A NEW APPROACH TO QUANTUM SCATTERING NEAR THE LOWEST LANDAU THRESHOLD FOR A SCHRÖDINGER OPERATOR WITH A CONSTANT MAGNETIC FIELD

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ABSTRACT. For fixed magnetic quantum number m results on spectral properties and scattering theory are given for the three-dimensional Schrödinger operator with a constant magnetic field and an axisymmetrical electric potential V. Asymptotic expansions for the resolvent of the Hamiltonian $H_m = H_{om} + V$ are deduced as the spectral parameter tends to the lowest Landau threshold E_0. In particular it is shown that E_0 can be an eigenvalue of H_m. Furthermore, asymptotic expansions of the scattering matrix associated with the pair (H_m, H_{om}) are derived as the energy parameter tends to E_0.

1. INTRODUCTION

Spectral and scattering theory for the three-dimensional Schrödinger operator with a constant magnetic field

$$H(\boldsymbol{A}) = H_0(\boldsymbol{A}) + V(\boldsymbol{x}) = (-i\nabla - \boldsymbol{A})^2 + V(\boldsymbol{x}), \quad \boldsymbol{A} = (1/2)\boldsymbol{B} \times \boldsymbol{x}, \tag{1.1}$$

has received substantial attention due to applications in astrophysics and solid-state physics (see the survey in ref. [21] and references therein) as well as mathematical interest. The basic mathematical aspects of the scattering theory for the pair $(H(\boldsymbol{A}), H_0(\boldsymbol{A}))$ have been studied in ref. [5] where the existence and completeness of the corresponding wave operators were proven for a large class of potentials V (see also ref. [20] for a more recent extension of these results).

This work concerns problems arising in the context of near-threshold scattering for the pair $(H(\boldsymbol{A}), H_0(\boldsymbol{A}))$, when the energy parameter approaches the lowest Landau threshold. A lot of work has been done in this field for Schrödinger operators without external fields. Classic results going back to the late forties and early fifties treat the radial symmetric case. In the late seventies, Newton [26] was the first to give detailed results on various threshold properties of three-dimensional Schrödinger operators with local (noncentral) potentials. His work was followed by the monumental work by Jensen and Kato [14]. Based on a

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detailed analysis of the zero-energy properties of the three-dimensional Schrödinger operator $-\Delta + V(x)$ with V satisfying an abstract short-range condition, Jensen and Kato deduce asymptotic expansions of the full resolvent as the spectral parameter tends to zero (the so-called low-energy limit). As an application they derive expansions of the scattering matrix as the energy parameter goes to zero. The closely related problem of coupling constant thresholds was studied by Klaus and Simon [17]. Threshold scattering in the three-dimensional case was then reconsidered in a very systematic way by Albeverio, Gesztesy and various co-workers [3, 2, 4], and in the two-dimensional case by Cheney [11] with a complete treatment provided later by Bollé, Gesztesy and Danneels [7]. The case of nonlocal interactions in three dimensions was first considered by Newton [27] and later completely resolved by Bollé, Gesztesy, Nessmann and Streit [9]. An excellent survey of threshold properties of Schrödinger operators in dimensions one, two and three can be found in [6].

Despite its obvious importance much less is known on such problems for the operator $H(\boldsymbol{A})$ in Eq. (1.1), which is probably explained by the additional complications that arise (see below).

We restrict ourselves to the case, where the electric potential is axisymmetric, i.e. $V(\boldsymbol{x}) = V(\rho, z)$, $\rho = (x^2+y^2)^{1/2}$, and decays like $V(\boldsymbol{x}) = \mathrm{O}(|\boldsymbol{x}|^{-\alpha})$ as $|\boldsymbol{x}| \to \infty$ for some $\alpha > 2$. Furthermore, we assume that the magnetic field has constant strength 2 and is aligned in the z direction. For fixed magnetic quantum number m the resulting Hamiltonian $H_m = H_{om} + V$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}, \rho\, d\rho\, dz)$ has the structure of an infinite-channel operator-valued matrix. With respect to the projection $\mathcal{P}_0$ onto the lowest Landau level we can represent H_m in a two-channel framework

$$H_m = \underbrace{\begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}}_{H'_{om}} + \begin{pmatrix} 0 & V_{01} \\ V_{10} & 0 \end{pmatrix} \tag{1.2}$$

on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0 = \operatorname{Ran} \mathcal{P}_0$ and $\mathcal{H}_1$ denotes its complement. By construction, H_0 and H_1 are self-adjoint operators in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Moreover $V_{01}^* = V_{10}$ and $V_{01} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$.

Due to the diagonal structure of the uncoupled Hamiltonian H'_{om} its spectrum is the union of the spectra of H_0 and H_1, respectively. We have $\sigma_{\mathrm{ac}}(H_0) = [E_0, \infty)$ and $\sigma_{\mathrm{ac}}(H_1) = [E_1, \infty)$, where $E_n = 2(|m| - m + 1 + 2n)$, $n = 0, 1, 2, \ldots$, are the Landau levels. There are several possible, mostly fairly 'singular' cases to treat, e.g. the one where we assume that E_0 is an isolated eigenvalue of H_1. Thus, H'_{om} has an eigenvalue embedded at E_0; the bottom of its continuous spectrum.

In ref. [25] we have derived asymptotic expansions of the resolvent $\boldsymbol{R}(\zeta) = (H_m - \zeta)^{-1}$ as the spectral parameter ζ tends to the lowest Landau threshold E_0. This is done in various, mostly fairly 'singular' situations, e.g. the afore-mentioned. As an application of these expansions we established scattering theory for the pair (H_m, H_{om}) and deduced asymptotic expansions of the scattering matrix as the energy parameter tends to the lowest Landau threshold. Let us state one of the main results (see ref. [25, Theorem 7.8]). Assume that $\alpha > 13$, $13/2 < s < \alpha - 13/2$ and that E_0 is a regular point of H_0, i.e. E_0 is neither an eigenvalue of H_0 nor a half-bound state (we have a half-bound state if there exists a solution to the equation $H_0\psi = E_0\psi$ belonging to a slightly larger space than $\mathcal{H}_0$; the case, where there is such a half-bound state of H_0, is also treated in ref. [25]). Under the latter assumption we have in the norm topology of $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ the following asymptotic expansion for the resolvent of H_0,

$$R_0(\zeta) = G_0^{(j)} + i(\zeta - E_0)^{1/2} G_1^{(j)} + \cdots \tag{1.3}$$

as $\zeta \to E_0$, $\zeta \in \mathbb{C}\backslash[E_0, \infty)$, provided the potential V decays sufficiently rapidly at infinity and s is chosen appropiately. Here $\mathcal{H}_0^s$ denotes the weighted space associated to $\mathcal{H}_0$ (see Eq. (5.6)). Assume, moreover, that E_0 is an isolated eigenvalue of H_1 with finite multiplicity. We denote by $P_{E_0}^{(1)}$ the eigenprojection onto the eigenspace associated with the eigenvalue E_0 of H_1. Assume, in addition, that the operator $P_{E_0}^{(1)} V_{10} G_0^{(0)} V_{01} P_{E_0}^{(1)}$ is strictly positive and invertible in $\mathcal{B}(P_{E_0}^{(1)}\mathcal{H}_1)$. This is a kind of effective interaction assumption. Then, generically, we have in the norm of $\mathcal{B}(\mathcal{H}_0^s \oplus \mathcal{H}_1^s, \mathcal{H}_0^{-s} \oplus \mathcal{H}_1^{-s})$ the asymptotic expansion

$$\boldsymbol{R}(\zeta) = R_0 + i(\zeta - E_0)^{1/2} R_1 + \mathrm{O}(|\zeta - E_0|) \tag{1.4}$$

as $|\zeta - E_0| \to 0$, where the coefficients R_0 and R_1 in Eq. (1.4) are given explicitly. Despite the singular nature of the problem, Eq. (1.4) reveals that, generically, the singularities cancel. In particular, the resolvent has a well-defined limit R_0 at the threshold point in the norm topology of $\mathcal{B}(\mathcal{H}_0^s \oplus \mathcal{H}_1^s, \mathcal{H}_0^{-s} \oplus \mathcal{H}_1^{-s})$. Moreover, under the afore-mentioned assumptions and via the expansion Eq. (1.4) we deduce that, in the norm topology of $\mathcal{B}(\mathbb{C}^2)$, the scattering matrix $S(\lambda)$ has the following leading order behaviour as the energy parameter λ tends to E_0:

$$S(\lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \mathrm{o}(1).$$

All the results in ref. [25] hold *generically*. Hence in all the various cases, e.g. the above-mentioned, we use in the proofs that a certain operator C (a different operator for each case) is compact and, consequently, the operator $I - C$ is invertible generically, i.e. if g is a real parameter g and we introduce the family $I - gC$ then $I - gC$ is invertible except for a discrete set of g's.

The aim of the present work is to deepen the analysis of the most simple case considered in ref. [25]. In the most simple case we assume that E_0 is a regular point of H_0 and $E_0 \in \rho(H_1)$. Then the expansion in Eq. (1.3) holds for the resolvent of H_0 and, furthermore, we have in the norm of $\mathcal{B}(\mathcal{H}_1)$ the usual Neumann expansion for the resolvent of H_1, viz.

$$R_1(\zeta) = \sum_{n=0}^{\infty} (\zeta - E_0)^n R_1(E_0)^{n+1} \tag{1.5}$$

for $|\zeta - E_0|$ sufficiently small. This is clearly the most simple case because we have no singular terms (negative powers in $(\zeta - E_0)^{1/2}$) neither in Eq. (1.3) nor in Eq. (1.5). In ref. [25] we derived an asymptotic expansion for $\boldsymbol{R}(\zeta)$ of the form in Eq. (1.4). As usual it holds generically. Within the context of ref. [25] this situation is trivial to treat in the generic case. (Of course the *singular cases* considered in ref. [25] are by no means trivial to treat in the generic sense).

