i defined by (5).

Having the representation π we can define a representation ρ of $U(sl(2,\mathbb{C})$ on $H(\rho) = V_\pi \otimes \mathbb{C}^2$ by setting $\rho(X_i) = \pi(x_i) \otimes \sigma_i$, $i = 1, 2, 3$ for the generators X_i of $U(sl(2,\mathbb{C})$ satisfying relations (6). It is known that ρ is unitarizable, i.e., there exists a scalar product $(\cdot,\cdot)_\rho$ on $H(\rho)$ such that $(\rho(X_i)\varphi, \psi)_\rho = -(\varphi, \rho(X_i)\psi)_\rho$.

Define a new representation $\tilde{\pi}$ of $\mathcal{Q}_{4,\lambda}$ on the Hilbert space $H(\rho) \otimes \mathbb{C}^2$ by $\tilde{\pi}(x_i) = \rho(X_i) \otimes \sigma_i$, $i = 1, 2, 3$. The scalar product, $(\cdot,\cdot)_{\tilde{\pi}}$ on $H(\rho) \otimes \mathbb{C}^2$ is defined on elementary tensors as $(\varphi_1\otimes\psi_1,\varphi_2\otimes\psi_2)_{\tilde\pi}=(\varphi_1,\varphi_2)_\rho(\psi_1,\psi_2)_{\mathbb{C}^2}$, for any $\varphi_i\in H(\rho)$, $\psi_i\in\mathbb{C}^2$, where $(\cdot,\cdot)_{\mathbb{C}^2}$ is the standard scalar product on $\mathbb{C}^2$. Let e_1, e_2 be the standard basis vectors $(1,0)$ and $(0,1)$ respectively in $\mathbb{C}^2$. For $\varphi,\psi\in V_\pi$ define

$$(\varphi,\psi)_\pi=((\varphi\otimes e_1)\otimes e_2+(\varphi\otimes e_2)\otimes e_1,(\psi\otimes e_1)\otimes e_2+(\psi\otimes e_2)\otimes e_1)_{\tilde\pi}.$$

It is easy to see that $(\cdot,\cdot)_\pi$ is a scalar product on V_π. Moreover, $(\pi(x_i)\varphi,\psi)=(\varphi,\pi(x_i)\psi)$ for any φ, $\psi\in V_\pi$, $i=1,2,3$. We restrict ourselves by showing the last formula for the generator x_1.

$$\begin{gathered}(\pi(x_1)\varphi,\psi)_\pi=\\
=((\pi(x_1)\varphi\otimes e_1)\otimes e_2+(\pi(x_1)\varphi\otimes e_2)\otimes e_1,(\psi\otimes e_1)\otimes e_2+(\psi\otimes e_2)\otimes e_1)_{\tilde\pi}=\\
=-((\pi(x_1)\otimes\sigma_1)(\varphi\otimes e_2)\otimes\sigma_1e_1+(\pi(x_1)\otimes\sigma_1)(\varphi\otimes e_1)\otimes\sigma_1e_2,\\
(\psi\otimes e_1)\otimes e_2+(\psi\otimes e_2)\otimes e_1)_{\tilde\pi}=\\
=-((\rho(X_1)\otimes\sigma_1)((\varphi\otimes e_1)\otimes e_2+(\varphi\otimes e_2)\otimes e_1),(\psi\otimes e_1)\otimes e_2+(\psi\otimes e_2)\otimes e_1)_{\tilde\pi}=\\
=-((\varphi\otimes e_1)\otimes e_2+(\varphi\otimes e_2)\otimes e_1,(\rho(X_1)\otimes\sigma_1)((\psi\otimes e_1)\otimes e_2+(\psi\otimes e_2)\otimes e_1))_{\tilde\pi}=\\
=-((\varphi\otimes e_1)\otimes e_2+(\varphi\otimes e_2)\otimes e_1,\\
(\pi(x_1)\otimes\sigma_1)(\psi\otimes e_2)\otimes\sigma_1e_1+(\pi(x_1)\otimes\sigma_1)(\psi\otimes e_1)\otimes\sigma_1e_2)_{\tilde\pi}=\\
=((\varphi\otimes e_1)\otimes e_2+(\varphi\otimes e_2)\otimes e_1,(\pi(x_1)\psi\otimes e_1)\otimes e_2+(\pi(x_1)\psi\otimes e_2)\otimes e_1)_{\tilde\pi}=\\
=(\varphi,\pi(x_1)\psi)_\pi.\end{gathered}$$

The proof is complete. □

As it follows from the previous proposition finite-dimensional indecomposable representations of $\mathcal{Q}_{4,\lambda}$ coincide with irreducible $*$-representations of the $*$-algebra $\mathcal{P}_{4,\lambda}$, $\lambda\neq 2$. For $\lambda=1+2/(2k+1)$ indecomposable representation of $\mathcal{Q}_{4,\lambda}$ is unique, up to equivalence, and acts in a $(2k+1)$-dimensional vector space. If $\lambda=1+2/(2k+2)$, there are four non-equivalent representations of $\mathcal{Q}_{4,\lambda}$ acting on $(k+1)$-dimensional space (see [OS2][Section 2.2.1]). The algebra $\mathcal{Q}_{4,2}$ is wild and the problem of describing of its indecomposable representations is very complicated, [Bo].

4.2 $*$-Algebras $\mathcal{Q}_{4,\lambda}(*)$ and their representations by bounded operators

Proposition 6. *For $\lambda\in\Lambda_{4,bd}$ the $*$-algebra $Q_{4,\lambda}(*)$ is $*$-wild.*

Proof. That $Q_{4,\lambda}(*)$ is $*$-wild for $\lambda=1,2$ follows from Proposition 2. Assume now that $\lambda\in\Lambda_{4,bd}\setminus\{1,2\}$. Define $p=(q_1+q_2)/2$, $q=(q_3+q_4)/2$, $r=(q_1-q_2)/2$, $s=(q_3-q_4)/2$. Direct computation shows that $Q_{4,\lambda}(*)$ is generated by p, q, r, s and their adjoint and the relations

$$\begin{gathered}pr=r(1-p),\quad ps=s(1-q),\\ r^2=p(1-p),\quad s^2=q(1-q),\\ p+q=\lambda/2e\end{gathered}\tag{7}$$

From the relation it follows that $ps=s(-1+\lambda-p)$ and $s^2=(\lambda/2-p)(1-\lambda/2+p)$. To prove the statement we will construct a $*$-homomorphism $\psi:\mathcal{Q}_{4,\lambda}(*)\to M_n(\mathbb{C})\otimes C^*(\mathcal{F}_2)$ $(\simeq M_n(C^*(\mathcal{F}_2))$ for some $n\in\mathbb{N}$ depending on λ.

1. Let $\lambda = 2 + 1/(2l)$, $l > 0$. Let $E_n = \underbrace{\begin{pmatrix} e & & 0 \\ & \ddots & \\ 0 & & e \end{pmatrix}}_{n \text{ times}}$, where e is the identity in $C^*(\mathcal{F}_2)$.

Let $J_3 = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$, $A_1 = \frac{1}{N} \begin{pmatrix} e & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 \\ 0 & 0 & 2e & 0 & 0 \\ 0 & 0 & 0 & 3e & 0 \end{pmatrix}$, $A_2 = \frac{1}{N} \begin{pmatrix} e & 0 & e & e & e \\ 0 & 2e & e & u_1 & 0 \\ 0 & 0 & e & 0 & u_2 \end{pmatrix}$, $A_3 = \sqrt{E_5 - A_1^* A_1 - A_2^* A_2}$, where u_1, u_2 are the free unitary generators of $C^*(\mathcal{F}_2)$, N is chosen so that $||A_1^* A_1 + A_2^* A_2||_{M_5(C^*(\mathcal{F}_2))} < 1$ ($||\cdot||_{M_5(C^*(\mathcal{F}_2))}$ is the C^*-norm on $M_5(C^*(\mathcal{F}_2))$).

Define a $*$-homomorphism $\psi : \mathcal{Q}_{4,\lambda}(*) \to M_{12\cdot 2l}(C^*(\mathcal{F}_2))$ as follows:

$$\psi(p) = diag(\lambda_0 E_{12}, \lambda_1 E_{12}, \dots, \lambda_{2l-1} E_{12}),$$
$$\psi(r) = diag(0 \cdot E_{12}, R_1, R_2, \dots, R_{l-1}, J_3 J_3^* - 1/2),$$
$$\psi(s) = diag(S_0, S_1, \dots, S_{l-1}),$$

where $\lambda_{2k} = 1 - \dfrac{k}{2l}$, $\lambda_{2k-1} = \dfrac{k}{2l}$, $R_k = \begin{pmatrix} 0 & \lambda_{2k-1}\lambda_{2k} E_{12} \\ E_{12} & 0 \end{pmatrix}$,

$$S_k = \begin{pmatrix} 0 & (\frac{\lambda}{2} - \lambda_{2k})(1 - \frac{\lambda}{2} + \lambda_{2k}) E_{12} \\ E_{12} & 0 \end{pmatrix} \ (k \neq l-1),\ S_{l-1} = \begin{pmatrix} 0 & s^{l-1} \\ s_{l-1} & 0 \end{pmatrix},$$

$$s_{l-1} = diag(x_1 E_4, x_2 E_3, x_3 E_5), \quad s^{l-1} = diag(y_1 E_4, y_2 E_3, y_3 E_5),$$

x_i, y_i are real numbers such that $x_i \neq x_j$, $y_i \neq y_j$ for $i \neq j$ and $x_i y_i = (4l^2 - 1)/16l^2$.

