On Time-dependent and Stationary Solutions to the Linear Boltzmann Equation

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Abstract

This paper considers the linear time-dependent Boltzmann equation in a bounded domain with general boundary conditions, together with an interior source term and an absorption term, both these terms with time-dependent coefficient going to zero, when time goes to infinity. First, mild L^1-solutions are constructed as limits of iterate functions. Then the problem of convergence to a stationary solution is studied by an H-theorem, using a relative entropy functional.

1 Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics, (e.g. for discribing the neutron distribution in reactor physics, cf. [1]-[4]).

One fundamental question concerns the large time behavior of the function $f(\mathbf{x}, \mathbf{v}, t)$, representing the distribution of particles; in particular, the problem of convergence to a stationary equilibrium solution, when time goes to infinity. In our earlier papers [5]-[8] we have studied such convergence to equilibrium for the space-dependent linear Boltzmann equation with general boundary conditions and general initial data, under the assumption of existence of a corresponding stationary solution. For the proofs we use iterate functions, defined by an exponential form of the equation together with the boundary conditions, and we also use a general relative entropy functional for the quotient of the time dependent and the stationary solutions.

Then a fundamental question in kinetics concerns the existence and uniqueness of stationary solutions to the space-dependent transport equation, with general collision mechanism (including the cases of inverse soft and hard power forces), together with general boundary conditions, (including the periodic, specular and diffuse cases). In our paper [9] we studied the stationary equation with a constant interior source term and an absorption term.

In this paper we consider the time-dependent linear Boltzmann equation for the function $f(\mathbf{x}, \mathbf{v}, t)$ in the case of an interior source term $\alpha(t)G(\mathbf{x}, \mathbf{v})$ where $\alpha(t) = 1/(1+t)$ and G is a given function, together with an absorption term $\alpha(t)f(\mathbf{x}, \mathbf{v}, t)$ and general boundary conditions. First, solutions are constructed as limits of iterate functions, $f(\mathbf{x}, \mathbf{v}, t) = \lim_{n\to\infty} f^n(\mathbf{x}, \mathbf{v}, t)$ in Section 3, where also global boundedness of higher velocity moments is discussed. Then the main result of this paper concerning an H-theorem for a relative entropy functional of our solution is proved in Section 4, together with some corollaries, when $t \to \infty$. Finally in Section 5, we discuss the problem concerning convergence of our solution to a stationary solution, when time goes to infinity.

For bounded gain operators the problem of existence and uniqueness for solutions to the linear Boltzmann equation has been studied earlier by a different technique, cf. ref. [3] and also [10]. But in our approach unbounded operators are also included, e.g. the case of hard power collision forces.

2 Preliminaries

We consider the transport equation for a distribution function $f(\mathbf{x}, \mathbf{v}, t)$, depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$ and a time variable $t \in \mathbb{R}_+$. The linear Boltzmann equation in the case of a given interior source $\alpha(t)G(\mathbf{x}, \mathbf{v})$, where $\alpha \geq 0$ is a given (continuous) function and $G \geq 0$ a given (measurable) function, together with an absorption term $\alpha(t)f(\mathbf{x}, \mathbf{v}, t)$, is in the strong form

$$\begin{aligned}&\frac{\partial f}{\partial t}(\mathbf{x}, \mathbf{v}, t) + \mathbf{v}\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t) + \alpha(t) f(\mathbf{x}, \mathbf{v}, t) = \\ &= \alpha(t)G(\mathbf{x}, \mathbf{v}) + (Qf)(\mathbf{x}, \mathbf{v}, t).\end{aligned} \tag{2.1}$$

The collision term can be written

$$(Qf)(\mathbf{x}, \mathbf{v}, t) = \iint_{V\Omega} [Y(\mathbf{x}, \mathbf{v}'_*)f(\mathbf{x}, \mathbf{v}', t) - Y(\mathbf{x}, \mathbf{v}_*)f(\mathbf{x}, \mathbf{v}, t)]B(\theta, |\mathbf{v} - \mathbf{v}_*|)d\theta d\zeta d\mathbf{v}_*, \tag{2.2}$$

where $Y \geq 0$ is a known distribution function, and $B \geq 0$ is given by the collision process. Here $\mathbf{v}, \mathbf{v}_*$ are the velocities before and $\mathbf{v}', \mathbf{v}'_*$ the velocities after a binary collision, and $\Omega = \{(\theta, \zeta) : 0 \leq \theta \leq \hat{\theta}, 0 \leq \zeta < 2\pi\}$ is the impact plane. In the angular cut-off case with $\hat{\theta} < \frac{\pi}{2}$ the gain and the loss term can be separated