In the present work we go beyond the generic case. We propose a new approach to the case, where E_0 is a regular point of H_0 and $E_0 \in \rho(H_1)$. We introduce an auxiliary one-dimensional Schrödinger operator $A' = -d^2/dz^2 + E_0 + W$ in $L^2(\mathbb{R})$ related to H_0 (see Eq. (6.4) for the precise relation). We show that the operator A' fits into the general framework of ref. [23]. In particular we have a complete classification of the E_0-energy properties of A'. A priori E_0 can be an eigenvalue of A' or a half-bound state (see definition above). In dimension one it is natural to assume that a half-bound state belongs to $L^\infty(\mathbb{R})$. It turns out that essentially three cases may occur: Case 0) A' has no eigenvalue E_0 and no half-bound state (*E_0 is a regular point*), Case 1) A' has no eigenvalue E_0 but has a half-bound state (*E_0 is an exceptional point of the 1st kind*), Case 2) A' has eigenvalue E_0 but has no half-bound state (*E_0 is an exceptional point of the 2nd kind*) and (the mixed) Case 3) A' has both eigenvalue E_0 and half-bound states (*E_0 is an exceptional point of the 3rd kind*). Furthermore, there are at most *two* half-bound states modulo L^2 functions. Due to this circumstance, additional subcases arise in the exceptional cases of 1st (three subcases) and 3rd kind (three subcases). In each case asymptotic expansions of the resolvent $R(A', \zeta)$ were deduced in ref. [23]. Generally, the expansions take the following form in the norm topology of $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$

$$\begin{aligned} R(A', \zeta) &= -(\zeta - E_0)^{-1} F_{-2}^{(l)} - i(\zeta - E_0)^{-1/2} F_{-1}^{(l)} \\ &\quad + F_0^{(l)} + i(\zeta - E_0)^{1/2} F_1^{(l)} + \cdots \end{aligned} \tag{1.6}$$

as $|\zeta - E_0| \to 0$, provided W decays sufficiently rapidly at infinity and s is chosen appropiately. Here $H^{t,s}(\mathbb{R})$ denotes the weighted Sobolev space (see the definition in Sect. 2).

In each of the possible cases characterized by the E_0-energy properties of A' we deduce asymptotic expansions of the resolvent $\boldsymbol{R}(\zeta)$ as ζ

tends to E_0. Generally, as $|\zeta - E_0| \to 0$ the expansions take the form

$$\begin{aligned} \boldsymbol{R}(\zeta) &= -(\zeta - E_0)^{-1} R_{-2}^{(l)} - i(\zeta - E_0)^{-1/2} R_{-1}^{(l)} \\ &\quad + R_0^{(l)} + i(\zeta - E_0)^{1/2} R_1^{(l)} + \cdots \end{aligned} \tag{1.7}$$

in the norm topology of $\mathcal{B}(\mathcal{H}_0^s \oplus \mathcal{H}_1, \mathcal{H}_0^{-s} \oplus \mathcal{H}_1)$ provided V decays sufficiently rapidly at infinity and s is chosen appropiately. We emphasize that the expansions in Eq. (1.7) reveal that any of the above-mentioned cases 0, 1 and 2 (and the mixtures in Case 3) may occur for the full Hamiltonian H_m. (Consequently, we have introduced an upper index l in the coefficients $R_k^{(l)}$ to differentiate between these cases). In particular, E_0 can be an eigenvalue of H_m.

As an application of these expansions, we derive asymptotic expansions of the scattering matrix associated with the pair (H_m, H_{om}) as the energy parameter tends to E_0.

The advantage of this work is that we are able to treat each of the cases which arise, *without* limiting ourselves to the generic cases.

The paper is organized as follows. In Sect. 2 we fix the notation. In Sects. 3 and 4 the magnetic Hamiltonians H_{om} and H_m are introduced. We fit the infinite-channel Hamiltonians H_{om} and H_m into a two-channel framework in Sect. 5. Auxiliary results on one-dimensional Schrödinger operators (with non-local potentials) are collected in Sect. 6 and in Sect. 7 we obtain the main results on asymptotic expansions for the resolvent of the Hamiltonian H_m as the spectral parameter tends to the lowest Landau threshold. In Sect. 8 we give applications to scattering theory.

We have decided to make the paper self-contained. Consequently, the contents of Sects. 3, 4 and 5 are identical with the contents of Sects. 3, 4 and 5 in ref. [25], wherein the Hamiltonians H_{om} and H_m are introduced and the two-channel model is constructed.

The present work and ref. [25] complement work by Kostrykin, Kvitsinsky and Merkuriev in ref. [18], who address the exact same problem. The authors restrict themselves to the most simple case, where E_0 is a regular point of H_0 and $E_0 \in \rho(H_1)$, which is precisely the assumption we make throughout the present paper. However, they limit themselves to the generic case. Their method is a direct generalization of a method developed by Bollé, Gesztesy and Wilk in ref. [10] in a study of the one-dimensional Schrödinger operator. The authors write that in order to incorporate the cases where some of the thresholds are exceptional points, one can adjust the technique of ref. [10]. In a sense, this has been one of the aims of the work and ref. [25], although we think that our methods go beyond what we understand as an adjustment.

In ref. [33] Tamura has recently studied low-energy scattering for the Schrödinger operator in dimension two with a compactly supported magnetic field. He is able to adapt the method developed by Jensen and Kato in ref. [14] to the problem. This is not possible in the present work.

Finally, we mention that this work has its origin in the author's Ph.D. thesis, ref. [22], where a preliminary version can be found in Part IV.

2. Preliminaries

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$. The spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, respectively. We use standard terminology for the various parts of the spectrum, see for example ref. [13]. The resolvent is $R(\zeta) = (T-\zeta)^{-1}$. The spectral family associated to T is denoted by $E_T(\lambda)$, $\lambda \in \mathbb{R}$.

For a complex number $z \in \mathbf{C} \setminus [0,\infty)$ we denote by $z^{1/2}$ the branch of the square root with positive imaginary part.

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space, denote points of $\mathbb{R}^d$ by $\boldsymbol{x} = (x_1, \dots, x_d)$ and let $|\boldsymbol{x}| = (\sum_{j=1}^d x_j^2)^{1/2}$. For $1 \le p \le \infty$ let $L^p(\mathbb{R}^d)$ be the space of (equivalence classes of) complex-valued functions ψ which are measurable and satisfy $\int_{\mathbb{R}^d} |\psi(\boldsymbol{x})|^p d\boldsymbol{x} < \infty$ if $p < \infty$ and $\|\psi\|_{L^\infty(\mathbb{R}^d)} = \text{ess sup } |\psi| < \infty$ if $p = \infty$. The measure $d\boldsymbol{x}$ is the Lebesgue measure. For any p the $L^p(\mathbb{R}^d)$ space is a Banach space with norm $\|\cdot\|_{L^p(\mathbb{R}^d)} = (\int_{\mathbb{R}^d} |\cdot|^p d\boldsymbol{x})^{1/p}$. In the case $p=2$, $L^2(\mathbb{R}^d)$ is a complex and separable Hilbert space with scalar product $(\phi,\psi)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \overline{\phi}\psi$ and corresponding norm $\|\psi\|_{L^2(\mathbb{R}^d)} = (\psi,\psi)_{L^2(\mathbb{R}^d)}^{1/2}$.

The space of infinite differentiable complex-valued functions with compact support will be denoted by $C_0^\infty(\mathbb{R}^d)$ or $\mathcal{D}(\mathbb{R}^d)$, the space of test functions. The adjoint space of $\mathcal{D}(\mathbb{R}^d)$, $\mathcal{D}'(\mathbb{R}^d)$, is the space of distributions on $\mathcal{D}(\mathbb{R}^d)$. The Schwarz space of rapidly decreasing functions and its adjoint space of tempered distributions are denoted by $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, respectively.

Let $\boldsymbol{p}$ denote the momentum operator $-i\nabla$ and let $\langle \boldsymbol{p} \rangle = (1+\boldsymbol{p}^2)^{1/2}$. We use the weighted Sobolev space $H^{m,s}(\mathbb{R}^d)$ given by

$$H^{m,s}(\mathbb{R}^d) = \{\psi \in \mathcal{S}'(\mathbb{R}^d) |\; \|\psi\|_{m,s} = \|\langle \boldsymbol{x}\rangle^s \langle \boldsymbol{p}\rangle^m \psi\|_{L^2} < \infty\}.$$

We use $\langle\cdot,\cdot\rangle$ to denote the inner product on $L^2(\mathbb{R}^d)$ and also the natural duality between $H^{m,s}(\mathbb{R}^d)$ and $H^{-m,-s}(\mathbb{R}^d)$. $\mathcal{B}(H^{m,s}(\mathbb{R}^d), H^{m',s'}(\mathbb{R}^d))$ denotes the space of bounded operators from $H^{m,s}$ to $H^{m',s'}$ with the operator norm. The Fourier transform is given by

$$(\mathcal{F}\psi)(\boldsymbol{\xi}) = \widehat{\psi}(\boldsymbol{\xi}) = (2\pi)^{-1/2} \int_{\mathbb{R}^d} e^{-i\boldsymbol{x}\cdot\boldsymbol{\xi}} \psi(\boldsymbol{x})\, d\boldsymbol{x}$$

and is a bounded map from $H^{m,s}(\mathbb{R}^d)$ to $H^{s,m}(\mathbb{R}^d)$.

A key ingredient in our approach is the Feshbach formula, which gives a convenient explicit representation of the resolvent $\boldsymbol{R}(\zeta)$ of a two-channel Hamiltonian $\boldsymbol{H}$ on the form

$$\boldsymbol{H} = \begin{pmatrix} H_a & V_{ab} \\ V_{ba} & H_b \end{pmatrix}.$$

There are two variants. We give only one of them. The other version is just an interchange of indices. Define

$$R_a(\zeta) = (H_a - \zeta)^{-1}, \tag{2.1}$$

$$T_b(\zeta) = H_b - \zeta - V_{ba} R_a(\zeta) V_{ab}. \tag{2.2}$$

Then for $\operatorname{Im} \zeta \neq 0$ we have

$$\boldsymbol{R}(\zeta) = \begin{pmatrix} R_a(\zeta) + R_a(\zeta) V_{ab} T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & -R_a(\zeta) V_{ab} T_b(\zeta)^{-1} \\ -T_b(\zeta)^{-1} V_{ba} R_a(\zeta) & T_b(\zeta)^{-1} \end{pmatrix} \tag{2.3}$$

