THE ESSENTIAL SPECTRUM FOR HOLOMORPHIC TOEPLITZ OPERATORS ON H^p SPACES AND $BMOA$

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Abstract. We compute the essential Taylor spectrum of a tuple of analytic Toeplitz operators T_g on $H^p(D)$, where D is a strictly pseudoconvex domain. We also provide specific formulas for the index of T_g provided that $g^{-1}(0)$ is a compact subset of D.

1. Introduction and main results

Let D be a bounded domain in $\mathbb{C}^n$ with C^3 boundary. The Hardy space H^p consists of holomorphic functions f such that

$$\|f\|_{H^p}^p = \limsup_{\epsilon\to 0} \int_{\partial D_\epsilon} |f|^p d\sigma < \infty ,$$

where $D_\epsilon = \{\rho < -\epsilon\}$ for some defining function ρ of D. A tuple $g_1, \dots, g_m$ of bounded holomorphic functions on D defines a tuple T_g of commuting Toeplitz operators on $H^p(D)$, i.e., $T_{g_j} f = g_j f$. For each fixed $w \in \mathbb{C}^m$ we have the Koszul complex

$$0 \leftarrow \Lambda^0 H^p \xleftarrow{\delta_{w-g}} \Lambda^1 H^p \xleftarrow{\delta_{w-g}} \dots \xleftarrow{\delta_{w-g}} \Lambda^m H^p \leftarrow 0, \tag{1.1}$$

where $\Lambda^\ell H^p(D)$ is the space of all formal expressions

$$f = \sum_{|I|=\ell}{}' f_I e_I,$$

$e_I = e_{I_1} \wedge \dots \wedge e_{I_l}$, $e_1, \dots, e_m$ being some nonsense basis, the prime means that the summation is performed over increasing multiindices I of length ℓ, and the mappings δ_{w-g} are contraction by the operator-valued dual object

$$\sum_1^m (w_j - g_j) e_j^*.$$

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Note that the next to leftmost arrow in (1.1) is just the mapping $(H^p)^m \to H^p$ defined by

$$(u_1, \ldots, u_m) \mapsto \sum (w_j - g_j)u_j. \tag{1.2}$$

The Taylor spectrum $\sigma(T_g, H^p)$ of the (commuting tuple of operators) T_g on H^p is defined as the set of $w \in \mathbb{C}^m$ such that (1.1) is not exact. The right spectrum $\sigma_r(g, H^p)$ is the set of w such that (1.1) is not exact at the next to leftmost point, i.e., such that (1.2) is not surjective. It is a consequence of the open mapping theorem that $\sigma_r(g, H^p)$ and $\sigma(g, H^p)$ are closed and hence compact sets. Obviously (1.2) is not surjective if $w \in g(D)$, and therefore we have the inclusions

$$\overline{g(D)} \subset \sigma_r(g, H^p) \subset \sigma(g, H^p). \tag{1.3}$$

The essential spectrum $\sigma_{ess}(T_g, H^p)$ is the set of $w \in \mathbb{C}^m$ such that not all the homology of (1.1) is finite dimensional, or stated in other words, that T_{w-g} is not a Fredholm tuple. Moreover, the right essential spectrum $\sigma_{ress}(T_g)$ as the set of w such that homology of (1.1) at $m = 0$ is infinite dimensional. In [3] it was proved

Theorem 1.1. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary and assume that $g_1, \ldots, g_m \in H^\infty$. Then $\sigma(g, H^p) = \sigma_r(g, H^p) = \overline{g(D)}$ for all $p < \infty$. If $n = 1$ it also holds for $p = \infty$.*

The main result in this note is

Theorem 1.2. *Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^3 boundary and assume that $g_1, \ldots, g_m \in H^\infty$. Then*

$$\sigma_{ess}(T_g, H^p) = \sigma_{ress}(T_g, H^p) = \bigcap_{\mathcal{U} \supset \partial D} \overline{g(D \cap \mathcal{U})} \tag{1.4}$$

for all $p < \infty$. If $n = 1$ it also holds for $p = \infty$.

