DUAL BIALGEBROIDS FOR DEPTH TWO RING EXTENSIONS

LARS KADISON AND KORNÉL SZLACHÁNYI

ABSTRACT. We introduce a general notion of depth two for ring homomorphism $N \to M$, and derive Morita equivalence of first and third relative commutants, $R = C_M(N)$ and $C = \mathrm{End}\,_{N-M}(M \otimes_N M)$, via dual bimodules and second relative commutants, $A = \mathrm{End}\,_N M_N$ and $B = (M \otimes_N M)^N$. Lu's bialgebroids $\mathrm{End}\,_k A'$ and $A' \otimes_k A'^{\mathrm{op}}$ over a k-algebra A' are generalized to left and right bialgebroids A and B with B the R-dual bialgebroid of A. We introduce Galois-type actions of A on M and B on $\mathrm{End}\,_N M$ when M_N is a balanced module. In the case of Frobenius extensions $M|N$, we prove an endomorphism ring theorem for depth two. Further in the case of irreducible extensions, we extend previous results on Hopf algebra actions in subfactor theory [38] and its generalizations [19] by methods other than nondegenerate pairing. As a result, we have concrete expressions for the Hopf algebra structures on the step two centralizers. In the presence of depth two, we show that biseparable extensions are QF.

1. INTRODUCTION

Poisson and symplectic groupoids were introduced by Weinstein in [41, 42] in the late eighties. The notions extend to noncommutative algebra via Lu's notion of Hopf algebroid [23] or bialgebroid with antipode. Some time before this, Takeuchi [39] introduced the notion of $\times_R$-bialgebras based on studies of isomorphism classes of simple algebras and earlier work by Sweedler [36]. A special case of this extended notion of bialgebra is Ravenel's commutative Hopf algebroid introduced in the study of stable homotopy groups of spheres [31]. Etingof and Varchenko [10] associated a Hopf algebroid to any dynamical twist [1].

There is a quite different motivation coming from physics. In algebraic quantum field theory [13] the quest for finding a 2 dimensional analogue of the Doplicher-Roberts theorem [7] (applies to quantum field theories in $d \geq 3$ spacetime dimension) has lead the authors of [2] to introduce weak C^*-Hopf algebras (called also quantum groupoids [28]). The basic theory of weak Hopf algebras have been developed in [3, 29, 4]. It turns out that bialgebroids and $\times_R$-bialgebras are equivalent[43, 44, 5], while weak Hopf algebras, Hayashi's face algebras, Maltsinotis's groupoid quantiques occur as special cases [9, 29, 28].

In this paper we will bring the notion of bialgebroid together with a notion of depth two in the classification of subfactors [30]. Finite depth is a property of the standard invariant of the Jones tower of subfactor pair $N \subset M$ [11, 16]. One forms

Date: September 26, 2001.

1991 *Mathematics Subject Classification.* 12F10,16W30, 22D30, 46L37.

The first author thanks the second and K.F.K.I. for a nice visit to Budapest in April 2001, B. Külshammer for discussions and a nice visit to Jena in May 2001, D. Nikshych, A.A. Stolin and O.A. Laudal for discussions and NorFA in Oslo for financial support. The second author was supported by the Hungarian Scientific Research Fund, OTKA T-034 512.

the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$ by iterating the basic construction $M_i = \langle M_{i-1}, e_i \rangle$ where the e_i are the braidlike idempotents. The tower of relative commutants are the finite dimensional semisimple algebras $V_n = \text{Hom}_{N-M_n}(M_n)$. Finite depth is then the condition that the generating function $\sum_{n\geq 0} \dim(V_n)x^n$ be rational; of depth n if $V_n = V_{n-1}e_nV_{n-1}$. Depth two inclusions are fundamental among finite depth finite index inclusions via a Galois correspondence with weak C^*-Hopf algebras and their coideal $*$-subalgebras [27] (in a role similar to Ocneanu's paragroups).

