APPROACH REGIONS FOR L^p POTENTIALS WITH RESPECT TO THE SQUARE ROOT OF THE POISSON KERNEL - A NEW PROOF.

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Abstract. If one replaces the Poisson kernel of the unit disc by its square root, then normalised Poisson integrals of L^p boundary functions converge along approach regions wider than the ordinary nontangential cones, as proved by Rönning. In this paper we present a new proof, characterising these regions.

1. Introduction

Let $P(z, \beta)$ be the standard Poisson kernel in the unit disc U,

$$P(z,\beta) = \frac{1}{2\pi} \cdot \frac{1-|z|^2}{|z-e^{i\beta}|^2}$$

where $z \in U$ and $\beta \in \partial U = \mathbb{T} \cong (-\pi, \pi]$.

Let

$$Pf(z) = \int_{\mathbb{T}} P(z,\beta) f(\beta)\, d\beta,$$

the Poisson integral of $f \in L^1(\mathbb{T})$.

For any function $h : \mathbb{R}_+ \to \mathbb{R}_+$ let

$$\mathcal{A}_h(\theta) = \{z \in U : |\arg z - \theta| \le h(1-|z|)\}. \tag{1}$$

We refer to $\mathcal{A}_h(\theta)$ as the (natural) approach region determined by h at $\theta \in \mathbb{T}$. Note that, even though we use the word "region", we have not imposed any openness assumptions on $\mathcal{A}_h(\theta)$.

Let

$$P_0 f(z) = \int_{\mathbb{T}} \sqrt{P(z,\beta)} f(\beta)\, d\beta,$$

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and denote the normalised operator by $\mathcal{P}_0$, i.e.

$$\mathcal{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

We now state some known convergence results for the operators P and $\mathcal{P}_0$. For more results about $\mathcal{P}_0$, see e.g. [MB].

Theorem (Schwarz, [HAS]). *Let $f \in C(\mathbb{T})$. Then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$, $z \in U$.*

Theorem (Fatou, [F]). *Let $f \in L^1(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, if $h(t) \sim t$.*

The theorem of Fatou was proved to be best possible, in the following sense:

Theorem (Littlewood, [L]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed Jordan curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathbb{T}$, the limit of f along γ_θ does not exist.*

Littlewood's result has been generalised in several directions. For instance, with the same assumptions as in Littlewood's theorem, Aikawa, [HA], disproves convergence at *all* points $\theta \in \mathbb{T}$.

Theorem (Sjögren, [PS1]). *Let $f \in L^1(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, if $h(t) \sim t \log 1/t$.*

Theorem (Rönning, [JOR]). *Let $1 \leq p < \infty$ be given and let $f \in L^p(\mathbb{T})$. Then, for a.e. $\theta \in \mathbb{T}$, one has that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$, if (and only if if h is assumed to be monotone) $h(t) \sim t(\log 1/t)^p$.*

The results by Sjögren and Rönning were proved via weak type estimates for the corresponding maximal operators, and approximation with continuous functions.

Theorem (Sjögren, [PS2]). *The following conditions are equivalent for any increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^\infty(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) *$h(t) = O(t^{1-\varepsilon})$ as $t \to 0$ for any $\varepsilon > 0$.*

In his proof, Sjögren never uses the assumption that h should be increasing. Thus, it remains valid for an even larger class of functions h. The proof of this result differs much from the L^p-case, since the continuous functions are not dense in L^∞. Sjögren instead used a result by Bellow and Jones, [B-J], "A Banach principle for L^∞". Following the same lines, the author proved the following ($L^{p,\infty}$ denotes weak L^p):

Theorem (Brundin, [MB]). *Let $1 < p < \infty$ be given. Then the following conditions are equivalent for any function $h : \mathbb{R}_+ \to \mathbb{R}_+$:*

(*i*) *For any $f \in L^{p,\infty}(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that*

$$\mathcal{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathcal{A}_h(\theta).$$

(*ii*) $\sum_{k=0}^{\infty} \sigma_k < \infty$, *where* $\sigma_k = \sup_{2^{-2^k} \leq s \leq 2^{-2^{k-1}}} \frac{h(s)}{s(\log 1/s)^p}$.