3. The Free Hamiltonian H_{om}

In $\mathbb{R}^3$ we consider a charged, spinless particle in a homogeneous magnetic field with no other forces present. Assume that the mass of the particle is 1/2 and its electric charge is 1, and that the magnetic field $\boldsymbol{B}$ has constant strength 2 and is aligned in the z direction: $\boldsymbol{B} = (0, 0, 2)$. The Hamiltonian of the particle is $H_0(\boldsymbol{A}) = (\boldsymbol{p} - \boldsymbol{A})^2$, where $\boldsymbol{p} = -i\nabla$ is the momentum operator and $\boldsymbol{A}$ is the vector potential associated with the field, viz. $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and defined up to a gauge transformation. We choose the gauge in which $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B} \times \boldsymbol{r})$ and denote the Hamiltonian by $H_0(\boldsymbol{B})$ to emphasize that the magnetic field is constant. The Hamiltonian $H_0(\boldsymbol{B})$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ (see ref. [30]). For convenience, we use the same notation $H_0(\boldsymbol{B})$ for its closure. With an appropiate choice of the system of units, $H_0(\boldsymbol{B})$ may be written in cylindrical coordinates (ρ, ϕ, z) in the form

$$H_0(\boldsymbol{B}) = -\Delta - 2i\frac{\partial}{\partial \phi} + \rho^2. \tag{3.1}$$

We rewrite Eq. (3.1) as

$$H_0(\boldsymbol{B}) = p_z^2 + H_{osc} - 2L_z, \tag{3.2}$$

where

$$H_{osc} = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) - \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \rho^2, \tag{3.3}$$

$$L_z = -i\frac{\partial}{\partial \phi}. \tag{3.4}$$

It is well-known (see ref. [5] for details), that H_{osc} and L_z acting on $L^2(\mathbb{R}^2)$ have a complete, joint set of eigenfunctions $\{f_{mn}\}_{m=0,\pm1,\pm2,\dots;n=0,1,2,\dots}$:

$$L_z f_{mn} = m f_{mn}, \quad H_{osc} f_{mn} = 2(|m| + 1 + 2n) f_{mn}.$$

The m and n are called the magnetic and radial quantum numbers, respectively. If we denote $H_\perp = H_{osc} - 2L_z$ then $H_\perp$ has eigenfunctions f_{mn} with corresponding eigenvalues

$$E_{mn} = 2(|m| - m + 1) + 4n. \tag{3.5}$$

The functions f_{mn} are known explicitly:

$$f_{mn}(\rho, \phi) = (2\pi)^{-1/2} e^{-im\phi} \Psi_{mn}(\rho), \tag{3.6}$$

$$\Psi_{mn}(\rho) = 2^{1/2}\{n!(n+|m|)!\}^{-1/2} \rho^{|m|} \exp(-\rho^2/2) L_n^{(|m|)}(\rho^2), \tag{3.7}$$

$$\int_0^\infty \Psi_{mn}^2(\rho)\rho d\rho = 1;\ m = 0, \pm 1, \pm 2, \ldots\ ;\ n = 0, 1, 2, \ldots \tag{3.8}$$

Here $L_n^{(|m|)}$ are the generalized Laguerre polynomials (see ref. [1]). It follows from the properties of $L_n^{(|m|)}$ that for any fixed m the set $\{\Psi_{mn}\}_{n=0}^\infty$ forms an orthonormal basis in $L^2((0,\infty), \rho d\rho)$. Often the eigenfunctions f_{mn} are referred to as Landau orbits, and the corresponding eigenvalues E_{mn} of $H_\perp$ are called the Landau energy levels. It is well-known from ref. [5] that

i): $\inf \operatorname{spec}(H_0(\boldsymbol{B})) = 2$.
ii): $\sigma(H_0(\boldsymbol{B})) = \sigma_{ess}(H_0(\boldsymbol{B})) = \sigma_{ac}(H_0(\boldsymbol{B})) = [2, \infty)$.

Let $\{P_{mn}\}$, $m \in \mathbb{Z}$, $n \geq 0$ be the set of orthonormal one-dimensional projections onto the corresponding eigenspaces of $H_\perp$. Hence, $H_\perp = \sum_{mn} E_{mn} P_{mn}$ in $L^2(\rho d\rho d\phi)$. Let $\mathcal{H}_{mn} = \operatorname{Ran} P_{mn} \otimes L^2(\mathbb{R})$. Decompose the space $L^2(\mathbb{R}^3)$ into the orthogonal sum of the subspaces $\mathcal{H}_{mn}$ corresponding to fixed magnetic and radial quantum numbers. Then we can express the free, magnetic Hamiltonian $H_0(\boldsymbol{B})$ as

$$H_0(\boldsymbol{B}) = I \otimes p_z^2 + \sum_{mn} E_{mn} P_{mn} \otimes I \tag{3.9}$$

in $\bigoplus_{mn} \mathcal{H}_{mn}$. One easily shows that $\mathcal{H}_{mn}$ is a reducing subspace of $H_0(\boldsymbol{B})$ and, in addition, for fixed m the orthogonal sum $\mathcal{H}_m = \bigoplus_{n=0}^\infty \mathcal{H}_{mn}$ is a reducing subspace of $H_0(\boldsymbol{B})$. Thus, the restriction of $H_0(\boldsymbol{B})$, denoted by H_{om}, to a fixed, magnetic quantum number m is a self-adjoint operator in $\mathcal{H}_m$ with domain $D(H_{om}) = D(H_0(\boldsymbol{B})) \cap \mathcal{H}_m$.

4. The Full Hamiltonian H_m

We make the following assumption on the potential V.

Assumption 4.1. Let $V = V(\boldsymbol{x})$ be a real-valued, measurable function on $\mathbb{R}^3$.
(i) Let V satisfy the estimate

$$|V(\boldsymbol{x})| \leq C(1 + |\boldsymbol{x}|)^{-\alpha} \tag{4.1}$$

for some constants $C > 0$ and $\alpha > 1$.
(ii) Let V be axisymmetric, i.e. $V(\boldsymbol{x}) = V(\rho, z)$ with $\rho^2 = x^2 + y^2$.

We refer to α as the *decay parameter*. Note that under Assumption 4.1 with $\alpha > 1$ we have that V is a compact map from $H^{1,0}(\mathbb{R}^3)$ to $H^{-1,\alpha'}(\mathbb{R}^3)$ for all $1 < \alpha' < \alpha$.
Let us state here as a lemma the following consequences of the Kato-Simon inequality $|e^{-tH_0(\boldsymbol{A})}\psi| \leq |e^{-t\Delta}||\psi|$ and a result of Dodds-Fremlin in ref. [12] and Pitt in ref. [29]:

Lemma 4.2. *Let W be a multiplication operator and let $\boldsymbol{A} \in L^2_{loc}(\mathbb{R}^n)$. Then*
(i) if W is Δ-bounded with relative bound a, W is $H_0(\boldsymbol{A})$-bounded with relative bound at most a;
(ii) if W is Δ-compact, it is $H_0(\boldsymbol{A})$- compact.

We have the following lemma.

Lemma 4.3. *In $L^2(\mathbb{R}^3)$ let V satisfy Assumption 4.1(i) with α chosen as below.*
(i) For $\alpha > 0$ V is $H_0(\boldsymbol{A})$-compact.
(ii) For $\alpha > 1$ $|x|V$ is $H_0(\boldsymbol{A})$-compact.
(iii) For $\alpha > 2$ $|x|^2V$ is $H_0(\boldsymbol{A})$-compact.

Proof. The assertions are easy to show for $H_0(\boldsymbol{A})$ replaced by Δ. Then the assertions for $H_0(\boldsymbol{A})$ follow by Lemma 4.2. □

Let V satisfy Assumption 4.1(i) with $\alpha > 0$ and let $\boldsymbol{B}$ be defined as in Sect. 3. It follows from Lemma 4.3 that the Schrödinger operator $H(\boldsymbol{B}) = H_0(\boldsymbol{B}) + V$ is self-adjoint in $L^2(\mathbb{R}^3)$ on the domain $D(H_0(\boldsymbol{B}))$ and, due to Weyl's essential spectrum theorem, we have that $\sigma_{ess}(H(\boldsymbol{B})) = \sigma_{ess}(H_0(\boldsymbol{B})) = [2, \infty)$.

In the sequel we assume that the electric potential V is axisymmetric. In particular the projection Q_m onto $\mathcal{H}_m$ commutes with V and, consequently, $\mathcal{H}_m$ reduces $H(\boldsymbol{B})$. Therefore, if Assumption 4.1 holds with $\alpha > 0$ then the operator $H_m = H(\boldsymbol{B})|_{\mathcal{H}_m} = H_{om} + V$ is a self-adjoint operator in $\mathcal{H}_m$ with domain $\mathcal{D}(H_m) = \mathcal{D}(H(\boldsymbol{B})) \cap \mathcal{H}_m$. Moreover, $\sigma_{ess}(H_m) = \sigma_{ess}(H_{om}) = [E_0, \infty)$, where $E_n := 2(|m| - m + 1 + 2n)$.

If Assumption 4.1 holds with $\alpha > 2$ then the number of eigenvalues of H_m below E_0 is finite (see ref. [32]). In particular, the eigenvalues of H_m below E_0 cannot accumulate at E_0. In general, for the full Hamiltonian $H(\boldsymbol{B})$ this is *not* necessarily the case (see ref. [5]).

For later purpose, we introduce the spaces $\mathcal{H}_m^{t,s} := Q_m H^{t,s}(\mathbb{R}^3)$. Then it follows from the mapping properties of V between weighted Sobolev spaces that V is a compact operator from $\mathcal{H}_m^{1,-s}$ to $\mathcal{H}_m^{-1,\alpha'-s}$ for all $1 < \alpha' < \alpha$.

We close this section by noting that the spaces $\mathcal{H}_m$ and $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}, \rho\, d\rho\, dz)$ are isomorphic. Let U denote the isomorphism between $\mathcal{H}_m$ and $\mathcal{H}$. We shall use the same notation for H_{om}, V and H_m, when we regard them as operators in $L^2(\mathbb{R}_+ \times \mathbb{R}, \rho\, d\rho\, dz)$. Furthermore, we shall use the spaces $\mathcal{H}^{t,s} := U\mathcal{H}_m^{t,s}$.