$$(Qf)(\mathbf{x},\mathbf{v},t) = \int_V K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) f(\mathbf{x},\mathbf{v}',t) d\mathbf{v}' - L(\mathbf{x},\mathbf{v}) f(\mathbf{x},\mathbf{v},t), \tag{2.3}$$

where L is the collision frequency

$$L(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} B(\theta,w) Y(\mathbf{x},\mathbf{v}_*) d\theta d\zeta d\mathbf{v}_*, w = |\mathbf{v} - \mathbf{v}_*|. \tag{2.4}$$

In the case of nonabsorbing body we have

$$L(\mathbf{x},\mathbf{v}) = \int_V K(\mathbf{x},\mathbf{v} \to \mathbf{v}'') d\mathbf{v}''. \tag{2.5}$$

One physically interesting case is that with inverse k-th power collision forces

$$B(\theta,w) = b(\theta) w^\gamma, \; \gamma = \frac{k-5}{k-1}, \tag{2.6}$$

with hard forces for $k > 5$, Maxwellian for $k = 5$, and soft forces for $3 < k < 5$.

The equation (2.1) is supplemented with the following initial conditions

$$f(\mathbf{x},\mathbf{v},0) = 0, \quad (\mathbf{x},\mathbf{v}) \in D \times V, \tag{2.7}$$

together with (general) boundary conditions

$$f_-(\mathbf{x},\mathbf{v},t) = (1-\beta(t)) \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v}) f_+(\mathbf{x},\tilde{\mathbf{v}},t) d\tilde{\mathbf{v}}, \tag{2.8}$$

where β is a given function , $0 \le \beta(t) \le 1$. The function $R \ge 0$ satisfies

$$\int_V R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v}) d\mathbf{v} \equiv 1, \tag{2.9}$$

and $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x} \in \Gamma$. The functions f_- and f_+ represent the ingoing and outgoing trace functions corresponding to f. Furthermore, in the specular reflection case, the function R is represented by Dirac measure $R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v}) = \delta(\mathbf{v} - \tilde{\mathbf{v}} + 2\mathbf{n}(\mathbf{n}\mathbf{v}))$, and in the diffuse reflection case $R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{n}\mathbf{v}| W(\mathbf{x},\mathbf{v})$ with some given function $W \ge 0$ (e.g. Maxwellian function).

Now using differentiation along the characteristics, the equation (2.1) can formally be written

$$\begin{aligned} &\frac{d}{dt}(f(\mathbf{x}+t\mathbf{v},\mathbf{v},t)) = \alpha(t) G(\mathbf{x}+t\mathbf{v},\mathbf{v}) + \\ &+ \int_V K(\mathbf{x}+t\mathbf{v},\mathbf{v}' \to \mathbf{v}) f(\mathbf{x}+t\mathbf{v},\mathbf{v}',t) d\mathbf{v}' - [\alpha(t) + L(\mathbf{x}+t\mathbf{v},\mathbf{v})] f(\mathbf{x}+t\mathbf{v},\mathbf{v},t). \end{aligned} \tag{2.10}$$

Let

$$t_b \equiv t_b(\mathbf{x}, \mathbf{v}) = \inf_{\tau \in \mathbb{R}_+} \{\tau : \mathbf{x} - \tau\mathbf{v} \notin D\}$$

and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x}, \mathbf{v}) = \mathbf{x} - t_b\mathbf{v}$. Here t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b = \mathbf{x} - t_b\mathbf{v}$ to the point $\mathbf{x}$.

Then we have the following *mild form* of the linear Boltzmann equation

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}, t) = f_-(\mathbf{x}_b, \mathbf{v}, t - t_b) + \int_0^t [Qf(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}, t - \tau) + \\ + \alpha(t - \tau)(G(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}) - f(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}, t - \tau))]d\tau \end{aligned} \tag{2.11}$$

and the *exponential* form

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}, t) = f_-(\mathbf{x}_b, \mathbf{v}, t - t_b)e^{-\int_0^{t_b}[\alpha(t-s)+L(\mathbf{x}-s\mathbf{v},\mathbf{v})]ds} + \\ + \int_0^t e^{-\int_0^\tau[\alpha(t-s)+L(\mathbf{x}-s\mathbf{v},\mathbf{v})]ds}[\alpha(t - \tau)G(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}) + \\ + \int_V K(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}' \to \mathbf{v})f(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}', t - \tau)d\mathbf{v}']d\tau. \end{aligned} \tag{2.12}$$

