Toeplitz Representations for Operators on Singular Manifolds*

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Abstract

A new approach to the construction of index formulas for elliptic operators on singular manifolds is suggested on the basis of K-theory of algebras and cyclic cohomology. The equivalence of Toeplitz and pseudodifferential quantizations, well known in the case of smooth compact manifolds, is extended to the singular case. An analog of the Toeplitz quantization for general symbol algebras arising in the analysis on singular manifolds is introduced and, as the first example, such a representation is constructed for the case of manifolds with conical singularities.

1 Introduction

1. In the last decade, the problem of finding index formulas for elliptic pseudodifferential operators on singular manifolds has been the subject of active research (e.g., see [6, 10, 12, 13, 16, 20] and other papers). Despite the numerous results obtained in this direction, the situation is far from

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being clear. Indeed, some of the formulas obtained earlier fail to express the index via the principal symbol (= an element of an appropriate Calkin algebra) alone (e.g., [10]), in other formulas, separate terms lack homotopy invariance (e.g., see [6]), and finally, the few formulas that combine both desirable properties (e.g., see [12, 20]) are valid only for the important but rather narrow class of operators satisfying certain symmetry conditions.

This situation is primarily caused by the complicated symbol structure for pseudodifferential operators on singular manifolds. It is well known that the appropriate notion of the symbol in the singular case involves many levels associated with various strata of the manifold. The components corresponding to adjacent strata satisfy certain matching conditions [18]. In a number of existing results, the index of the operator is expressed as a sum of contributions from these symbol components. These contributions are usually noninteger, and in view of the fact that there are matching conditions, it is not at all surprising that they lack homotopy invariance and have no straightforward topological or algebraic meaning. In particular, the usual relation between the index and characteristic classes of some vector bundle constructed from the symbol fails. The homotopy invariance can be ensured, but only by imposing some additional conditions (like the symmetry condition), which prevents one from writing out a meaningful index formula for *all* elliptic pseudodifferential operators. A detailed analysis of the symmetry type conditions and their role in obtaining invariant index formulas can be found in [12, 17], and we do not dwell on the topic here.

In the present paper, we propose another approach to the construction of index formulas in the singular case. This approach is based on K-theory of algebras and cyclic cohomology. In the framework of this approach, one has to take a slightly different viewpoint as to what a "good" index formula must be.

Instead of trying to use topological invariants of separate components of the symbol, we consider the symbol as a whole, that is, as an element of an appropriate symbol algebra. Thus, instead of topological objects, one deals with algebraic objects like the Chern character viewed as an element of the cyclic cohomology group of the symbol algebra. In the abstract framework, the scheme is well known. The pseudodifferential quantization (the term "extension" is also used), that is, the mapping taking symbols to pseudodifferential operators is a unital linear mapping

$$\tau : \mathcal{A} \longrightarrow \mathrm{End}(\mathfrak{H})$$

of the symbol algebra $\mathcal{A}$ into the algebra of operators in a Hilbert space[1] $\mathfrak{H}$ such that

$$\tau(a)\tau(b) - \tau(ab) = \rho(a,b) \tag{1}$$

is a compact operator for any $a, b \in \mathcal{A}$.

The quantization τ over $\mathcal{A}$ extends, in a natural way, to the matrix algebra $\mathrm{M}(\mathcal{A})$ (the mapping is applied to each entry of a matrix); if an element $\mathfrak{a} \equiv \left[a_{jk}\right] \in \mathrm{M}(\mathcal{A})$ is elliptic (i.e., invertible), then $\tau(\mathfrak{a})$ is Fredholm.

There exist standard algebraical procedures for constructing index formulas for a given quantization; see, e.g., [3]. There are, however, two requirements for these procedures to be actually applicable. The first, formal one requires the quantization to be of finite order, i.e., to have the operator $\rho(a,b)$ in (1) not only compact but belonging to some Schatten class. Another, informal condition requires the quantization τ to be expressed in sufficiently simple terms, presumably as a *Fredholm module*.

The standard pseudodifferential quantization does not fit into this framework. Therefore, the need arises to find a nicer equivalent representation.

Let us recall the notion of equivalence of quantizations. A quantization is said to be *trivial* if the corresponding mapping τ is a homomorphism. One says that two quantizations of $\mathcal{A}$ defined by almost homomorphisms

$$\tau_j : \mathcal{A} \longrightarrow \mathrm{End}(\mathfrak{H}_j), \quad j = 1, 2,$$

are *equivalent* if, possibly after adding trivial direct summands, there exists an (almost) unitary operator

$$U : \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$$

such that

$$\tau_1(a) = U^{-1}\tau_2(a)U$$

modulo compact operators for every $a \in \mathcal{A}$, where U^{-1} is an arbitrary almost inverse of U. This definition is in particular justified by the fact that in this case, obviously,

$$\operatorname{ind}\tau_1(\mathfrak{a}) = \operatorname{ind}\tau_2(\mathfrak{a})$$

for every invertible $\mathfrak{a} \in \mathrm{M}(\mathcal{A})$.

[1]We consider only bounded operators corresponding to zero-order symbols.

Now we can describe what kind of quantizations we would like to have. A *Fredholm module* is a quantization

$$\tau : \mathcal{A} \longrightarrow \mathrm{End}(\mathfrak{H})$$

of the form

$$\tau(a) = P\mu(a), \quad a \in \mathcal{A},$$

where

$$\mu : \mathcal{A} \longrightarrow \mathrm{End}(\mathfrak{W})$$

is a representation of $\mathcal{A}$ by bounded operators in $\mathfrak{W} = \mathfrak{H} \oplus \mathfrak{V}$ ($\mathfrak{V}$ is an auxiliary Hilbert space) and

$$P : \mathfrak{W} \longrightarrow \mathfrak{H}$$

is the orthogonal projection on $\mathfrak{H}$ in $\mathfrak{W}$. Moreover, the commutator $[P, \mu(a)]$ is required to be compact for every $a \in \mathcal{A}$. Furthermore, to apply the non-commutative geometry machinery, one has to assume that these commutators not only are compact, but also belong to some Neumann–Schatten class $\mathfrak{S}_p$ whenever a belongs to some dense local subalgebra $\mathcal{A}_\infty \subset \mathcal{A}$ (see [2]; in this case, one says that the Fredholm module is p-summable).

Although in the abstract setting every quantization is equivalent to some Fredholm module (see, e.g.,[2]), the general construction, being fairly complicated, is unsuitable for concrete index calculations; moreover, it does not guarantee the Schatten class property. To obtain a nicer Fredholm module representation, we recall the Toeplitz-pseudodifferential connection found in [8]. Given a closed n-dimensional manifold M (which we assume to be real-analytic), one can equip the co-ball bundle B^*M with a structure of a complex-analytic manifold with boundary S^*M, so that B^*M is strictly pseudoconvex. In this environment, the following Fredholm module over the algebra of continuous functions $C(S^*M)$ is considered. The Hilbert space $\mathfrak{W}$ is $L_2(S^*M)$, the representation $\mu : C(S^*M) \to \mathrm{End}(\mathfrak{W})$ is defined as the multiplication by the function in $C(S^*M)$, and P is the orthogonal projection on the subspace $\mathfrak{H}$ consisting of boundary values of functions analytic in the interior of B^*M. This means that $P\mu$ describes a *Toeplitz* quantization of the algebra $C(S^*M)$. The main result in [8], saying that this latter Fredholm module is equivalent to the pseudodifferential quantization, is also expressed as "pseudodifferential operators are Toeplitz operators in disguise." This equivalence played an important part in finding new proofs of the Atiyah–Singer formula as well as in understanding the algebraic structure of the pseudodifferential quantization.