5. Two-Channel Framework

The basic Hilbert space is $\mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}, \rho\, d\rho\, dz)$. To represent the magnetic Hamiltonians H_{om} and H_m as two-channel Hamiltonians we need two additional Hilbert spaces defined via the following projections. For any $\phi \in \mathcal{H}$ define the projection $(\mathcal{P}_0\phi)(\rho, z) = \langle \phi, \Psi_0 \rangle_{L^2(\mathbb{R}_+, \rho d\rho)} \Psi_0(\rho)$, where $\Psi_0(\rho) := \Psi_{m0}(\rho)$ is defined in Sect. 3. Let the complement of $\mathcal{P}_0$ in $\mathcal{H}$ be denoted by $\mathcal{P}_1$. Introduce the Hilbert spaces $\mathcal{H}_0 = \operatorname{Ran} \mathcal{P}_0$, $\mathcal{H}_1 = \operatorname{Ran} \mathcal{P}_1$ and $\widehat{\mathcal{H}}_0 = L^2(\mathbb{R})$. Then we have the basic decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. As for the adjoints we have that $\mathcal{P}_j^* = \mathcal{P}_j$, $j = 0, 1$. We want to fit the magnetic Hamiltonian H_m into a two-channel Hamiltonian via the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. For this purpose, we define the following matrix elements, where $i, j = 0, 1$:

$$H_{ij} = \mathcal{P}_i H_{om} \mathcal{P}_j^* : \mathcal{H}_j \longrightarrow \mathcal{H}_i, \qquad V_{ij} = \mathcal{P}_i V \mathcal{P}_j^* : \mathcal{H}_j \longrightarrow \mathcal{H}_i. \quad (5.1)$$

With respect to this decomposition the magnetic Hamiltonian H_{om} is represented as a 2×2 matrix given by

$$H_m = H_{om} + V = \begin{pmatrix} H_{00} & 0 \\ 0 & H_{11} \end{pmatrix} + \begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix}. \quad (5.2)$$

in $\mathcal{H}_0 \oplus \mathcal{H}_1$.The operators $\mathcal{P}_0$ and $\mathcal{P}_1$ are spectral projections and therefore they commute with H_{om}. Since $\mathcal{P}_0$ and $\mathcal{P}_1$ are orthogonal, the elements H_{01} and H_{10} vanish in the representation of H_{om}. We have the following results.

Lemma 5.1. *Let Assumption 4.1 be satisfied with $\alpha > 0$. Then V_{0j} and V_{1j} are H_{jj}-compact, $j = 0, 1$.*

Proof. We prove the assertions for $j = 0$. Under Assumption 4.1 with $\alpha > 0$, the potential V is H_{om}-compact, i.e. $V(H_{om} + i)^{-1}$ is compact in $\mathcal{H}$. Since $\mathcal{P}_0$ commutes with H_{om}, $\mathcal{P}_0(H_{om} + i)^{-1}\mathcal{P}_0^* = (H_{00} + i)^{-1}$ and $\mathcal{P}_0$ is bounded from $\mathcal{H}$ to $\mathcal{H}_0$, it follows that $V_{00}(H_{00} + i)^{-1}$ is compact in $\mathcal{H}_0$. Similarly, we prove that V_{10} is H_{00}-compact from $\mathcal{H}_0$ to $\mathcal{H}_1$. □

Formally, let $H_0 = H_{00} + V_{00}$ and $H_1 = H_{11} + V_{11}$. Under Assumption 4.1 with $\alpha > 0$ the operators H_0 and H_1 are self-adjoint in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Moreover, we have that

$$\sigma_{ess}(H_j) = \sigma_{ess}(H_{jj}) = [E_j, \infty), \quad j = 0, 1. \quad (5.3)$$

For later purpose we make the following decomposition of H_m in terms of H_0 and H_1:

$$H_m = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} + \begin{pmatrix} 0 & V_{01} \\ V_{10} & 0 \end{pmatrix}. \quad (5.4)$$

We wish to give conditions, which quarantees that $\sigma_d(H_0)$ consists of finitely many eigenvalues below E_0. For this purpose we begin by expressing V_{00} in an explicit way. If $f \in \mathcal{H}_0$ then there exists $g(z) \in$

$L^2(\mathbb{R})$ such that $f(z,\rho) = g(z)\Psi_0(\rho)$. It is easily seen that $V_{00}f(z,\rho) = \tilde{V}_{00}(z)f(z,\rho)$, where

$$\tilde{V}_{00}(z) = \int_{\mathbb{R}_+} V(z,\rho')|\Psi_0(\rho')|^2\rho' d\rho'. \tag{5.5}$$

The representation $H_0 = (p_z^2 + E_0 + \tilde{V}_{00}) \otimes P_0'$ implies that $\sigma_d(H_0) = \sigma_d(p_z^2 + E_0 + \tilde{V}_{00})$, hence we need to put conditions on V such that $\sigma_d(p_z^2 + E_0 + \tilde{V}_{00})$ consists of finitely many eigenvalues below E_0. If Assumption 4.1 holds with $\alpha > 2$ then it follows from the estimate $|\tilde{V}_{00}(z)| \leq C(1+|z|)^{-\alpha}$ and ref. [28, Equation (5)] that the number of eigenvalues below E_0 for the self-adjoint operator $p_z^2 + E_0 + \tilde{V}_{00}$ in $L^2(\mathbb{R})$ is finite.

Define $\sigma_0(f(z)\Psi_0) = f(z)$ for any element $f(z)\Psi_0 \in \mathcal{H}_0$. Then σ_0 is an isomorphism from $\mathcal{H}_0$ onto $\widehat{\mathcal{H}}_0$ and its adjoint acts as $\sigma_0^* f(z) = f(z)\Psi_0(\rho)$ for any $f(z) \in \widehat{\mathcal{H}}_0$. Moreover, let $\widehat{\mathcal{H}}_1 = \mathcal{H}_1$ and $\sigma_1 = I$. We define the projections $P_0 = \sigma_0 \mathcal{P}_0$ and $P_1 = \sigma_1 \mathcal{P}_1$.

We have the following result.

Lemma 5.2. *Let Assumption 4.1 be satisfied with $\alpha > 1$. Then the operator $P_0 H_m P_0^* = p_z^2 + E_0 + \tilde{V}_{00}$ in $\widehat{\mathcal{H}}_0 = L^2(\mathbb{R})$ has no eigenvalues larger than E_0.*

Proof. It follows immediately from the estimate $|\tilde{V}_{00}(z)| \leq C(1+|z|)^{-\alpha}$ and ref. [31, Theorem XIII.56]. □

In the sequel we also need the following spaces

$$\mathcal{H}_0^{t,s} = \mathcal{P}_0\mathcal{H}^{t,s}, \quad \mathcal{H}_1^{t,s} = \mathcal{P}_1\mathcal{H}^{t,s}. \tag{5.6}$$

We use the short-hand notation $\mathcal{H}_j^s = \mathcal{H}_j^{0,s}$, $j = 0, 1$.

6. Auxiliary One-dimensional Schrödinger Operator

In this section we introduce an auxiliary one-dimensional Schrödinger operator A and collect some results on the spectral properties of A near the bottom of its continuous spectrum, viz. the threshold 0. The main results are asymptotic expansions for the resolvent $R(A,\zeta) = (A-\zeta)^{-1}$ as the parameter ζ tends to 0. The results are crucial in Sect. 7, where we return to the study of the magnetic Hamiltonian H_m. Throughout this section we assume that $E_0 \in \rho(H_1)$, where the operator H_1 was introduced in Sect. 5. The auxiliary one-dimensional Schrödinger operator A is formally defined as $A = h_0 + W$ in $L^2(\mathbb{R})$, where h_0 denotes the self-adjoint realization of $-d^2/dz^2$ in $L^2(\mathbb{R})$ with domain $D(h_0) = H^2(\mathbb{R})$ and spectrum $\sigma(h_0) = \sigma_{ess}(h_0) = [0,\infty)$ and, furthermore, the perturbation W is given by $W = \tilde{V}_{00} - P_0 V \mathcal{P}_1^* R_1(E_0) \mathcal{P}_1 V P_0^*$. The definitions of the various operators in W can be found in Sect. 5. In this section we first define A in a rigorous way via the Kato-Rellich theorem and establish some mapping properties of W between weighted

Sobolev spaces. Secondly we collect low-energy results for the resolvent of A near the threshold 0.

Lemma 6.1. *Let Assumption 4.1 with $\alpha > 0$ hold. Then A is a self-adjoint operator in $L^2(\mathbb{R})$.*

Proof. It follows easily that W is a bounded self-adjoint operator in $L^2(\mathbb{R})$ since the various operators in the definition of W are bounded, V is real-valued, H_1 is a self-adjoint operator in $\mathcal{H}_1$ (for $\alpha > 0$) and $E_0 \notin \sigma(H_1)$ is real. Consequently, W is infinitesimally small with respect to h_0 and the operator sum $h_0 + W$ is self-adjoint according to the Kato-Rellich theorem. □

Next we determine under which condition W is relatively compact with respect to h_0.

Lemma 6.2. *Let Assumption 4.1 hold with $\alpha > 1$. Then W is relatively compact with respect to h_0.*

Proof. First we show that $\tilde{V}_{00}(h_0 + E_0)^{-1}$ is compact in $L^2(\mathbb{R})$. If $\tilde{V}_{00} \in L^2(\mathbb{R})$, then $\chi_{[-n,n]}\tilde{V}_{00}(h_0 + E_0)^{-1}$ is Hilbert-Schmidt (use the explicit kernel of the resolvent), and moreover, this operator converges in norm sense to $\tilde{V}_{00}(h_0 + E_0)^{-1}$ as $n \to \infty$. This yields the compactness of $\tilde{V}_{00}(h_0 + E_0)^{-1}$. From the estimate $|\tilde{V}_{00}(z)| \le C(1 + |z|)^{-\alpha}$ it follows immediately that $\tilde{V}_{00} \in L^2(\mathbb{R})$ for $\alpha > 1/2$. For the term $\sigma_0 V_{01} R_1(E_0) V_{10} \sigma_0^*$ we rely entirely on mapping properties. Indeed, for $\alpha > 1$, it follows from the closing remarks of Sect. 4 that V_{ij} is a compact operator from $\mathcal{H}_j^{2,0}$ to $\mathcal{H}_i^{0,\alpha'}$ for all $1 < \alpha' < \alpha$. The latter property in combination with the mapping properties of $R_1(E_0)$ and σ_0 imply that $\sigma_0 V_{01} R_1(E_0) V_{10} \sigma_0^*$ is a compact operator from the Sobolev space $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ as desired. □

Remark 6.3. In Lemma 6.2 we do not aim at the optimal conditions on α because we shall put much stronger conditions on α in the sequel.

Under Assumption 4.1 with $\alpha > 1$ we immediately obtain from Lemma 6.2, the Kato-Rellich theorem and Weyl's essential spectrum theorem that A is a self-adjoint operator in $L^2(\mathbb{R})$ with domain $D(A) = H^2(\mathbb{R})$ and $\sigma_{ess}(A) = \sigma_{ess}(h_0) = [0, \infty)$.

Next we establish mapping properties of W between weighted Sobolev spaces.