3 Construction of solutions

We construct mild L^1-solutions to our problem as limits of iterate functions f^n, when $n \to \infty$. Let first $f^{-1}(\mathbf{x}, \mathbf{v}, t) \equiv 0$ for all $\mathbf{x}, \mathbf{v} \in \mathbb{R}^3, t \in \mathbb{R}_+$. Then define, for given function f^{n-1} the next iterate f^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation);

$$\begin{aligned} f^n_-(\mathbf{x}, \mathbf{v}, t) = (1 - \beta(t)) \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) f^{n-1}_+(\mathbf{x}, \tilde{\mathbf{v}}, t)d\tilde{\mathbf{v}}, \\ \mathbf{n}\mathbf{v} < 0, \mathbf{x} \in \Gamma = \partial D, \ \mathbf{v} \in V = \mathbb{R}^3, t \in \mathbb{R}_+; \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} f^n(\mathbf{x}, \mathbf{v}, t) = f^n_-(\mathbf{x} - t_b\mathbf{v}, \mathbf{v}, t - t_b)e^{-\int_0^{t_b}[\alpha(t-s)+L(\mathbf{x}-s\mathbf{v},\mathbf{v})]ds} + \\ + \int_0^t e^{-\int_0^\tau[\alpha(t-s)+L(\mathbf{x}-s\mathbf{v},\mathbf{v})]ds}[\alpha(t - \tau)G(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}) + \\ + \int_V K(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}' \to \mathbf{v})f^{n-1}(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}', t - \tau)d\mathbf{v}']d\tau, \\ \mathbf{x} \in D \setminus \Gamma_-(\mathbf{v}), \ \mathbf{v} \in V = \mathbb{R}^3, t \in \mathbb{R}_+. \end{aligned} \tag{3.2}$$

Let also $f^n(\mathbf{x}, \mathbf{v}, t) \equiv 0$ for $\mathbf{x} \in \mathbb{R}^3 \setminus D$.

Now we get a monotonicity lemma, which is essential in the following, and which can be proved by induction.

Lemma 3.1. $f^n(\mathbf{x}, \mathbf{v}, t) \geq f^{n-1}(\mathbf{x}, \mathbf{v}, t), \mathbf{x} \in D, \mathbf{v} \in V, t \in \mathbb{R}_+, n \in \mathbb{N}.$

Remark. The iterate function $f^n(\mathbf{x}, \mathbf{v}, t)$ represents the distribution of particles undergone at most n collisions (inside D or at the boundary $\Gamma = \partial D$) in the time interval $(0, t)$.

Using differentiation along the characteristics, we get by (3.2) that

$$\begin{aligned}
&\frac{d}{dt}[f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)] = \alpha(t)[G(\mathbf{x}+t\mathbf{v}, \mathbf{v}) - f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)] + \\
&+ \int_V K(\mathbf{x}+t\mathbf{v}, \mathbf{v}' \to \mathbf{v}) f^{n-1}(\mathbf{x}+t\mathbf{v}, \mathbf{v}', t) d\mathbf{v}' - \\
&- L(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t) f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t).
\end{aligned} \tag{3.3}$$

Let now $\alpha(t) = 1/(1+t)$, and $\beta(t) = \rho/(1+t)$ with a constant $\rho \geq 0$, be the absorption coefficients used in the following.

Then multiplying (3.3) by $(1+t)$ and differentiating the product, we get

$$\begin{aligned}
&\frac{d}{dt}[(1+t) f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t)] = G(\mathbf{x}+t\mathbf{v}, \mathbf{v}) + \\
&+ \int_V K(\mathbf{x}+t\mathbf{v}, \mathbf{v}' \to \mathbf{v})(1+t) f^{n-1}(\mathbf{x}+t\mathbf{v}, \mathbf{v}', t) d\mathbf{v}' - \\
&- L(\mathbf{x}+t\mathbf{v}, \mathbf{v})(1+t) f^n(\mathbf{x}+t\mathbf{v}, \mathbf{v}, t).
\end{aligned} \tag{3.4}$$

