On the maximal rate of decay of solutions to nonlinear Klein-Gordon equations

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The Klein-Gordon equation is the equation for relativistic wave-propagation

\[(\text{KG}) \quad \partial_t^2 u_o - \Delta u_o + m^2 u_o = 0 \quad x \in \mathbb{R}^n, t \geq 0
\]
\[u_o|_{t=0} = \varphi, \partial_t u_o|_{t=0} = \psi \quad x \in \mathbb{R}^n\]

where \( m > 0, \Delta = \sum_{j=1}^{n} \partial_{x_j}^2 \), \((n \geq 3)\). The nonlinear counterpart, extensively studied since early 1960’s, is

\[(\text{NLKG}) \quad \partial_t^2 u - \Delta u + m^2 u + f(u) = 0 \quad x \in \mathbb{R}^n, t \geq 0
\]
\[u|_{t=0} = \varphi, \partial_t u|_{t=0} = \psi \quad x \in \mathbb{R}^n\]

where \( f(u) \) is a nonlinear function, \( f(u) \equiv |u|^{p-1}u \); modified at 0 if necessary, to be smooth enough, and with

\[1 + \frac{4}{n} < \rho < \frac{n+2}{n-2} = \rho^*\]

The conditions on \( f \) will be made precise below.

**Energy:** Let \( F(u) = \int_0^u f(v)dv \geq 0 \). The energy

\[E(t) = \frac{1}{2} \int (|\partial_t u|^2 + |\partial_x u|^2 + m^2 |u|^2)dx + \int F(u)dx\]

is a conserved quantity, \( E(t) = E(0) \). Let \( X_c = H^1 \times L^2 \) with norm \( ||.||_c \) defined by

\[||u(t)||_c^2 = ||u(t)||_{H^1}^2 + ||\partial_t u(t)||_{L^2}^2\]
Our assumptions on $\rho$ and $f$ imply that

$$E(t) \leq C\|u(t)\|_E^2$$

**Global** estimates in space-time (Strichartz estimates) for the KG (Strichartz [15], Segal [12]): If the data $\varphi, \psi$ belong to $X_e$, then

$$\|u_0\|_{L^p_t L^\frac{2}{p}(\mathbb{R}^n)} \leq C(\|\varphi\|_{H^1} + \|\psi\|_{L^2}) \leq C\|u_0(0)\|_{e}$$

where $p \geq 2$, $\delta_p = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$. More general, but also much more complex, estimates that bound $u_0$ in $L_q(\mathbb{R}, H^\sigma_p(\mathbb{R}^n))$ are available (Strichartz [15], Marshall-Strauss-Wainger [9]; cf also Brenner [2]. A good exposition is given by Ginibre and Velo 1995 [6]).

**Space-time integrals of solutions of NLKG.** Let $u_o$ be a solution of KG with the same data at $t = 0$ as $u$, the solution of NLKG. Assume that the data has finite energy (i.e. $u(0)$ belongs to $X_e$). Then one example of a Strichartz-type estimate for the NLKG due to the author is that if $\sigma < \rho + \delta, \delta = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$, then

$$\text{if } u_0 \in L_q(\mathbb{R}, L^\sigma_p(\mathbb{R}^n)) \text{ then } u \in L_q(\mathbb{R}, L^\sigma_p(\mathbb{R}^n))$$

For more results of this type for the NLKG see Brenner [4] and Ginibre and Velo [5].

Such a time-space estimate implies a decay estimate in the following sense: Under the assumptions of finite energy data, let $X = L^\sigma_p(\mathbb{R}^n)$. We then get

$$\mathcal{M}u(t) = \mathcal{M}^X_{\rho, \delta} u(t) = \left( \frac{1}{T} \int_t^{t+T} \|u(\tau)\|_X^\delta \, d\tau \right)^\frac{1}{\delta} \to 0,$$

for $t \to \infty$, and for any $T > 0$.

**How fast can $\mathcal{M}u(t)$ tend to $0$?** For nontrivial solutions $u_o \in X_e$ of the Klein-Gordon equation (KG) it is known (Glassy [7]) that in case $X = L^\rho_p(\mathbb{R}^n)$ and with $\delta = \frac{1}{2} - \frac{1}{p} \geq 0$ and $t \geq T > 0$,

$$\mathcal{M}u_o(t) \geq c t^{-n\delta}$$

The next result will answer the question about a bound for the rate of maximal decay for solutions of the NLKG.