The aim of this paper is to introduce an analog of the Toeplitz quantization for general symbol algebras arising in analysis on singular manifolds and, as the first example, construct such a representation for the case of manifolds with conical singularities.

2. To give a general definition of a Toeplitz quantization, one must have a deeper insight into the structure of symbol algebras on singular manifolds. Usually, this structure is as follows. Let M be a singular manifold. To each stratum Y of M, one associates a topological space $\widetilde{Y}$ (which is usually the cospheric bundle S^*Y or the cotangent bundle $T*Y$), and an element of the Calkin algebra of symbols of pseudodifferential operators is a collection $a = \{\mathbf{a}_Y\}_{Y\in\mathcal{Y}}$, where $\mathcal{Y}$ is the set of all strata of M and each $\mathbf{a}_Y$ is a continuous function on $\widetilde{Y}$ ranging in the space of bounded operators in a Hilbert space $\mathfrak{H}_Y$. For adjacent strata Y, Y', the components $\mathbf{a}_Y$ and $\mathbf{a}_{Y'}$ must satisfy certain matching conditions. Another way to describe this structure is in the terms of *solvable* algebras. The algebra $\mathcal{A}$ is said to be solvable if it admits a composition series of closed ideals

$$0 = \mathcal{J}_0 \subset \mathcal{J}_1 \subset \cdots \subset \mathcal{J}_k = \mathcal{A}$$

such that each factor $\mathcal{J}_{s+1}/\mathcal{J}_s$ is isomorphic to the algebra of continuous, tending to zero at infinity functions on some locally compact space, with values being compact operators in some Hilbert space. It is shown in [14] that for a rather general class of manifolds with singularities, the Calkin algebra is solvable, with structure governed by the geometry of singularities.

Here examples are in order.

Example 1. Let M be a smooth manifold. Then there is only one stratum $Y = M$, the symbol algebra is $C(S^*M)$, and there are no matching conditions.

Example 2. Let M be an n-dimensional manifold with conical singularities $\alpha_1, \dots, \alpha_N$ (we recall the definition in the next section). Then M consists of the n-dimensional stratum $Y_0 = \overset{\circ}{M} \equiv M \setminus \{\alpha_1, \dots, \alpha_N\}$ and the 0-dimensional strata $Y_j = \{\alpha_j\}$, $j = 1, \dots, N$. The symbol algebra $\mathcal{A}$ consists of elements of the form $a = (A_0, \mathbf{A}_1, \dots, \mathbf{A}_N)$, where $A_0 \in C(S^*Y_0)$ is the *interior symbol*, which must be extendable by continuity to the cospheric bundle $S^*M^\wedge$ of the *stretched manifold* $M^\wedge$ obtained from M by blowing up the conical points (cf. [18]), and the $\mathbf{A}_j$, $j = 1, \dots, N$, known as the *conormal symbol*, is a dilation-invariant operator in the space $L_2(K_j)$, where K_j is the cone at the point α_j (see below). It suffices to state the matching condition on the dense subset of $\mathcal{A}$ formed by elements $a = (A_0, \mathbf{A}_1, \dots, \mathbf{A}_N)$ such that $A_0 \in C^\infty(S^*M^\wedge)$ and $\mathbf{A}_j$, $j = 1, \dots, N$, is a *pseudodifferential operator* in $L_2(K_j)$ (or, equivalently, a pseudodifferential operator with parameter on the base Ω_j of the cone K_j). The matching condition then says that the principal symbol of the conormal symbol must coincide with the restriction of the interior symbol to $\partial S^*M^\wedge$.

Example 3. Let M be an n-dimensional manifold with edge X of dimension m. Thus, near X the manifold M locally has the structure $M \simeq \mathbb{R}^m \times \Omega \times \mathbb{R}_+$, where Ω is a smooth compact manifold of dimension $n - m - 1$. The symbol algebra consists of pairs $a = (A_0, \mathbf{A})$, where $A_0 \in C(S^*(M \setminus X))$ is the interior symbol and $\mathbf{A} \in C\big(S^*X, \operatorname{End}(L_2(K_\Omega))\big)$ is the edge symbol. Here K_Ω is the cone with base Ω. Again, there is a matching condition saying (in terms of a dense subset of the symbol algebra) that the principal symbol of the edge symbol must be consistent with the values of the interior symbol when approaching the edge.

Analytic expressions of matching conditions in various cases may be different, but all of them have the same meaning. If in some small region in M the operator can be defined via symbols associated with two adjacent strata, then the two definitions must differ by a compact operator. The structure of the spaces $\widetilde{Y}_j$ is actually determined by irreducible representations of the symbol algebra. These representations are studied in detail in [14, 15]. Associated with each stratum is a series of representations. For the main stratum $\overset{\circ}{M}$, they are labelled by points of $S^*\overset{\circ}{M}$ and are one-dimensional. For the singular strata, one usually has infinite-dimensional representations (or finite-dimensional but nonscalar representations, in the case of one-dimensional M). The topology of the space of representations is very complicated and reflects the matching conditions in a nontrivial way.

By passing to the direct integral over each stratum, we obtain a representation μ of the symbol algebra $\mathcal{A}$ in the Hilbert space

$$\mathfrak{W}_{\mathcal{A}} = \bigoplus_{Y \in \mathcal{Y}} L_2(\widetilde{Y}, \mathfrak{H}_Y), \tag{2}$$

where an element $\mathbf{a} = \{a_Y\}_{Y \in \mathcal{Y}} \in \mathcal{A}$ acts in each component $L_2(\widetilde{Y}, \mathfrak{H}_Y)$ by pointwise multiplication (on the left) by the corresponding scalar or operator function $\mathbf{a}_Y$. In view of this, we shall omit μ and write a instead of $\mu(a)$.

Definition 4. Suppose that $\mathcal{A}$ is a C*-algebra realized as a subalgebra in $\oplus_{Y \in \mathcal{Y}} C(\widetilde{Y}, \mathrm{End}(\mathfrak{H}_Y))$. A *Toeplitz* quantization of $\mathcal{A}$ is a mapping $\tau : \mathcal{A} \to \mathrm{End}(\mathfrak{H})$, $\tau(a) = Pa$, where $\mathfrak{H}$ is a subspace in (2), P is the orthogonal projection $P : \mathfrak{H} \to \mathfrak{W}$ and $[P, a]$ is a compact operator in $\mathfrak{W}$ for any $a \in \mathcal{A}$. If, moreover, the operator $[P, a]$ belongs to the Schatten class $\mathfrak{S}_p$ for all a in a dense local subalgebra $\mathcal{A}_\infty$ in $\mathcal{A}$ for some $p \in (0, \infty)$, then we say that τ is a p-summable Toeplitz quantization.