Thus, roughly speaking, $\sigma_{ess}(T_g)$ is the image of the boundary of ∂D under the mapping $g \colon D \to \mathbb{C}^m$. It follows that $\sigma_{ess}(T_g) = \sigma(T_g)$ if $m < n$. For two generators these results were proved in [9]. The analog of Theorem 1.1 for the Bergman space $L^2_a(D) = \mathcal{O}(D) \cap L^2(D)$ was proved in [5] for any bounded pseudoconvex domain as well as the inclusion

$$\sigma_{ess}(T_g, L^2_a(D)) \subset \bigcap_{\mathcal{U} \supset \partial D} \overline{g(D \cap \mathcal{U})}, \tag{1.5}$$

with equality at least if D is strictly pseudoconvex.

Remark 1. Theorem 1.1 was also proved for $BMOA$ and $L^p_a(D) = \mathcal{O}(D) \cap L^p(D)$, $1 \le p < \infty$ in [3], The equality $\sigma_r(T_g, X) = \overline{g(D)}$ has been proved for a large number of other function spaces X in strictly pseudoconvex domains such as various Besov spaces ([7] and [8]), and $\mathcal{Q}_p$ spaces ([13] and [4]). In each case here, the g_j are assumed to be multipliers on the space X in question. The proofs in these papers can be adapted to yield the complete analogues of Theorem 1.1 and the corresponding inclusions (1.5). Moreover, Theorem 1.1 holds for all $p \leq 2$ if D admits a C^2 plurisubharmonic defining function, see [12], [1], and [2], and again one can prove the corresponding inclusion (1.5) in this case. □

2. Proofs

Clearly it is enough to prove that

$$\sigma_{ess}(T_g, H^p) = \bigcap_{\mathcal{U} \supset \partial D} \overline{g(D \cap \mathcal{U})}. \tag{2.1}$$

The inclusion $\supset$ follows verbatimly as for the Bergman space $L^2_a(D)$, see Theorem 8.2.6 in in [6]. For the rest of the paper we therefore concentrate on the inclusion $\subset$ of (2.1). A moment of thought reveals that it is equivalent to the statement that (1.1) has finite dimensional homology for each w such that $g^{-1}(w)$ is a compact subset of D. In what follows we can with no loss of generality assume that $w = 0$. Let us introduce the notation

$$H_r(T_g, H^p(D)) = \frac{\mathrm{Ker}\,(\Lambda^r H^p(D) \xrightarrow{\delta_g} \Lambda^{r-1} H^p(D))}{\mathrm{Im}\,(\Lambda^{r+1} H^p(D) \xrightarrow{\delta_g} \Lambda^r H^p(D))}.$$

and define $H_r(T_g, \mathcal{O}(V))$ in the analogous way, for $V \subset\subset D$. The restriction mapping $H^p(D) \to \mathcal{O}(V)$ then induces a mapping of complexes from (1.1) (for $w = 0$) to the complex

$$0 \leftarrow \Lambda^0 \mathcal{O}(V) \xleftarrow{\delta_g} \Lambda^1 \mathcal{O}(V) \to \ldots,$$

and hence we get a mapping on homology

$$H_r(T_g, H^p(D)) \to H_r(T_g, \mathcal{O}(V)). \tag{2.2}$$

We now have

Theorem 2.1. *Suppose that D is a strictly pseudoconvex domain with C^3 boundary, and g is a tuple of bounded holomorphic functions on D such that $g^{-1}(0)$ is a compact subset of D, and let V be a pseudoconvex open set such that $g^{-1}(0) \subset V \subset\subset D$. If $n \geq 1$ and $1 \leq p < \infty$ or $n = 1$ and $= \infty$, then the mapping (2.2) is an isomorphism. The same holds for $BMOA$ provided g_j are multipliers on $BMOA$.*

If $g^{-1}(0)$ is a compact subset of D it is clearly a discrete set.