Now integrating (3.4), it follows by Green's formula that

$$\begin{aligned}
&(1+t) \iint_{DV} f^n(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v} + \int_0^t (1+\tau) \iint_{\Gamma V} f_+^n(\mathbf{x}, \mathbf{v}, \tau) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma d\tau = \\
&= t \iint_{DV} G(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} + \int_0^t (1+\tau) \iint_{\Gamma V} f_-^n(\mathbf{x}, \mathbf{v}, \tau) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma d\tau + \\
&+ \int_0^t (1+\tau) \iint_{DV} L(\mathbf{x}, \mathbf{v})[f^{n-1}(\mathbf{x}, \mathbf{v}, \tau) - f^n(\mathbf{x}, \mathbf{v}, \tau)] d\mathbf{x} d\mathbf{v} d\tau,
\end{aligned} \tag{3.5}$$

where by (2.9)

$$\int_V f_-^n(\mathbf{x}, \mathbf{v}, \tau) |\mathbf{n}\mathbf{v}| d\mathbf{v} = (1 - \beta(\tau)) \int_V f_+^{n-1}(\mathbf{x}, \mathbf{v}, \tau) |\mathbf{n}\mathbf{v}| d\mathbf{v}.$$

So by Lemma 3.1 and (3.5) it follows that

$$\begin{aligned} &\iint_{DV} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} + \frac{\rho}{1+t}\int_0^t \iint_{\Gamma V} f_+^{n-1}(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma d\tau \\ &\leq \frac{t}{1+t}\iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}. \end{aligned} \tag{3.6}$$

Then Levi's theorem gives existence of mild (defined by (2.11)) L^1-solutions $f(\mathbf{x},\mathbf{v},t) = \lim_{n\to\infty} f^n(\mathbf{x},\mathbf{v},t)$ to our problem. Furthermore, if $L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t) \in L^1(D\times V)$, then we get equality in (3.6) for the limit function f, giving mass conservation together with uniqueness in the relevant function space, cf. [6] and also [3],

$$\begin{aligned} &\iint_{DV} f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} + \frac{\rho}{1+t}\int_0^t \iint_{\Gamma V} f_+(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma d\tau \\ &= \frac{t}{1+t}\iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}. \end{aligned} \tag{3.7}$$

In summary, we have the following existence and uniqueness theorem for solutions to our time-dependent linear Boltzmann equation with general boundary reflections.

Theorem 3.2. *Assume that K, L and R are nonnegative, measurable functions, such that (2.5) and (2.9) hold, and $L(\mathbf{x},\mathbf{v}) \in L^1_{loc}(D\times V)$. Let $\alpha(t) = 1/(1+t)$ and $\beta(t) = \rho/(1+t)$ be the absorption coefficients in the interior and at the boundary of the body D respectively, and let $G(\mathbf{x},\mathbf{v}) \in L^1(\mathbf{x},\mathbf{v})$ with $\int\int G d\mathbf{x}d\mathbf{v} > 0$.*

a) Then there exists a mild L^1 solution $f(\mathbf{x},\mathbf{v},t)$ to the problem (2.1)-(2.4) with (2.7) and (2.8). This solution, depending on G and ρ, satisfies the corresponding inequality in (3.7).

b) Moreover, if $L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t) \in L^1(D\times V)$, then the trace of the solution f satisfies the boundary condition (2.8) for a.e. $(\mathbf{x},\mathbf{v}) \in \Gamma\times V$. Furthermore, mass conservation, giving equality in (3.7), holds together with uniqueness in the relevant L^1-space.

Remarks.

1) The statement in Theorem 3.2 (b) on existence of traces follows e.g. from Proposition 3.3, Chapter XI, in [3].

2) The assumption $Lf \in L^1(D\times V)$ is for instance, satisfied for the solution f in the case of inverse power collision forces, cf. (2.6), together with specular or diffuse

boundary reflections. This follows from a statement on global boundedness (in time) of higher velocity moments, cf. Theorem 4.1 in [9],

$$\begin{aligned}&\iint_{DV}(1+v^2)^{\sigma/2}f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}\leq C_\sigma<\infty,\\&\sigma\geq\max(0,\gamma),-1<\gamma=(k-5)/(k-1)<1,t>0,\end{aligned}\tag{3.8}$$

if $(1+v^2)^{\sigma/2}G(\mathbf{x},\mathbf{v})\in L^1(D\times V)$.