Lemma 6.4. *Let Assumption 4.1 hold with $\alpha > 1$. Then the perturbation W is a compact operator from $H^{1,0}(\mathbb{R})$ to $H^{-1,\alpha}(\mathbb{R})$, and W extends to a compact operator from $H^{1,-\alpha}(\mathbb{R})$ to $H^{-1,0}(\mathbb{R})$.*

Proof. Due to Assumption 4.1 $\tilde{V}_{00}$ satisfies the estimate $|\tilde{V}_{00}(z)| \le C(1 + |z|)^{-\alpha}$. Equipped with the latter estimate it is a standard result that $\tilde{V}_{00}$ is a compact map from $H^{1,-s}(\mathbb{R})$ to $H^{-1,\alpha-s}(\mathbb{R})$ when $\alpha > 1$ and $s \in \mathbb{R}$. Under Assumption 4.1 with $\alpha > 1$ it follows from the closing remarks of Sect. 4 that V_{ij} is a compact operator from $\mathcal{H}_j^{1,0}$ to $\mathcal{H}_i^{-1,\alpha}$, and in addition V_{ij} extends to a compact operator from $\mathcal{H}_j^{1,-\alpha}$ to $\mathcal{H}_i^{-1,0}$. The latter properties in combination with the mapping properties of $R_1(E_0)$ and σ_0 imply that the operator $\sigma_0 V_{01} R_1(E_0) V_{10} \sigma_0^*$ maps as desired. □

We have shown that $A = h_0 + W$ is a self-adjoint (Schrödinger) operator in $L^2(\mathbb{R})$ and, moreover, Lemma 6.4 asserts that the (non-local) potential W satisfies the following abstract short-range condition.

Abstract short-range condition: Let $\tilde{W}$ be a symmetric operator in $L^2(\mathbb{R})$. Assume that $\tilde{W}$ is a compact operator from $H^{1,0}(\mathbb{R})$ to $H^{-1,\alpha}(\mathbb{R})$ for some $\alpha > 1$, and $\tilde{W}$ extends to a compact operator from $H^{1,-\alpha}(\mathbb{R})$ to $H^{-1,0}(\mathbb{R})$.

Under this abstract condition a complete classification of the zero-energy properties of A was established in ref. [23]. A priori zero can be an eigenvalue of A or a half-bound state for A, or both. We have a half-bound state (or zero resonance) if $A\psi = 0$ has a solution ψ in a space slightly larger than $L^2(\mathbb{R})$. In dimension one it is natural to assume that a half-bound state belongs to L^∞. It turns out that essentially three cases may occur: Case 0) A has no eigenvalue zero and no half-bound state (*zero is a regular point*), Case 1) A has no eigenvalue zero but has a half-bound state (*zero is an exceptional point of 1st kind*), A has eigenvalue zero but has no half-bound state (*zero is an exceptional point of 2nd kind*), or A has both eigenvalue zero and a half-bound state (*zero is an exceptional point of the 3rd kind*). Furthermore, there are at most *two* half-bound states modulo L^2-functions. Due to this circumstance, additional subcases arise in the exceptional cases of 1st (three subcases) and 3rd kind (three subcases) (see ref. [23] for details). In all cases asymptotic expansions of the resolvent $R(A, \zeta)$ were deduced in ref. [23] as ζ tends to zero. Here we just state the results for the most significant cases (ignoring the exceptional case of 3rd kind). We refer to ref. [23] for the rather lengthy proofs.

Theorem 6.5. *Suppose zero is a regular point of A. Assume $\alpha > 9$ and let s satisfy $9/2 < s < \alpha - 9/2$. For some $\delta > 0$ we have in the norm of $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$ the asymptotic expansion*

$$R(A,\zeta) = B_0^{(0)} + i\zeta^{1/2} B_1^{(0)} - \zeta B_2^{(0)} + O(\zeta^{3/2}) \tag{6.1}$$

for $|\zeta| < \delta$, $\operatorname{Im} \zeta^{1/2} > 0$, where $(B_k^{(0)})^* = B_k^{(0)}$, $k = 0, 1, 2$, as operators in $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$.

Theorem 6.6. *Suppose zero is an exceptional point of the 1st kind for A (Type 1, 2 or 3). Assume $\alpha > 13$ and let s satisfy $13/2 < s < \alpha - 13/2$. For some $\delta > 0$ we have in the norm of $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$*

the asymptotic expansion

$$R(A,\zeta) = -i\zeta^{-1/2}B_{-1}^{(1,j)} + B_0^{(1,j)} + i\zeta^{1/2}B_1^{(1,j)} - \zeta B_2^{(1,j)} + O(\zeta^{3/2}) \tag{6.2}$$

for $|\zeta| < \delta$, $\operatorname{Im}\zeta^{1/2} > 0$, *where* $(B_k^{(1,j)})^* = B_k^{(1,j)}$, $k = -1, 0, 1, 2$, *as operators in* $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$. *Here the upper index* j *indicates the subcases 1, 2 or 3.*

Theorem 6.7. *Suppose zero is an exceptional point of the 2nd kind for* A. *Assume* $\alpha > 17$ *and let* s *satisfy* $17/2 < s < \alpha - 17/2$. *For some* $\delta > 0$ *we have in the norm of* $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$ *the asymptotic expansion*

$$R(A,\zeta) = -\zeta^{-1}B_{-2}^{(2)} - i\zeta^{-1/2}B_{-1}^{(2)} + B_0^{(2)} + i\zeta^{1/2}B_1^{(2)} - \zeta B_2^{(2)} + O(\zeta^{3/2}) \tag{6.3}$$

for $|\zeta| < \delta$, $\operatorname{Im}\zeta^{1/2} > 0$, *where* $(B_k^{(2)})^* = B_k^{(2)}$, $k = -2, -1, 0, 1, 2$, *as operators in* $\mathcal{B}(H^{-1,s}(\mathbb{R}), H^{1,-s}(\mathbb{R}))$.

Theorems 6.5, 6.6 and 6.7 are found in ref. [23] under the assumption that $\langle 1, W1\rangle \neq 0$. This assumption is a natural solvability condition but has no physical explanation. Expansions with a similar structure can be derived if $\langle 1, W1\rangle = 0$ but in this case the coefficients are different (see ref. [24] for details). By stating the theorems as above, we do not differentiate between the two situations. The first few coefficients in each of the expansions are computed explicitly in ref. [23]. In principle the method allow us to compute terms to any order but in practice the computations of higher order terms become very complicated and tedious to perform. Similar expansions have been derived by Bollé, Gesztesy, Wilk and Klaus in ref. [10, 8] for local potentials (/multiplication operators) having exponential decay at infinity but their methods do not work for non-local potentials satisfying, e.g., the above-mentioned abstract short-range condition.

The results in Theorems 6.5, 6.6 and 6.7 are valid for the Schrödinger operator $A = h_0 + W$ in $\widehat{\mathcal{H}}_0 = L^2(\mathbb{R})$. If we introduce $A' := A + E_0$ and

$$\tilde{A} := H_0 - V_{01}R_1(E_0)V_{10} = P_0^* A' P_0 \tag{6.4}$$

then we obtain the following general expression for the asymptotic expansions of the resolvent of $\tilde{A}$:

$$\begin{aligned} R(\tilde{A},\zeta) = &-(\zeta - E_0)^{-1}G_{-2}^{(l)} - i(\zeta - E_0)^{-1/2}G_{-1}^{(l)} + \\ &+ G_0^{(l)} + i(\zeta - E_0)^{1/2}G_1^{(l)} - (\zeta - E_0)G_2^{(l)} + \mathrm{O}(|\zeta - E_0|^{3/2}), \end{aligned} \tag{6.5}$$

in the norm of $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ as $\zeta \to E_0$, $\zeta \in \mathbb{C}\backslash[E_0, \infty)$, and $G_k^{(l)} = P_0^* B_k^{(l)} P_0$, $l = 0, 1$, $k = -1, 0, 1, 2$. The expansions in Eq. (6.5) are used explicitly in the following section.

7. Asymptotic Expansions of the Resolvent at the Lowest Landau Threshold

In this section we deduce asymptotic expansions of the resolvent $\boldsymbol{R}(\zeta)$ as the spectral parameter ζ tends to the threshold E_0. We take into account explicitly the E_0-energy properties of $\tilde{A}$.

Assumption 7.1.
(i) Let $E_0 \in \rho(H_1)$.
(ii) Suppose that E_0 is a regular point of $\tilde{A}$.

Lemma 7.2. *Let Assumption 4.1 hold with $\alpha > 7$ and let Assumption 7.1 be fulfilled. Let s satisfy $7/2 < s < \alpha - 7/2$. Then we have in the norm of $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ the expansion*

$$T_0(\zeta)^{-1} = t_0^{(0)} + i(\zeta - E_0)^{1/2} t_1^{(0)} + O(|\zeta - E_0|) \tag{7.1}$$

as $|\zeta - E_0| \to 0$, where

$$t_0^{(0)} = G_0^{(0)}, \quad t_1^{(0)} = G_1^{(0)}. \tag{7.2}$$

Here $G_k^{(0)} = P_0^* B_k^{(0)} P_0$, where the coefficients $B_k^{(0)}$ appeared in Theorem 6.5.

Proof. The idea of the proof is to factor the operator $T_0(\zeta)$ in order to show that the inverse of $T_0(\zeta)$ exists and admits an asymptotic expansion in the norm topology of $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ for suitable s and $|\zeta - E_0|$ small enough.

Under Assumption 7.1 it follows from Theorem 6.6 that we have in the norm of $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ the expansion

$$R(\tilde{A}, \zeta) = G_0^{(0)} + i(\zeta - E_0)^{1/2} G_1^{(0)} + \mathrm{O}(|\zeta - E_0|) \tag{7.3}$$

as $|\zeta - E_0| \to 0$, $\alpha > 7$ and $7/2 < s < \alpha - 7/2$. We use the factorization

$$T_0(\zeta) = (\tilde{A} - \zeta)[I - (\zeta - E_0) R(\tilde{A}, \zeta) V_{01} \tilde{R}_1(\zeta) V_{10}], \tag{7.4}$$

where, for $|\zeta - E_0|$ small enough,

$$\tilde{R}_1(\zeta) = \sum_{n=0}^{\infty} (\zeta - E_0)^n R_1(E_0)^{n+2}. \tag{7.5}$$

From Eq. (7.3) and Eq. (7.5) we have the following asymptotic expansion in $\mathcal{B}(\mathcal{H}_0^{-s})$.

$$\begin{aligned} I &- (\zeta - E_0) R(\tilde{A}, \zeta) V_{01} \tilde{R}_1(\zeta) V_{10} \\ &= I - (\zeta - E_0) G_0^{(0)} V_{01} R_1(E_0)^2 V_{10} \\ &- i(\zeta - E_0)^{3/2} G_1^{(0)} V_{01} R_1(E_0)^2 V_{10} + \mathrm{O}(|\zeta - E_0|^2). \end{aligned}$$

Via the Neumann series we invert the latter expression and find that

$$
\begin{aligned}
&[I-(\zeta-E_0)R(\tilde{A},\zeta)V_{01}\tilde{R}_1(\zeta)V_{10}]^{-1}\\
&\quad = I-(\zeta-E_0)G_0^{(0)}V_{01}R_1(E_0)^2V_{10}\\
&\quad -i(\zeta-E_0)^{3/2}G_1^{(0)}V_{01}R_1(E_0)^2V_{10}+\mathrm{O}(|\zeta-E_0|^2). \qquad (7.6)
\end{aligned}
$$

Combining Eq. (7.3) and Eq. (7.6) we obtain the desired expansion in Eq. (7.1). □

From Lemma 7.2 and the Feshbach formula (See Eq. (2.3)) we immediately obtain the following result.