Once we succeed in constructing this representation, we can apply the general index formula in terms of cyclic cohomology. Specifically, suppose that this Toeplitz representation is p-summable on a dense local subalgebra $\mathcal{A}_\infty \subset \mathcal{A}$. Then for an arbitrary elliptic element $\mathfrak{a} \in \mathrm{M}(\mathcal{A}_\infty)$ one has [3]

$$\mathrm{ind}(\tau(\mathfrak{a})) = b_N \big(\mathrm{ch}[\tau] \otimes \mathrm{tr}\big) \big(\underbrace{\mathfrak{a}^{-1}, \mathfrak{a}, \mathfrak{a}^{-1}, \mathfrak{a}, \dots, \mathfrak{a}^{-1}, \mathfrak{a}}_{N+1 \text{ arguments}}\big), \tag{3}$$

where $N > p^{-1}$ is an odd integer, tr is the matrix trace, the *Chern character* $\mathrm{ch}[\tau]$ of τ is the cyclic homology class of $\mathcal{A}_\infty$ given by the formula

$$\mathrm{ch}[\tau](a_0, a_1, \dots, a_N) = c_N \,\mathrm{Tr}\{a_0 [P, a_1][P, a_2] \cdots [P, a_N]\} \tag{4}$$

(note that the operator trace Tr on the right-hand side is well defined in view of the p-summability and the condition imposed on N), and the normalising constants b_N and c_N have the form

$$b_N = (2i)^{-1/2} 2^{-N+1} \Gamma(N/2 + 1), \quad c_N = (2i)^{1/2} (-1)^{N(N-1)/2} \Gamma(N/2 + 1).$$

This normalization of constants was proposed by Connes [3] in order to satisfy the functoriality and S-periodicity conditions.

This algebraic index formula has several advantages:

- first, it expresses the index via the principal symbol alone and is homotopy invariant;
- second, it is valid for arbitrary invertible symbols, not just for some subclass singled out by certain homotopic conditions;

- third, most importantly, it is expressed in terms of the cyclic homology class $\mathrm{ch}[\tau]$ of the algebra $\mathcal{A}_\infty$. *It is the cyclic (co)homology of the symbol algebra that replaces the* (co)*homology of a manifold as one passes from the algebra of functions on a smooth manifold to more general symbol algebras.*

In what follows we construct a Toeplitz representation for the simplest species of singular manifolds, namely, for manifolds with isolated singularities (Example 2). (As far as the index theory is concerned, the type of isolated singularities makes no difference, and we can always assume that they are conical; e.g., see [19].)

2 Manifolds with Isolated Singularities and Function Spaces

Manifolds with isolated singularities. We consider pseudodifferential operators on a compact manifold M with isolated (e.g., conical) singularities $\{\alpha_1, \dots, \alpha_N\}$. Let us recall the definition of a manifold with singularities for the case of conical points (details can be found, e.g., in [19].)

Definition 5. A *manifold M with conical singularities* $\alpha_1, \dots, \alpha_N$ is a compact Hausdorff topological space M with distinguished points $\alpha_1, \dots, \alpha_N$ such that

1) the set $\overset{\circ}{M} = M \setminus \{\alpha_1, \dots, \alpha_n\}$ is equipped with the structure of a C^∞ manifold compatible with the topology;

2) for each point α_j, a homeomorphism
$$\varphi_j : U_j \simeq K_{\Omega_j} \equiv \bigl\{\Omega \times [0,1)\bigr\} \Big/ \bigl\{\Omega \times \{0\}\bigr\} \tag{5}$$
of a neighborhood $U_j \subset M$ of α_j on the cone K_{Ω_j} with smooth compact base Ω_j is given;

3) the neighborhoods U_j are disjoint;

4) the mapping φ_j takes α_j to the vertex $\widetilde{\alpha}_j$ of the cone K_{Ω_j}, and the restriction of φ_j to $\overset{\circ}{U}_j = U_j \setminus \alpha_j$ is a diffeomorphism of $\overset{\circ}{U}_j$ onto $\overset{\circ}{K}_{\Omega_j} = K_{\Omega_j} \setminus \widetilde{\alpha}_j$.

The coordinate on the interval $[0,1)$ in the representation of the cone will be denoted by r and referred to as the *conical coordinate*. It is often convenient to deal with the infinite cone, where $r \in [0,\infty)$.

The definitions for other types of singular points can be found in [19].

To simplify the exposition, we assume that $N = 1$ and hence there is only one singular point, which will be denoted by α. However, all our considerations can readily be generalized to the case in which $N > 1$. Moreover, one can glue together all vertices, so that there is just one conical point, with the base of the cone being the disjoint union of Ω_j.

Cylindrical representation. We shall use the *cylindrical representation*, which is actually the same for all types of isolated singularities (see [11]). In this representation, the smooth open n-dimensional manifold $\overset{\circ}{M} = M \setminus \{\alpha\}$ looks like a manifold with a cylindrical end; more specifically, the punctured neighborhood $\overset{\circ}{U}$ of the conical point is represented as the direct product

$$\overset{\circ}{U} = \Omega \times \mathbb{R}_+,$$

where Ω is a smooth compact manifold without boundary (the *base of the singularity*) and $\mathbb{R}_+$ is the positive real half-line with coordinate t. It is assumed that $t \longrightarrow \infty$ corresponds to approaching the singular point. The generic point of Ω will be denoted by ω.

Remark 6. The function $t : \overset{\circ}{U} \longrightarrow \mathbb{R}_+$ defined on $\overset{\circ}{U}$ by virtue of the direct product structure, extends to a smooth function $t : \overset{\circ}{M} \longrightarrow \mathbb{R}$ nonpositive outside $\overset{\circ}{U}$, so that $\overset{\circ}{U} = \bigl\{x \in M \bigm| t(x) > 0\bigr\}$. In the sequel, we assume that some choice of the extension has been made. We adopt the convention that this function is defined on the entire M (respectively, on the stretched manifold $M^\wedge$, see below) and is equal to $+\infty$ at α (respectively, on the boundary $\partial M^\wedge$).

Sobolev spaces. We use the direct product structure on $\overset{\circ}{U}$ to introduce a Riemannian metric dx^2 on M such that in $\overset{\circ}{U}$ it has the form

$$dx^2 = dt^2 + d\omega^2,$$

where $d\omega^2$ is some Riemannian metric on Ω. Accordingly, the Beltrami–Laplace operator $\triangle_M$ on M has in $\overset{\circ}{U}$ the form

$$\triangle_M = \frac{\partial^2}{\partial t^2} + \triangle_\Omega,$$

where $\triangle_\Omega$ is the Beltrami–Laplace operator on Ω corresponding to the chosen metric and the measure $d\mu$ on M associated with the metric has in $\overset{\circ}{U}$ the form

$$d\mu = dt \wedge (d\omega)^{n-1},$$

where $(d\omega)^{n-1}$ is the measure on Ω associated with the metric $d\omega^2$. The operator $\triangle_M$ is essentially self-adjoint in $L_2(M, d\mu)$ on $C_0^\infty(\overset{\circ}{M})$.