Proof of the inclusion $\subset$ of Theorem 1.2. It is wellknown that $H_r(T_g, \mathcal{O}(V))$ are all finite dimensional and therefore $H_r(T_g, H^p(D))$ are. Thus the inclusion $\subset$ of (1.4) follows. □

In case that T_g is a Fredholm tuple we define the index as

$$\mathrm{Ind}\,(T_g, H^p(D)) = \sum_{r=0}^{m} (-1)^r \dim H_r(T_g, H^p(D)),$$

and $\mathrm{Ind}\,(T_g, \mathcal{O}(V))$ in the analogous way. Then we have

Corollary 2.2. *Under the assumptions in Theorem 2.1 we have that*

$$\operatorname{Ind}(T_g, H^p(D)) = \operatorname{Ind}(T_g, \mathcal{O}(V)).$$

In particular, if $m = n$ it is wellknown that $H_r(T_g, \mathcal{O}(V) = 0$ for $r > 0$ and that

$$\dim H_0(T_g, \mathcal{O}(V)) = \sum_{j=1}^{M} \nu_{z_j}(g),$$

if $z_1, \dots, z_M$ is the zeros of g and $\nu_{z_j}(g)$ is the order of the zero of g at the point z_j. Thus we have

Corollary 2.3. *Under the assumptions in Theorem 2.1 and $m = n$ we have that $H_r(T_g, H^p(D)) = 0$ for $r > 0$ and*

$$\operatorname{Ind}(T_g, H^p(D)) = \sum_{j=1}^{M} \nu_{z_j}(g). \tag{2.3}$$

The formula (2.3) appeared in [5] for the Bergman space $L^2_a(D)$ instead of $H^p(D)$.

3. Proof of Theorem 2.1

We only write down the proof of Theorem 2.1 for $H^p(D)$, $1 \le p < \infty$; the case $p = \infty$ when $n = 1$ follows with minor modifications. The proof relies on the technique and results in [10]. Let us fix a p, $1 \le p < \infty$, and let $X = H^p(D)$. There are, cf., [10], Frechet spaces B_k, such that $\mathcal{E}_{0,k}(\overline{D}) \subset B_k \subset \mathcal{E}_{0,k}(D)$,

$$0 \to X \to B_0 \xrightarrow{\bar\partial} B_1 \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} B_n \to 0 \tag{3.1}$$

is a complex, and such that, moreover, T_{g_j} are bounded on B_k, and if χ is a cutoff function in D which is identically 1 in a neighborhood of $g^{-1}(0)$, then also $(1-\chi)\bar{g}_j/|g|^2$ are bounded operators on B_k. In fact, we just take the Banach spaces B_k from [10] intersected with $\mathcal{E}_{0,k}(D)$; the last statement follows from [10] since the B_k condition is just a boundedness condition close to ∂D. To see that these new spaces are complete, let $\{f_j\}$ be a Cauchy sequence in the space of B_k from [10] intersected with $\mathcal{E}_{0,k}(D)$. Let f be the limit of $\{f_j\}$ in $\mathcal{E}_{0,k}(D)$. Fatou's lemma applied twice to each term in the definition of $\|\cdot\|_{B_k}$ yields that $\|f - f_j\|_{B_k} \le \liminf_l \|f_l - f_j\|_{B_k}$. Thus $\{f_j\}$ converge to f in B_k and hence B_k intersected with $\mathcal{E}_{0,k}(D)$ is complete.

Let us consider the exterior algebra over the vector bundle $T^*_{0,1} \oplus E$ over D and let $B^\ell_k(D)$ denote sections of degree k in $T^*_{0,1}$ (i.e., $(0,k)$ forms) and degree ℓ in e_j; thus $f \in B^\ell_k(D)$ can be written

$$f = \sum_{|I|=\ell}{}' f_I \wedge e_I,$$

where $f_I \in B_k$. The operators $\delta = \delta_g$ and $\bar\partial$ anticommute so $B_k^\ell(D)$ is a double complex, and we have the corresponding total complex

$$(3.2)\qquad \xrightarrow{\delta-\bar\partial} \mathcal{L}^r(\mathrm{Tot}\, B_k^\ell(D)) \xrightarrow{\delta-\bar\partial} \mathcal{L}^{r+1}(\mathrm{Tot}\, B_k^\ell(D)) \xrightarrow{\delta-\bar\partial}$$

where

$$\mathcal{L}^r(\mathrm{Tot}\, B_k^\ell(D)) = \bigoplus_{k=0}^{n} B_k^{k-r}(D),$$