Theorem 7.3. *Let the assumptions in Lemma 7.2 hold. Then we have in the norm of* $\mathcal{B}(\mathcal{H}_0^s\oplus\mathcal{H}_1,\mathcal{H}_0^{-s}\oplus\mathcal{H}_1)$ *the expansion*

$$
\begin{aligned}
\boldsymbol{R}(\zeta) &= \begin{pmatrix} t_0^{(0)} & -t_0^{(0)}V_{01}R_1(E_0)\\ -R_1(E_0)V_{10}t_0^{(0)} & R_1(E_0)+R_1(E_0)V_{10}t_0^{(0)}V_{01}R_1(E_0)\end{pmatrix}\\
&\quad +i(\zeta-E_0)^{1/2}\begin{pmatrix} t_1^{(0)} & -t_1^{(0)}V_{01}R_1(E_0)\\ -R_1(E_0)V_{10}t_1^{(0)} & R_1(E_0)V_{10}t_1^{(0)}V_{01}R_1(E_0)\end{pmatrix}\\
&\quad +O(|\zeta-E_0|).
\end{aligned}
$$

as $|\zeta-E_0|\to 0$, *where the operators* $t_k^{(0)}$ *are given in Lemma 7.2.*

Next we consider the following case.

Assumption 7.4.
(i) Let $E_0\in\rho(H_1)$.
(ii) Suppose that E_0 is an exceptional point of 1st kind for $\tilde{A}$ (type 1, 2 or 3).

Lemma 7.5. *Let Assumption 4.1 hold with* $\alpha>11$ *and let Assumption 7.4 be fulfilled. Let* s *satisfy* $11/2<s<\alpha-11/2$. *Then we have in the norm of* $\mathcal{B}(\mathcal{H}_0^s,\mathcal{H}_0^{-s})$ *the expansion*

$$
T_0(\zeta)^{-1} = -i(\zeta-E_0)^{-1/2}t_{-1}^{(1,j)}+t_0^{(1,j)}+i(\zeta-E_0)^{1/2}t_1^{(1,j)}+O(|\zeta-E_0|) \qquad (7.7)
$$

as $|\zeta-E_0|\to 0$, $\operatorname{Im}\zeta>0$, *where*

$$
t_{-1}^{(1,j)}=G_{-1}^{(1,j)},\quad t_0^{(1,j)}=G_0^{(1,j)}-G_{-1}^{(1,j)}V_{01}R_1(E_0)^2V_{10}G_{-1}^{(1,j)}, \qquad (7.8)
$$

$$
\begin{aligned}
t_1^{(1,j)} &= G_1^{(1,j)}-G_{-1}^{(1,j)}V_{01}R_1(E_0)^2V_{10}G_0^{(1,j)}\\
&\quad -G_0^{(1,j)}V_{01}R_1(E_0)^2V_{10}G_{-1}^{(1,j)}+(G_{-1}^{(1,j)}V_{01}R_1(E_0)^2V_{10})^2G_{-1}^{(1,j)}. \qquad (7.9)
\end{aligned}
$$

Here $G_k^{(1,j)}=P_0^*B_k^{(1,j)}P_0$, *where the coefficients* $B_k^{(1,j)}$ *appeared in Theorem 6.6.*

Proof. The proof is similar to the proof of Lemma 7.2. □

From Lemma 7.5 and the Feshbach formula we obtain the following result.

Theorem 7.6. *Let the assumptions in Lemma 7.5 hold. Then we have in the norm of* $\mathcal{B}(\mathcal{H}_0^s \oplus \mathcal{H}_1, \mathcal{H}_0^{-s} \oplus \mathcal{H}_1)$ *the expansion*

$$
\begin{aligned}
\boldsymbol{R}(\zeta) &= -i(\zeta - E_0)^{-1/2} \times \\
&\times \begin{pmatrix} t_{-1}^{(1,j)} & -t_{-1}^{(1,j)} V_{01} R_1(E_0) \\ -R_1(E_0) V_{10} t_{-1}^{(1,j)} & R_1(E_0) V_{10} t_{-1}^{(1,j)} V_{01} R_1(E_0) \end{pmatrix} \\
&+ \begin{pmatrix} t_0^{(1,j)} & -t_0^{(1,j)} V_{01} R_1(E_0) \\ -R_1(E_0) V_{10} t_0^{(1,j)} & R_1(E_0) + R_1(E_0) V_{10} t_0^{(1,j)} V_{10} R_1(E_0) \end{pmatrix} \\
&+ i(\zeta - E_0)^{1/2} \begin{pmatrix} t_1^{(1,j)} & t_{-1}^{(1,j)} V_{01} R_1(E_0)^2 \\ R_1(E_0)^2 V_{10} t_{-1}^{(1,j)} & -R_1(E_0)^2 V_{10} t_{-1}^{(1,j)} V_{01} R_1(E_0) - R_1(E_0) V_{10} t_{-1}^{(1,j)} V_{01} R_1(E_0)^2 \end{pmatrix} \\
&+ O(|\zeta - E_0|)
\end{aligned}
$$

as $|\zeta - E_0| \to 0$*, where the operators* $t_k^{(1,j)}$ *are given in Lemma 7.5.*

Finally we consider the following case.

Assumption 7.7.
(i) Let $E_0 \in \rho(H_1)$.
(ii) Suppose that E_0 is an exceptional point of 2nd kind for $\tilde{A}$.
(iii) Suppose that E_0 is a simple eigenvalue of $\tilde{A}$. We denote the associated normalized eigenfunction ψ.

We use the notation $P_{\{E_0\}} = \langle \cdot, \psi \rangle \psi$ for the projection along ψ and note that $G_{-2}^{(2)} = P_{\{E_0\}}$ according to Theorem 6.7. Suppose, in addition, that

$$\alpha_0 := \langle V_{01} R_1(E_0)^2 V_{10} \psi, \psi \rangle \neq 0.$$

Having introduced the constant α_0 we define the projections

$$J_1 = \alpha_0^{-1} \langle V_{01} R_1(E_0)^2 V_{10} \cdot, \psi \rangle \psi, \quad J_0 = 1 - J_1.$$

Define also $\tilde{J} = (1 + \alpha_0 J_1)^{-1} = J_0 + \frac{1}{1+\alpha_0} J_1$. Then we have the following result.

Lemma 7.8. *Let Assumption 4.1 hold with* $\alpha > 15$ *and let Assumption 7.7 be fulfilled. Let* s *satisfy* $15/2 < s < \alpha - 15/2$*. Assume, moreover, that* $\alpha_0 \neq 0$*. Then we have in the norm of* $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ *the expansion*

$$
\begin{aligned}
T_0(\zeta)^{-1} &= -(\zeta - E_0)^{-1} t_{-2}^{(2)} - i(\zeta - E_0)^{-1/2} t_{-1}^{(2)} + t_0^{(2)} \\
&\quad + i(\zeta - E_0)^{1/2} t_1^{(2)} + O(|\zeta - E_0|) \qquad (7.10)
\end{aligned}
$$

as $|\zeta - E_0| \to 0$*, where*

$$t_{-2}^{(2)} = \tilde{J} G_{-2}^{(2)}, \quad t_{-1}^{(2)} = \tilde{J} G_{-1}^{(2)} - \tilde{J} G_{-1}^{(2)} V_{01} R_1(E_0)^2 V_{10} \tilde{J} G_{-2}^{(2)}, \qquad (7.11)$$

$$t_0^{(2)} = \tilde{J}G_0^{(2)} + \tilde{J}G_0^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-2}^{(2)} - \tilde{J}G_{-2}^{(2)}V_{01}R_1(E_0)^3V_{10}\tilde{J}G_{-2}^{(2)}$$
$$+\tilde{J}(G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10})^2\tilde{J}G_{-2}^{(2)} - \tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-1}^{(2)} \quad (7.12)$$

$$\begin{aligned} t_1^{(2)} &= \tilde{J}G_1^{(2)} - \tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_0^{(2)} \\ &+\tilde{J}G_0^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-1}^{(2)} - \tilde{J}G_{-2}^{(2)}V_{01}R_1(E_0)^3V_{10}\tilde{J}G_{-1}^{(2)} \\ &+\tilde{J}(G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10})^2\tilde{J}G_{-1}^{(2)} + \tilde{J}G_1^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-2}^{(2)} \\ &-\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^3V_{10}\tilde{J}G_{-2}^{(2)} + \tilde{J}(G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10})^3\tilde{J}G_{-2}^{(2)} \\ &+\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_0^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-2}^{(2)} \\ &-\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-2}^{(2)}V_{01}R_1(E_0)^3V_{10}\tilde{J}G_{-2}^{(2)} \\ &+\tilde{J}G_0^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-2}^{(2)} \\ &-\tilde{J}G_{-2}^{(2)}V_{01}R_1(E_0)^3V_{10}\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10}\tilde{J}G_{-2}^{(2)}. \end{aligned} \quad (7.13)$$

Here $G_k^{(2)} = P_0^* B_k^{(2)} P_0$, *where the coefficients* $B_k^{(2)}$ *appeared in Theorem 6.7.*

Proof. The proof follows the lines of the proof of Lemma 7.2 with some additional manipulations.