It is a routine to check the functor F_ψ is full and we leave it to the reader. The construction is similar for $\lambda = 2 - 1/2l$, $l \in \mathbb{N}$: there exists a $*$-homomorphism $\psi : \mathcal{Q}_{4,\lambda}(*) \to M_{12\cdot 2l}(C^*(\mathcal{F}_2))$ with $\psi(p) = diag(\lambda_0 E_{12}, \lambda_1 E_{12}, \dots, \lambda_{2l-1} E_{12})$, where $\lambda_{2k} = \dfrac{k}{2l}$, $\lambda_{2k-1} = 1 - \dfrac{k}{2l}$.

2. Let $\lambda = 2 + 2/(2l+1)$, $l > 0$. Define $\psi : \mathcal{Q}_{4,\lambda}(*) \to M_{12(2l+1)}(C^*(\mathcal{F}_2))$ as follows:

$$\psi(p) = diag(\lambda_1 E_{12}, \lambda_2 E_{12}, \dots, \lambda_{2l+1} E_{12}),$$
$$\psi(r) = diag(0 \cdot E_{12}, R_1, R_2, \dots, R_{l-1}, R_l),$$
$$\psi(s) = diag(S_1, S_2, \dots, S_l, 0 \cdot E_{12}),$$

where $\lambda_{2k} = \dfrac{2k}{2l+1}$, $\lambda_{2k+1} = 1 - \dfrac{2k}{2l+1}$, $R_k = \begin{pmatrix} 0 & \lambda_{2k}\lambda_{2k+1} E_{12} \\ E_{12} & 0 \end{pmatrix}$, $(k \neq 1)$,

$$R_1 = \begin{pmatrix} 0 & (2J_3 J_3^* - 1)\sqrt{\lambda_2 \lambda_3} \\ (2J_3 J_3^* - 1)\sqrt{\lambda_2 \lambda_3} & 0 \end{pmatrix},$$

$$S_k = \begin{pmatrix} 0 & (\frac{\lambda}{2} - \lambda_{2k-1})(1 - \frac{\lambda}{2} + \lambda_{2k-1}) E_{12} \\ E_{12} & 0 \end{pmatrix},\ (k \neq 1, 2),\ S_k = \begin{pmatrix} 0 & s^k \\ s_k & 0 \end{pmatrix},\ (k = 1, 2),$$

with

$$s_k = diag(x_1^k E_4, x_2^k E_3, x_3^k E_5), \quad s^k = diag(y_1^k E_4, y_2^k E_3, y_3^k E_5)$$

x_i^k, y_i^k are real numbers such that $x_i^k \neq x_j^k$, $y_i^k \neq y_j^k$ for $i \neq j$ and $x_i^k y_i^k = (\frac{\lambda}{2} - \lambda_{2k-1})(1 - \frac{\lambda}{2} + \lambda_{2k-1})$.

One can check that the functor F_ψ is full. The construction is similar for $\lambda = 2-2/(2l+1)$, $l \in \mathbb{N}$. In this case there exists a $*$-homomorphism $\psi : \mathcal{Q}_{4,\lambda} \to M_{12(2l+1)}(C^*(\mathcal{F}_2))$ with $\psi(p) = diag(\lambda_1 E_{12}, \lambda_2 E_{12}, \ldots, \lambda_{2l+1} E_{12})$, where $\lambda_{2k} = 1 - \dfrac{2k}{2l+1}$, $\lambda_{2k+1} = \dfrac{2k}{2l+1}$.

3. Let $\lambda = 2 + 1/(2l+1)$, $l > 0$. Define $\psi : \mathcal{Q}_{4,\lambda}(*) \to M_{12(2l+1)}(C^*(\mathcal{F}_2))$ as follows:

$$\psi(p) = diag(\lambda_1 E_{12}, \lambda_2 E_{12}, \ldots, \lambda_{2l+1} E_{12}),$$
$$\psi(r) = diag(0 \cdot E_{12}, R_1, R_2, \ldots, R_{l-1}, R_l),$$
$$\psi(s) = diag(S_1, S_2, \ldots, S_l, J_3 J_3^* - 1/2),$$

where $\lambda_{2k} = \dfrac{k}{2l+1}$, $\lambda_{2k+1} = 1 - \dfrac{k}{2l+1}$, $R_k = \begin{pmatrix} 0 & \lambda_{2k}\lambda_{2k+1}E_{12} \\ E_{12} & 0 \end{pmatrix}$, $(k \neq l)$, $R_l = \begin{pmatrix} 0 & r^l \\ r_l & 0 \end{pmatrix}$ with $r_l = diag(x_1 E_4, x_2 E_3, x_3 E_5)$, $\quad r^l = diag(y_1 E_4, y_2 E_3, y_3 E_5)$, where x_i, y_i are real numbers such that $x_i \neq x_j$, $y_i \neq y_j$ for $i \neq j$ and $x_i y_i = \frac{l}{2l+1}(1 - \frac{l}{2l+1})$.

$$S_k = \begin{pmatrix} 0 & (\frac{\lambda}{2} - \lambda_{2k-1})(1 - \frac{\lambda}{2} + \lambda_{2k-1})E_{12} \\ E_{12} & 0 \end{pmatrix}$$

The functor F_ψ is full. The construction is similar for $\lambda = 2 - 2/(2l+1)$, $l \in \mathbb{N}$. In this case there exists a $*$-homomorphism $\psi : \mathcal{Q}_{4,\lambda} \to M_{12(2l+1)}(C^*(\mathcal{F}_2))$ with $\psi(p) = diag(\lambda_1 E_{12}, \lambda_2 E_{12}, \ldots, \lambda_{2l+1} E_{12})$, where $\lambda_{2k} = 1 - \dfrac{k}{2l+1}$, $\lambda_{2k+1} = \dfrac{k}{2l+1}$.

□

The problem of classification of all representations of $\mathcal{Q}_{4,\lambda}(*)$, $\lambda \in \Lambda_{4,bd}$ is very difficult as it follows from Proposition 6. However, if we restrict ourself to representations such that the images of the generators q_i, $i = 1,2,3,4$ are selfadjoint, this problem reduces to the problem of describing representations of the $*$-algebra $\mathbb{C}\langle x_1, x_2, x_3, e \mid \{x_1, x_2\} = x_3, \{x_1, x_3\} = x_2, \{x_2, x_3\} = x_1, (\lambda-2)^2(x_1^2 + x_2^2 + x_3^2 + 1/4e) = 1, x_i^* = x_i\rangle$, $\lambda \neq 2$, a factor $*$-algebra of the graded analogue of the Lie algebra $so(3)$ (see section 4.1). Representations of the graded $so(3)$ are classified in [GoP], see also [OS2]. According to this result there exists a unique, up to unitary equivalence, irreducible representation of $\mathcal{P}_{4,\lambda}$ for $\lambda = 1 + 2/(2k+1)$, acting in a $(2k+1)$-dimensional vector space and there are four non-equivalent representations of $\mathcal{P}_{4,\lambda}$ acting on $(k+1)$-dimensional space if $\lambda = 1 + 2/(2k+2)$. If $\lambda = 2$, $\mathcal{P}_{4,2}$ has uncountable set of irreducible unitarily non-equivalent representations which are one or two-dimensional (see [OS2][Section 2.2.1]).

4.3 Representations of $\mathcal{Q}_{4,\lambda}(*)$ by unbounded operators

As we already know, for each $\lambda \in \mathbb{C}$ the algebra $\mathcal{Q}_{4,\lambda}$ and therefore $\mathcal{Q}_{4,\lambda}(*)$ is non-zero. Representations of $\mathcal{Q}_{4,\lambda}$ or $\mathcal{Q}_{4,\lambda}(*)$ by bounded operators on a Hilbert space exist, however, not for all $\lambda \in \mathbb{C}$ (see Proposition 4).

Proposition 7. $\Lambda_{4,unbd} = \mathbb{C}$

Proof. In order to prove the statement it is enough for each $\lambda \in \mathbb{C}$ to give a concrete construction of unbounded representation of $\mathcal{Q}_{4,\lambda}(*)$. We follow [RSS].

