CONTINUED FRACTIONS AND INCREASING SUBSEQUENCES IN PERMUTATIONS

PETTER BRÄNDÉN, ANDERS CLAESSON, AND EINAR STEINGRÍMSSON

ABSTRACT. We point out a generalization of a theorem of Robertson, Wilf and Zeilberger, giving a continued fraction expansion for the joint distribution of the numbers e_k of increasing subsequences of length $k+1$ in 132-avoiding permutations, for all k simultaneously. This shows that any Stieltjes continued fraction with monomial denominators is the generating function of a statistic consisting of a linear combination of the e_k's. Some applications are given, one of which relates fountains of coins to 132-avoiding permutations according to number of inversions. Another relates ballot numbers to such permutations according to number of right-to-left maxima.

1. INTRODUCTION AND MAIN RESULTS

The purpose of this paper is to point out a generalization of a theorem of Robertson, Wilf and Zeilberger [9], and some interesting consequences of this generalization. The theorem of Robertson, Wilf and Zeilberger gives a simple continued fraction that records the joint distribution of the patterns 12 and 123 (increasing subsequences of lengths two and three, respectively) on permutations avoiding the pattern 132.

Generalizations of this theorem have already been given, by Krattenthaler [4], by Mansour and Vainshtein [5] and by Jani and Rieper [3]. However, in none of these papers is there explicit mention of the *joint* distribution of the statistics under consideration, namely the number of increasing subsequences of length k in a permutation.

That generalization gives a continued fraction of Stieltjes type, with monomial generators, and we show that any such continued fraction is the generating function for some combination of the patterns $12\cdots k$ for various k. Moreover, there is a single invertible linear transformation that translates between combinations of pattern statistics and the corresponding continued fractions.

Let $\pi = a_1a_2\cdots a_n$ be a permutation on $\{1,2,\ldots,n\}$ and let $\tau = b_1b_2\cdots b_k$ be a permutation on $\{1,2,\ldots,k\}$. We say that π has j occurrences of the *pattern* τ if there are exactly j different sequences $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that the numbers $a_{i_1}a_{i_2}\cdots a_{i_k}$ are in the same relative order as $b_1b_2\cdots b_k$. We indicate this by $\tau(\pi) = j$. If $\tau(\pi) = 0$ we say that π is τ*-avoiding*. For convenience we let $\mathcal{S}_\infty = \bigcup_{n\ge 0}\mathcal{S}_n$, where $\mathcal{S}_n$ is the set of permutations of length n, and we let $\mathcal{S}_\infty(132)$ be the set of all 132-avoiding

Date: 29th January 2001.

permutations. For $k \geq 1$, we denote by e_{k-1} the pattern $1\,2 \cdots k$. Thus $e_0\pi$ is the length of π.

We now give the main theorem. Its proof is a straightforward generalization of the proof in [9], and implicit in the proof in [5], so we only sketch it here.

Theorem 1. *The following continued fraction expansion holds:*

$$\sum_{\pi \in \mathcal{S}_\infty(132)} \prod_{k \geq 0} x_k^{e_k \pi} = \cfrac{1}{1 - \cfrac{x_0^{\binom{0}{0}}}{1 - \cfrac{x_0^{\binom{1}{0}} x_1^{\binom{1}{1}}}{1 - \cfrac{x_0^{\binom{2}{0}} x_1^{\binom{2}{1}} x_2^{\binom{2}{2}}}{1 - \cfrac{x_0^{\binom{3}{0}} x_1^{\binom{3}{1}} x_2^{\binom{3}{2}} x_4^{\binom{3}{3}}}{\ddots}}}}}$$

in which the $(n+1)$*st denominator is* $\displaystyle\prod_{k=0}^{n} x_k^{\binom{n}{k}}$.