and we let $H^r(\mathrm{Tot}\, B_k^\ell(D))$ denote the cohomology of the complex (3.2). In the same way we have the double complex $\mathcal{E}_{0,k}^\ell(V)$, and its corresponding total complex $\mathcal{L}^r(\mathrm{Tot}\, \mathcal{E}_{0,k}^\ell(V))$, and cohomology groups $H^r($ $\mathrm{Tot}\, \mathcal{E}_{0,k}^\ell(V))$. The restriction mapping induces a mapping of double complexes, thus a mapping of the total complexes, and hence on cohomology, i.e., we have natural mappings

$$(3.3)\qquad H^r(\mathrm{Tot}\, B_k^\ell(D)) \to H^r(\mathrm{Tot}\, \mathcal{E}_{0,k}^\ell(V)),$$

Lemma 3.1. *If $g^{-1}(0) \subset V \subset\subset D$, then (3.3) is an isomorphism.*

Proof. Both $B_k^\ell(D)$ and $\mathcal{E}_{0,k}^\ell(V)$ are modules over $\mathcal{E}(\overline{D})$. Let

$$\gamma = \sum_{j=1}^{m} \frac{\bar g_j}{|g|^2} e_j,$$

and let $\Gamma f = \gamma \wedge f$. Then Γ is bounded on $B_k(D \setminus g^{-1}(0)))$ and $\mathcal{E}_{0,k}^\ell(V \setminus g^{-1}(0))$, and $(\delta\Gamma + \Gamma\delta)f = f$ so the double complexes $B_k^\ell(D \setminus g^{-1}(0))$ and $\mathcal{E}_{0,k}^\ell(V \setminus g^{-1}(0))$ are exact in ℓ and therefore the lemma follows from a standard homology argument.

However, for us it is worthwhile to consider a more concrete version of the argument. Let χ be a cutoff function in V that is 1 in a neighborhood of $g^{-1}(0)$; then $f \mapsto (1-\chi)\gamma \wedge f$ is a bounded operator on B_k. Moreover, if

$$\alpha = \sum_{1}^{m} \gamma \wedge (\bar\partial\gamma)^{j-1},$$

then $(\delta - \bar\partial)\alpha = 1$ and $f \mapsto (1-\chi)\alpha \wedge f$ is bounded $\mathcal{L}^r(\mathrm{Tot}\, B_k^\ell(D)) \to \mathcal{L}^{r-1}(\mathrm{Tot}\, B_k^\ell(D))$. If now $[f] \in H^r(\mathrm{Tot}\, B_k^\ell(D))$ and there is a $v \in \mathcal{L}^r(\mathrm{Tot}\, \mathcal{E}_{0,k}^\ell(V))$ such that $(\delta - \bar\partial)v = f$ in V, then

$$(\delta - \bar\partial)\big(\chi v + (1-\chi)\alpha \wedge f - \bar\partial\chi \wedge \alpha \wedge v\big) = f$$

in D, which shows that (3.3) is injective. On the other hand, if $v \in \mathcal{L}^r(\mathrm{Tot}\, \mathcal{E}_{0,k}^\ell(V))$ and $(\delta - \bar\partial)v = 0$, then

$$(\delta - \bar\partial)\big((1-\chi)\alpha \wedge v\big) = v - (\chi v - \bar\partial\chi \wedge \alpha \wedge v)$$

which shows that v then is cohomologous in V to the form $\chi v - \bar\partial\chi \wedge \alpha \wedge v \in \mathcal{L}^r(\mathrm{Tot}\, B_k^\ell(D))$, which shows that (3.3) is surjective. $\square$