Let $\varphi(\cdot)$, $\psi(\cdot)$ be the following idempotent matrix-functions from $\mathbb{C}$ to $M_2(\mathbb{C})$:

$$\varphi(t) = \begin{pmatrix} t & t-t^2 \\ 1 & 1-t \end{pmatrix}, \psi(t) = \begin{pmatrix} t & -(t-t^2) \\ -1 & 1-t \end{pmatrix}.$$

Consider a sequence of complex numbers $x_j = j(\lambda/2 - 1)$, $j \in \mathbb{N}$. Let $H = l_2$ and fix an orthonormal basis, $\{e_i, i \in \mathbb{N}\}$ in H. Define operators Q_i, Q_i^+ on the set, Φ, of finite vectors, i.e. finite linear combinations of e_i, so that their matrix representations with respect to this fixed basis are given by

$$\begin{aligned} Q_1 &= diag\{\varphi(x_1), \varphi(x_3), \varphi(x_5), \dots\}, \\ Q_2 &= diag\{\psi(x_1), \psi(x_3), \psi(x_5), \dots\}, \\ Q_3 &= diag\{1, \varphi(x_2), \varphi(x_4), \varphi(x_6), \dots\}, \\ Q_4 &= diag\{1, \psi(x_2), \psi(x_4), \psi(x_6), \dots\}, \end{aligned}$$

$$\begin{aligned} Q_1^+ &= diag\{\varphi(x_1)^*, \varphi(x_3)^*, \varphi(x_5)^*, \dots\}, \\ Q_2^+ &= diag\{\psi(x_1)^*, \psi(x_3)^*, \psi(x_5)^*, \dots\}, \\ Q_3^+ &= diag\{1, \varphi(x_2)^*, \varphi(x_4)^*, \varphi(x_6)^*, \dots\}, \\ Q_4^+ &= diag\{1, \psi(x_2)^*, \psi(x_4)^*, \psi(x_6)^*, \dots\}, \end{aligned}$$

where A^* is the adjoint matrix to the matrix A. Clearly, Φ is invariant with respect to Q_i, Q_i^+, $i = 1,2,3,4$. Moreover, direct calculation shows that $Q_i^2 = Q_i$, $(Q_i^+)^2 = Q_i^+$, $\sum_{i=1}^4 Q_i = \lambda I$, $\sum_{i=1}^4 Q_i^+ = \overline{\lambda} I$. Setting $\pi(q_i) = Q_i$, $\pi(q_i^*) = Q_i^+$, $D(\pi) = \Phi$, and then extending π to the whole algebra $\mathcal{Q}_{4,\lambda}(*)$ we obtain a $*$-representation of $\mathcal{Q}_{4,\lambda}(*)$. □

We remark that the construction given in the proof can be derived from one given in section 4.1.

In the same way as we obtained, in sections 4.1, 4.2, some bounded representations of $\mathcal{Q}_{4,\lambda}$ and $\mathcal{Q}_{4,\lambda}(*)$ from representations and, respectively, unitary representations of the compact group $SU(2)$ (or representations of the corresponding Lie algebra), unbounded representations of $\mathcal{Q}_{4,\lambda}$ and $\mathcal{Q}_{4,\lambda}(*)$ can be obtained from representations and, respectively, unitary representations of the Lie group $SL(2,\mathbb{R})$.

Let $\lambda \neq 2$ and let $U(sl(2,\mathbb{C}))$ be the universal enveloping algebra of $sl(2,\mathbb{C})$ with the basis X_1, X_2, X_3 and the relations

$$[X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2. \tag{8}$$

Denote by Δ the Casimir operator $X_1^2 + X_2^2 + X_3^2$. In the algebra $U(sl(2,\mathbb{C})) \otimes M_2(\mathbb{C})$ consider the elements $x_1 = X_1 \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $x_2 = X_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $x_3 = X_3 \otimes \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

It is easy to check that they satisfy the relations $\{x_1, x_2\} = x_3$, $\{x_2, x_3\} = x_1$, $\{x_3, x_1\} = x_2$. We set

$$\begin{aligned} Q_1 &= \frac{\lambda-2}{2}(-x_1+x_2+x_3)+\frac{\lambda}{4}, \\ Q_2 &= \frac{\lambda-2}{2}(x_1-x_2+x_3)+\frac{\lambda}{4}, \\ Q_3 &= \frac{\lambda-2}{2}(x_1+x_2-x_3)+\frac{\lambda}{4}, \\ Q_4 &= \frac{\lambda-2}{2}(-x_1-x_2-x_3)+\frac{\lambda}{4}. \end{aligned} \tag{9}$$

Then, $Q_1 + Q_2 + Q_3 + Q_4 = \lambda I$. Moreover, Q_i, $i = 1, 2, 3, 4$, are idempotents iff

$$\hat{\Delta} = x_1^2 + x_2^2 + x_3^2 = \frac{1}{(\lambda-2)^2} - \frac{1}{4}.$$

It follows from the representation theory for $sl(2, \mathbb{C})$ that for any $\lambda \in \mathbb{C}$ there exists a representation of $U(sl(2, \mathbb{C}))$ such that the range of the Casimir operator is $(\frac{1}{4} - \frac{1}{(\lambda-2)^2})I$ (see [V]). Namely, let $\chi = (l, \varepsilon)$, where l is a complex number and $\varepsilon \in \{0, 1/2\}$. With each such pair χ we associate a space

$$\mathcal{D}_\chi = \{f \in C^\infty(\mathbb{R}) \mid \hat{f}(x) = |x|^{2l}(sgn x)^{2\varepsilon} f(\frac{1}{x}) \in C^\infty(\mathbb{R})\}.$$

Consider now representations T_χ of $SL(2, \mathbb{R})$ on $\mathcal{D}_\chi$ given by

$$T_\chi(g)f(x) = |\beta x + \delta|^{2l} sgn^{2\varepsilon}(\beta x + \delta) f(\frac{\lambda x + \gamma}{\beta x + \delta}),$$

where $g = \begin{pmatrix} \lambda & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R})$ (see [GGV, V]). The infinitesimal operators of these representations are

$$\begin{aligned} A_1 &= 2lx + (1-x^2)\tfrac{d}{dx}, \\ A_2 &= -2lx + (1+x^2)\tfrac{d}{dx}, \\ A_3 &= 2l - 2x\tfrac{d}{dx}. \end{aligned}$$

They satisfy the relations $A_1A_2 - A_2A_1 = -2A_3$, $A_2A_3 - A_3A_2 = -2A_1$, $A_3A_1 - A_1A_2 = 2A_2$. Let $X_1 = iA_1/2$, $X_2 = A_2/2$, $X_3 = -iA_3/2$. Then X_1, X_2, X_3 satisfy (8) on $\mathcal{D}_\chi$ with $\Delta = X_1^2 + X_2^2 + X_3^2 = -l(l+1)$. Obviously, a solution $l \in \mathbb{C}$ of $1/(\lambda-2)^2 - 1/4 = l(l+1)$ exists for any $\lambda \in \mathbb{C}$. Therefore there exist linear operators Q_1, Q_2, Q_3, Q_4 on $\mathcal{D}_\chi$ such that $Q_1 + Q_2 + Q_3 + Q_4 = \lambda I$ and $Q_i^2 = Q_i$. We have the following expressions for Q_i, $i = 1, 2, 3, 4$:

$$Q_1 = \frac{\lambda-2}{4}\begin{pmatrix} -A_3 & A_1+A_2 \\ A_1-A_2 & A_3 \end{pmatrix} + \frac{\lambda}{4}, \quad Q_2 = \frac{\lambda-2}{4}\begin{pmatrix} -A_3 & -A_1-A_2 \\ A_2-A_1 & A_3 \end{pmatrix} + \frac{\lambda}{4},$$

$$Q_3 = \frac{\lambda-2}{4}\begin{pmatrix} A_3 & A_2-A_1 \\ -A_1-A_2 & -A_3 \end{pmatrix} + \frac{\lambda}{4}, \quad Q_4 = \frac{\lambda-2}{4}\begin{pmatrix} A_3 & A_1-A_2 \\ A_1+A_2 & -A_3 \end{pmatrix} + \frac{\lambda}{4}.$$

The operators Q_i, $i = 1, 2, 3, 4$ define a representation of $\mathcal{Q}_{4,\lambda}$ on $D_\chi \otimes \mathbb{C}^2$.

It is known that for some values of χ one can introduce a scalar product $(\cdot,\cdot)$ in D_χ which is invariant with respect to the representation T_χ, i.e., $(\varphi,\psi) = (T_\chi(g)\varphi, T_\chi(g)\psi)$, $\varphi,\psi \in D_\chi$. The completion of D_χ with respect to the norm $||\varphi||^2 = (\varphi,\varphi)$ gives us a Hilbert space H_χ and the continuous extension of T_χ to H_χ gives a unitary representation of $SL(2,\mathbb{R})$. In this case the infinitesimal operators A^χ of T_χ will be skew-selfadjoint, i.e. $(A^\chi)^* = -A^\chi$.

Recall that D_χ possesses an invariant scalar product in the following cases:

a) $l = -1/2 + i\rho$, $\rho \in \mathbb{R}$, $\varepsilon \in \{0, 1/2\}$ and $\lambda = 2 + is$, $s \in \mathbb{R}$, $s \neq 0$, the corresponding representation T_χ is called a representation of the principal series;

b) $-1 < l < 0$, $l \neq -1/2$, $\varepsilon = 0$ and $\lambda \in (-\infty, 0) \cup (4, +\infty)$, the corresponding representation T_χ is called a representation of a supplementary series.