For a proof of (3.8) we can multiply equation (3.4) by $(1+v^2)^{\sigma/2}$, and integrate, (using Green's formula) getting an equation analogous to (3.5). Then the gain and loss terms can be estimated, using an inequality for the velocities in a binary collision, cf. [5],

$$\begin{aligned}&(1+(v')^2)^{\sigma/2}-(1+v^2)^{\sigma/2}\leq\\&\leq C_1 w\cos\theta(1+v_*)^{\max(1,\sigma-1)}(1+v^2)^{\frac{\sigma-2}{2}}-\\&-C_2 w\cos^2\theta(1+v^2)^{\frac{\sigma-1}{2}},\end{aligned}$$

with constants $C_1, C_2>0$ and $\sigma>0$, together with some elementary estimate,

$$-w^{\gamma+1}\leq(1+v_*)^{\gamma+1}-2^{-1}(1+v^2)^{\frac{\gamma+1}{2}}.$$

The function Y in (2.2) is here assumed to satisfy the following conditions

$$\int_V(1+v_*)^{\gamma+\max(2,\sigma)}\sup_{x\in D}(Y(\mathbf{x},\mathbf{v}_*))d\mathbf{v}_*<\infty,$$

$$\int_V\inf_{x\in D}(Y(\mathbf{x},\mathbf{v}_*))d\mathbf{v}_*>0.$$

For further details on boundedness of higher velocity moments, see [9], and also our earlier papers [5]-[8].

4 An H-theorem for a relative entropy functional

In this section we will prove an entropy theorem for the quotient of two solutions, f and $\bar{f}$, from Section 3, with a time shifting in the absorption coefficients for one of the solutions.

$$\begin{aligned}&\text{Let } g(\mathbf{x},\mathbf{v},t)=(1+t)f(\mathbf{x},\mathbf{v},t),\\&\text{and } \bar{g}(\mathbf{x},\mathbf{v},t)=(T+t)\bar{f}(\mathbf{x},\mathbf{v},t),\end{aligned}\tag{4.1}$$

where f and $\bar{f}$ are solutions from Section 3 with absorption coefficients $\alpha(t)=1(1+t),\beta(t)=\rho/(1+t)$, and $\bar{\alpha}(t)=1/(T+t),\bar{\beta}(t)=\rho/(T+t),T>0$, respectively.

Then we can start from the following equations, cf. (3.4),

$$
\begin{aligned}
&\frac{d}{dt}[g(\mathbf{x}+t\mathbf{v},\mathbf{v},t)] = G(\mathbf{x}+t\mathbf{v},\mathbf{v}) + (Qg)(\mathbf{x}+t\mathbf{v},\mathbf{v},t),\\
&\qquad g(\mathbf{x},\mathbf{v},0) \equiv 0,\\
&\qquad g_-(\mathbf{x},\mathbf{v},t) = \Big(1-\frac{\rho}{1+t}\Big)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) g_+(\mathbf{x},\tilde{\mathbf{v}},t)d\tilde{\mathbf{v}},
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}[\bar{g}(\mathbf{x}+t\mathbf{v},\mathbf{v},t)] = G(\mathbf{x}+t\mathbf{v},\mathbf{v}) + (Q\bar{g})(\mathbf{x}+t\mathbf{v},\mathbf{v},t),\\
&\qquad \bar{g}(\mathbf{x},\mathbf{v},0) \equiv 0,\\
&\qquad \bar{g}_-(\mathbf{x},\mathbf{v},t) = \Big(1-\frac{\rho}{T+t}\Big)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) \bar{g}_+(\mathbf{x},\tilde{\mathbf{v}},t)d\tilde{\mathbf{v}}.
\end{aligned}
\tag{4.3}
$$

To prove an H-theorem for the convex function $\varphi(z) = (z-1)^2, z = \bar{g}/g$, we begin with the following calculations, cf. (4.2), (4.3),