In this paper we prove the following theorem, with a method different from Rönning's:

Theorem 1. *Let $1 \leq p < \infty$ be given and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be any function. Then the following conditions are equivalent:*

(*i*) *For any $f \in L^p(\mathbb{T})$ one has for almost all $\theta \in \mathbb{T}$ that $\mathcal{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathcal{A}_h(\theta)$.*

(*ii*) $\displaystyle \limsup_{t \to 0} \frac{h(t)}{t(\log 1/t)^p} < \infty.$

2. The proof of Theorem 1

We shall prove the implication (*ii*) $\Rightarrow$ (*i*) in Theorem 1 via Proposition 1 below. The implication (*i*) $\Rightarrow$ (*ii*) is proved by via contraposition. We begin by introducing a convenient notation.

Let $t = 1 - |z|$ and $z = (1-t)e^{i\theta}$. Then

$$\mathcal{P}_0 f(z) = R_t * f(\theta),$$

where the convolution is taken in $\mathbb{T}$ and

$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)e^{i\theta} - 1|} \frac{1}{P_0 1(1-t)}.$$

If f and g are positive functions we say that $f \lesssim g$ provided that there exists some positive constant C such that $f(x) \leq C g(x)$. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Since we are interested only in small values of t, we might as well from now on assume that $t < 1/2$. Since $P_0 1(1-t) \sim \sqrt{t} \log 1/t$, the order of magnitude of R_t is given by

$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$

Now let τ_η denote the translation $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (*i*) in Theorem 1 above means

$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

The relevant maximal operator for our problem is

$$M_0 f(\theta) = \sup_{\substack{|\arg z - \theta| < h(1-|z|) \\ |z| > 1/2}} |\mathcal{P}_0 f(z)|.$$

Notice that $M_0 f(\theta)$ is dominated by a constant times

$$(2) \qquad Mf(\theta) = \sup_{\substack{|\eta| < h(t) \\ t < 1/2}} \tau_\eta Q_t * |f|(\theta).$$

Proposition 1. *Assume that condition (ii) in Theorem 1 holds. Then $Mf \leq C(M_{HL} f^p)^{1/p}$, where M_{HL} denotes the classical Hardy-Littlewood maximal operator.*

Note that this result in particular shows that M is of weak type (p, p). Convergence follows immediately with a standard approximation argument with continuous functions.

Proof. (*Proposition 1*) We may assume that $f \geq 0$, without loss of generality. We also assume that $1 \leq p < \infty$ is fixed and that $q = p/(p-1)$ (where $q = \infty$ if $p = 1$).

Note, first of all, that

$$(3) \qquad \|Q_t\|_q \leq C_q \frac{1}{t^{1/p} \log 1/t}$$

We also have that

$$(4) \qquad \|Q_t\|_1 \leq C,$$

for all $t \in (0, 1/2)$. By (4) it follows, as is well known, that

$$(5) \qquad Q_t * f(\theta) \leq C M_{HL} f(\theta),$$

independently of t.

Now, let $t \in (0, 1/2)$ be fixed and let $f(\varphi) = f_1(\varphi) + f_2(\varphi)$, where

$$f_1(\varphi) = f(\varphi) \chi_{\{|\varphi - \theta| \leq 2h(t)\}}.$$

Then, by (3) and by assumption, we have

$$\begin{aligned}
\tau_\eta Q_t * f_1(\theta) &\le \|Q_t\|_q \|f_1\|_p \\
&\le \frac{C}{t^{1/p}\log 1/t}\cdot\left(\int_{|\varphi-\theta|\le 2h(t)} f(\varphi)^p\,d\varphi\right)^{1/p} \\
&= C\left(\frac{4h(t)}{t(\log 1/t)^p}\cdot\frac{1}{4h(t)}\int_{|\varphi-\theta|\le 2h(t)} f(\varphi)^p\,d\varphi\right)^{1/p} \\
&\le C(M_{HL}f^p(\theta))^{1/p}.
\end{aligned}$$

Furthermore, by (5) and since $|\eta| < h(t)$, we have

$$\begin{aligned}
\tau_\eta Q_t * f_2(\theta) &\le C\cdot M_{HL}f_2(\theta-\eta) \\
&= C\sup_{r>0}\frac{1}{2r}\int_{|\varphi-(\theta-\eta)|<r} f_2(\varphi)\,d\varphi \\
&\le C\sup_{r>h(t)}\frac{1}{2r}\int_{|\varphi-\theta|<2r} f(\varphi)\,d\varphi \\
&\le C\cdot M_{HL}f(\theta) \\
&\le C(M_{HL}f^p(\theta))^{1/p}.
\end{aligned}$$

Hence, for all $t\in(0,1/2)$ and $|\eta|<h(t)$ we have

$$\tau_\eta Q_t * f(\theta) \le C(M_{HL}f^p(\theta))^{1/p},$$

and the proposition is established.