For $l \in \frac{1}{2}\mathbb{Z}, l \leq -1$ and ε satisfying the condition $l + \varepsilon \in \mathbb{Z}$ the corresponding space D_χ has two subspaces F_l^+, F_l^- which are invariant with respect to the operators $T_\chi(g)$. It is not possible to introduce an invariant scalar product on D_χ, but it is possible to do it on each of these subspaces. The corresponding subrepresentations of T_χ are called representations of the discrete series. In this case λ takes values $2 \pm \frac{2}{k}$, $k \in \mathbb{N}$.

Taking the infinitesimal representation of a unitary representation of $SL(2,\mathbb{R})$ in a Hilbert space H we define the (unbounded) operators Q_i, $i = 1,2,3,4$ on $H \oplus H$ as above. These operators are densely defined and clearly, there exist a dense invariant domain D such that the the equalities $(Q_1 + Q_2 + Q_3 + Q_4)\varphi = \lambda\varphi$, $Q_i^2\varphi = Q_i\varphi$ hold for any $\varphi \in D$. Moreover, if λ is real then $Q_1^* \supseteq Q_3$, $Q_2^* \supseteq Q_4$.

4.3.1 Unbounded idempotents the sum of which is zero

The rest of the section is devoted to a detailed discussion of unbounded representations of $\mathcal{Q}_{4,0}(*)$. Note that idempotents whose sum is zero were studied in [BES] in connection with their investigation of the concept of logarithmic residues in Banach algebras. We will study representations of $\mathcal{Q}_{4,0}(*)$ under some additional conditions which will allow us to classify them up to unitary equivalence.

Having these unbounded representations we construct a C^*-algebra $\mathfrak{A}$ and unbounded elements q_1,q_2, q_3, q_4 which are affiliated with $\mathfrak{A}$ such that any non-degenerate representation of $\mathfrak{A}$ extended to its affiliated elements gives us an integrable representation of $\mathcal{Q}_{4,0}(*)$ defined below. Moreover, any integrable representation with some extra condition can be obtained this way.

1. Hilbert space level.

Consider new generators in $\mathcal{Q}_{4,0}(*)$ given by $p = (q_1 + q_2)/2$, $q = (q_3 + q_4)/2$, $r = (q_1 - q_2)/2$, $s = (q_3 - q_4)/2$ and their adjoint. Direct computation shows that relations $q_1 + q_2 + q_3 + q_4 = 0$, $q_i^2 = q_i$, $i = 1, \ldots, 4$, are equivalent to the following ones:

$$\begin{array}{ll} pr = r(1-p), & ps = s(-1-p), \\ r^2 = p(1-p), & s^2 = -p(p+1). \end{array} \tag{10}$$

Let $\mathcal{A}_{4,0}$ be the quotient of $\mathcal{Q}_{4,0}(*)$ by the two-sided $*$-ideal generated by the elements $pr^* - rp$, $ps^* - sp$ and $p - p^*$. So we have additional relations in $\mathcal{A}_{4,0}$, namely,

$$pr^* = rp,\ ps^* = sp,\ p = p^*. \tag{11}$$

In what follows we will study representations of $\mathcal{A}_{4,0}$. Obviously, any $*$-representation of $\mathcal{A}_{4,0}$ is a representation of $\mathcal{Q}_{4,0}(*)$.

Henceforth, $H_a(A_1, \dots, A_n)$ will denote the set of joint analytic vectors for selfadjoint operators $A_1, \dots, A_n$ (see [S]). Also, a linear set Φ will be called a core for a closed operator A if $\Phi \subseteq D(A)$ and the closure of the operator A restricted to the domain Φ is equal A.

Definition 1. *We say that closed operators ($p = p^*$, $q = q^*$, r, s, r^*, s^*) is a representation of commutation relations* $(10)-(11)$ *on a Hilbert space H if there exists a linear dense subset $\Phi \subset H$ such that*

1. $\Phi \subseteq H_a(p, q, r^*r, s^*s)$;
2. *Φ is a core for the operators r, r^*, s and s^*;*
3. *relations* $(10) - (11)$ *hold on Φ.*

A family $\{A_j \mid j \in J\}$ of closed unbounded operators on a Hilbert space H is called *irreducible* if decomposition $A_j = B_j \oplus C_j$ for all $j \in J$ with respect to an orthogonal direct sum $H = H_1 \oplus H_2$ is only possible when either $H_1 = \{0\}$ or $H_2 = \{0\}$, or equivalently if

$$\{C \in B(H) \mid CA_j \subseteq A_jC \text{ and } C^*A_j \subseteq A_jC^*, j \in J\} = \mathbb{C}I.$$

These and other definitions and facts from the general theory of unbounded representations of algebras and relations can be found, for example, in [S].

Remark 4. The operators $p = p^*$, $q = q^*$, r, s, r^*, s^* satisfying the conditions of Definition 1 define a representation, π, of $\mathcal{A}_{4,0}$ and $\mathcal{Q}_{4,0}(*)$ on the domain Φ in the sense of definition given section 2.3. This domain is not unique and therefore there are many $*$-representations of $\mathcal{A}_{4,0}$ corresponding to the closed operators $p = p^*$, $q = q^*$, r, s, r^*, s^*. Among them there is a unique selfadjoint representation π^*. This representation is irreducible iff the family of the closed operators is irreducible, two such representations are unitarily equivalent iff the corresponding families of closed operators are unitarily equivalent (see Remark 1).

In what follows we mean these selfadjoint representations when we talk about integrable representation of the $*$-algebra $\mathcal{A}_{4,0}$.

Let O_x be the trajectory of the point x with respect to the mappings $F_1(x) = 1 - x$, $F_2(x) = -1 - x$, i.e., $O_x = \{F_{i_1} \dots F_{i_n}(x) \mid i_k \in \{1,2\}, n \in \mathbb{N}\} = \{(-1)^n(x-n), (-1)^n(-x-n) \mid n \in \mathbb{N} \cup \{0\}\}$.

Let $O_0^+ = \{F_{i_1} \dots F_1(0) \mid i_k \in \{1,2\}, n \in \mathbb{N}\} = \{(-1)^{k+1}k \mid k \in \mathbb{N}\}$ and $O_0^- = \{F_{i_1} \dots F_2(0) \mid i_k \in \{1,2\}, n \in \mathbb{N}\} = \{(-1)^k k \mid k \in \mathbb{N}\}$.

Denote by $l_2(K)$ the separable Hilbert space with the orthonormal basis $\{e_\mu\}_{\mu \in K}$

Theorem 1. *Any irreducible integrable representation π of the $*$-algebra $\mathcal{A}_{4,0}$ in a Hilbert space H such that* $\ker p \neq \{0\}$ *is unitarily equivalent to one of the following:*

I. $H = l_2(O_\lambda)$

$$\begin{aligned} pe_\mu &= \mu e_\mu \\ qe_\mu &= -\mu e_\mu \\ re_\mu &= (1-\mu)e_{1-\mu} \\ se_\mu &= -(1+\mu)e_{-1-\mu} \end{aligned} \tag{12}$$

where $\lambda \in (-1/2, 1/2) \setminus \{0\}$.

II. $H = l_2(O_{1/2})$

$$\begin{array}{rcl} pe_\mu & = & \mu e_\mu \\ qe_\mu & = & -\mu e_\mu \\ re_\mu & = & \left\{ \begin{array}{ll} ae_{1/2} & \mu = 1/2 \\ (1-\mu)e_{1-\mu} & \mu \neq 1/2 \end{array} \right. \\ se_\mu & = & -(1+\mu)e_{-1-\mu} \end{array} \qquad (13)$$

where $a = \pm 1/2$, $s = \pm 1$.

III. $H = l_2(O_{-1/2})$

$$\begin{array}{rcl} pe_\mu & = & \mu e_\mu \\ qe_\mu & = & -\mu e_\mu \\ re_\mu & = & (1-\mu)e_{1-\mu} \\ se_\mu & = & \left\{ \begin{array}{ll} ae_{-1/2} & \mu = -1/2 \\ -(1+\mu)e_{-1-\mu} & \mu \neq -1/2 \end{array} \right. \end{array} \qquad (14)$$

where $a = \pm 1/2$, $s = \pm 1$.