$$
\begin{aligned}
&\frac{d}{dt}\Big[\Big(\frac{\bar{g}}{g}-1\Big)^2 g\Big](\mathbf{x}+t\mathbf{v},\mathbf{v},t) =\\
&= \Big[2\frac{\bar{g}}{g}\frac{d\bar{g}}{dt} - \Big(\frac{\bar{g}}{g}\Big)^2\frac{dg}{dt} - 2\frac{d\bar{g}}{dt} + \frac{dg}{dt}\Big](\mathbf{x}+t\mathbf{v},\mathbf{v},t) =\\
&= \Big[2\frac{\bar{g}}{g} - \Big(\frac{\bar{g}}{g}\Big)^2 - 1\Big]G +\\
&+ 2\Big(\frac{\bar{g}}{g}-1\Big)\Big[\int K\bar{g}(\mathbf{v}')d\mathbf{v}' - L\cdot\bar{g}\Big] +\\
&+ \Big[1-\Big(\frac{\bar{g}}{g}\Big)^2\Big]\Big[\int Kg(\mathbf{v}')d\mathbf{v}' - L\cdot g\Big] =\\
&= -\Big(\frac{\bar{g}}{g}-1\Big)^2 G + 2\frac{\bar{g}(\mathbf{v})}{g(\mathbf{v})}\int K\frac{\bar{g}(\mathbf{v}')}{g(\mathbf{v}')}g(\mathbf{v}')d\mathbf{v}' -\\
&- \int K\Big(\frac{\bar{g}}{g}\Big)^2(\mathbf{v})g(\mathbf{v}')d\mathbf{v}' - L\cdot\Big(\frac{\bar{g}}{g}\Big)^2(\mathbf{v})g(\mathbf{v}) -\\
&- 2[\int K\bar{g}(\mathbf{v}')d\mathbf{v}' - L\cdot\bar{g}(\mathbf{v})] +\\
&+ \int Kg(\mathbf{v}')d\mathbf{v}' - L\cdot g(\mathbf{v}),
\end{aligned}
\tag{4.4}
$$

where $L = \int K(\mathbf{v}\to\mathbf{v}')d\mathbf{v}'$, (and where we have shortened the notations to the essential variables).

Assume now that $Lf, L\bar{f} \in L^1(D\times V)$, so also $Lg, L\bar{g} \in L^1(D\times V)$ for $t>0$. Then, $\iint Kg(\mathbf{v}')d\mathbf{v}'d\mathbf{v} = \int Lg(\mathbf{v})d\mathbf{v}$, and the same relation holds for $\bar{g}$.

Then integration of (4.4) (from t_0 to t in the time variable) using Green's formula, gives (with some shortened notations)

$$\begin{aligned}
&\iint_{DV}\Big(\frac{\bar g}{g}-1\Big)^2 g(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}-\iint_{DV}\Big(\frac{\bar g}{g}-1\Big)^2 g(\mathbf{x},\mathbf{v},t_0)d\mathbf{x}d\mathbf{v}\\
&+\int_{t_0}^{t}\iint_{\Gamma V}\Big(\frac{\bar g_+}{g_+}-1\Big)^2 g_+(\mathbf{x},\mathbf{v},\tau)|\mathbf{nv}|d\mathbf{v}d\Gamma d\tau-\\
&-\int_{t_0}^{t}\iint_{\Gamma V}\Big(\frac{\bar g_-}{g_-}-1\Big)^2 g_-(\mathbf{x},\mathbf{v},\tau)|\mathbf{nv}|d\mathbf{v}d\Gamma d\tau=\\
&=-\int_{t_0}^{t}\iint_{DV}\Big(\frac{\bar g(\mathbf{x},\mathbf{v},\tau)}{g(\mathbf{x},\mathbf{v},\tau)}-1\Big)^2 G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}d\tau-\\
&-\int_{t_0}^{t}\iiint_{DVV}K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\Big[\Big(\frac{\bar g}{g}\Big)^2(\mathbf{v})-2\frac{\bar g(\mathbf{v})}{g(\mathbf{v})}\frac{\bar g(\mathbf{v}')}{g(\mathbf{v}')}+\Big(\frac{\bar g}{g}\Big)^2(\mathbf{v}')\Big]\\
&\times g(\mathbf{x},\mathbf{v}',\tau)d\mathbf{x}d\mathbf{v}d\mathbf{v}'d\tau.
\end{aligned}\tag{4.5}$$

For the boundary terms we use, if $\rho=0$, a Darrozes-Guiraud inequality, cf. ref [2],

$$\int_V\Big(\frac{\bar g_-}{g_-}-1\Big)^2 g_-|\mathbf{nv}|d\mathbf{v}\le\int_V\Big(\frac{\bar g_+}{g_+}-1\Big)^2 g_+|\mathbf{nv}|d\mathbf{v}.\tag{4.6}$$