In this paper we investigate the fundamentals behind this trend beginning in operator algebras and noncommutative Galois theory of associating groups and quantum algebras to certain finite depth algebra extensions. Noncommutative Galois theory for operator algebras was the point of departure for Vaughan Jones's theory of subfactors [15]. In [38] finite dimensional Hopf C^*-algebras or Kac algebras are associated to finite index irreducible subfactors of depth two. In [26] certain weak Hopf C^*-algebras [3] are associated to non-integer index subfactors of finite depth with Galois correspondence [27]. In [19] the depth two notion and results of [38] are extended to an algebraic analog without trace: certain semisimple Hopf algebras are shown to have a Galois action on split separable Frobenius extensions with trivial centralizer. As we show in Section 8 of this paper, a similar Hopf algebra H may be associated to an irreducible Frobenius extension $M|N$ of depth two; semisimplicity or cosemisimplicity of H being equivalent to $M|N$ being a split or separable extension respectively. Even the assumption of triviality in [18] for the centralizer may be relaxed to separability (or absolute semisimplicity) at the price of obtaining weak Hopf algebras, or quantum groupoids [20]. In each of these papers, it was essential to establish the quantum algebra properties of B together with its dual A via a nondegenerate pairing.

In this paper we propose a completely general notion of depth two for a ring extension $M|N$ which allows the construction of bialgebroid structures on the centralizers A and B directly without a nondegenerate pairing. In Section 2 we extend the theory of bialgebroids to cover left and right bialgebroids and their duals, actions and smash products. We define depth 2 ring extension in Section 3 and derive from a certain extension of Morita theory (cf. Section 1.1) the basic classical properties among the step 1, step 2, and step 3 centralizers in a Jones tower above a depth 2 extension $M|N$: the large centralizer C is Morita equivalent to the small centralizer R (with no conditions imposed on it) while the step 2 centralizers A and B are the Morita bimodules dual to one another and implementing the equivalence. In Section 4 we show directly that A is a left bialgebroid over R with left action on M: if M_N is balanced, the invariant subalgebra is N. We show that $\text{End}\, M_N$ is isomorphic to a smash product of M with the bialgebroid A over R, which is a basic step toward a Galois theory for bialgebroid actions. In Section 5 we show directly that B is a right bialgebroid with right action on $\text{End}\,_N M$ and subalgebra of invariants M. A and B are generalizations of Lu's bialgebroids in [23, Section 3] to noncommutative ring extensions, and are shown to be R-dual to one another in Sections 2, 3 and 5. In Section 6 we specialize to the case where $M|N$ is a Frobenius extension. We answer a question in [19] by showing that depth two passes up to the endomorphism ring extension. In Section 7 we show that A and B specialize to isomorphic copies of the dual Hopf algebras in [19] in case R is trivial in a depth two strongly separable extension of algebras. We also provide an answer to a question in [6] in the presence of depth two by showing that a biseparable (i.e. split + sepa-

Depth 2 Frobenius Extension	*Centralizer*	*Dual Quantum Algebras with Galois Actions*
biseparable algebra extension	trivial	semisimple, cosemisimple Hopf algebra
algebra extension	trivial	Hopf algebra
algebra extension	separable	weak Hopf algebra
unrestricted	unrestricted	bialgebroid

rable + f.g. projective) extension is quasi-Frobenius (QF). In Section 8 we extend the results in [19] to an irreducible depth two Frobenius extension by finding an antipode $S : A \to A$ for a Frobenius bialgebroid over trivial centralizer. We prove that the action of A on M in [19] is given by the analogous formula for the action of B on M_1. We summarize the algebraic results to date in a table — with a remark that there is in principle room for many more entries in future investigations.