The Sobolev spaces $H^s(M) \equiv H^s(M, d\mu)$, $s \in \mathbb{R}$, are the completions of $C_0^\infty\big(\overset{\circ}{M}\big)$ with respect to the norms

$$\|u\|_s = \left\{ \int \left|(1 - \triangle_M)^{s/2} u\right|^2 \, d\mu \right\}^{1/2}$$

(see [18, 19]). One also considers the weighted spaces $H^{s,\gamma}(M)$ with the norm $\|u\|_{s,\gamma} = \|e^{\gamma t} u\|_s$, where the function $e^{\gamma t}$ is assumed to be continued from $\overset{\circ}{U}$ to the entire M as a smooth nonvanishing function. This more general case can be reduced to the one considered here with the use of the isomorphism $e^{-\gamma t} : H^s \longrightarrow H^{s,\gamma}$; cf. [19]. The case $\gamma = 0$ is convenient in that the covariable p (dual to t) in symbols of pseudodifferential operators varies on the real axis rather than on the weight line $\mathfrak{L} = \{\operatorname{Im} P = \gamma\}$.

Since we mostly deal with pseudodifferential operators of degree 0, the space $L_2(M) \equiv L_2(M, d\mu) \equiv H^0(M)$ will be of primary importance to us. Occasionally, we use other spaces from this Sobolev scale.

Relationship with the usual representation. The original representation (5) of a neighborhood of a singular point is reduced to the cylindrical representation by a change of variables $r = f(t)$, where r is the radial variable (essentially, the distance from the singular point). The specific form of the function $f(t)$ depends on the type of the singular point. For example, for the case of a conical point the change of variables has the form $r = e^{-t}$. We point out that regardless of the type of the singular point, the corresponding change of variables reduces everything to the cylindrical representation; in particular, the weighted Sobolev spaces (with weight exponent $\gamma = 0$) naturally associated [19] with any type of the singular point are transformed by this change of variables into $H^s(M)$. (Accordingly, for $\gamma \neq 0$ they are taken to $H^{s,\gamma}(M)$.)

The cospheric bundle. In the cylindrical representation, one can define the cospheric bundle of M in the most convenient way without resorting to the (very important and highly geometric) intrinsic constructions like the "compressed cotangent bundle" (see [9]). Namely, we consider the cospheric bundle $S^*\overset{\circ}{M}$ and compactify it by attaching the manifold

$$\big[(T^*\Omega \times \mathbb{R}) \setminus \{0\}\big]/\mathbb{R}_+$$

(where $\{0\}$ is the zero section of the vector bundle $T^*\Omega \times \mathbb{R} \longrightarrow \Omega$) at $t = \infty$ as follows. In view of the direct product structure on $\overset{\circ}{U}$, for each point $(\omega, t) \in \overset{\circ}{U}$ we have the canonical isomorphism

$$S^*_{(\omega,t)} M \simeq \big[(T^*_\omega\Omega \times \mathbb{R}) \setminus \{0\}\big]/\mathbb{R}_+.$$

Now we say that a sequence $(x_k, \xi_k) \in S^*\overset{\circ}{M}$, where $x_k = (\omega_k, t_k)$ and $\xi_k \in \big[(T^*_{\omega_k}\Omega \times \mathbb{R}) \setminus \{0\}\big]/\mathbb{R}_+$, tends to $(\omega, \xi) \in \big[(T^*\Omega \times \mathbb{R}) \setminus \{0\}\big]/\mathbb{R}_+$ if $t_k \longrightarrow \infty$, $\omega_k \longrightarrow \omega$, and $\xi_k \longrightarrow \xi$. The compactified space S^*M thus obtained is a C^0 manifold with boundary

$$\partial S^*M = \big[(T^*\Omega \times \mathbb{R}) \setminus \{0\}\big]/\mathbb{R}_+. \tag{6}$$

Both the boundary and the interior $S^*\overset{\circ}{M}$ of this manifold carry a smooth structure compatible with the C^0 structure of the entire manifold. Needless to say, if we specify the type of the singular point (i.e., the mapping $t = f(r)$), then S^*M becomes a C^∞ manifold; however, this C^∞ structure depends on the type of the singular point. For brevity, we say that a function F on the cospheric bundle S^*M is C^∞ if it is continuous, its restrictions to ∂S^*M and $S^*\overset{\circ}{M}$ are both C^∞, and in the cylindrical coordinates all derivatives of F are bounded. The cospheric bundle is a (real) vector bundle over the *stretched manifold* $M^\wedge$, which is obtained from $\overset{\circ}{M}$ by a similar compactification procedure. Using the projection, we can lift the function t (see Remark 6) to S^*M; the resulting function is smooth on $S^*\overset{\circ}{M}$ and infinite on the boundary.

The double. In the subsequent argument, it is now and then convenient to interpret operators on M whose integral kernels are compactly supported in $\overset{\circ}{M} \times \overset{\circ}{M}$ as operators on some closed manifold. To obtain this closed manifold, we must essentially cut away a small "cap" near the conical point (which gives a manifold with boundary) and patch the "hole" with something smooth. There is a very old recipe for doing this: one simply attaches a second copy of the manifold with boundary along the boundary, thus obtaining what is called the double. We point out that this is a purely technical tool in our considerations, and all operators we treat this way will essentially vanish on the attached second part; in fact, one can attach anything that fits. (This is in contrast with the papers [12, 20], dealing with index problems for operators with symmetry conditions, where the passage to the double has a topological meaning and is restricted by topological obstructions.) To make things certain, let us assume that we work in cylindrical coordinates and cut away the cylindrical end along $t = 10$. The smooth structure on the double after attaching the second copy is then obtained automatically, since a direct product structure is given in a neighborhood of the cut (we leave details to the reader). The double will be denoted by $\mathcal{M}$. We can perform a similar cut-and-paste procedure with S^*M, which results in $S^*\mathcal{M}$. (Note, however, that the fibres of the cospheric bundle must be glued together via the involution $p \longmapsto -p$ rather than identically, where p is the dual variable of t. This is because the directions of the t-axis on the first and second copies are opposite.)