Notice that $\mathcal{E}^\ell_{0,k}(V)$ is exact in k (recall that V is assumed to be pseudoconvex) except on level $k=0$, where the kernels are $\mathcal{O}^\ell(V) = \Lambda^\ell\mathcal{O}(V)$. By standard homological algebra it follows that the natural mapping $H_r(T_g, \mathcal{O}(V)) \to H^{-r}(\mathrm{Tot}\, \mathcal{E}^\ell_{0,k}(V))$ is an isomorphism (an explicit argument is contained in the proofs of Lemma 3.2 and Theorem 2.1 below). The mapping $X \to B_0$ also induces natural mappings $H_r(T_g, X) \to H^{-r}(\mathrm{Tot}\, B^\ell_k(D))$ and thus we have the following picture,

$$H_r(T_g, X) \to H^{-r}(\mathrm{Tot}\, B^\ell_k(D)) \simeq H^{-r}(\mathrm{Tot}\, \mathcal{E}^\ell_{0,k}(V)) \simeq H_r(T_g, \mathcal{O}(V)), \tag{3.4}$$

where the composed mappings $H_r(T_g, X) \to H_r(T_g, \mathcal{O}(V))$ are the natural ones, i.e., (3.3). If also the complex (3.1) were exact, then it would follow immediately, by the same argument, that the leftmost arrow in (3.4) were an isomorphism, and hence Theorem 2.1 would have been proved.

Unfortunately, we cannot find such a complex with the stated properties which is also exact. The main point in [10] is that one can do without exactness.

Remark 2. The idea was used, though not formalized, in [3] and goes back to Wolff's proof of the corona problem. To see this, let $n=1$ and $p=\infty$. Then our space B_1 is the space of $(0,1)$-forms f such that $r|f|^2 + r|\partial f|$ is a Carleson measure, r being the distance to the boundary, and B_0 is the space of functions u such that $\bar\partial u \in B_1$. Then $0 \to H^\infty \to B_0 \to B_1 \to 0$ is a complex and B_j are closed under T_g. Although not exact, any $f \in B_1$ admits a solution $u \in L^\infty(\partial D)$ to $\bar\partial_b u = f$, this is Wolff's theorem, and this turns out to imply the missing isomorphism in (3.4) (for $r=0$). Assuming that $g^{-1}(0)$ is empty, as in the corona theorem, one concludes that $H_r(T_g, X) = 0$ which precisely is the corona theorem. □

It turns out that it is enough to find spaces B'_k, $\mathcal{E}_{0,k}(\overline{D}) \subset B_k \subset \mathcal{E}_{0,k}(D)$, with the following properties:
(i) The spaces B'_k are closed under T_g.
(ii) If $k \ge 0$ and $f \in B_{k+1} + B'_{k+1}$ and $\bar\partial f = 0$, then there is a solution $u \in B'_k$ to $\bar\partial u = f$.
(iii) If $f \in B_0 + B'_0$ and $\bar\partial f = 0$, then $f \in X$.

To be precise, in [10] is defined spaces B'_k such that (i) and (ii) holds for $k \ge 1$ (for $k \ge 2$ actually $B'_k = B_k$). Moreover, B'_0 is chosen as $L^p(\partial D)$ and the exact statement of (ii) for $k=0$ is: If $f \in B_1 + B'_1$ then there is a solution $u \in B'_0$ to $\bar\partial_b u = f$; for this use of the symbol $\bar\partial_b$, see [10]. Moreover, it turns out that there is a trace mapping $\tau\colon B_0 \to \tau B_0 \subset L^p(\partial D)$, thus the elements in B_0 has boundary values in $L^p(\partial D)$, and the precise statement of (iii) is: If $f \in \tau B_0 + B'_0$ and $\bar\partial_b f = 0$, then $f \in \tau H^p \simeq H^p$.

In the proof below, for simplicity, assume that we actually have access to spaces satisfying (i) to (iii), and leave it to the reader to fill in the small formal modifications that are needed; see also [10].

Lemma 3.2. *If* $u \in \mathcal{L}^{-r}(\mathrm{Tot}\, B_k^\ell(D))$ *and*

$$(\delta - \bar\partial)u = f \in \Lambda^{r+1}X,$$

then there is $v \in \mathcal{L}^{-r-1}(\mathrm{Tot}\,(B_k')^\ell(D))$ *such that*

$$u + (\delta - \bar\partial)v = h \in \Lambda^r X.$$

Proof. In fact, if $u = \sum u_{r+k,k}$, $u_{r+k,k} \in B_k^{r+k}$, then $\bar\partial u_{r+n,n} = 0$ so we can solve $\bar\partial v_{r+n,n-1} = u_{r+n,n}$ in $(B_{n-1}')^{r+n}$. Then $\delta v_{r+n,n-1} \in (B_{n-1}')^{r+n-1}$ and $\bar\partial(u_{r+n-1,n-1} + \delta v_{r+n,n-1}) = 0$, so we can inductively find $v_{r+k+1,k} \in (B_k')^{r+k+1}$ such that $\bar\partial v_{r+k,k-1} = u_{r+k,k} + \delta v_{r+k+1,k}$. Finally, $h = u_{r,0} + \delta v_{r+1,0} \in \Lambda^r X$. □