Contents

In this paper rings are unital and ring homomorphisms preserve the units. A *ring extension* $M|N$ in its most general sense is a ring homomorphism $\iota : N \to M$, which induces a natural N-N-bimodule structure on M via $n \cdot m \cdot n' := \iota(n)m\iota(n')$; the ring extension is *proper* if $N \hookrightarrow M$. The ι is suppressed in the language of ring extensions. ${}_MP_M$ denotes an M-M-bimodule, and P^M the centralizer subgroup $\{p \in P | \, pm = mp, \forall m \in M\}$, with the centralizer $C_M(N) = M^N$ being a special case. A ring extension $M|N$ is said to have a property like (left) finitely generated (f.g.) if ${}_NM$ and M_N have this property (respectively, just ${}_NM$ is f.g.). We denote P being isomorphic to a direct summand of another M-M-bimodule Q by ${}_MP_M \oplus * \cong {}_MQ_M$.

1.1. H-equivalence and Morita theory.

Recall that two rings T and U are said to be Morita equivalent if the category of left (or right) T-modules is equivalent to the category of left (or right) U-modules. The following statements due to Hirata [14], generalize Morita's main theorems in a very useful and simplifying way. Let S be a ring with right modules V and W.

Lemma 1.1. *If $W_S \oplus * \cong (V \oplus \cdots \oplus V)_S$, and we set $T := \operatorname{End} V_S$, $U := \operatorname{End} W_S$, then the natural bimodules ${}_T V_S$, ${}_U W_S$, ${}_U \operatorname{Hom}(V_S, W_S)_T$, ${}_T \operatorname{Hom}(W_S, V_S)_U$ are related by the following isomorphisms:*

1. *a U-U-isomorphism $\mu_T : \operatorname{Hom}(V_S, W_S) \otimes_T \operatorname{Hom}(W_S, V_S) \xrightarrow{\cong} U$ via composition;*
2. *a U-T-isomorphism $\psi : \operatorname{Hom}(V_S, W_S) \xrightarrow{\cong} \operatorname{Hom}({}_T\operatorname{Hom}(W_S, V_S), {}_T T)$ defined by $f \mapsto (g \mapsto g \circ f)$.*
3. *a U-S-isomorphism $\tau : \operatorname{Hom}(V_S, W_S) \otimes_T V \xrightarrow{\cong} W$ given by $f \otimes v \mapsto f(v)$.*
4. *a U-S-isomorphism $\iota : W \xrightarrow{\cong} \operatorname{Hom}({}_T\operatorname{Hom}(W_S, V_S), {}_T V)$ via $w \mapsto (f \mapsto f(w))$.*
5. $\operatorname{Hom}(V_S, W_S)$ *is a finitely generated projective right T-module and a generator left U-module, while* $\operatorname{Hom}(W_S, V_S)$ *is a finite projective left T-module and generator right U-module.*
6. $\operatorname{Hom}(\operatorname{Hom}(V_S, W_S)_T, \operatorname{Hom}(V_S, W_S)_T) \cong U$ *and*
   $\operatorname{Hom}({}_U\operatorname{Hom}(W_S, V_S), {}_U\operatorname{Hom}(W_S, V_S)) \cong U$.

Proof. We observe that there are a finite number of $f_i \in \operatorname{Hom}(V_S, W_S)$ and $g_i \in \operatorname{Hom}(W_S, V_S)$ such that $\sum_i f_i \circ g_i = \mathrm{id}_W$. Define $\mu_T^{-1}(u) = \sum_i u f_i \otimes g_i$. Define $\psi^{-1}(F) = \sum_i f_i \circ F(g_i)$. The rest of the proof is quite similar and left to the reader [14]. □

The lemma has the following easy converse: if μ_T is epi, then $W_S \oplus * \cong \oplus^n V_S$. The lemma leads directly to Hirata's result [14]:

Proposition 1.2. *If both $V \oplus * \cong \oplus^n W$ and $W \oplus * \cong \oplus^m V$ (in which case we say V and W are H-equivalent S-modules), then T and U are Morita equivalent rings with Morita context given by*

$$({}_U\operatorname{Hom}(V_S, W_S)_T,\ {}_T\operatorname{Hom}(W_S, V_S)_U, \mu_T, \mu_U).$$

If V_S is f.g. projective and a generator, then V_S and S_S are H-equivalent, and $\operatorname{End} V_S$ is Morita equivalent to S via V and its right S-dual V^*. This recovers Morita's theorem.