3 Pseudodifferential Operators

Pseudodifferential operators and symbols. For details concerning the definition of pseudodifferential operators on singular manifolds, we refer the reader to the book [19] and references therein. Further specification of this construction in the cylindrical representation can be found in [11]. We start by considering scalar pseudodifferential operators, for constructing the Toeplitz quantization. As was explained in the introduction, this quantization leads to index formulas for operators with matrix symbols as well. Moreover, we deal only with classical pseudodifferential operators of order 0. The algebra of such operators will be denoted by $\Psi(M)$. It is well known that every pseudodifferential operator $\widehat{a} : L_2(M) \longrightarrow L_2(M)$ is determined modulo compact operators by its *principal symbol*, which is a pair

$$\sigma(\widehat{a}) = \big(A(x,\xi), \mathbf{A}(p)\big)$$

consisting of an *interior symbol* $A(x,\xi)$ that is a C^∞ function on S^*M and an order zero *conormal symbol* $\mathbf{A}(p)$ that is a family of classical pseudodifferential operators with parameter $p \in \mathbb{R}$ on the manifold Ω in the sense of Agranovich–Vishik. Recall that by saying that a pseudodifferential operator is *classical* we mean that its complete symbol in any local coordinates admits an asymptotic expansion in homogeneous functions with step 1; for operators with parameter, the variable p, as well as all covariables, is included in the definition of homogeneity. In particular, in our case the requirement that $\mathbf{A}(p)$ is a family of classical pseudodifferential operators with parameter is equivalent to the following: the operator

$$\mathbf{A}\left(-i\frac{\partial}{\partial t}\right) : L_2(C_\Omega) \longrightarrow L_2(C_\Omega)$$

on the cylinder $C_\Omega = \Omega \times \mathbb{R}$ is a translation-invariant zero-order pseudodifferential operator. Note also that $\mathbf{A}(p)$ is defined and smooth in p for *all* $p \in \mathbb{R}$ (including $p = 0$) and that for every $k \geq 0$ the derivative $\mathbf{A}^{(k)}(p)$ satisfies the estimates

$$\begin{aligned} \left\|\mathbf{A}^{(k)}(p)\right\|_{L_2(\Omega)\longrightarrow L_2(\Omega)} &\leq C_k(1+|p|)^{-k} \\ \left\|\mathbf{A}^{(k)}(p)\right\|_{H^s(\Omega)\longrightarrow H^{s+k}(\Omega)} &\leq C_{sk}, \quad s \in \mathbb{R}. \end{aligned}$$

The elements of this pair must satisfy the matching condition

$$\sigma(\mathbf{A}(p)) = A(x,\xi)\big|_{\partial S^*M},$$

where $\sigma(\mathbf{A}(p))$ is the principal symbol of $\mathbf{A}(p)$ as an operator with parameter. (Note that the principal symbol $\sigma(\mathbf{A}(p))$ of the conormal symbol is defined on $(T^*\Omega \times \mathbb{R}) \setminus \{0\}$, which is identified with ∂S^*M by (6).)

The Calkin algebra. Ellipticity. The principal symbols $\sigma(\widehat{a})$ thus defined are just elements of the Calkin algebra

$$\mathcal{A}_\infty = \Psi(M)\Big/\{\mathcal{K} \cap \Psi(M)\}$$

of the algebra of zero-order classical pseudodifferential operators in $L_2(M)$. (Here $\mathcal{K}$ is the ideal of compact operators in $L_2(M)$.) The algebra $\mathcal{A}_\infty$ is a dense local subalgebra of the symbol algebra $\mathcal{A}$ described in Example 2. The multiplication law in $\mathcal{A}$ is defined separately on each component; it is given by pointwise multiplication of functions on the first component (interior symbols) and by multiplication of operator families on the second component. There exists a linear mapping

$$\tau_\Psi : \mathcal{A}_\infty \longrightarrow \Psi(M)$$

such that $\sigma \circ \tau_\Psi = \mathrm{id}$. The construction of this mapping, which uses partitions of unity, can be found in numerous places; we refer the reader to [18]. This mapping (unique and homomorphic modulo compact operators) will be called the *pseudodifferential quantization* on M.

The invertibility of the principal symbol (that is, the nonvanishing of $A(x, \xi)$ and the invertibility of $\mathbf{A}(p)$ for each $p \in \mathbb{R}$) is a sufficient (and in fact necessary) condition for the operator $\widehat{a}$ to be Fredholm. Invertible elements in $\mathcal{A}$ (and in $\mathrm{M}(\mathcal{A})$ are also said to be *elliptic*.

Remark 7. We nowhere require that $\mathbf{A}(p)$ be analytic. This requirement is only needed if one wishes to pass from one weight exponent γ to another (which necessitates considering complex values of p) or prove that the singularities of $\mathbf{A}^{-1}(p)$ are isolated, whereby the ellipticity condition can be achieved by an arbitrarily small shift in the weight exponent. Here we are interested in neither of the topics. The Toeplitz representation constructed in this paper is valid for arbitrary conormal symbols, and accordingly, so are the K-theoretic algebraic index formulas.

Toeplitz quantizations. Statement of the problem. Our aim is to find an index formula for elliptic pseudodifferential operators with symbols in $\mathrm{M}(\mathcal{A}_\infty)$ by explicitly constructing some Toeplitz quantization of $\mathcal{A}_\infty$ equivalent to the pseudodifferential quantization.

It will be convenient to us (and pretty sufficient) to solve the problem of constructing an equivalent Toeplitz representation not for the entire algebra $\mathcal{A}_\infty$ but rather for a subalgebra $\widetilde{\mathcal{A}}_\infty$ such that each elliptic element of $\mathrm{M}(\mathcal{A}_\infty)$ is homotopic via elliptic elements to an element elliptic in $\mathrm{M}(\widetilde{\mathcal{A}}_\infty)$. (In fact, it will turn out that ellipticity in $\mathrm{M}(\widetilde{\mathcal{A}}_\infty)$ and $\mathrm{M}(\mathcal{A}_\infty)$ is the same). This will provide index formulas for elliptic elements of $\mathrm{M}(\widetilde{\mathcal{A}}_\infty)$ and hence (by homotopy) for all elements of $\mathrm{M}(\mathcal{A}_\infty)$.

4 Construction of the Toeplitz quantization for the algebra $\mathcal{A}$

The representation space $\mathfrak{W}$. As was already mentioned, irreducible representations of $\mathcal{A}$ fall into two series [14, 15]. One series consists of one-dimensional representations numbered by points $(x, \xi) \in S^*\overset{\circ}{M}$ and given by the formula

$$\mu_{(x,\xi)}(a) = A(x, \xi), \quad a = (A(x, \xi), \mathbf{A}(p)) \in \mathcal{A}.$$

It corresponds to the first component of a. The other series consists of the infinite-dimensional representations

$$\mu_p(a) = \mathbf{A}(p), \quad p \in \mathbb{R},$$

in the Hilbert space $L_2(\Omega)$. It corresponds to the second component of a. This suggests that the representation μ occurring in the definition of Toeplitz quantization can be obtained as a direct integral of these representations.

