On the maximal rate of decay of solutions to nonlinear Klein-Gordon equations

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August 31, 2001

The Klein-Gordon equation is the equation for relativistic wave-propagation

(KG)
$$\partial_t^2 u_o - \Delta u_o + m^2 u_o = 0 \quad x \in \mathbf{R}^n, t \geq 0$$
$$u_o|_0 = \varphi,\ \partial_t u_o|_0 = \psi \quad x \in \mathbf{R}^n$$

where $m > 0, \Delta = \sum_{j=1}^n \partial_{x_j}^2$, $(n \geq 3)$. The nonlinear counterpart, extensively studied since early 1960's, is

(NLKG)
$$\partial_t^2 u - \Delta u + m^2 u + f(u) = 0 \quad x \in \mathbf{R}^n, t \geq 0$$
$$u|_0 = \varphi,\ \partial_t u|_0 = \psi \quad x \in \mathbf{R}^n$$

where $f(u)$ is a nonlinear function, $f(u) \cong |u|^{\rho-1}u$; modified at 0 if necessary, to be smooth enough, and with

$$1 + \tfrac{4}{n} < \rho < \tfrac{n+2}{n-2} = \rho^*$$

The conditions on f will be made precise below.

Energy:Let $F(u) = \int_0^u f(v)dv \geq 0$. The energy

$$E(t) = \tfrac{1}{2} \int (|\partial_x u|^2 + |\partial_t u|^2 + m^2|u|^2)dx + \int F(u)dx$$

is a conserved quantity, $E(t) = E(0)$.Let $X_e = H_2^1 \times L_2$ with norm $\|.\|_e$ defined by

$$\|u(t)\|_e^2 = \|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L_2}^2$$

Our assumptions on ρ and f imply that

$$E(t) \leq C\|u(t)\|_e^2$$

Global estimates in space-time (Strichartz estimates) for the KG (Strichartz [15], Segal [12]): If the data φ, ψ belong to X_e, then

$$\|u_o\|_{L_p(L_p^{\frac{1}{2}})} \leq C(\|\varphi\|_{H^1} + \|\psi\|_{L_2}) \leq C\|u_o(0)\|_e$$

where $p \geq 2$, $\delta_p = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$. More general, but also much more complex, estimates that bound u_0 in $L_q(\mathbf{R}, H_p^s(\mathbf{R}^n))$ are available (Strichartz [15], Marshall-Strauss-Wainger [9];cf also Brenner [2].A good exposition is given by Ginibre and Velo 1995 [6])

Space-time integrals of solutions of NLKG. Let u_o be a solution of KG with the same data at $t = 0$ as u, the solution of NLKG. Assume that the data has finite energy (i.e. $u(0)$ belongs to X_e). Then one example of a Strichartz-type estimate for the NLKG due to the author is that if $\sigma < \rho + \delta, \delta = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$, then

$$if\, u_o \in L_q(\mathbf{R}, L_p^\sigma(\mathbf{R}^n))\, then\, u \in L_q(\mathbf{R}, L_p^\sigma(\mathbf{R}^n))$$

For more results of this type for the NLKG see Brenner [4] and Ginibre and Velo [5].

Such a time-space estimate implies a decay estimate in the following sense: Under the assumptions of finite energy data, let $X = L_p^\sigma(\mathbf{R}^n)$. We then get

$$\mathcal{M}u(t) = \mathcal{M}_{q,X}^T u(t) = (\frac{1}{T}\int_t^{t+T} \|u(\tau)\|_X^q d\tau)^{\frac{1}{q}} \to 0 \ ,$$

for $t \to \infty$, and for any $T > 0$.

How fast can $\mathcal{M}u(t)$ tend to 0 ? For nontrivial solutions $u_o \in X_e$ of the Klein-Gordon equation (KG) it is known (Glassey [7]) that in case $X = L_p(\mathbf{R}^n)$ and with $\delta = \frac{1}{2} - \frac{1}{p} \geq 0$ *and* $t \geq T > 0$,

$$\mathcal{M}u_o(t) \geq ct^{-n\delta}$$

The next result will answer the question about a bound for the rate of maximal decay for solutions of the NLKG.