IV. $H = l_2(O_0^-)$

$$\begin{array}{rcl} pe_\mu & = & \mu e_\mu \\ qe_\mu & = & -\mu e_\mu \\ re_\mu & = & (1-\mu)e_{1-\mu} \\ se_\mu & = & \left\{ \begin{array}{ll} 0 & \mu = -1 \\ -(1+\mu)e_{-1-\mu} & \mu \neq -1 \end{array} \right. \end{array} \qquad (15)$$

V. $H = l_2(O_0^+)$

$$\begin{array}{rcl} pe_\mu & = & \mu e_\mu \\ qe_\mu & = & -\mu e_\mu \\ re_\mu & = & \left\{ \begin{array}{ll} 0 & \mu = 1 \\ (1-\mu)e_{1-\mu} & \mu \neq 1 \end{array} \right. \\ se_\mu & = & -(1+\mu)e_{-1-\mu} \end{array} \qquad (16)$$

Proof. Let p, q, r, s be closed operators satisfying the conditions $(1)-(3)$ of Definition 1, $E_p(\cdot)$ the resolution of the identity for the selfadjoint operator p and $r = u_r|r|$, $s = u_s|s|$ the polar decompositions of the closed operators r, s. Here $|r| = (r^*r)^{1/2}$, $|s| = (s^*s)^{1/2}$, $\ker u_r = \ker r = \ker |r|$ and $\ker u_s = \ker s = \ker |s|$. By [OS2], we conclude that

$$p,\ |r| \text{ and } p,\ |s| \text{ commute strongly,}$$

(i.e. in the sense of resolutions of the identities)

$$E_p(\Delta)u_r = u_r E_p(1-\Delta),\ E_p(\Delta)u_s = u_s E_p(-1-\Delta),\ \forall \Delta \in \mathfrak{B}(\mathbb{R}). \qquad (17)$$

Here $\mathfrak{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. Assume first that $\ker p(1-p)(1+p) = \{0\}$. Since $r^2\varphi = p(1-p)\varphi$ and $s^2\varphi = -p(1+p)\varphi$, $\varphi \in \Phi$ we have that $\ker r \subset \ker r^2 = \{0\}$, $\ker s \subset \ker s^2 = \{0\}$ and u_r, u_s are unitary operators. The equality $pr^*\varphi = rp\varphi$ gives $pr^*r\varphi = r^2(1-p)\varphi$ which implies $|r| = |1-p|$. From (17) one can easily derive that $u_r\mathcal{D}(|1-p|) \in \mathcal{D}(|p|)$ and

$$|p|u_r\psi = u_r|1-p|\psi, \quad \psi \in \mathcal{D}(|1-p|) = \mathcal{D}(p).$$

From $r^2\varphi = p(1-p)\varphi$ we have $u_r|r|r\varphi = p(1-p)\varphi$ for any $\varphi \in \Phi$ and, since $r\varphi \in \mathcal{D}(p)$, $|p|u_r^2|1-p|\varphi = p(1-p)\varphi$ and $u_r^2 = \text{sgn}(p(1-p))$. Setting $u_1 = \text{sgn}(p)u_r$, we get $r = u_1(1-p)$. Similarly, $|s| = |1+p|$, $u_s^2 = \text{sgn}(-p(1+p))$ and $s = -u_2(1+p)$, where $u_2 = \text{sgn}(p)u_s$.

It follows from (17) that if $\Delta \in \mathfrak{B}(\mathbb{R})$ is invariant with respect to the mappings $F_1(\lambda) = 1-\lambda$, $F_2(\lambda) = -1-\lambda$ then $E_p(\Delta)$ commutes with p, r, s, r^*, s^* in the sense $E_p(\Delta)T \subseteq TE_p(\Delta)$ for $T = p, q, r, s, r^*, s^*$. Therefore, if $(p,\ q,\ r,\ s,\ r^*,\ s^*)$ is irreducible then $E_p(\Delta) = cI$, where $c = 0, 1$. One can easily check that the set $\tau = [-1/2, 1/2]$ intersects every trajectory O_λ exactly in one points which implies that the spectral measure $E_p(\cdot)$ is concentrated on an orbit O_λ for some $\lambda \in [-1/2, 1/2]$ if the representation is irreducible. Searching irreducible representation we can assume now that the representation space H is a direct sum $\oplus_{\mu\in O_\lambda} H_\mu$, where H_μ is an eigenspace of p corresponding to the eigenvalue μ.

If $\lambda \neq 0, \pm 1/2$ one can easily check that the linear span of the vectors $\{u_{i_1}\dots u_{i_k}e \mid i_l \in \{r,s\}, k \in \mathbb{N}\cup\{0\}\}$, where $e \in H_\lambda$, is invariant with respect to the operators p, q, r, s, r^*, s^* and moreover the operators restricted to the closure of this subspace define an irreducible representation which is given by formulae (12).

If $\lambda = 1/2$ then $u_r H_{1/2} \subset H_{1/2}$. It is not difficult to see that, given an irreducible representation, the operator u_r has an eigenvector $e \in H_{1/2}$ and the vectors $\{u_{i_1}\dots u_{i_l}u_s e \mid i_k \in r, s, k \in \mathbb{N}\cup\{0\}\}$ build a basis of the representation space The corresponding irreducible representation is given by (13).

Representations related to the orbit $O_{-1/2}$ can be obtained in a similar way. Note that there is no representation related with the trajectory O_0 such that $\ker p(1-p)(1+p) = \{0\}$.

If $\ker(1-p)(1+p) \neq \{0\}$ and $\ker p = \{0\}$ then using the same arguments one can show that $r^*r = |1-p|$ $s^*s = |1+p|$, $u_r^2 = \text{sgn}(p(1-p))$, $u_s^2 = \text{sgn}(-p(1+p))$ and $u_r^*|_{ker(1-p)} = 0$, $u_s^*|_{ker(1+p)} = 0$. Moreover $W_1 = \oplus_{k\in\mathbb{N}} u_s(u_r u_s)^k \ker(1-p) \oplus \oplus_{k\in\mathbb{N}}(u_r u_s)^k \ker(1-p)$ and $W_2 = \oplus_{k\in\mathbb{N}} u_r(u_s u_r)^k \ker(1+p) \oplus \oplus_{k\in\mathbb{N}}(u_s u_r)^k \ker(1+p)$ are invariant with respect to the operators p, q, r, s, r^*, s^* and the corresponding irreducible representation are given by (16) and (15) respectively.

Clearly, $W_1 \perp W_2$ and any representation space H can be decomposed into a direct sum of invariant with respect to the representation subspaces, namely, $H = W_1 \oplus W_2 \oplus W_3$, where $W_3 = (W_1 \oplus W_2)^\perp$. Moreover, if $\ker p = \{0\}$, we obtain $\ker p(1-p)(1+p)|_{W_3} = \{0\}$. Setting $u_1 = \text{sgn}(p)u_r$ on $(\ker(1-p))^\perp$, $u_2 = \text{sgn}(p)u_s$ on $(\ker(1+p))^\perp$ and extending them to $\ker(1-p)$ and $\ker(1+p)$ in a way that u_1, u_2 are unitary and satisfying (17) we get that the operators $\tilde{p} = pp_1$, $\tilde{q} = qp_1$, $\tilde{r} = u_1(1-p)p_1$, $\tilde{s} = -u_2(1+p)p_1$, where p_1 is the projection onto W_1, define a representation of $\mathcal{A}_{4,0}$ on W_1, and $\hat{p} = pp_2$, $\hat{q} = qp_2$ $\hat{r} = u_1(1-p)p_2$, $\hat{s} = -u_2(1+p)p_2$, where p_2 is the projection onto W_2, define a representation of $\mathcal{A}_{4,0}$ on W_2. Moreover, any representation on W_1 and W_2 can be obtained this way. The proof is finished. □

2. C^*-algebra level

In the sequel, we use the following notation. The set of multiplier of a C^*-algebra A is denoted by $M(A)$. The notation $T\eta A$ means T is affiliated with the algebra A and z_T denotes its z-transform. We write $Mor(A, B)$ for the set of morphisms from A to another C^*-algebra B. For the definition and facts related to these notions we refer the reader to [W1].

It follows from the proof of Theorem 1 that any representation $(p,\ q,\ r,\ s)$ in a Hilbert space H provided $\ker p = \{0\}$ is of the form: $a = a^1 \oplus a^2 \oplus a^3$, where $a \in \{p, q, r, s, r^*, s^*\}$, a^i are operators on W_i , $i = 1, 2, 3$ described in the proof. Moreover, $p^i = (p^i)^*$, $q^i = -p^i$,

$\ker(1-p^3)p^3(1+p^3)=\{0\}$, $Sp(p^1)\subset O_0^+$, $Sp(p^2)\subset O_0^-$, $r^i=u_1^i(1-p^i)$, $s^i=-u_2^i(1+p^i)$, where u_1^i, u_2^i are unitary operators such that

$$(u_1^i)^2=1,\ (u_2^i)^2=1,\ (u_1^i)^*p^iu_1^i=1-p^i,\ (u_2^i)^*pu_2^i=-1-p^i,\ i=1,2,3. \tag{18}$$

Our aim now is to define a C^*-algebra $\mathfrak{A}$ generated by selfadjoint element $p=p^*$ and unitary elements u_1, u_2 satisfying (18) and affiliated with $\mathfrak{A}$. Namely, we look for C^*-algebra with the following universal property: for any C^*-algebra $\mathfrak{A}'$ and any U_1, U_2, $P\ \eta\ \mathfrak{A}'$ such that U_1, U_2 are unitary, P is selfadjoint and $U_1^*PU_1=1-P$, $U_2^*PU_2=-1-P$, $U_1^2=1$, $U_2^2=1$, there exists unique $\Phi\in Mor(\mathfrak{A},\mathfrak{A}')$ such that $\Phi(u_1)=U_1$, $\Phi(u_2)=U_2$, $\Phi(p)=P$.

Let $\mathfrak{A}=C_\infty(\mathbb{R})\ltimes_\alpha(\mathbb{Z}_2\times\mathbb{Z}_2)$, where $C_\infty(\mathbb{R})$ is the algebra of all continuous, vanishing at infinity functions on $\mathbb{R}$. The action α of $\mathbb{Z}_2\times\mathbb{Z}_2$ on $C_\infty(\mathbb{R})$ is defined by

$$(gf)(\xi)=f(g\xi),$$

where the action on the real line $\mathbb{R}$ is given by

$$g_1\xi=1-\xi,\ g_2\xi=-1-\xi$$

for the generators $g_1\in\mathbb{Z}_2$, $g_2\in\mathbb{Z}_2$.