Under Assumption 7.7 it follows from Theorem 6.7 that we have in the norm of $\mathcal{B}(\mathcal{H}_0^s, \mathcal{H}_0^{-s})$ the expansion

$$\begin{aligned} R(\tilde{A}, \zeta) &= -(\zeta - E_0)^{-1}G_{-2}^{(2)} - i(\zeta - E_0)^{-1/2}G_{-1}^{(2)} \\ &+G_0^{(2)} + i(\zeta - E_0)^{1/2}G_1^{(2)} + \mathrm{O}(|\zeta - E_0|) \end{aligned} \quad (7.14)$$

as $|\zeta - E_0| \to 0$, $\alpha > 15$ and $15/2 < s < \alpha - 15/2$. Again we use the factorization (7.4). From Eq. (7.14) and Eq. (7.5) we have the following asymptotic expansion in $\mathcal{B}(\mathcal{H}_0^{-s})$.

$$\begin{aligned} &I - (\zeta - E_0)R(\tilde{A}, \zeta)V_{01}\tilde{R}_1(\zeta)V_{10} = (I + \alpha_0 J_1)\times \\ &\Big(I + i(\zeta - E_0)^{1/2}\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^2V_{10} - (\zeta - E_0)\tilde{J}G_0^{(2)}V_{01}R_1(E_0)^2V_{10} \\ &\quad +(\zeta - E_0)\tilde{J}G_{-2}^{(2)}V_{01}R_1(E_0)^3V_{10} - i(\zeta - E_0)^{3/2}\tilde{J}G_1^{(2)}V_{01}R_1(E_0)^2V_{10} \\ &\quad + i(\zeta - E_0)^{3/2}\tilde{J}G_{-1}^{(2)}V_{01}R_1(E_0)^3V_{10}\Big) + \mathrm{O}(|\zeta - E_0|^2). \end{aligned}$$

Above we have rewritten the term $G_{-2}^{(2)}V_{01}R_1(E_0)^2V_{10}$ via $G_{-2}^{(2)} = P_{\{E_0\}} = \langle\cdot, \psi\rangle\psi$ and the operator J_1. Furthermore, we have used that the operator $I + \alpha_0 J_1$ has the inverse $\tilde{J} = J_0 + \frac{1}{1+\alpha_0}J_1$. Via the Neumann series we invert the latter expression and in conjunction with Eq. (7.4) and Eq. (7.14) we obtain the desired expansion in Eq. (7.10). □

Lemma 7.8 and the Feshbach formula yield the following result.

Theorem 7.9. *Let the assumptions in Lemma 7.8 hold. Then we have in the norm of* $\mathcal{B}(\mathcal{H}_0^s \oplus \mathcal{H}_1, \mathcal{H}_0^{-s} \oplus \mathcal{H}_1)$ *the expansion*

$$\begin{aligned}
\boldsymbol{R}(\zeta) = &-(\zeta - E_0)^{-1} \begin{pmatrix} t_{-2}^{(2)} & -t_{-2}^{(2)} V_{01} R_1(E_0) \\ -R_1(E_0) V_{10} t_{-2}^{(2)} & R_1(E_0) V_{10} t_{-2}^{(2)} V_{01} R_1(E_0) \end{pmatrix} \\
&-i(\zeta - \lambda)^{-1/2} \begin{pmatrix} t_{-1}^{(2)} & -t_{-1}^{(2)} V_{01} R_1(E_0) \\ -R_1(E_0) V_{10} t_{-1}^{(2)} & R_1(E_0) V_{10} t_{-1}^{(2)} V_{01} R_1(E_0) \end{pmatrix} \\
&+ \begin{pmatrix} t_0^{(2)} & -t_0^{(2)} V_{01} R_1(E_0) + t_{-2}^{(2)} V_{01} R_1(E_0)^2 \\ -R_1(E_0) V_{10} t_0^{(2)} + R_1(E_0)^2 V_{10} t_{-2}^{(2)} & R_1(E_0)(V_{10} t_0^{(2)} V_{01} - V_{10} t_{-2}^{(2)} V_{01} R_1(E_0) - R_1(E_0) V_{10} t_{-2}^{(2)} V_{01}) R_1(E_0) \end{pmatrix} \\
&+i(\zeta - E_0)^{1/2} \begin{pmatrix} t_1^{(2)} & t_{-1}^{(2)} V_{01} R_1(E_0)^2 - t_1^{(2)} V_{01} R_1(E_0) \\ R_1(E_0)^2 V_{10} t_{-1}^{(2)} - R_1(E_0) V_{10} t_1^{(2)} & R_1(E_0)(V_{01} t_1^{(2)} V_{01} - R_1(E_0) V_{01} t_{-1}^{(2)} V_{01} - V_{01} t_{-1}^{(2)} V_{01} R_1(E_0)) R_1(E_0) \end{pmatrix} \\
&+O(|\zeta - E_0|) \qquad (7.15)
\end{aligned}$$

as $|\zeta - E_0| \to 0$*, where the operators* $t_k^{(2)}$ *are given in Lemma 7.8.*

As a consequence of Theorem 7.9 we have the following result.

Corollary 7.10. *Let the hypotheses of Lemma 7.8 hold. Then* E_0 *is an eigenvalue of* H_m.

8. The Scattering Matrix near the Lowest Landau Threshold

The scattering theory for the pair (H_m, H_{om}) was established in ref. [25] by means of the abstract short range scattering theory developed by Jensen, Mourre and Perry in ref. [15] (See also ref. [16]). Here we just recapitulate the main result. The essential spectrum of H_{om} is the union of infinitely many semilines starting at E_n, $n = 0, 1, 2, \dots$, respectively. The points E_n constitute the threshold set $\mathsf{T}_m = \{E_n : n = 0, 1, 2, \dots\}$ which underpins the definition of the intervals

$$I_n = (E_n, E_{n+1}), \;\; n = 0, 1, 2, \dots .$$

We summarize the contents of ref. [25, Lemma 8.5 and Proposition 8.6] in the following proposition.

Proposition 8.1. *Let Assumption 4.1 hold with* $\alpha > 1$*. Then the (local) wave operators* $W_\pm(H_m, H_{om}; I_n)$ *exist and are strongly asymptotically complete. Furthermore,* $\sigma_s(H_m) \cap I_n$ *is discrete in* I_n*,* $n = 0, 1, 2, \dots$.

It follows from Proposition 8.1 that the local scattering operator S_n, defined by

$$S_n = W_+^*(H_m, H_{om}; I_n)W_-(H_m, H_{om}; I_n), \ \ n = 0, 1, 2, \dots$$

is a unitary operator on $E_{I_n}(H_{om})P_{ac}(H_{om})\mathcal{H}$. Let $\tilde{S}_0$ denote the unitary representation of S_0 in $L^2(I_0; \mathbb{C}^2)$. There is a general theorem asserting that $\tilde{S}_0$ admits a diagonal representation $(\tilde{S}_0\psi)(\lambda) = \tilde{S}_0(\lambda)\psi(\lambda)$ (see, e.g., [19, Theorem 6.2]). Here λ denotes the energy parameter. We restrict ourselves to $\tilde{S}_0$, since we are only interested in the lowest Landau threshold E_0. In this section we give an explicit representation of $\tilde{S}_0(\lambda)$ (in the sequel we suppress the tilde character and the lower index) and derive asymptotic expansions of $S(\lambda)$ as $\lambda \downarrow E_0$.

We need several definitions. For $j = 0, 1, 2, \dots$, we define $k_j(\lambda) = (\lambda - E_j)^{1/2}$ and $\nu_j(\lambda) = (2k_j(\lambda))^{-1}$. In addition, define $j(\lambda)$ as the largest integer satisfying $2(m+|m|+2j(\lambda)) \leq \lambda$, i.e. $j(\lambda)$ is the number of Landau thresholds open at the energy λ. Introduce the layer $\widehat{\mathcal{H}}(\lambda)$ as the direct sum of a finite number of two-dimensional linear spaces $\mathbb{C}^2$ with elements

$$\widehat{\mathcal{H}}(\lambda) \ni g = \bigoplus_{j=0}^{j(\lambda)} \nu_j(\lambda) g_j, \quad g_j = \begin{pmatrix} g_j^{(+)} \\ g_j^{(-)} \end{pmatrix}.$$

Via the functions

$$\tilde{F}(j, \xi; \rho, z) = \frac{e^{-i\xi z}}{\sqrt{2\pi}} \Psi_j(\rho), \ \ j = 0, 1, 2, \dots,$$

we can introduce the trace operator for any $\psi \in E_{H_{om}}(I_0)\mathcal{H} = E_{H_{00}}(I_0)\mathcal{H}_0$ as

$$\begin{aligned} &\tilde{\gamma}(\lambda)\psi = \sqrt{\nu_0(\lambda)} \times \\ &\times \int_{\mathbb{R}}\int_{\mathbb{R}_+} \left\{ \tilde{F}(0, k_0(\lambda); \rho, z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right. \\ &\qquad \left. + \tilde{F}(0, -k_0(\lambda); \rho, z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \psi(\rho, z)\rho d\rho dz \\ &= \ 2^{-1/2}(\lambda - E_0)^{-1/4} \begin{pmatrix} \widehat{\sigma_0 \mathcal{P}_0 \psi}((\lambda - E_0)^{1/2}) \\ \widehat{\sigma_0 \mathcal{P}_0 \psi}(-(\lambda - E_0)^{1/2}) \end{pmatrix}, \ \ \lambda \in I_0, \quad (8.1) \end{aligned}$$

where the widehat symbol denotes the one-dimensional Fourier transform with respect to the z variable. The trace operator maps from $\mathcal{H}^{0,s} \oplus \mathcal{H}^{0,s}$ to $\widehat{\mathcal{H}}(\lambda)$ for $s > 1/2$.

Having introduced the necessary objects, we are ready to formulate the result. We refer ref. [22] for the rather lengthy proof. The proof imitates (but is more involved than) the proof of the representation of the scattering operator associated with Schrödinger operators found in, e.g. ref. [19].

Proposition 8.2. *Let Assumption 4.1 hold with $\alpha > 1$. Then the operator S is represented as*

$$(S\psi)(\lambda) = S(\lambda)\psi(\lambda), \ \ a.e. \ \ \lambda \in I_0\backslash\sigma_{pp^+}(H_m), \ \ \psi \in L^2(I_0;\mathbb{C}^2),$$

where

$$S(\lambda) = 1 - 2\pi i\tilde{\gamma}(\lambda)(1 - VR(\lambda + i0))V\tilde{\gamma}(\lambda)^*, \ \ \lambda \in I_0\backslash\sigma_{pp^+}(H_m). \tag{8.2}$$

The operator $S(\lambda)$ is unitary for all $\lambda \in I_0\backslash\sigma_{pp^+}(H_m)$. The operator $S(\lambda) - 1$ is compact for all $\lambda \in I_0\backslash\sigma_{pp^+}(H_m)$.