More precisely, we take

$$\mathfrak{W} = L_2(S^*\overset{\circ}{M}) \oplus L_2(C_\Omega),$$

where the measure on $S^*\overset{\circ}{M}$ is smooth, translation invariant on the cylindrical end, and otherwise arbitrary; the representation μ is given by the formula

$$\mu(a)(u \oplus v) = \left(A(x, \xi)u, \mathbf{A}\left(-i\frac{\partial}{\partial t}\right)v\right).$$

Remark 8. The components of a are not independent: they are related by the matching condition. Hence it might be possible to obtain a narrower representation space by passing to a suitable subspace of the direct integral. However, this is by no means necessary: all we need is essentially to ensure that all irreducible representations of $\mathcal{A}$ be present in μ. Moreover, the consideration of irreducible representations only gives some motivation and serves no other purpose.

In the following, we often omit μ and write a instead of $\mu(a)$. This is harmless, since μ is faithful.

The subalgebra $\widetilde{\mathcal{A}}$. It is convenient not to construct a Toeplitz quantization for the entire algebra $\mathcal{A}$ using the above representation. Instead, we pass to the subalgebra $\widetilde{\mathcal{A}} \subset \mathcal{A}$ consisting of elements $a = (A(x,\xi), \mathbf{A}(p))$ such that $A(x,\xi) \equiv A(\omega,t,\xi)$ is independent of t for $t \geq 0$ (where the notation $x = (\omega,t)$ is well defined). Thus, actually $A(x,\xi) = A(\omega,\xi)$ for $t \geq 0$. The subalgebra $\widetilde{\mathcal{A}}$ obviously has the following properties:

- an element is invertible in $\mathrm{M}(\widetilde{\mathcal{A}})$ if and only if it is invertible in $\mathrm{M}(\mathcal{A})$;
- each elliptic element $a \in \mathrm{M}(\mathcal{A})$ is homotopic in $\mathrm{M}(\mathcal{A})$ via elliptic elements with the same conormal symbol to an element of $\mathrm{M}(\widetilde{\mathcal{A}})$.

The homotopy "sweeps" all the variation of $A(\omega,t,\xi)$ in t on the cylindrical end into a small half-neighborhood $(-\varepsilon < t < 0)$ of the section $\{t = 0\}$ of M.

The subspace $\mathfrak{H}$ and the pseudo-Guillemin transform. The Toeplitz representation and the main theorem. In this subsection we shall define a subspace $\mathfrak{H} \subset \mathfrak{W}$ and an almost isomorphism

$$\Gamma : L_2(M) \longrightarrow \mathfrak{W}$$

of $L_2(M)$ on $\mathfrak{H}$. This mapping will be called the pseudo-Guillemin transform (or the Guillemin transform for manifolds with singularities), since, as we shall see shortly, it is an analog of the ordinary Guillemin transform [4, 5, 8] for smooth compact manifolds. The orthogonal projection on $\mathfrak{H}$, together with the representation μ, will define the desired Toeplitz quantization of $\widetilde{\mathcal{A}}$, and the mapping Γ itself will be an equivalence (an almost intertwining operator) between this Toeplitz quantization and the pseudodifferential quantization.

We start from the description of Γ, while $\mathfrak{H}$ will be defined merely as the range of Γ. On the real line $\mathbb{R}$, we consider a partition of unity

$$1 = \left(\chi_1(t)\right)^2 + \left(\chi_2(t)\right)^2$$

such that the $\chi_j(t)$ are smooth real-valued functions and

$$\chi_1(t) = \begin{cases} 0, & t \leq 1, \\ 1, & t \geq 3. \end{cases}$$

Next, let $\psi(t)$ be a real-valued function such that $\operatorname{supp}\psi \subset \{t < 4\}$ and $\psi(t)\chi_2(t) = \chi_2(t)$. All these functions can be viewed as functions on M and C_Ω. Moreover, we can also treat them as functions on the double $\mathcal{M}$ if we extend them as continuous functions beyond the cut $t = 10$ by constant values on the newly attached second copy of M. Using the natural projections, we also lift these functions to S^*M, S^*C_Ω, and $S^*\mathcal{M}$.

Let

$$\mathrm{T} : L_2(\mathcal{M}) \longrightarrow L_2(S^*\mathcal{M})$$

be the Guillemin transform [8] for the compact closed C^∞ manifold $\mathcal{M}$. This is a Fourier integral operator associated with a certain positive complex canonical relation in $T_0^*\mathcal{M} \times T_0^*(S^*\mathcal{M})$. However, we do not need the explicit structure of T, but are only interested in the following properties of this operator.

Proposition 9 ([8]). *The following assertions hold:*

(1) T *is an operator of order* 0 *in Sobolev scales, that is,* $\mathrm{T} : H^s(\mathcal{M}) \longrightarrow H^s(S^*\mathcal{M})$ *is bounded for all* s;

(2) *the range* $R(\mathrm{T})$ *of* T *is closed*;

(3) $\mathrm{T}^*\mathrm{T} = \mathrm{id}$, *and* $\Pi = \mathrm{T}\mathrm{T}^*$ *is the projection on* $R(\mathrm{T})$;

(4) *for every smooth function* b *on* $S^*\mathcal{M}$, *the commutator* $[\Pi, b]$ *is an operator of order* -1 *in the Sobolev scale on* $S^*\mathcal{M}$ (*here and in the following,* b *is interpreted as the operator of point-wise multiplication by* b);

(5) *for every smooth function* b *on* $S^*\mathcal{M}$, *one has*

$$\mathrm{T}^* b \mathrm{T} = \widehat{b} + K,$$

where $\widehat{b}$ *is an arbitrary pseudodifferential operator of order zero on* M *with principal symbol* b *and* K *is an operator* (*actually, pseudodifferential*) *of order* -1 *in the Sobolev scale on* $\mathcal{M}$;

(6) *the preceding assertion can be equivalently restated as*

$$\mathrm{T}b - \widehat{b}\mathrm{T} = K_1,$$

where K_1 *is an operator of order* -1 *between Sobolev scales on* $\mathcal{M}$ *and* $S^*\mathcal{M}$.

Remark 10. The operators of order -1 arising in assertions 4–6 of this proposition are actually compact in L_2 (since both $\mathcal{M}$ and $S^*\mathcal{M}$ are compact manifolds) and belong to the respective Neumann–Schatten classes $\mathfrak{S}_k$, where $k > n$ or $k > 2n-1$ on $\mathcal{M}$ and $S^*\mathcal{M}$, respectively. Thus, the preceding theorem gives a Toeplitz representation of pseudodifferential operators on $\mathcal{M}$ (which, in fact, was used in [4, 5]). In the following, we adopt the convention that, when writing $B \in \mathfrak{S}_k$, we imply that k is an arbitrary number greater than the dimension of the manifold where the functions on which B acts are defined.

We define a mapping

$$\Gamma : L_2(M) \longrightarrow \mathfrak{W} = L_2(S^*M) \oplus L_2(C_\Omega)$$

by setting

$$\Gamma\varphi = \psi \mathrm{T} \chi_2 \varphi \oplus \chi_1 \varphi, \quad \varphi \in L_2(M). \tag{7}$$

This is well defined. Indeed, the function $\chi_2\varphi$ is supported in $\{t < 3\}$ and hence can be treated as a function on $\mathcal{M}$, whereby we can apply the Guillemin transform T. Next, the multiplication by ψ permits us to understand $\psi\mathrm{T}\chi_2\varphi$ as a function on S^*M (supported in $\{t < 4\}$) rather than $S^*\mathcal{M}$. Likewise, $\chi_1\varphi$ can be viewed as a function on the cylinder C_Ω.