We can now conclude the proof of Theorem 2.1.

Proof of Theorem 2.1. We start with the injectivity. In view of (3.4) it is enough to show that the leftmost mapping actually is an isomorphism. Take $[f] \in H_r(T_g, X)$ and assume that it is 0 in $H^{-r}(\mathrm{Tot}\, B_k^\ell(D))$. Thus there is a $u \in \mathcal{L}^{-r-1}(\mathrm{Tot}\, B_k^\ell(D))$ such that $(\delta - \bar\partial)u = f$. But then, by Lemma 3.2, there is $v \in \mathcal{L}^{-r-2}(\mathrm{Tot}\,(B_k')^\ell)$ such that

$$u + (\delta - \bar\partial)v = h \in \Lambda^{r+1}X.$$

Thus

$$\delta h = (\delta - \bar\partial)h = (\delta - \bar\partial)u = f$$

so that $[f] = 0$ in $H_r(T_g, X)$. Thus (2.2) is injective.

For the surjectivity, take $[u] \in H^{-r}(\mathrm{Tot}\, B_k^\ell)$; then (since $(\delta-\bar\partial)u = 0$) Lemma 3.2 gives a $v \in \mathcal{L}^{-r-1}(\mathrm{Tot}\,(B_k')^\ell)$ such that

$$u + (\delta - \bar\partial)v = h \in \Lambda^r X;$$

then

$$\delta h = (\delta - \bar\partial)h = (\delta - \bar\partial)u = 0,$$

so h really defines a class $[h]$ in $H_r(T_g, X)$. Now,

$$u - h = -(\delta - \bar\partial)v$$

in V, i.e., $[u]$ and $[h]$ coincide as elements in $H^{-r}(\mathrm{Tot}\, \mathcal{E}_{0,k}^\ell(V))$, and by (3.4) it follows that they actually coincide in $H^{-r}(\mathrm{Tot}\, B_k^\ell(D))$, and thus $[u]$ is in the image of (2.2). This concludes the proof. □

For the reader who wants a more concrete argument for the isomorphism (2.2), let us dissect the argument; we restrict to the injectivity part. So let us again start with $[f] \in H_r(T_g, X)$ and assume that its image in $H_r(T_g, \mathcal{O}(V))$ vanishes. Thus there is a holomorphic solution in V to $\delta v = f$. With the notation from the proof of Lemma 3.1 we get the form

$$w = \chi v + (1-\chi)\alpha \wedge f - \bar\partial\chi \wedge \alpha \wedge v \in \mathcal{L}^{-r-1}(\mathrm{Tot}\, B_k^\ell(D)),$$

such that $(\delta - \bar{\partial})w = f$. It remains to analyze what happens in the proof of Lemma 3.2. If K is a linear homotopy operator for $\bar{\partial}$ in D that admits the stated estimates, then the function $h \in \Lambda^{r+1}X$ such that $\delta h = f$ is obtained as

$$h = \sum_{j=0}^{n} (\delta K)^j w_j,$$

where w_j is the component of w which is in B_j^{-r-1+j}. The surjectivity part can be explained in very a similar way.

One can also for simplicity make out the whole procedure in the smaller domains $D_\epsilon = \{\rho < -\epsilon\}$, so that g is holomorphic in a neighborhood, and everything is smooth (or at least continuous) up to the boundary. Noticing that we have uniform B_k estimates in each step, we end up with h_ϵ with uniform $H^p(D_\epsilon)$ estimates such that $\delta_g h_\epsilon = f$ in D_ϵ. The conclusion then follows by a normal family argument.

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