2. Bialgebroids

Lu's original definition [23] of a bialgebroid corresponds to what is called a left bialgebroid below. The necessity to introduce a left and a right version comes from the assymmetry of the bialgebroid axioms under the switch to the opposite ring structure. The axioms we use here for the (right) bialgebroid are those of [37] which can be easily seen to be equivalent to (the right handed versions of) Lu's original axioms.

Let R be a ring. A **right bialgebroid** over R consists of the data and axioms:

1. a ring A and two ring homomorphisms $R^{op} \xrightarrow{t} A \xleftarrow{s} R$ such that $s(r')t(r) = t(r)s(r')$ for $r, r' \in R$. Thus A can be made into an R-R-bimodule by setting $r \cdot a \cdot r' := at(r)s(r')$.
2. R-R-bimodule maps $\Delta \colon A \to A \otimes_R A$ and $\varepsilon \colon A \to R$ such that the triple $\langle A, \Delta, \varepsilon \rangle$ is a comonoid in the category ${}_R\mathcal{M}_R$. (Another name is R-coring.)
3. Δ is multiplicative in the following sense. Although $A \otimes_R A$ has no ring structure in general, its sub-bimodule

$$A \times_R A := \{X \in A \otimes_R A \,|\, (s(r) \otimes 1)X = (1 \otimes t(r))X,\ \forall r \in R\}$$

is a ring with multiplication $(a \otimes a')(a'' \otimes a''') = aa'' \otimes a'a'''$. Now we require that

$$\Delta\colon A \to A \times_R A$$

be a ring homomorphism.

4. ε preserves the unit: $\varepsilon(1) = 1_R$
5. ε is compatible with multiplication in the sense of the axioms

$$\varepsilon(t(\varepsilon(a))b) \;=\; \varepsilon(ab) \;=\; \varepsilon(s(\varepsilon(a))b)\,, \qquad a, b \in A\,.$$

When discussing duals of bialgebroids in Subsection 2.4 we shall see that property (5) is dual to the unitalness of the coproduct, $\Delta(1) = 1 \otimes 1$, which is part of property (3) above.

Without much comment we can now list the axioms of a **left bialgebroid** over R. It consists of

- a pair $R \xrightarrow{s} A \xleftarrow{t} R^{op}$ of ring homomorphisms such that $s(r)t(r') = t(r')s(r)$, $r, r' \in R$
- R-R-bimodule maps $\Delta\colon A \to A \otimes_R A$ and $\varepsilon\colon A \to R$ where A is given the bimodule structure $r \cdot a \cdot r' := s(r)t(r')a$

such that

i. $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$
ii. $(\varepsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \varepsilon) \circ \Delta$
iii. $\Delta(a)(t(r) \otimes 1) = \Delta(a)(1 \otimes s(r))$, $a \in A$, $r \in R$
iv. $\Delta(ab) = \Delta(a)\Delta(b)$, $a, b \in A$
v. $\Delta(1) = 1 \otimes 1$
vi. $\varepsilon(1) = 1_R$
vii. $\varepsilon(as(\varepsilon(b))) = \varepsilon(ab) = \varepsilon(at(\varepsilon(b)))$, $a, b \in A$.

It follows from the R-linear property of Δ that(for a left bialgebroid)

$$\Delta(s(r)) = s(r) \otimes 1, \tag{1}$$

$$\Delta(t(r)) = 1 \otimes t(r). \tag{2}$$

Notice our convention of s being a homomorphism and t being an anti-homomorphism from R both in the case of left and right bialgebroids. In the language of weak Hopf algebras [3] $s(R)$ corresponds to A^L in the case of left bialgebroids but corresponds to A^R in the case of right bialgebroids. So the formulas (1) and (2) are interchanged for a right bialgebroid.