We first need to be more precise about the nonlinearity $f$. The following are the conditions we impose on $f$

Let $f(u) \in C^4$ with $f(\mathbb{R}) \subseteq \mathbb{R}$ and assume that
(i)  
\[ F(u) = \int_0^u f(v)dv \geq 0. \]

(ii)  
\[
\begin{align*}
|f'(u)| &< |u|^{p_0 - 1}, \quad |u| \leq 1 \\
|f'(u)| &< |u|^{p_1 - 1}, \quad |u| \geq 1
\end{align*}
\]

where
\[
1 + \frac{4}{\pi} < \rho_0, \quad \rho_1 < \frac{n+2}{n-2} = \rho^*
\]

(iii)  
\[
u f(u) - 2F(u) \geq \alpha F(u), \text{ some } \alpha > 0
\]

and \( F \) is not flat at 0 or \( \infty \).

The last condition ensures that we avoid local concentration of energy.

Here is now our main result:

**Main Theorem.** Let \( u \in X^c \) be a solution of the NLKG, \( n \geq 3 \) and with \( f \) satisfying (i) through (iii). Assume that \( u \in L^\infty_q(\mathbb{R}, L^\infty_p(\mathbb{R}^n)) \), with \( q, p \geq 2 \), and \( \delta = \frac{1}{p} - \frac{1}{p} \). Let

\[ X_t = L_p(|x| \leq t) \supset X = L_p(\mathbb{R}^n) \]

Then there is a constant \( c > 0 \) such that

\[ M^T_{q, X} u(t) \geq \left( \frac{1}{T} \int_t^{t+T} \|u(\tau)\|^2_{X^c} d\tau \right)^{\frac{1}{2}} \geq ct^{-n\delta}, \]

for \( t \geq 1 \) and \( t \geq T > 0 \).

**Comment.** We may replace \( X_t \) in our Main Theorem by \( Y_t = L_p(|x| \leq t) \leq (1 - \epsilon(t)) \) where \( \epsilon(t) \) denotes any positive function that tends to 0 as \( t \to \infty \).

The question remains about the rate of decay in \( L_p(|x| \leq \epsilon(t) t) \), since by the Energy decay theorem the corresponding \( L_2 \)-norm tends to 0 as \( t \to \infty \). In fact, a result due to Morawetz [10] shows that for any fixed compact subset \( \Omega \) of \( \mathbb{R}^n \) we have \( \|u(t)\|_{L_2(\Omega)} \in L_2 \), where as above \( u \) is a solution of the NLKG.

**Comment.** Corresponding pointwise (i.e. \( q = \infty \)) were previously given in the case of smooth (and small) data ([17], [1] - also large data, and [8]).
The following gives an example of a case when the maximal rate of $\mathcal{M}u(t)$ is attained. Let $X = L^p(\mathbb{R}^n)$ where now

$$
\delta = \frac{1 - \frac{1}{p}}{\theta} , \theta \in [0, 1] , 0 < \delta < \frac{1}{n-1}
$$

$$
(n-1-\theta)\delta < 1 < (n-1+\theta)\delta
$$

(1)

$L_p$-Decay (Brenner, to appear). Assume that $u \in X_\epsilon$ is a solution of the NLKG, and let $u_\circ$ be corresponding solution of the Klein-Gordon equation. Let (1) be satisfied, and assume that $\mathcal{M}_\delta^{t_0} u_\circ(t)$ has maximal rate of decay. Then $\mathcal{M}_\delta^{t_0} u(t)$ also attains the maximal rate of decay, that is decays as $O(t^{-n\delta})$.

Similar results hold in the other cases when Strichartz’ estimates are known to hold for the NLKG. For smooth data decay results decay results are also given by e.g. [1], [4], [11], and for small data by [17].