□

Proof. (*Proof of the implication* $(i) \Rightarrow (ii)$) Assume that condition (ii) in Theorem 1 is false. We show that this implies that (i) is false also. We assume that $p>1$, since the result for $p=1$ is already established by Sjögren.

Assume that

$$\limsup_{t\to 0}\frac{h(t)}{t(\log 1/t)^p} = \infty, \tag{6}$$

Pick any decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$1 \le \frac{h(t_i)}{t_i(\log 1/t_i)^p} \uparrow \infty, \tag{7}$$

as $i\to\infty$. Let

$$f_i(\varphi) = t_i^{1/(p-1)}\log 1/t_i\cdot\left(\frac{1}{t_i+|\varphi|}\right)^{1/(p-1)}\cdot\chi_{\{|\varphi|<h(t_i)\}},$$

Now,

$$\begin{aligned}\|f_i\|_p^p &= C_p t_i^{p/(p-1)} (\log 1/t_i)^p \int_0^{h(t_i)} \left(\frac{1}{t_i + |\varphi|}\right)^{p/(p-1)} d\varphi \\ &\leq C_p' t_i^{p/(p-1)} (\log 1/t_i)^p t_i^{1-p/(p-1)} \\ &= C_p' t_i (\log 1/t_i)^p\end{aligned}$$

It follows that

$$\frac{h(t_i)}{\|f_i\|_p^p} \geq C(p) \cdot \frac{h(t_i)}{t_i (\log 1/t_i)^p}.$$

By (7) the right hand side tends to ∞ as $i \to \infty$. Thus, by standard techniques, we can pick a subsequence of $\{t_i\}$, with possible repetitions, for simplicity denoted $\{t_i\}$ also, such that

$$\sum_1^\infty h(t_i) = \infty, \tag{8}$$

and

$$\sum_1^\infty \|f_i\|_p^p < \infty. \tag{9}$$

Let $A_1 = h(t_1)$, and for $n \geq 2$ let $A_n = h(t_n) + \sum_{j=1}^{n-1} 2h(t_j)$. By (8) one has that $\lim_{n\to\infty} A_n = \infty$.

Define (on $\mathbb{T}$) $F_j(\varphi) = \tau_{A_j} f_j(\varphi)$, and let

$$F^{(N)}(\varphi) = \sup_{j \geq N} F_j(\varphi).$$

It is clear by construction that any given $\varphi \in \mathbb{T}$ lies in the support of infinitely many F_j:s.

Since $[F^{(N)}(\varphi)]^p = \sup_{j\geq N} [F_j(\varphi)]^p \leq \sum_{j \geq N} [F_j(\varphi)]^p$, it follows that

$$\begin{aligned}\|F^{(N)}\|_p^p &\leq \sum_{j=N}^\infty \|F_i\|_p^p \\ &= \sum_{j=N}^\infty \|f_i\|_p^p \to 0\end{aligned}$$

as $N \to \infty$, by (9). Thus, in particular, $F^{(N)} \in L^p$ for any $N \geq 1$.

For $\theta \in \mathbb{T}$ and a given $\xi_0 > 0$ we can, by construction, find $j \in \mathbb{N}$ so that $\theta \in \operatorname{supp}(F_j)$ and so that $t_j \in (0, \xi_0)$. We can then choose η, with $|\eta| < h(t_j)$, so that $\theta - \eta \equiv A_j \mod 2\pi$. It follows that

$$\limsup_{t\to 0,\ |\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq \limsup_{j\to\infty} \mathcal{P}_0 F_j((1-t_j)e^{iA_j}).$$

We have

$$\begin{aligned}
\mathcal{P}_0 F_j((1-t_j)e^{iA_j}) &\geq \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{F_j(A_j-\varphi)}{t_j+|\varphi|}\, d\varphi \\
&= \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{f_j(\varphi)}{t_j+|\varphi|}\, d\varphi \\
&= 2Ct_j^{1/(p-1)} \int_0^{h(t_j)} \left(\frac{1}{t_j+\varphi}\right)^{1+1/(p-1)} d\varphi \\
&\geq C_p'' \\
&> 0.
\end{aligned}$$

To sum up, we have shown that for any $\theta \in \mathbb{T}$ one has

$$\limsup_{t\to 0,\ |\eta|<h(t)} \mathcal{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq C_p'' > 0.$$

Take N so large so that the measure of $\{F^{(N)} > C_p''/2\}$ is small, and a.e. convergence to $F^{(N)}$ is disproved.

□

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