Proof. If $\pi \in \mathcal{S}_\infty(132)$ then each letter in π to the left of n must be greater than any letter to the right of n. Thus, if $\pi = \pi_1 n \pi_2$ (where both π_1 and π_2 must necessarily be 132-avoiding) then

$$e_k\pi = e_k\pi_1 + e_{k-1}\pi_1 + e_k\pi_2.$$

It follows that the generating function

$$C(x_0, x_1, \dots) := \sum_{\pi \in \mathcal{S}_\infty(132)} x_0^{e_0} x_1^{e_1} \cdots$$

satisfies

$$C(x_0, x_1, \dots) = 1 + x_0 C(x_0x_1, x_1x_2, \dots) C(x_0, x_1, \dots).$$

Equivalently,

$$C(x_0, x_1, \dots) = \frac{1}{1 - x_0 C(x_0x_1, x_1x_2, \dots)}$$

and the theorem follows by induction. □

By considering statistics defined in terms of linear combinations of the e_ks we obtain a more general version of Theorem 1. Let $\mathbf{q} = (q_0, q_1, \dots)$, where the q_i are indeterminates. Let $\mathbf{e} = (e_0, e_1, \dots)$, $\Lambda_k = (\lambda_{0k}, \lambda_{1k}, \dots) \in \mathbb{Z}^{\mathbb{N}}$, and let Λ be the infinite matrix $[\lambda_{ij}]$. Also, let $\langle \Lambda_k, \mathbf{e} \rangle = \sum_{j \geq 0} \lambda_{jk} e_j$. Then we have

$$\begin{aligned} \mu(\pi, \Lambda; \mathbf{q}) &:= \prod_{k \geq 0} q_k^{\langle \Lambda_k, \mathbf{e} \rangle \pi} \\ &= \prod_{k \geq 0} \prod_{j \geq 0} q_k^{\lambda_{jk} e_j \pi} \\ &= \prod_{j \geq 0} \left(\prod_{k \geq 0} q_k^{\lambda_{jk}} \right)^{e_j \pi}. \end{aligned}$$

By letting $x_j = \prod_{k\geq 0} q_k^{\lambda_{jk}}$ and applying Theorem 1 we get a continued fraction in which the $(n+1)$st denominator is

$$\prod_{j\geq 0} x_j^{\binom{n}{j}} = \prod_{j\geq 0}\left(\prod_{k\geq 0} q_k^{\lambda_{jk}}\right)^{\binom{n}{j}} = \prod_{k\geq 0} q_k^{\langle \Lambda_k, \mathbf{b}_n\rangle},$$

where $\mathbf{b}_n = \left(\binom{n}{0}, \binom{n}{1}, \dots\right)$. This proves the following corollary:

Corollary 2. *With definitions as above, we have*

$$\sum_{\pi\in\mathcal{S}_\infty(132)} \mu(\pi,\Lambda;\mathbf{q}) = \cfrac{1}{1-\cfrac{\prod q_k^{\lambda_{0k}}}{1-\cfrac{\prod q_k^{\lambda_{0k}+\lambda_{1k}}}{1-\cfrac{\prod q_k^{\lambda_{0k}+2\lambda_{1k}+\lambda_{2k}}}{1-\cfrac{\prod q_k^{\lambda_{0k}+3\lambda_{1k}+3\lambda_{2k}+\lambda_{3k}}}{\ddots}}}}}$$

in which the $(n+1)$*st denominator is* $\prod_{k\geq 0} q_k^{\langle \Lambda_k, \mathbf{b}_n\rangle}$, *where* $\mathbf{b}_n = \left(\binom{n}{0}, \binom{n}{1}, \dots\right)$.

A *Stieltjes continued fraction* is a continued fraction of the form

$$C = \cfrac{1}{1-\cfrac{c_1}{1-\cfrac{c_2}{1-\cfrac{c_3}{\ddots}}}}.$$

We say that C has *monomial denominators* if each c_i is a monomial in some set of variables.

Corollary 2 allows us to interpret every Stieltjes continued fraction with monomial denominators as the generating function for the distribution of a statistic on $\mathcal{S}_\infty(132)$ consisting of a (possibly infinite) linear combination of e_k's. Let $B = [\binom{i}{j}]$ be the infinite matrix with $\binom{i}{j}$ in the ith row and jth column, for $i, j \geq 0$. Its inverse is $B^{-1} = [(-1)^{i-j}\binom{i}{j}]$.