Next we deduce asymptotic expansions of the scattering matrix $S(\lambda)$ as the energy parameter λ tends to the lowest Landau threshold E_0. The scattering matrix has the diagonal representation in Eq. (8.2), which can be rewritten as

$$S(\lambda) = 1 - \pi i(\lambda - E_0)^{-1/2}\gamma_0(\lambda)(1 - \boldsymbol{R}(\lambda + i0)V)V\gamma_0(\lambda)^*, \tag{8.3}$$

where the definition of $\gamma_0(\lambda)$ is obvious from Eq. (8.2) and Eq. (8.3). To derive asymptotic expansions for $S_0(\lambda)$ as $\lambda \downarrow E_0$ we need expansions for the operators $\gamma_0(\lambda)$ and $\gamma_0(\lambda)^*$. Formally, we have

$$\gamma_0(\lambda) = \sum_{j=0}^{\infty} i^j(\lambda - E_0)^{j/2}\Gamma_j, \tag{8.4}$$

where

$$\Gamma_j \ : \ (2\pi)^{-1/2}(j!)^{-1} \begin{pmatrix} (-z)^j\Psi_0(\rho) \\ z^j\Psi_0(\rho) \end{pmatrix}. \tag{8.5}$$

This follows from a formal expansion of

$$\gamma_0(\lambda) \ : \ (2\pi)^{-1/2} \begin{pmatrix} \exp(-i(\lambda - E_0)^{1/2}z)\Psi_0(\rho) \\ \exp(i(\lambda - E_0)^{1/2}z)\Psi_0(\rho) \end{pmatrix}.$$

From $\Psi_0 \in L^2(\mathbb{R}_+, \rho d\rho)$ and Hölder's inequality we see that

$$\Gamma_j \in \mathcal{B}(\mathcal{H}^{0,s}, \mathbb{C}^2), \ \ s > j + 1/2. \tag{8.6}$$

Moreover, it follows from Taylor's formula with remainder that the expansion in Eq. (8.4) is valid as $\lambda \downarrow E_0$ in the sense that if $\gamma_0(\lambda)$ is approximated by a finite series up to $j = k$, k being the largest integer satisfying $s > k + 1/2$, then the remainder is $\mathrm{o}((\lambda - E_0)^{k/2})$ in the norm of $\mathcal{B}(\mathcal{H}^{0,s}, \mathbb{C}^2)$.

We establish the following leading order behaviour of the scattering matrix in the limit $\lambda \downarrow E_0$. The results are based on Eq. (8.3) and the asymptotic expansions of the resolvent $\boldsymbol{R}(\zeta)$ in Sect. 7.

Theorem 8.3. *Let Assumption 4.1 hold with $\alpha > 7$. Moreover, let Assumption 7.1 be satisfied. Then we have in the norm of $\mathcal{B}(\mathbb{C}^2)$ the* *following leading order expansion*

$$S(\lambda)=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}+o(1). \tag{8.7}$$

as $\lambda\downarrow E_0$.

Proof. Let s satisfy $7/2<s<\alpha-7/2$ and let $R_j^{(0)}$, $j=0,1$, denote the coefficients in Theorem 7.3. From Eq. (8.3), Eq. (8.4) and Theorem 7.3 we have the expansion $S(\lambda)=-i(\lambda-E_0)^{-1/2}S_{-1}^{(0)}+S_0^{(0)}+\mathrm{o}(1)$ in $B(\mathbb{C}^2)$, where

$$\begin{aligned} S_{-1}^{(0)} &= \pi\Gamma_0(V-VR_0^{(0)}V)\Gamma_0^*, \\ S_0^{(0)} &= 1_\lambda+\pi\Gamma_1(V-VR_0^{(0)}V)\Gamma_0^*-\pi\Gamma_0VR_1^{(0)}V\Gamma_0^* \\ &\quad -\pi\Gamma_0(V-VR_0^{(0)}V)\Gamma_1^*. \end{aligned}$$

Using $1=S(\lambda)S(\lambda)^*$ and the simple fact that $T^2=0$ implies that $T=0$ for any self-adjoint operator T, we obtain that $\Gamma_0(V-VR_0^{(0)}V)\Gamma_0^*=0$. Thus $S_{-1}^{(0)}=0$. As for $S_0^{(0)}$ we begin by rewriting the term $\pi\Gamma_0VR_1^{(0)}V\Gamma_0^*$ via the expression for Γ_0 in Eq. (8.5) and the expression for $R_1^{(0)}$ given in Theorem 7.3. For any $(z_1,z_2)\in\mathbb{C}^2$, the operator acts as

$$\pi\Gamma_0VR_1^{(0)}V\Gamma_0^*\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}=c\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where

$$\begin{aligned} c \;=\; & \frac{1}{2}\langle V_{00}t_1^{(0)}V_{00}\Psi_0-V_{00}t_1^{(0)}V_{01}R_1(E_0)V_{10}\Psi_0-V_{01}R_1(E_0)V_{10}t_1^{(0)}V_{00}\Psi_0 \\ & +V_{01}R_1(E_0)V_{10}t_1^{(0)}V_{01}R_1(E_0)V_{10}\Psi_0+V_{00}t_1^{(0)}V_{01}\Psi_0 \\ & -V_{01}t_1^{(0)}V_{01}R_1(E_0)V_{11}\Psi_0-V_{00}R_1(E_0)V_{10}t_1^{(0)}V_{01}\Psi_0 \\ & +V_{01}R_1(E_0)V_{10}t_1^{(0)}V_{01}R_1(E_0)V_{11}\Psi_0,\Psi_0\rangle. \end{aligned} \tag{8.8}$$

The operator $\Gamma_1(V-VR_0^{(0)}V)\Gamma_0^*$ can be written as a matrix with real elements. Therefore, for some real number a we find that

$$\left\{\pi\Gamma_1(V-VR_0^{(0)}V)\Gamma_0^*-\pi\Gamma_0(V-VR_0^{(0)}V)\Gamma_1^*\right\}=\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix},$$

since the terms on the left-hand side are each other adjoints. Hence,

$$S_0^{(0)}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-c\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}+\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}.$$

By the unitarity of $S_0^{(0)}$, we infer that $a=0$ and either $c=0$ or $c=1$. We show that $c=1$. First, we observe that c depends continuously on V_{00}, V_{01}, V_{10} and V_{11}, hence it suffices to consider the case where $V_{01}=V_{10}=0$. Moreover, only the first term on the right-hand side of Eq. (8.8) remains. In order to compute c we need the expression for $B_1^{(0)}$ (bear in mind that $t_1^{(0)}=G_1^{(0)}=P_0^*B_1^{(0)}P_0$) which is given in [23, Theorem 4]. Using this expression we find that $\langle V_{00}t_1^{(0)}V_{00}\Psi_0, \Psi_0\rangle = 2$. Hence $c = 1$ as desired. $\square$

In a similar way we obtain the following two theorems.

Theorem 8.4. *Let Assumption 4.1 hold with* $\alpha > 11$. *Moreover, let Assumption 7.4 be satisfied. Then we have in the norm of* $\mathcal{B}(\mathbb{C}^2)$ *the following leading order expansion*

$$\begin{aligned} S^{(1,j)}(\lambda) &= 1 - \pi\Gamma_0 V R_1^{(1,j)} V\Gamma_0^* - \pi\Gamma_0(V - VR_0^{(1,j)}V)\Gamma_1^* \\ &+\pi\Gamma_1(V - VR_0^{(1,j)}V)\Gamma_0^* - \pi\Gamma_0 V R_{-1}^{(1,j)} V\Gamma_2^* \\ &-\pi\Gamma_2 V R_{-1}^{(1,j)} V\Gamma_0^* + \pi\Gamma_1 V R_{-1}^{(1,j)} V\Gamma_1^* + o(1), \quad (j = 1,2,3), \end{aligned} \tag{8.9}$$

as $\lambda \downarrow E_0$.

Theorem 8.5. *Let Assumption 4.1 hold with* $\alpha > 15$ *and let Assumption 7.7 be satisfied. Assume moreover that* $\alpha_0 \neq 0$. *Then we have in the norm of* $\mathcal{B}(\mathbb{C}^2)$ *the following leading order expansion*

$$\begin{aligned} S^{(2)}(\lambda) &= 1 - \pi\Gamma_0 V R_1^{(2)} V\Gamma_0^* - \pi\Gamma_0(V - VR_0^{(2)}V)\Gamma_1^* \\ &+\pi\Gamma_1(V - VR_0^{(2)}V)\Gamma_0^* - \pi\Gamma_0 V R_{-1}^{(2)} V\Gamma_2^* - \pi\Gamma_2 V R_{-1}^{(2)} V\Gamma_0^* \\ &+\pi\Gamma_0 V R_{-2}^{(2)} V\Gamma_3^* - \pi\Gamma_3 V R_{-2}^{(2)} V\Gamma_0^* - \pi\Gamma_1 V R_{-2}^{(2)} V\Gamma_2^* \\ &+\pi\Gamma_2 V R_{-2}^{(2)} V\Gamma_1^* + \pi\Gamma_1 V R_{-1}^{(2)} V\Gamma_1^* + o(1) \end{aligned} \tag{8.10}$$

as $\lambda \downarrow E_0$.

In principle, the method allows us to derive an asymptotic expansion of the scattering matrix as $\lambda \downarrow E_0$ in each of the cases we consider. In practice, however, the computations become extremely complicated. In Theorem 8.3 we derived an explicit expression for the leading term and in particular we were able to simplify the leading coefficient (see Eq. (8.7)). Similarly, we obtained explicit expressions for the leading terms in Theorems 8.4 and 8.5. In the latter situations, however, we are not able to simplify the expressions in Eq. (8.9) and Eq. (8.10) since the expressions are quite complicated. For instance we observe that in order to simplify the expression in Eq. (8.9) it is necessary to determine the coefficient $G_1^{(1,j)}$, since this coefficient appears in the expression for $t_1^{(1,j)}$ (see Eq. (7.9)). In order to determine $G_1^{(1,j)}$ we need to compute explicitly the coefficient $B_1^{(1,j)}$ in the expansion in Eq. (6.2) of the resolvent for the one-dimensional Schrödinger operator A by means of the method in ref. [23]. This task turns out to be extremely tedious and complicated to do and, as a consequence, we have not succeeded in doing so.

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