Then we get the following H-theorem for $\bar g/g$:

$$\begin{aligned}
&\iint_{DV}\Big[\frac{\bar g(\mathbf{x},\mathbf{v},t)}{g(\mathbf{x},\mathbf{v},t)}-1\Big]^2 g(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}+\\
&+\int_{t_0}^{t}\iint_{DV}\Big[\frac{\bar g(\mathbf{x},\mathbf{v},\tau)}{g(\mathbf{x},\mathbf{v},\tau)}-1\Big]^2 G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}d\tau+\\
&+\int_{t_0}^{t}\iiint_{DVV}K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\Big[\frac{\bar g(\mathbf{x},\mathbf{v},\tau)}{g(\mathbf{x},\mathbf{v},\tau)}-\frac{\bar g(\mathbf{x},\mathbf{v}',\tau)}{g(\mathbf{x},\mathbf{v}',\tau)}\Big]^2\\
&\times g(\mathbf{x},\mathbf{v}',\tau)d\mathbf{x}d\mathbf{v}d\mathbf{v}'d\tau\le\\
&\le\iint_{DV}\Big[\frac{\bar g(\mathbf{x},\mathbf{v},t_0)}{g(\mathbf{x},\mathbf{v},t_0)}-1\Big]^2 g(\mathbf{x},\mathbf{v},t_0)d\mathbf{x}d\mathbf{v}.
\end{aligned}\tag{4.7}$$

But $\bar g=(T+t)\bar f$ and $g=(1+t)f$, so

$$\begin{aligned}
&(1+t)\iint_{DV}\Big[\frac{T+t}{1+t}\cdot\frac{\bar f(\mathbf{x},\mathbf{v},t)}{f(\mathbf{x},\mathbf{v},t)}-1\Big]^2 f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}+\\
&+\int_{t_0}^{t}\iint_{DV}\Big[\frac{T+\tau}{1+\tau}\cdot\frac{\bar f(\mathbf{x},\mathbf{v},\tau)}{f(\mathbf{x},\mathbf{v},\tau)}-1\Big]^2 G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}d\tau+\\
&+\int_{t_0}^{t}(1+\tau)\Big(\frac{T+\tau}{1+\tau}\Big)^2\iiint_{DVV}K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\Big[\frac{\bar f(\mathbf{x},\mathbf{v},\tau)}{f(\mathbf{x},\mathbf{v},\tau)}-\frac{\bar f(\mathbf{x},\mathbf{v}',\tau)}{f(\mathbf{x},\mathbf{v}',\tau)}\Big]^2\\
&\times f(\mathbf{x},\mathbf{v}',\tau)d\mathbf{x}d\mathbf{v}d\mathbf{v}'d\tau\le\\
&\le(1+t_0)\iint_{DV}\Big[\frac{T+t_0}{1+t_0}\cdot\frac{\bar f(\mathbf{x},\mathbf{v},t_0)}{f(\mathbf{x},\mathbf{v},t_0)}-1\Big]^2 f(\mathbf{x},\mathbf{v},t_0)d\mathbf{x}d\mathbf{v}.
\end{aligned}\tag{4.8}$$

Now divide by $(1+t)$, and let $t \to \infty$. Then

$$\begin{aligned}
&\lim_{t\to\infty}\Big\{\iint_{DV}\Big[\frac{\bar f(\mathbf{x},\mathbf{v},t)}{f(\mathbf{x},\mathbf{v},t)}-1\Big]^2 f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}\Big\}+\\
&+\lim_{t\to\infty}\Big\{\frac{1}{(1+t)}\int_{t_0}^{t}\Big[\frac{\bar f(\mathbf{x},\mathbf{v},\tau)}{f(\mathbf{x},\mathbf{v},\tau)}-1\Big]^2 G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}d\tau\Big\}+\\
&+\lim_{t\to\infty}\Big\{\frac{1}{(1+t)}\int_{t_0}^{t}(1+\tau)\Big(\frac{T+\tau}{1+\tau}\Big)^2\iiint_{DVV}K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\\
&\times\Big[\frac{\bar f(\mathbf{x},\mathbf{v},\tau)}{f(\mathbf{x},\mathbf{v},\tau)}-\frac{\bar f(\mathbf{x},\mathbf{v}',\tau)}{f(\mathbf{x},\mathbf{v}',\tau)}\Big]^2 f(\mathbf{x},\mathbf{v}',\tau)d\mathbf{x}d\mathbf{v}d\mathbf{v}'d\tau\Big\}\\
&\le 0.
\end{aligned}\tag{4.9}$$

We find that this inequality also holds in the case $\rho > 0$, because then we have, instead of (4.6), that

$$\int\Big(\frac{\bar g_-}{g_-}-1\Big)^2 g_-|\mathbf{nv}|d\mathbf{v}\le\int\Big[\frac{1-\bar\beta}{1-\beta}\cdot\frac{\bar g_+}{g_+}-1\Big]^2(1-\beta)g_+|\mathbf{nv}|d\mathbf{v}$$

with $\beta=\rho/(1+t), \bar\beta=\rho/(T+t)$.