Theorem 3. *There is a one-to-one correspondence between Stieltjes continued fractions with monomial denominators and statistics consisting of linear combinations of* e_k*'s on* $\mathcal{S}_\infty(132)$. *Moreover, if* $F(q_0, q_1, \dots)$, *is a continued fraction with* $(n+1)$*st denominator* $\prod_{k\geq 0} q_k^{\gamma_{nk}}$ *then the corresponding* λ_{nk}*s in Corollary 2 are given by the infinite matrix relation*

$$[\lambda_{nk}] = B^{-1}[\gamma_{nk}].$$

Proof. The kth exponent of the $(n+1)$st denominator in the continued fraction of Corollary 2 is $\gamma_{nk} = \langle \mathbf{b}_n, \Lambda_k\rangle$. In other words, the relationship between the λ_{nk} and the γ_{nk} is given by

$$[\gamma_{nk}] = [\langle \mathbf{b}_n, \Lambda_k\rangle] = B[\lambda_{nk}]$$

and the Theorem follows. □

2. Dyck paths

Before giving applications of Theorem 3 we review some theory on Dyck paths and their relation to 132-avoiding permutations.

A *Dyck path* of length $2n$ is a path in the integral plane from $(0,0)$ to $(2n,0)$, consisting of steps of type $u=(1,1)$ and $d=(1,-1)$ and never going below the x-axis. We call the steps of type u *up-steps* and those of type d we call *down-steps*. The *height* of a step in a Dyck path is the height above the x-axis of its left point.

A nonempty Dyck path w can be written uniquely as uw_1dw_2 where w_1 and w_2 are Dyck paths. This decomposition is called the *first return decomposition* of w, because the d in uw_1dw_2 corresponds to the first place, after $(0,0)$, where the path touches the x-axis.

In [4] a bijection Φ between $\mathcal{S}_\infty(132)$ and the set of Dyck paths of length $2n$ is studied. This bijection can also be defined recursively as we now explain.

For a permutation π on $S\subset\mathbb{N}$ let $\overline{\pi}$ be the corresponding permutation in $\mathcal{S}_n$, where $n=|S|$, such that the order relations among the elements are preserved. For example if $\pi=3627$ then $\overline{\pi}=2314$. A permutation π of length $n>0$ is easily seen to be 132-avoiding if and only if it can be written as $\pi_1 n\pi_2$ where π_1 and π_2 are 132-avoiding and each letter in π_1 is larger than each letter in π_2. The bijection Φ is then defined recursively by:

$$\Phi(\varepsilon)=\varepsilon \quad\text{and}\quad \Phi(\pi)=u\Phi(\overline{\pi_1})d\Phi(\overline{\pi_2}),$$

where ε is the empty permutation/word. For example, letting Φ operate on the permutation 41253 we successively obtain

$$41253 \;\to\; u412d3 \;\to\; uud12dud \;\to\; uudu1ddud \;\to\; uuduudddud.$$

In what follows, when we talk about a correspondence between a Dyck path and a 132-avoiding permutation, we will always mean the correspondence afforded by Φ.

Using Φ we can interpret $e_k\pi$ in terms of the Dyck path corresponding to π. Namely (see [4]), we have

$$e_k\pi=\sum_{d \text{ in } \Phi(\pi)}\binom{h(d)-1}{k}, \tag{1}$$

where the sum is over all down-steps d in $\Phi(\pi)$ and $h(d)$ is the height of the left point of d. This can also be shown by induction over the length of π, since for a nonempty permutation $\pi=\pi_1n\pi_2$ we have

$$e_k\pi=e_k\overline{\pi_1}+e_{k-1}\overline{\pi_1}+e_k\overline{\pi_2}.$$

Thus, if $f_k w = \sum_{d \text{ in } w} \binom{h(d)-1}{k}$, we have

$$\begin{aligned} f_k w &= \sum_{d \text{ in } w} \binom{h(d)-1}{k} \\ &= \sum_{d \text{ in } w_1} \binom{h(d)}{k} + \sum_{d \text{ in } w_2} \binom{h(d)-1}{k} \\ &= \sum_{d \text{ in } w_1} \binom{h(d)-1}{k} + \sum_{d \text{ in } w_1} \binom{h(d)-1}{k-1} + f_k w_2 \\ &= f_k w_1 + f_{k-1} w_1 + f_k w_2, \end{aligned}$$

for $w = uw_1 dw_2$. Since $f_0 \Phi(\pi) = e_0 \pi$, it follows by induction over the length of π that $f_k \circ \Phi = e_k$.

3. Applications

We now give some applications of Theorem 3. Some of these relate known continued fractions to the statistics e_k but others relate these statistics to various other combinatorial structures.