Then there exist unitary operators u_1, $u_2\in M(\mathfrak{A})$ such that

$$u_1^*fu_1=g_1f,\ u_2^*fu_2=g_2f, u_1^2=1,\ u_2^2=1.$$

Let now p be the function defined by $p(\xi)=\xi$ for all $\xi\in\mathbb{R}$. Clearly, $p\ \eta\ C_\infty(\mathbb{R})$, $(1-p)\ \eta\ C_\infty(\mathbb{R})$, $(-1-p)\ \eta\ C_\infty(\mathbb{R})$ and since the embedding $C_\infty(\mathbb{R})\hookrightarrow C_\infty(\mathbb{R})\ltimes_\alpha(\mathbb{Z}_2\times\mathbb{Z}_2)$ is in $Mor(C_\infty(\mathbb{R}),\mathfrak{A})$ we have $p,(1-p),(-1-p)\ \eta\ \mathfrak{A}$ and $u_1^*pu_1=g_1p$, $u_2^*pu_2=g_2p$.

Clearly, $\mathfrak{A}$ possesses the universality property defined above.

Proposition 8. *The elements $r=u_1(1-p)$, $s=-u_2(1+p)$ are affiliated with $\mathfrak{A}$. Moreover, there is a dense domain, D, of $\mathfrak{A}$ such that relations (10) hold on D.*

Proof. The first statement follows from [W1, Example 2] and the fact that u_1, $u_2\in M(\mathfrak{A})$ are invertible and $(1-p),(1+p)\ \eta\ \mathfrak{A}$. In this case $D(r)=D(s)=D(p)$. One can easily check also that the relations hold on $D(p^2)$ which is dense in $\mathfrak{A}$. □

By [W2][Theorem 3.3], the affiliated elements p, q, r, s generate the C^*-algebra $\mathfrak{A}$: for any Hilbert space H, any C^*-subalgebra, B, of $B(H)$ and any non-degenerate representation π of $\mathfrak{A}$ on H we have $\pi(X_i)\ \eta\ B$ for $X_i=p,q,r,s$ implies $\pi\in Mor(\mathfrak{A},B)$.

We see also that any representation of $\mathfrak{A}$ generates a representation of $\mathcal{A}_{4,0}$ satisfying the conditions of Definition 1. Moreover, any such irreducible representation is unitarily equivalent either to one from Theorem 1 or to one-dimensional zero representation $\pi(x)=0$, $x=p$, q, r, s. Conversely, for any representation P, Q, R, S, R^*, S^* of $\mathcal{A}_{4,0}$ defined in Definition 1 and such that $\ker P=\{0\}$ there exists a representation π of $\mathfrak{A}$ having the property $X=\pi(x)$, $(X,x)=(P,p),(Q,q),(R,r),(S,s)(R^*,r^*),(S^*,s^*)$, where $\pi(x)$ is the unique extension of π to the affiliated elements.

One can also define idempotents q_1, q_2, q_3, q_4 with the zero sum in a way that all of them are affiliated with $\mathfrak{A}$. Further we will use the following statement from [W1]:

Let A be a C^*-algebra; a, b, c, $d\in M(A)$ and $Q=\begin{pmatrix} d, & -c^* \\ b & a^*\end{pmatrix}$. Assume that (1) $ab=cd$, (2) a^*A is dense in A, (3) dA is dense in A, (4) $Q(A\oplus A)$ is dense in $A\oplus A$. Then there exists $T \ \eta \ A$ such that 1. dA is a core for T and $Tdx = bx$ for any $x \in A$. 2. For any $x, y \in A$

$$\left(\begin{array}{c} x \in D(T) \text{ and} \\ y = Tx \end{array} \right) \Leftrightarrow (ay = cx)$$

If Q is invertible then $D(T) = dA$.

Using this statement we prove the following

Proposition 9. *There exists elements q_1, q_2, q_3, $q_4 \ \eta \ \mathfrak{A}$ and a dense domain D such that D is invariant with respect to q_i, q_i^*, D is a core for any q_i and q_i^* and the relations $\sum_{i=1}^4 q_i = 0$, $\sum_{i=1}^4 q_i^* = 0$ and $q_i^2 = q_i$, $(q_i^*)^2 = q_i^*$ hold on D. Moreover, $q_1 x = px + rx$, $q_2 x = px - rx$, $q_3 x = -px + sx$, $q_4 x = -px - sx$ for any $x \in D$.*

Proof. First we will prove the existence of q_1. Let

$$a = d = (1 - z_p^2)^{1/2}(1 - z_{1-p}^2)^{1/2}, \ b = c = z_p(1 - z_{1-p}^2)^{1/2} + u_1 z_{1-p}(1 - z_p^2)^{1/2}$$

Then, clearly, $ab = cd$ and $a^*\mathfrak{A} = d\mathfrak{A}$ is dense in $\mathfrak{A}$. We have also

$$Q^*Q = \left(\begin{array}{cc} d^*d + b^*b, & 0 \\ 0, & cc^* + aa^* \end{array} \right).$$

We state that $Q^*Q(\mathfrak{A} \oplus \mathfrak{A})$ is dense in $\mathfrak{A} \oplus \mathfrak{A}$. It is enough to see that $(d^*d + b^*b)\mathfrak{A}$ and $(cc^* + aa^*)\mathfrak{A}$ are dense in $\mathfrak{A}$. Assume that $\overline{(d^*d + b^*b)\mathfrak{A}} \neq \mathfrak{A}$. Then there exists a pure state w on $\mathfrak{A}$ such that $w((d^*d + b^*b)x) = 0$ for any $x \in \mathfrak{A}$. Let π be the GNS representation of $\mathfrak{A}$ acting on a Hilbert space H_π and $\Omega \in H_\pi$ be the corresponding cyclic vector such that

$$w(x) = (\Omega, \pi(x)\Omega)$$

for any $x \in \mathfrak{A}$. This gives that the range $R(\pi(d)^*\pi(d) + \pi(b)\pi(b)^*)$ belongs to the set $\{\varphi \in H_\pi \mid (\Omega, \varphi) = 0\}$. Since the operators $\pi(d)^*\pi(d)$ and $\pi(b)\pi(b)^*$ are positive and commute with each other, we have that $\Omega \in \ker \pi(b)\pi(b)^* \cap \ker \pi(d)^*\pi(d)$. This contradicts the statement that $d\mathfrak{A}$ is dense in $\mathfrak{A}$. Using the same arguments one can show that $\overline{(cc^* + aa^*)\mathfrak{A}} = \mathfrak{A}$. The statement about density of $Q(\mathfrak{A} \oplus \mathfrak{A})$ can be easily derived from the density of $Q^*Q(A \oplus A)$.

Let $q_1 \ \eta \ \mathfrak{A}$ be the operator from the previous statement. Then $d\mathfrak{A}$ is a core for q_1 and $q_1 x = px + rx$ for any $x \in d\mathfrak{A}$. Similarly, we can construct $q_i \ \eta \ \mathfrak{A}$, $i = 2, 3, 4$ such that $d\mathfrak{A}$ is a core for q_2, $q_2 x = px - rx$ for any $x \in d\mathfrak{A}$ and $d'\mathfrak{A} := (1 - z_p^2)^{1/2}(1 - z_{1+p}^2)^{1/2}\mathfrak{A}$ is a core for q_3 and q_4, $q_3 x = -px + sx$, $q_4 x = -px - sx$ for any $x \in d'\mathfrak{A}$. Set $D = D(p^2)$. Then $D = d\mathfrak{A} = d'\mathfrak{A}$ and D is a core for all idempotents q_1, q_2, q_3, q_4. Moreover, the relations $q_1 + q_2 + q_3 + q_4 = 0$, $q_i^2 = q_i$ hold on D. □

Remark 5. It was proved in [BES] that $\mathcal{Q}_{4,0}$ and therefore $\mathcal{Q}_{4,0}(*)$ is not trivially B-representable, i.e., there exists no non-trivial isomorphism of $\mathcal{Q}_{4,0}$ into a subalgebra of a Banach algebra and respectively $*$-isomorphism of $\mathcal{Q}_{4,0}(*)$ into a $*$-subalgebra of an involutive Banach algebra . We have shown that there exist a C^*-algebra $\mathfrak{A}$ and unbounded elements q_1, q_2, q_3, q_4 which are affiliated with $\mathfrak{A}$ and such that $q_1 + q_2 + q_3 + q_4 = 0$, $q_i^2 = q_i$, $q_1^* + q_2^* + q_3^* + q_4^* = 0$ and $(q_i^*)^2 = q_i^*$ on a dense invariant domain of $\mathfrak{A}$.

3. Again representations

Next result shows that the class of unbounded representations of $\mathcal{A}_{4,0}$ satisfying the conditions of Definition 1 is $*$-wild (see [T] for the definition of $*$-wild unbounded representations).