Then further elementary calculations, using (4.9), give the following H-theorem for $\bar f/f$, when $t\to\infty$.

Theorem 4.1. *Let $\bar f$ and f be solutions from Theorem 3.2(b) corresponding to absorption coefficients $\bar\alpha(t)=1/(T+t), \bar\beta(t)=\rho/(T+t)$, and $\alpha(t)=1/(1+t), \beta(t)=\rho/(1+t)$ respectively, and assume that $L(\mathbf{x},\mathbf{v})\bar f(\mathbf{x},\mathbf{v},t)$ and $L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t)\in L^1(D\times V), t>0$. Then*

$$\begin{aligned}
&\lim_{t\to\infty}\Big\{\iint_{DV}\Big[\frac{\bar f(\mathbf{x},\mathbf{v},t)}{f(\mathbf{x},\mathbf{v},t)}-1\Big]^2 f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}\Big\}=0,\\
&\lim_{t\to\infty}\Big\{\iint_{DV}\Big[\frac{\bar f(\mathbf{x},\mathbf{v},t)}{f(\mathbf{x},\mathbf{v},t)}-1\Big]^2 G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}\Big\}=0,\\
&\lim_{t\to\infty}\Big\{(1+t)\iiint_{DVV}K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\Big[\frac{\bar f(\mathbf{x},\mathbf{v},t)}{f(\mathbf{x},\mathbf{v},t)}-\frac{\bar f(\mathbf{x},\mathbf{v}',t)}{f(\mathbf{x},\mathbf{v}',t)}\Big]^2\\
&\times f(\mathbf{x},\mathbf{v}',t)d\mathbf{x}d\mathbf{v}d\mathbf{v}'\Big\}=0.
\end{aligned}\tag{4.10}$$

Remark. Analogous H-theorems can be proved for general convex functions $\varphi(z), z=\bar f/f$, cf. ref. [8].

Furthermore, if $G(\mathbf{x},\mathbf{v})>0$ a.e. in $D\times V$, (e.g. with Maxwellian G), then we have the following corollary.

Corollary 4.2. *For the functions $\bar{f}$ and f in Theorem 4.1 it holds a.e. in $D \times V$ that*

$$\lim_{t\to\infty} \left\{ \frac{\bar{f}(\mathbf{x},\mathbf{v},t)}{f(\mathbf{x},\mathbf{v},t)} \right\} = 1. \tag{4.11}$$

5 On existence of stationary solutions

In this section we will discuss the problem concerning convergence of our solution $f(\mathbf{x},\mathbf{v},t)$, (constructed in Section 3), to a corresponding stationary solution $F(\mathbf{x},\mathbf{v})$, when time goes to infinity. Let

$$\bar{f}(\mathbf{x},\mathbf{v},t) = f(\mathbf{x},\mathbf{v},t+\Delta) \tag{5.1}$$

with $\Delta = T - 1$.

Then it follows, by e.g. (2.1), that $\bar{f}$ satisfies an equation with absorption coefficients $\bar{\alpha}(t) = \alpha(t+\Delta) = 1/(T+t)$ and $\bar{\beta}(t) = \beta(t+\Delta) = \rho/(T+t)$, used in Section 4. And the results, concerning the relative entropy functional from that section, can be applied in the following theorem.

Corollary 5.1. *The solution f from Theorem 3.2 satisfies (for arbitrary $\Delta = T - 1$)*

$$\lim_{t\to\infty} \frac{f(\mathbf{x},\mathbf{v},t+\Delta)}{f(\mathbf{x},\mathbf{v},t)} = 1, \; a.e. (\mathbf{x},\mathbf{v}) \in D \times V. \tag{5.2}$$

Then, concerning convergence to equilibrium for our time-dependent solution (constructed in Section 3), it follows that $f(\mathbf{x},\mathbf{v},t)$ converges (at least for a time sequence) to some (stationary) measure solution, which (in some cases) can be a L^1 solution, for instance in the case, when f also is a L^∞ solution. We hope to come back to this question in a forthcoming paper, and we know (by an H-theorem analogous to that in Section 4), that we have uniqueness, if there exists a mild L^1 solution, cf. ref. [8]. Compare also other papers concerning the problem of convergence to equilibrium for the linear Boltzmann equation, e.g. ref. [11] and also ref. [12].

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