3.1. **A continued fraction of Ramanujan.** The following continued fraction $R(q,t)$ was studied by Ramanujan (see [7, p. 126]). It was shown in [2] that the coefficient to $t^n q^k$ in the expansion of $R(q,t)$ is the number of Dyck paths of length $2n$ and area k:

$$R(q,t) = \cfrac{1}{1 - \cfrac{qt}{1 - \cfrac{q^3 t}{1 - \cfrac{q^5 t}{1 - \cfrac{q^7 t}{\ddots}}}}}$$

Using Theorem 3, we give an interpretation in terms of the patterns e_k. We have $\Gamma = (\gamma_0, \gamma_1, \dots) = (1, 3, 5, \dots)$. Thus, according to Theorem 3, we have $(\lambda_0, \lambda_1, \dots) = [(-1)^{n-k}\binom{n}{k}]\Gamma = (1, 2, 0, 0, \dots)$, since

$$\lambda_n = \sum_{k \geq 0} (2k+1)(-1)^{n-k} \binom{n}{k} = \delta_{n0} + 2\delta_{n1},$$

where δ_{ij} is the Kronecker delta. Thus

$$R(q,t) = \sum_{\pi \in \mathcal{S}_\infty(132)} q^{e_0 \pi + 2e_1 \pi} t^{|\pi|},$$

so $R(q,t)$ records the statistic $e_0 + 2e_1$ on 132-avoiding permutations. In fact, the bijection Φ translates the statistic $e_0 + 2e_1$ into the sum of the heights of the steps in the corresponding Dyck path, which in turn is easily seen to equal area.

3.2. **Fountains of coins.** A *fountain of coins* is a two dimensional array of circles where each circle (except for those in the bottom row) rests on two adjacent ones in the row below (see Figure 1). Let $F(x,t) = \sum_{n,k} f(n,k)x^k t^n$, where $f(n,k)$ counts the number of fountains with n coins in the bottom row and k coins in total. In [6] it is shown that

$$F(x,t) = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{x^2 t}{1 - \cfrac{x^3 t}{1 - \cfrac{x^4 t}{\ddots}}}}}$$

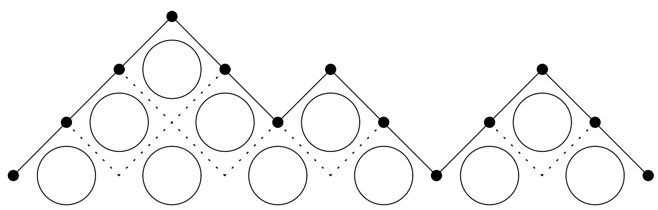
A straightforward application of Theorem 3 gives the following result. Note that $e_1\pi$ is the number of *non-inversions* in π, that is, pairs of letters in π that are in increasing order.

Proposition 4. *The number $f(n,k)$ equals the number of permutations $\pi \in \mathcal{S}_n(132)$ with $(e_0 + e_1)\pi = k$. Equivalently, $f(n,k)$ equals the number of permutations in $\mathcal{S}_n(132)$ with $k-n$ non-inversions.*

If we reverse each permutation in $\mathcal{S}_n(132)$ we see that $f(n,k)$ also equals the number of 231-avoiding permutations in $\mathcal{S}_n$ with exactly $k-n$ inversions.

We also give a combinatorial proof of Proposition 4, by constructing a bijection between the set of Dyck paths of length $2n$ and the set of fountains with n coins in the bottom row. Let Ψ be the bijection that maps a Dyck path on the fountain obtained by placing coins at the centre of all lattice squares inside the path, in the way that Figure 1 suggests.

FIGURE 1. A fountain of coins and the corresponding Dyck path



The ith *slant line* in a fountain is the sequence of coins starting with the ith coin from the left in the bottom row and continuing in the northeast direction. The height of a down-step thus corresponds to the number of coins in the slant line ending at the down-step d. Now, e_0 is the number of coins in the bottom row and $\binom{h(d)-1}{1}$ is one less than the number of coins in the corresponding slant line (see the end of Section 2). Thus we have that $e_0 + e_1$ counts the total number of coins in the fountain.