Now we can state our main theorem.

Theorem 11. *The following assertions hold.*

1. *The range* $\mathfrak{H} = R(\Gamma)$ *of the mapping* Γ *is closed.*

2. *The mapping* $\Gamma : L_2(M) \longrightarrow \mathfrak{H}$ *is Fredholm and almost unitary.* (*The latter assertion means that*

$$\Gamma^*\Gamma = \mathrm{id}_{L_2(M)} + K_1, \quad \Gamma\Gamma^* = \mathrm{id}_{\mathfrak{H}} + K_2,$$

where K_1 *and* K_2 *are compact operators.*)

3. *The orthogonal projection* P *on* $\mathfrak{H}$ *in* $\mathfrak{W}$ *satisfies the condition that* $[P, a] \in \mathfrak{S}_k$, $k > n$, *for arbitrary* $a \in \widetilde{\mathcal{A}}$. *Thus,* P *determines a Toeplitz quantization of* $\widetilde{\mathcal{A}}$.

4. *One has*
$$\Gamma^* a \Gamma = \tau_\Psi(a)$$
modulo compact operators (in fact, modulo operators belonging to the Neumann–Schatten class $\mathfrak{S}_k$), so that Γ determines an equivalence between this Toeplitz quantization and the pseudodifferential quantization.

Thus, for the pseudodifferential quantization on a manifold with isolated singularities Γ defines a representation as a Toeplitz Fredholm module of order $k > n$.

Proof. One has
$$\Gamma^*\Gamma = \chi_2 \mathrm{T}^* \psi^2 \mathrm{T} \chi_2 + \chi_1^2 = 1 + K_1,$$
where K_1 is a self-adjoint pseudodifferential operator of order -1 on M with integral kernel supported in the compact set $\{t \le 3\} \times \{t \le 3\}$. It follows that $\Gamma^*\Gamma$ is Fredholm and $\operatorname{Ker}\Gamma^*\Gamma = \operatorname{Ker}\Gamma$ is finite-dimensional and consists of functions supported in $\{t \le 3\}$. By standard argument, we find that $R(\Gamma)$ is closed. Let Q be the orthogonal projection on $\operatorname{Ker}\Gamma$. Then the operator $B = \Gamma^*\Gamma + Q$ is an invertible pseudodifferential operator of order 0 and the inverse B^{-1} has the form $B^{-1} = 1 + K_2$ with K_2 a self-adjoint pseudodifferential operator of order -1 on M whose integral kernel is supported in the compact set $\{t \le 3\} \times \{t \le 3\}$. (Hence both K_1 and K_2 belong to $\mathfrak{S}_k$.) Now the projection on $R(\Gamma)$ has the form
$$P = \Gamma B^{-1} \Gamma^* = \begin{pmatrix} \psi \mathrm{T} \chi_2 B^{-1} \chi_2 \mathrm{T}^* \psi & \psi \mathrm{T} \chi_2 B^{-1} \chi_1 \\ \chi_1 B^{-1} \chi_2 \mathrm{T}^* \psi & \chi_1 B^{-1} \chi_1 \end{pmatrix}.$$
Next,
$$\begin{aligned} \{\Gamma\Gamma^*\}|_{\mathfrak{H}} &= \{\Gamma B^{-1}\Gamma^*\Gamma\Gamma^*\}|_{\mathfrak{H}} \\ &= \{\Gamma B^{-1}(1+K_1)\Gamma^*\}|_{\mathfrak{H}} \\ &= (1+K)|_{\mathfrak{H}}, \end{aligned}$$
where $K = \Gamma B^{-1} K_1 \Gamma^*$ is compact, and we have proved assertions 1 and 2. Since B^{-1} differs from the identity operator by an operator of the Neumann–Schatten class $\mathfrak{S}_k$, in the proof of assertion 3 we can safely replace the operator P by
$$\widetilde{P} = \Gamma\Gamma^* = \begin{pmatrix} \psi \mathrm{T} \chi_2^2 \mathrm{T}^* \psi & \psi \mathrm{T} \chi_2 \chi_1 \\ \chi_1 \chi_2 \mathrm{T}^* \psi & \chi_1^2 \end{pmatrix}$$

and estimate the commutator $[\widetilde{P}, a]$. Moreover, owing to the presence of the cutoff function ψ we can assume that the interior symbol A vanishes for $t > 9$ and hence $\widetilde{P}$ can be treated as an operator acting in $L_2(S^*\mathcal{M}) \oplus L_2(C_\Omega)$ when we proceed with the estimates. We have

$$[\widetilde{P}, a] = \begin{pmatrix} [\psi \mathrm{T} \chi_2^2 \mathrm{T}^* \psi, A] & \psi \mathrm{T} \chi_2 \chi_1 \widehat{\mathbf{A}} - A \psi \mathrm{T} \chi_2 \chi_1 \\ \chi_1 \chi_2 \mathrm{T}^* A - \widehat{\mathbf{A}} \chi_1 \chi_2 \mathrm{T}^* \psi & [\chi_1^2, \widehat{\mathbf{A}}] \end{pmatrix},$$

where we for brevity write

$$\widehat{\mathbf{A}} = \mathbf{A}\left(-i\frac{\partial}{\partial t}\right).$$

Now, by the properties of T,

$$[\psi \mathrm{T} \chi_2^2 \mathrm{T}^* \psi, A] \equiv [\Pi, A] \equiv 0.$$

(here and in the following $\equiv$ stands for equality modulo operators of the Neumann–Schatten class $\mathfrak{S}_k$). Next, we shall estimate $[\chi_1^2, \widehat{\mathbf{A}}]$. (Similar estimates of commutators of pseudodifferential operators and cutoff functions on the infinite cylinder can be found in [13].) To this end, we introduce a smooth partition of unity

$$\varphi_0(t) + \varphi_1(t) + \varphi_2(t) \equiv 1$$

on the real line such that the following properties hold:

- $\chi_1^2 \varphi_1 = \varphi_1$;
- $\chi_1^2 \varphi_2 = 0$;
- $\operatorname{supp} \varphi_0$ is contained in the interval $[0, 10]$ and $\varphi_0 = 1$ on $\operatorname{supp} \chi_1'$).