We first need to be more precise about the nonlinearity f: The following are the conditions we impose on f

Let $f(u) \in C^1$ with $f(\mathbf{R}) \subseteq \mathbf{R}$ and assume that

(i)

$$F(u) = \int_0^u f(v)dv \geq 0.$$

(ii)

$$|f'(u)| < |u|^{\rho_0 - 1}, \qquad |u| \leq 1$$
$$|f'(u)| < |u|^{\rho_1 - 1}, \qquad |u| \geq 1$$

where

$$1 + \tfrac{4}{n} < \rho_0,\ \rho_1 < \tfrac{n+2}{n-2} = \rho^*$$

(iii)

$$uf(u) - 2F(u) \geq \alpha F(u), \text{ some } \alpha > 0$$
$$\text{and } F \text{ is not flat at 0 or } \infty$$

The last condition ensures that we avoid local concentration of energy

Here is now our main result:

Main Theorem. *Let $u \in X_e$ be a solution of the NLKG, $n \geq 3$ and with f satisfying (i) through (iii). Assume that $u \in L_q^{loc}(\mathbf{R}, L_p^{loc}(\mathbf{R}^n))$, with $q, p \geq 2$,and $\delta = \frac{1}{2} - \frac{1}{p}$. Let*

$$X_t = L_p(\{|x| \leq t\}) \supseteq X = L_p(\mathbf{R}^n)$$

Then there is a constant $c > 0$ such that

$$\mathcal{M}_{q,X}^T u(t) \geq (\frac{1}{T}\int_t^{t+T} \|u(\tau)\|_{X_\tau}^q d\tau)^{\frac{1}{q}} \geq ct^{-n\delta} \ ,$$

for $t \geq 1$,and $t \geq T > 0$.

Comment. We may replace X_t in our Main Theorem by $Y_t = L_p(\{\epsilon(t)t \leq |x| \leq (1-\epsilon(t))t\})$ where $\epsilon(t)$ denotes any positive function that tends to 0 as $t \to \infty$.

The question remains about the rate of decay in $L_p(\{|x| \leq \epsilon(t)t\})$, since by the Energy decay theorem the corresponding L_2-norm tends to 0 as $t \to \infty$. In fact, a result due to Morawetz [10] shows that for any fixed compact subset Ω of $\mathbf{R^n}$ we have $\|u(t)\|_{L_2(\Omega)} \in L_2$, where as above u is a solution of the NLKG.

Comment. Corresponding pointwise (i.e. $q = \infty$) were previously given in the case of smoooth (and small) data ([17], [1] - also large data, and [8]).

The following gives an example of a case when the maximal rate of $\mathcal{M}u(t)$ is attained. Let $X = L_p(\mathbf{R}^n)$ where now

$$\begin{aligned} &\delta = \tfrac{1}{2} - \tfrac{1}{p} \ , \theta \in [0,1] \ , \ 0 \le \delta < \tfrac{1}{n-1} \\ (1) \qquad &(n-1-\theta)\delta < 1 < (n-1+\theta)\delta \end{aligned}$$

L_p-Decay (Brenner, to appear). *Assume that $u \in X_e$ is a solution of the NLKG, and let u_o be corresponding solution of the Klein-Gordon equation. Let (1) be satisfied, and assume that $\mathcal{M}^t_{q,X} u_o(t)$ has maximal rate of decay. Then $\mathcal{M}^t_{q,X} u(t)$ also attains the maximal rate of decay, that is decays as $\mathcal{O}(t^{-n\delta})$.*

Similar results hold in the other cases when Strichartz' estimates are known to hold for the NLKG.For smooth data decay results decay results are also given by e.g. [1], [4], [11], and for small data by [17].