3.3. **Increasing subsequences.** The total number of increasing subsequences in a permutation is counted by $e_0 + e_1 + \cdots$. An application of Theorem 3 gives the following continued fraction for the distribution of $e_0 + e_1 + \cdots$:

$$\sum_{\pi \in \mathcal{S}_\infty(132)} x^{e_0\pi + e_1\pi + \cdots} t^{|\pi|} = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{x^2t}{1 - \cfrac{x^4t}{1 - \cfrac{x^8t}{\ddots}}}}}$$

3.4. **Right-to-left maxima.** We say that an increasing subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of $\pi \in \mathcal{S}_n$ is *right maximal* if $\pi(i_k) < \pi(j)$ implies $j < i_k$ (so that the sequence can not be extended to the right).

Proposition 5. *Let $\pi \in \mathcal{S}_n(132)$ and let $m_k\pi$ be the number of right maximal increasing subsequences of π of length $k+1$. Then*

$$m_k\pi = e_k\pi - e_{k+1}\pi + e_{k+2}\pi - \cdots .$$

In particular, the number of right-to-left maxima in π equals

$$e_0\pi - e_1\pi + e_2\pi - e_3\pi + \cdots .$$

Proof. It suffices to prove that for all $\pi \in \mathcal{S}_\infty(132)$ and $k \geq 0$ we have $m_k\pi + m_{k+1}\pi = e_k\pi$. The statistic e_k counts all increasing sequences of length k in π. If such a sequence is right maximal, it is counted by m_k. It therefore suffices to show that every increasing subsequence of length k that is not right maximal can be associated to a unique right maximal subsequence of length $k+1$, and conversely.

If an increasing subsequence of length k is not right maximal, it can be extended to a right maximal one of length $k+1$ and we show that this can only be done in one way. Suppose x is the last letter of the original sequence and that the sequence can be extended to a right maximal one by adjoining either y or z, where y comes before z in π. Then y must be greater than z, so x, y, z form a 132-sequence which is contrary to the assumption that π is 132-avoiding.

Conversely, deleting the last letter in a right maximal sequence of length $k+1$ clearly gives a non-right maximal sequence of length k. $\square$

Let $M_k(x,t) := \sum_{\pi \in \mathcal{S}_\infty(132)} x^{m_k(\pi)} t^{|\pi|}$. If $\gamma_i = 0$ for $i < k$ and $\gamma_{k+j} = (-1)^j$ for $j \geq 0$, then the n-th coordinate in $B[\gamma_i]$ is

$$\binom{n}{k} - \binom{n}{k+1} + \binom{n}{k+2} - \cdots ,$$

which is easily shown to equal $\binom{n-1}{k-1}$. That, together with Theorem 1, implies that $M_k(x,t)$ is the continued fraction whose $(n+1)$st denominator is the monomial $tx^{\binom{n-1}{k-1}}$. Let $F_k(x,t) = \sum_{\pi \in \mathcal{S}_\infty(132)} x^{e_k\pi} t^{|\pi|}$ and define $e_{-1}\pi$ to be 1 for all permutations π (that is, we declare all permutations to have exactly one increasing subsequence of length 0). Applying Theorem 1 we then have, for all $k \geq -1$, that $F_k(x,t)$ is the continued fraction with $(n+1)$st denominator $tx^{\binom{n}{k}}$, since $\binom{n-1}{-1}$ is naturally defined to be δ_{n0}. This leads to the following observation.

Proposition 6. *For all $k \geq 0$ we have*

$$M_k(x,t) = \frac{1}{1 - tF_{k-1}(x,t)}.$$

The *ballot number* $b(n,k)$ is the number of paths from $(0,0)$ to $(n+k, n-k)$ that do not cross the x-axis. It is well known that the ballot number $b(n,k)$ is equal to $\frac{n+1-k}{n+1}\binom{n+k}{n}$. Let $B(x,t) := \sum_{n,k} b(n,k)x^k t^n$. We have the following identity for $B(x,t)$ (see [8, p 152]):

$$B(x,t) = \frac{C(xt)}{1 - tC(xt)},$$

where $C(x)$ is the generating function for the Catalan numbers.

Proposition 7. *The number of permutations of length n with k right-to-left maxima equals the ballot number*

$$b(n-1, n-k) = \frac{k}{2n-k}\binom{2n-k}{n},$$

and

$$b(n-1,k) = \frac{n-k}{n+k}\binom{n+k}{k}$$

counts the number of permutations of length n with k right maximal increasing subsequences of length two.