Now we represent $\widehat{\mathbf{A}}$ in the form

$$\widehat{\mathbf{A}} = \sum_{j,k=0}^{2} \varphi_j \widehat{\mathbf{A}} \varphi_k$$

and compute the commutator as follows:

$$
\begin{aligned}
[\chi_1^2, \widehat{\mathbf{A}}] &= \sum_{j,k=0,1,2} [\chi_1^2, \varphi_j \widehat{\mathbf{A}} \varphi_k] \\
&= [\chi_1^2, \varphi_0 \widehat{\mathbf{A}} \varphi_0] - [\chi_2^2, \varphi_0 \widehat{\mathbf{A}} \varphi_1] + [\chi_1^2, \varphi_0 \widehat{\mathbf{A}} \varphi_2] \\
&\quad - [\chi_2^2, \varphi_1 \widehat{\mathbf{A}} \varphi_0] + [\chi_1^2, \varphi_2 \widehat{\mathbf{A}} \varphi_0] + \sum_{j,k=1,2} (-1)^j [\chi_{3-j}^2, \varphi_j \widehat{\mathbf{A}} \varphi_k].
\end{aligned}
$$

Here we have used the fact that $\chi_1^2 + \chi_2^2 = 1$.

Now the first term can be interpreted as a pseudodifferential operator of order -1 on $\mathcal{M}$ (the support of its Schwartz kernel with respect to the variable t is contained in the square $[0,10] \times [0,10]$) and hence belongs to the desired Neumann–Schatten class $\mathfrak{S}_k$. All other terms have the form $a\widehat{\mathbf{A}}b$, where $a = a(t)$ and $b = b(t)$ are smooth functions constant at infinity with disjoint supports. For example,

$$
[\chi_2^2, \varphi_0 \widehat{\mathbf{A}} \varphi_1] = \chi_2^2 \varphi_0 \widehat{\mathbf{A}} \varphi_1 - \varphi_0 \widehat{\mathbf{A}} \varphi_1 \chi_2^2 = \chi_2^2 \varphi_0 \widehat{\mathbf{A}} \varphi_1;
$$

here we have $a = \chi_2^2 \varphi_0$ and $b = \varphi_1$.

The integral kernel of such an operator has the form

$$
K(x, x') = a(t) b(t') K_0(\omega, \omega', t, t'),
$$

where $K_0(\omega, \omega', t, t')$ is the integral kernel of $\widehat{\mathbf{A}}$. The kernel $K_0(\omega, \omega', t, t')$ is smooth outside the diagonal $t = t'$ and decays more rapidly than an arbitrary power of $|t - t'|^{-1}$. With regard for the arrangement of supports of $a(t)$ and $b(t)$, we have

$$
1 + |t| + |t'| \leq C|t - t'|
$$

with some constant C on

$$
\operatorname{supp} K(x, x') \subset \operatorname{supp} a(t) \times \operatorname{supp} b(t') \times \Omega \times \Omega.
$$

It follows that $K(x, x')$ is everywhere smooth and satisfies the estimates

$$
|K(x, x')| \leq C_N (1 + |t| + |t'|)^{-N}
$$

for all N; similar estimates are valid for the derivatives of $K(x, x')$. We conclude that the operator $a\widehat{\mathbf{A}}b$ belongs to all Neumann–Schatten classes. Thus, we have obtained the desired estimate for the commutator $[\chi_1^2, \widehat{\mathbf{A}}]$.

Consider the operator $\psi \mathrm{T}\chi_2\chi_1\widehat{\mathbf{A}} - A\psi\mathrm{T}\chi_2\chi_1$. We have

$$\psi\mathrm{T}\chi_2\chi_1\widehat{\mathbf{A}} - A\psi\mathrm{T}\chi_2\chi_1 \equiv \psi\mathrm{T}\chi_2\chi_1\widehat{\mathbf{A}} - \psi\mathrm{T}\widehat{A}\chi_2\chi_1,$$

where $\widehat{A}$ is some pseudodifferential operator with symbol A, and the desired assertion follows (since $\sigma(\widehat{\mathbf{A}}) = A$ on $\operatorname{supp}\chi_2\chi_1$) by an argument similar to the preceding. The estimate for the lower left entry of the commutator is similar.

It remains to prove assertion 4. We have

$$\Gamma^* a\Gamma = \chi_2\mathrm{T}^*\psi A\psi\mathrm{T}\chi_2 + \chi_1\widehat{\mathbf{A}}\chi_1 \equiv \chi_2^2\widehat{A} + \chi_1^2\widehat{\mathbf{A}} \equiv \tau_\Psi(a)$$

with regard for our conditions on the symbols in $\widetilde{\mathcal{A}}$. The proof is complete. $\square$

5 Conclusion

Let us summarise our results and discuss possible further directions of research. The main results are contained in the last theorem, where we construct a generalisation

$$\Gamma : L_2(M) \longrightarrow L_2(S^*M) \oplus L_2(C_\Omega)$$

of the Guillemin transform for manifolds with isolated singularities and establish the equivalence of the Toeplitz quantization related to this transform with the usual pseudodifferential quantization. It follows that

$$[\tau_\Psi] = [\tau_{\mathrm{Toeplitz}}] \in K^1(\mathcal{A}), \tag{8}$$

where $\mathcal{A}$ is the Calkin algebra of the algebra of pseudodifferential operators and $K^1(A)$ is the K-group of the operator algebra K-theory.

With this theorem at hand, we can readily write out the index formula for elliptic pseudodifferential operators on M. It has the form (3), where the order N of the Chern character (4) of our Toeplitz representation satisfies $N > 1/n$.

Although the index formulas provided by this equivalence may prove complicated in specific examples, they are undoubtedly of theoretical importance, since they express the index in an appropriate way via the principal symbol. We would also like to mention possible further directions of research in connection with this new transform.

First, one can try to generalize it to manifolds with nonisolated singularities. In this topical field, the situation with index formulas and, more generally, with topological invariants seems to be considerably more complicated, which, in particular, reflects itself in the absence of more or less general results.

Second, one can try to construct an analog of Toeplitz quantization on an arbitrary contact space with isolated singularities rather than on S^*M (for a smooth closed phase space, this was done by Boutet de Monvel [4] as early as in 1988).

Third, the following question is of interest. On a closed manifold, Eq. (8) can be continued as follows (e.g., see [1, 7]):

$$[\tau_\Psi] = [\tau_{\mathrm{Toeplitz}}] = [\tau_{\mathrm{Dirac}}], \tag{9}$$

where $[\tau_{\mathrm{Dirac}}]$ is the Toeplitz quantization on S^*M corresponding to the projection on the positive spectral subspace of the (self-adjoint) Dirac operator naturally defined on S^*M. This quantization is advantageous in that the Dirac operator is a first-order *differential* operator, and hence, after some manipulations, one can manage to obtain a purely local index formula containing only the symbol and its first-order derivatives. Thus, the question is whether one can obtain a reduction similar to (9) for a manifold with singularities and what kind of Dirac operator occurs in this case. If one manages to obtain the answers and the Dirac operator proves not to be too weird (in the sense that an analog of the Atiyah–Patodi–Singer formula still will be valid for this operator), then for the index of an elliptic operator $\widehat{a}$ on a manifold with isolated singularities one will obtain a relation of the form

$$\operatorname{ind} \widehat{a} = \int_M AS(A) + f(\mathbf{A}),$$

where the integrand $AS(A)$ is determined (in contrast, say, with the Melrose–Nistor formulas [10]) by the principal symbol alone. (Needless to say, neither term in this formula will be homotopy invariant.)

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