The proof of the Theorem is based on the following three results:

Scattering (Brenner 1983-86,[2],[3],[4]). *There exisits an everywhere defined scattering operator on X_e for the NLKG. In particular, for any finite energy solution $u \in X_e$ there is a solution u_+ of the Klein-Gordon equation with the same energy as u, such that*

$$\|u(t) - u_+(t)\|_e \to 0 \ , as \ t \to \infty$$

Let $\Omega_t = \{\epsilon(t)t \le |x| \le (1-\epsilon(t))t\}$, where $0 < \epsilon(t) < 1 \ , \epsilon(t) \to 0 \ , as \ t \to \infty$.Let $Y_t = H^1_2(\mathbf{R}^n \setminus \Omega_t)$. Then

Energy decay (Strichartz 1981, [16]). *Let u_o be a finite energy solution of the Klein-Gordon equation.Then*

$$\|u_o(t)\|_{Y_t} \to 0 \ , as \ t \to \infty$$

Proposition. *Let u_o be a non-trivial finite energy solution of the Klein-Gordon equation. Then there is a constant $c_o = c_o(data) > 0$ such that*

$$\|u_o(t)\|_{L_2(\mathbf{R}^n)} \to c_o > 0$$

Using these results we otain the following

Lemma. *Let u be a finite energy solution of the NLKG. Then there are constants $c_o = c_o(data) > 0$ as above, and $t_* \ge 1$ such that*

$$\|u(t)\|_{H^1_2(|x|>t)} \to 0 \ , as \ t \to \infty$$

and

$$\tfrac{1}{8} c_o \le \|u(t)\|^2_{L_2(|x|\le t)} \ , t \ge t_*$$

Proof. The first statement follows from the Scattering and Energy decay theorems. The second follows from that and the Proposition. □

The steps of the proof of the Main Theorem are now obvious(following Glassey's proof for the Klein-Gordon equation ,[7]):

$$\tfrac{1}{8}c_o \leq \|u(t)\|_{L_2(|x|\leq t)} \leq \|u(t)\|_{L_p(|x|\leq t)} t^{n\delta}$$

and the results follows by taking the meanvalue over (t,t+T) for $t \geq T$. □

It remains to prove the Proposition. Let $\mathcal{F}$ denote the Fourier transform, let $B(\xi) = (|\xi|^2 + m^2)^{\frac{1}{2}}$ and $Bu(x) = \mathcal{F}^{-1}_{\xi\to x}(B(\xi)\mathcal{F}u(\xi))$. Define

$$\Phi = \tfrac{1}{2}(\phi + iB^{-1}\psi) \text{ and } \Psi = \tfrac{1}{2}(\phi - iB^{-1}\psi)$$

where

$$\phi = u(0) \quad , \ \psi = \partial_t u(0)$$

Then the solution u of the Klein-Gordon equation can be written in the form

$$u(t) = exp(itB)\Phi + exp(-itB)\Psi$$

and, using duality, we have

$$\int |u(t)|^2 dx = \int |\Phi|^2 dx + \int |\Psi|^2 dx + 2Re \int exp(2itB)\Phi\bar{\Psi} dx$$

Now, by Parseval's formula, using the notation $\hat{v} = \mathcal{F}v$,

$$\int exp(2itB)\Phi\bar{\Psi} dx = \int exp(2itB(\xi))\hat{\Phi}(\xi)\bar{\hat{\Psi}}(\xi) d\xi$$

Since $grad_\xi B(\xi) \neq 0$ for $\xi \neq 0$, and Φ , Ψ belong to L_2, as well as their Fourier transforms (so that the products belong to L_1 , respectively), we can apply the (generalized) Riemann-Lebesgue lemma to see that the right hand side tends to 0 as $t \to \infty$.

Since

$$\int |\Phi|^2 dx + \int |\Psi|^2 dx = \int |\phi|^2 dx + \int |B^{-1}\psi|^2 dx$$

the Proposition is proved. □

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