The proof of the Theorem is based on the following three results:

Scattering (Brenner 1983-86, [2],[3],[4]). There exists an everywhere defined scattering operator on $X_\epsilon$ for the NLKG. In particular, for any finite energy solution $u \in X_\epsilon$ there is a solution $u_+ \in X_\epsilon$ of the Klein-Gordon equation with the same energy as $u$, such that

$$
\|u(t) - u_+(t)\|_c \to 0 , \text{ as } t \to \infty
$$

Let $\Omega = \{t \leq |x| < (1-\epsilon(t))t\}$, where $0 < \epsilon(t) < 1 , \epsilon(t) \to 0 , \text{ as } t \to \infty$. Let $Y = H^2_v(\mathbb{R}^n \setminus \Omega)$. Then

Energy decay (Strichartz 1981, [16]). Let $u_\circ$ be a finite energy solution of the Klein-Gordon equation. Then

$$
\|u_\circ(t)\|_{Y} \to 0 , \text{ as } t \to \infty
$$

Proposition. Let $u_\circ$ be a non-trivial finite energy solution of the Klein-Gordon equation. Then there is a constant $c_\circ = c_\circ(\text{data}) > 0$ such that

$$
\|u_\circ(t)\|_{L^2(\mathbb{R}^n)} \to c_\circ > 0
$$

Using these results we obtain the following

Lemma. Let $u$ be a finite energy solution of the NLKG. Then there are constants $c_\circ = c_\circ(\text{data}) > 0$ as above, and $t_* \geq 1$ such that

$$
\|u(t)\|_{H^2(|x| > t)} \to 0 , \text{ as } t \to \infty
$$

and

$$
\|u(t)\|_{L^2(|x| \leq t)}^2 \geq \frac{c_\circ}{t} , \text{ for } t \geq t_*
$$
Proof. The first statement follows from the Scattering and Energy decay theorems. The second follows from that and the Proposition.

The steps of the proof of the Main Theorem are now obvious (following Glassey’s proof for the Klein-Gordon equation [7]):

\[
\frac{1}{t}\|u(t)\|_{L^2(\mathbb{R}^n)} \leq \|u(t)\|_{L^p(\mathbb{R}^n)} \leq \|u(t)\|_{L^p(\mathbb{R}^n)} t^\frac{n}{p}
\]

and the results follow by taking the meanvalue over \((t,t+T)\) for \(t \geq T\).

It remains to prove the Proposition. Let \(\mathcal{F}\) denote the Fourier transform, let \(B(\xi) = (|\xi|^2 + m^2)^{\frac{1}{2}}\) and \(Bu(x) = \mathcal{F}^{-1}_{\xi \rightarrow x}(B(\xi) \mathcal{F}u(\xi))\). Define

\[
\Phi = \frac{1}{t}(\phi + iB^{-1}\psi) \quad \text{and} \quad \Psi = \frac{1}{t}(\phi - iB^{-1}\psi)
\]

where

\[
\phi = u(0) \quad , \quad \psi = \partial_t u(0)
\]

Then the solution \(u\) of the Klein-Gordon equation can be written in the form

\[
u(t) = \exp(itB)\Phi + \exp(-itB)\Psi
\]

and, using duality, we have

\[
\int |u(t)|^2 dx = \int |\Phi|^2 dx + \int |\Psi|^2 dx
\]

\[
+ 2Re \int \exp(2itB)\Phi\overline{\Psi} dx
\]

Now, by Parseval’s formula, using the notation \(\check{v} = \mathcal{F} v\),

\[
\int \exp(2itB)\Phi\overline{\Psi} dx = \int \exp(2itB(\xi))\hat{\Phi}(\xi)\overline{\hat{\Psi}}(\xi) d\xi
\]

Since \(\text{grad}_x B(\xi) \neq 0\) for \(\xi \neq 0\), and \(\Phi\), \(\Psi\), belong to \(L_2\), as well as their Fourier transforms (so that the products belong to \(L_1\), respectively), we can apply the (generalized) Riemann-Lebesgue lemma to see that the right hand side tends to 0 as \(t \rightarrow \infty\).

Since

\[
\int |\Phi|^2 dx + \int |\Psi|^2 dx = \int |\phi|^2 dx + \int |B^{-1}\psi|^2 dx
\]

the Proposition is proved. \(\square\)
References


nichtlineare Wellengleichungen bei kleinen Anfangswerten und das asympto-