Proposition 10. *The class of integrable representations of $\mathcal{A}_{4,0}$ is $*$-wild.*

Proof. Let $\alpha, \beta > 0$ and let $\mathfrak{S}_2$ be the $*$-algebra generated by selfadjoint elements a and b. Consider the set $\mathfrak{R}$ of all representations π of $\mathfrak{S}_2$ such that $||\pi(a)|| \leq \alpha$, $||\pi(b)|| \leq \beta$. Denote by $\mathfrak{A}_{\alpha,\beta}$ the completion of $\mathfrak{S}_2/\{z : |||z||| = 0\}$ under $|||z||| = sup\{||\rho(z)||; \rho \in \mathfrak{R}\}$.

Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$, let P_k be the orthoprojection onto $\mathbb{C}\langle e_k\rangle$, $k \in \mathbb{Z}$. We consider operators v, w defined by $ve_k = e_{k+1}$, $ve_{k+1} = e_k$ if k is even and $we_k = e_{k+1}$, $we_{k+1} = e_k$ if k is odd. Clearly, $(P_{2k} + P_{2k+1})H$ (respectively $(P_{2k+1} + P_{2k+2})H$) is invariant with respect to v (respectively w).

Let now

$$\tilde{p} = \textstyle\sum_{k\neq 0}(-1)^{k+1}kP_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad \tilde{q} = \textstyle\sum_{k\neq 0}(-1)^{k}kP_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

$$\tilde{r} = (\textstyle\sum_{k\neq 0}(2k+1)vP_{2k} - 2kvP_{2k+1}) \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + vP_0 \otimes \begin{pmatrix} e & 0 & 0 \\ 0 & 2e & 0 \end{pmatrix},$$

$$\tilde{s} = (\textstyle\sum_{k\neq 0}(2k+1)wP_{2k+2} - (2k+2)P_{2k+1}) \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + wP_0 \otimes \begin{pmatrix} e & e & a+ib \\ 0 & e & e \end{pmatrix}.$$

Here e is the identity element in $\mathcal{A}_{\alpha,\beta}$. We write $\mathcal{H}$ for the Hilbert space $P_0H \oplus P_0H \oplus P_0H \oplus ((I-P_0)H \oplus (I-P_0)H)$ Let $CB(\mathcal{H})$ be the C^*-algebra of compact operators on $\mathcal{H}$. Direct verification shows that $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ are affiliated with the C^*-algebra $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ (the completion of the algebraic tensor product of $CB(\mathcal{H})$ and $\mathcal{A}_{\alpha,\beta}$ with respect to a C^*-norm, it does not depend which one). Moreover, since any representation of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ is of the form $V^{-1}(id \otimes \pi)V$, where V is a unitary operator, id is the identical representation of $CB(\mathcal{H})$ and π is a representation of $\mathcal{A}_{\alpha,\beta}$, one can show that $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$, $\tilde{r}^*$, $\tilde{s}^*$ separate representations of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$, i.e., if π_1, π_2 are different non-degenerate representations of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ then $\pi_1(x) \neq \pi_2(x)$, where x is one of p, q, r, s. In fact, if $V_1^{-1}(id \otimes \pi_1)(x)V_1 = V_2^{-1}(id \otimes \pi_2)(x)V_2$, $x = p, q, r, s, r^*, s^*$, direct verification shows that $V_2V_1^{-1} = I \otimes V$, where $V\pi_1 = \pi_2 V$ and therefore $V_1^{-1}(id \otimes \pi_1)V_1 = V_2^{-1}(id \otimes \pi_2)V_2$. Besides, since $(I + \tilde{p}^2)^{-1} = \sum_{k\neq 0}(1+k^2)^{-1}P_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$, $(I + \tilde{p}^2)^{-1} \in CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$. Therefore, by [W2, Theorem 3.3], $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ generate the C^*-algebra $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$.

Let $D = l.s.\ \{a \otimes b \mid a \in CB(\mathcal{H}), a \in \mathcal{F}, b \in \mathcal{A}_{\alpha,\beta}\}$, where $\mathcal{F}$ is the space of finite-dimensional operators in $\mathcal{H}$. Then D is dense in $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ and invariant with respect to $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$, D is a core for the elements $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ and $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ satisfy relations (10)–(11) on D. Moreover, with $\psi(p) = \tilde{p}$, $\psi(q) = \tilde{q}$, $\psi(r) = \tilde{r}$, $\psi(s) = \tilde{s}$ the representation $(\pi(\psi)(p))$, $\pi(\psi(q))$, $\pi(\psi(r)$, $\pi(\psi(s)))$ of $\mathcal{A}_{4,0}$ satisfies the condition of Definition 1 for any representation π of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$. From this it follows that the class R is $*$-wild.

The mapping ψ defines a functor F_ψ from the category $\mathcal{R}ep(\mathcal{A}_{\alpha,\beta})$ of non-degenerated representations of $\mathcal{A}_{\alpha,\beta}$ to the category $\mathcal{R}ep(\mathcal{A}_{4,0})$ as follows:

- $F_\psi(\pi)(x) = (id \otimes \pi)(\psi(x))$ for any $\pi \in \mathcal{R}ep(\mathcal{A}_{\alpha,\beta})$, $x = p, q, r, s$,
- $F_\psi(A) = E \otimes A$ for any operator A intertwining π_1 and $\pi_2 \in \mathcal{R}ep(\mathcal{A}_{\alpha,\beta})$.

Since $id \otimes \pi$ is a representation of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ it can be uniquely extended to affiliated elements $\psi(p)$, $\psi(q)$, $\psi(r)$, $\psi(s)$. It follows from [T] that the functor F_ψ is full. □

5 Representations of algebras $\mathcal{Q}_{n,\lambda}$ and $*$-algebras $\mathcal{Q}_{n,\lambda}(*)$, $n \geq 5$

5.1 Algebras $\mathcal{Q}_{n,\lambda}$, $n \geq 5$, and their representations

For each $n \geq 5$ and $\lambda \in \mathbb{C}$ the algebra $\mathcal{Q}_{n,\lambda}$ is non-zero and contains as a subalgebra the free algebra with two generators, [RSS].

In this paper we do not give the description of the whole set $\Lambda_{n,fd}$ for $n \geq 5$ but some facts concerning this set. For other results see, for example, [Wu2, Wa].

As it was noticed before, $\Lambda_{n,fd} \subset \mathbb{Q}$. On the other hand, $\Lambda_{n,fd}$ contains the set $\Sigma_{n,fd} = \{\alpha \in \mathbb{R} \mid \exists H, dimH < \infty$, orthoprojections $P_1, \dots, P_n$ such that $\sum P_k = \alpha I\}$, the last being studied in [KRS]. By [KRS], the following statement holds.

Proposition 11.

$$\mathbb{Q} \supset \Lambda_{n,fd} \supset \Lambda_n^1 \cup \Lambda_n^2$$

where

$$\Lambda_n^1 = \{0, 1 + \frac{1}{(n-1)}, 1 + \frac{1}{(n-2) - \frac{1}{(n-1)}}, \dots, 1 + \frac{1}{(n-2) - \frac{1}{(n-2) - \frac{1}{\ddots - \frac{1}{(n-1)}}}}, \dots\},$$

$$\Lambda_n^2 = \{1, 1 + \frac{1}{(n-2)}, 1 + \frac{1}{(n-2) - \frac{1}{(n-2)}}, \dots, 1 + \frac{1}{(n-2) - \frac{1}{(n-2) - \frac{1}{\ddots - \frac{1}{(n-2)}}}}, \dots, \}$$

As to the description of finite-dimensional representations of $\mathcal{Q}_{n,\Lambda}$, $\lambda \in \Lambda_{n,fd}$, up to similarity, this is an open question now.

Concerning the set $\Lambda_{n,bd}$, $n \geq 5$, we have the following

Proposition 12.

$$\Lambda_{n,bd} = \mathbb{C}, \quad (n \leq 5).$$

Proof. For each $\lambda \in \mathbb{C}$, we give, following [RS], a concrete construction of five idempotents $Q_i \in B(H)$, whose sum is equal to λI. Let $H = l_2 \oplus l_2 \oplus l_2$ and let $\mathcal{I}$ denote the identity operator on l_2. Define Q_i, $i = 1, \dots, 5$, in the following way:

$$Q_1 = \begin{pmatrix} a\mathcal{I} & 3a\mathcal{I} & b\mathcal{I} \\ a\mathcal{I} & 3a\mathcal{I} & b\mathcal{I} \\ a\mathcal{I} & 3a\mathcal{I} & b\mathcal{I} \end{pmatrix}, Q_2 = \begin{pmatrix} a\mathcal{I} & -3a\mathcal{I} & b\mathcal{I} \\ -a\mathcal{I} & 3a\mathcal{I} & -b\mathcal{I} \\ a\mathcal{I} & -3a\mathcal{I} & b\mathcal{I} \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} 4a\mathcal{I} & 0 & -2b\mathcal{I} \\ 0 & 0 & 0 \\ -2a\mathcal{I} & 0 & b\mathcal{I} \end{pmatrix}, Q_4 = \begin{pmatrix} 2c\mathcal{I} & 0 & 2dcS_1^* \\ 0 & 2c\mathcal{I} & 2dcS_2^* \\ S_1 & S_2\mathcal{I} & d\mathcal{I} \end{pmatrix}$$

$$Q_5 = \begin{pmatrix} 2c\mathcal{I} & 0 & -2dcS_1^* \\ 0 & 2c\mathcal{I} & -2dcS_2^* \\ -S_1 & -S_2\mathcal{I} & d\mathcal{I} \end{pmatrix}$$

where $a = (5-2\lambda)/6$, $b = (4\lambda - 7)/3$, $c = (3\lambda - 5)/4$, $d = (7 - 3\lambda)/2$, and S_1, S_2 are operators of a representation of the Cuntz algebra $\mathcal{O}_2$ ([Cu]). Recall that $\mathcal{O}_2$ is a unital

$*$-algebra generated by s_1, s_2, s_1^*, s_2^* and relations $s_1^*s_2 = 0$, $s_1^*s_1 = e = s_2^*s_2 = s_1s_1^* + s_2s_2^*$. Direct verification shows that $Q_i = Q_i^2$, $i = 1, \dots, 5$, and $Q_1 + Q_2 + Q_3 + Q_4 + Q_5 = \lambda I$. $\square$

Note that the construction which is given in the proposition is a generalization of an example in [BES] of five idempotents with zero sum.