Proof. By Proposition 6 we have that

$$M_0(x,t) = \frac{1}{1 - xtC(t)}$$

records the distribution of right-to-left maxima. Since $B(x^{-1}, xt) = \frac{C(t)}{1-xtC(t)}$ we have that:

$$M_0(x,t) = 1 + xtB(x^{-1}, xt) = 1 + \sum_{n,k} b(n-1, n-k)x^k t^n$$

and the first assertion follows. For the second assertion, observe that by Proposition 6 we have

$$M_1(x,t) = \frac{1}{1 - tC(xt)}.$$

Furthermore, we have

$$M_1(x,t) = M_0(x^{-1}, xt) = 1 + tB(x,t),$$

which concludes the proof. □

The first assertion of Proposition 7 can be proved bijectively using the map Φ in Section 2. In fact, the number of right-to-left maxima of π is equal to the *number of returns* in $\Phi(\pi)$, that is, the number of times the path $\Phi(\pi)$ intersects the x-axis. This number is known to have a distribution given by $b(n-1, k-1)$ (see [1]).

3.5. **Narayana numbers.** The generating function for the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ satisfies the following functional equation (see for example [1]):

$$N(x,t) := \sum_{n,k} N(n,k)x^k t^n = 1 + xtN^2(x,t) - xtN(x,t) + tN(x,t).$$

This allows us to express $N(x,t)$ as a continued fraction:

$$N(x,t) = \cfrac{1}{1 - \cfrac{t}{1 - xtN(x,t)}} = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{tx}{1 - \cfrac{t}{1 - \cfrac{tx}{\ddots}}}}}.$$

Since the sequence of exponents to x is $(0,1,0,1,\dots)$ we set $(\lambda_0, \lambda_1, \dots) = [(-1)^{n-k}\binom{n}{k}](0,1,0,1,\dots)$ in order to apply Theorem 3. Now, for $n \neq 0$ we have

$$\lambda_n = \sum_{k\ odd} (-1)^{n-k}\binom{n}{k} = (-2)^{n-1},$$

so $(\lambda_0, \lambda_1, \dots) = (0, 1, -2, 4, -8, \dots)$, which leads to the following Proposition.

Proposition 8. *Let $a\pi = (e_1 - 2e_2 + 4e_3 - \cdots)\pi$. Then the statistic a has the Narayana distribution over $\mathcal{S}_\infty(132)$, that is,*

$$\sum_{\pi \in \mathcal{S}_\infty(132)} x^{a\pi} t^{|\pi|} = \sum_{n,k} N(n,k)x^k t^n.$$

Now

$$\sum_{k\geq 1}(-2)^{k-1} f_k w = \sum_{k\geq 1}\sum_{d\ in\ w} (-2)^{k-1}\binom{h(d)-1}{k} = \sum_{d\ in\ w} \frac{1+(-1)^{h(d)}}{2}$$

so the interpretation of $e_1 - 2e_2 + 4e_3 - \cdots$ in terms of Dyck-paths is the number of even down-steps whose distribution is known to be given by the Narayana numbers.

References

[1] Emeric Deutsch. A bijection on Dyck paths and its consequences. *Discrete Math.*, 179(1-3):253–256, 1998.

[2] P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.*, 32(2):125–161, 1980.

[3] Mahendra Jani and Robert G. Rieper. Continued fractions and Catalan problems. *Electron. J. Combin.*, 7(1):Research Paper 45, 8 pp. (electronic), 2000.

[4] Christian Krattenthaler. Permutations with restricted patterns and Dyck paths.

[5] Toufik Mansour and Alek Vainshtein. Restricted permutations, continued fractions, and Chebyshev polynomials. *Electron. J. Combin.*, 7(1):Research Paper 17, 9 pp. (electronic), 2000.

[6] A. M. Odlyzko and H. S. Wilf. The editor's corner: n coins in a fountain. *Amer. Math. Monthly*, 95:840–843, 1988.

[7] Oskar Perron. *Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche.* B. G. Teubner Verlagsgesellschaft, Stuttgart, 1954. 3te Aufl.
[8] John Riordan. *Combinatorial identities.* John Wiley & Sons Inc., New York, 1968.
[9] Aaron Robertson, Herbert S. Wilf, and Doron Zeilberger. Permutation patterns and continued fractions. *Electron. J. Combin.*, 6(1):Research Paper 38, 6 pp. (electronic), 1999.

Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

E-mail address: branden@math.chalmers.se, claesson@math.chalmers.se, einar@math.chalmers.se