5.2 $*$-Algebras $\mathcal{Q}_{n,\lambda}(*)$, $n \geq 5$ and their $*$-representations by bounded operators

As we already know, $\Lambda_{n,bd} = \mathbb{C}$ if $n \geq 5$. As to representation of $\mathcal{Q}_{n,\lambda}(*)$ we have the following

Proposition 13. *$*$-Algebra $\mathcal{Q}_{n,\lambda}(*)$ is not of type I for each $\lambda \in \mathbb{C}$ and $n \geq 5$, i.e. for each $\lambda \in \mathbb{C}$ it has a factor-representation which is not of type I.*

Proof. It is enough to show that the $*$-algebras $\mathcal{Q}_{5,\lambda}(*)$ is not of type I for each $\lambda \in \mathbb{C}$. It is known that the Cuntz algebra $\mathcal{O}_2$ is of not type I. So, there exists a factor-representation, ρ, of $\mathcal{O}_2$ such that the double commutant $\rho(\mathcal{O}_2)''$ is not of type I. Consider now a representation, π, of $\mathcal{Q}_{5,\lambda}(*)$ given in Proposition 12 with $S_i = \rho(s_i)$, $i = 1, \dots, 5$. Direct calculations show that the commutant $\pi(\mathcal{Q}_{5,\lambda}(*))'$ coincides with $\{diag(C, C, C) \mid C \in \rho(\mathcal{O}_2)'\}$ and $\pi(\mathcal{Q}_{5,\lambda}(*))'' = M_3(\mathcal{N})$, where $\mathcal{N} = \rho(\mathcal{O}_2)''$. Since $\mathcal{N}$ is not of type I, $M_3(\mathcal{N})$ is not of type I, completing the proof. $\square$

For many $\alpha \in \mathbb{R}$ we can say even more: there exist $\alpha \in \mathbb{R}$ such that $\mathcal{Q}_{n,\lambda}(*)$ and even $\mathcal{P}_{n,\alpha}$ is $*$-wild.

Proposition 14. *The $*$-algebras $\mathcal{Q}_{n,\alpha}(*)$ ($n \geq 5$) are $*$-wild for α from the following sets:*
(a) $\Lambda_{4,bd} = \{2 \pm 2/k (k \in \mathbb{N}), 2\}$,
(b) $\Lambda_{n,orb(2)} = \{\alpha_0 = 2, \alpha_k = (n-1) - 1/(\alpha_{k-1} - 1), k \in \mathbb{Z}\}$,
(c) $\Lambda_{n,orb(n/2)} = \{\alpha_0 = n/2, \alpha_k = (n-1) - 1/(\alpha_{k-1} - 1), k \in \mathbb{Z}\}$.

Proof. (a) $\mathcal{Q}_{n,\alpha}(*)$ ($n \geq 5$) is $*$-wild for $\alpha \in \Lambda_{4,bd}$ because $Q_{4,\alpha}(*)$ is $*$-wild for every $\alpha \in \Lambda_{4,bd}$ by Proposition 6.

(b) By [OS2][Theorem 57], [KS2][Theorem 4] the unital $*$-algebra $\mathcal{P}_{3,\perp 2} = \mathbb{C}\langle r, r_1, r_2 \mid r = r^*, r^2 = r, r_i^* = r_i, r_i^2 = r_i, r_1r_2 = 0\rangle$ is $*$-wild. Setting $\psi(p_1) = r$, $\psi(p_2) = e - r$, $\psi(p_3) = r_1$, $\psi(p_4) = r_2$, $\psi(p_5) = e - r_1 - r_2$ for the generators p_i, $i = 1, \dots, 5$, of $\mathcal{P}_{5,2}$, we obtain a $*$-epimorphism from $\mathcal{P}_{5,2}$ to $\mathcal{P}_{3,\perp 2}$ so that $\mathcal{P}_{3,\perp 2}$ is a factor $*$-algebra of $\mathcal{P}_{5,2}$. This shows that $\mathcal{P}_{5,2}$ and therefore $\mathcal{P}_{n,2}$, $n \geq 5$, is $*$-wild. Then the Coxter functors $F : \mathcal{P}_{n,\alpha} \to \mathcal{P}_{n,1+1/(n-\alpha-1)}$ and $R : \mathcal{P}_{n,\alpha} \to \mathcal{P}_{n,n-1-1/(\alpha-1)}$, constructed in [KRS], spread out the $*$-wildness to all points

$$\alpha \in \Lambda_{n,orb(2)} = \{\dots, \alpha_{-1} = 1 + \frac{1}{n-3}, \alpha_0 = 2, \alpha_1 = n-2, \dots, \alpha_k = (n-1) - \frac{1}{\alpha_{k-1} - 1}, \dots\}.$$

The set $\Lambda_{n,orb(2)}$ is the two-sided orbit of the point $\{2\}$ with respect to the dynamical system $\alpha \to f(\alpha) = (n-1) - 1/(\alpha - 1)$.

(c) By [OS2][Theorem 55], the unital $*$-algebra $\mathcal{P}_{3,2anti} = \mathbb{C}\langle w_i, i = 1,2,3 \mid w_i^* = w_i, w_i^2 = e, w_1w_2 = w_2w_1 = 0\rangle$ is $*$-wild. The $*$-algebra $\mathcal{P}_{5,5/2}$ is itsfactor $*$-algebra with corresponding $*$-epimorphism ψ given by $\psi(p_1) = (w_1 + \sqrt{3}w_2 + 2e)/4$, $\psi(p_2) = (w_1 - \sqrt{3}w_2 + 2e)/4$, $\psi(p_3) = (-w_1 + e)/2$, $\psi(p_4) = (w_3 + e)/2$, $\psi(p_5) = (-w_3 + e)/2$. Since $\mathcal{P}_{5,5/2}$ is a factor-$*$-algebra of $\mathcal{P}_{2n+1,(2n+1)/2}$ for each $n \geq 5$, we obtain that $\mathcal{P}_{2n+1,(2n+1)/2}$ is $*$-wild for any $n \geq 2$. Since the $*$-algebra $\mathcal{P}_{5,2}$ is $*$-wild, by the same arguments, we obtain that $\mathcal{P}_{6,3}$ and $\mathcal{P}_{2n,n}$, $n \geq 3$ are $*$-wild. Then, using the Coxter functors F and R we get that the $*$-algebras $\mathcal{P}_{n,\alpha}$ are $*$-wild for any

$$\alpha \in \Lambda_{n,orb(n/2)} = \{\dots, \alpha_{-1} = 1 + \frac{2}{n-2}, \alpha_0 = \frac{n}{2}, \dots, \alpha_k = (n-1) - \frac{1}{\alpha_{k-1} - 1}, \dots\}.$$

The set $\Lambda_{n,orb(n/2)}$ is the two-sided orbit of the point $\{n/2\}$ with respect to the dynamical system $\alpha \to f(\alpha) = (n-1) - 1/(\alpha - 1)$. □

Restricting ourselves to $*$-representations of $\mathcal{P}_{n,\alpha}$, or, equivalently, $*$-representations of $\mathcal{Q}_{n,\alpha}(*)$ with the condition that the images of q_i are selfadjoint, we can give the full classification of such $*$-representations for $\alpha \in \Lambda_n^1 \cup \Lambda_n^2$. If $\alpha \in \Lambda_n^1$ there exists a unique, up to unitary equivalence, irreducible representation of $\mathcal{P}_{n,\alpha}$, but if $\alpha \in \Lambda_n^2$ there are n unitarily non-equivalent irreducible representations of $\mathcal{P}_{n,\alpha}$ (see [KRS]).

5.3 Representations of $\mathcal{Q}_{n,\lambda}$, $n \geq 5$, by unbounded operators

We do not study here unbounded representations of $\mathcal{Q}_{n,\lambda}$, $n \geq 5$ as the structure of bounded representations of $\mathcal{Q}_{n,\lambda}$ ($n \geq 5$) is already very complicated.

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