As for the relation of left and right bialgebroids we note that if $\langle A, R, s, t, \Delta, \varepsilon\rangle$ is a left bialgebroid then $\langle A', R, s', t', \Delta, \varepsilon\rangle$ is a right bialgebroid where

$$A' \;=\; A^{op}, \quad s' = t^{op}\colon R \to A^{op}, \quad t' = s^{op}\colon R^{op} \to A^{op}. \tag{3}$$

On the other hand, passing to the opposite coring structure does not change "handedness". As a matter of fact if $\langle A, R, s, t, \Delta, \varepsilon\rangle$ is a left bialgebroid then $A^{cop} := \langle A, R', s', t', \Delta', \varepsilon\rangle$ is also a left bialgebroid where

$$R' = R^{op}, \quad s' = t, \quad t' = s \tag{4}$$

thus the bimodule structure of ${}_{R'}A'_{R'}$ is the opposite of ${}_RA_R$, i.e.,

$$r_1 \cdot' a \cdot' r_2 \;=\; t(r_1)s(r_2)a \;=\; r_2 \cdot a \cdot r_1\,, \quad r_1, r_2 \in R,\ a \in A\,. \tag{5}$$

Applying the Sweedler notation $a_{(1)} \otimes a_{(2)}$ for $\Delta(a)$ the coproduct of A^{cop} is

$$\Delta' = \Delta^{op}\colon a \mapsto a_{(2)} \otimes_{R^{op}} a_{(1)} \tag{6}$$

for which ε is the counit.

The same construction yields a right bialgebroid A^{cop} from a right bialgebroid A.

Example 2.1. If A is an algebra over a commutative ring R with $s = t = u : R \to A$, the unit map, then a bialgebroid structure on A is a bialgebra [24].

2.1. **Module algebroids.** The extra structure on a ring A which makes it a left (right) bialgebroidover R is precisely a monoidal structure on its category ${}_A\mathcal{M}$ ($\mathcal{M}_A$) of modules together with a strictly monoidal forgetful functor to ${}_R\mathcal{M}_R$ (cf. [37]). Therefore the natural candidate for a "module algebra" over a bialgebroid A is a monoid in the category of A-modules. More explicitely a **left A-module algebroid** over a left bialgebroid $\langle A, R, s, t, \Delta, \varepsilon\rangle$ consists of

- a left A-module ${}_AM$ inheriting an R-R bimodule structure from the A-action: $r \cdot m \cdot r' = (r \cdot 1_A \cdot r') \triangleright m = s(r)t(r') \triangleright m$
- an associative multiplication $\mu_M : M \otimes_R M \to M$, $m \otimes m' \mapsto mm'$ satisfying

$$a \triangleright (mm') = (a_{(1)} \triangleright m)(a_{(2)} \triangleright m'), \quad a \in A, \ m, m' \in M \tag{7}$$

- and a unit $\eta_M : R \to M$, $r \mapsto r \cdot 1_M \equiv 1_M \cdot r$ for the multiplication μ_M which satisfies

$$a \triangleright 1_M = \varepsilon(a) \cdot 1_M\,, \qquad a \in A\,. \tag{8}$$

Note then that

$$(m \cdot r)m' = m(r \cdot m'), \tag{9}$$

as well as $r \cdot (mm') = (r \cdot m)m'$ and $(mm') \cdot r = m(m' \cdot r)$. Notice that Eqns. (7) and (8) express the fact that $\langle M, \mu_M, \eta_M\rangle$ is not only a monoid in ${}_R\mathcal{M}_R$ but also in ${}_A\mathcal{M}$.

A **right A-module algebroid** over a right bialgebroid $\langle A, R, s, t, \Delta, \varepsilon\rangle$ consists of

- a right A-module M_A inheriting an R-R bimodule structure from the A-action: $r \cdot m \cdot r' = m \triangleleft (r \cdot 1_A \cdot r') = m \triangleleft t(r)s(r')$
- an associative multiplication $\mu_M : M \otimes_R M \to M$, $m \otimes m' \mapsto mm'$ satisfying

$$(mm') \triangleleft a = (m \triangleleft a_{(1)})(m' \triangleleft a_{(2)}), \quad a \in A, \ m, m' \in M \tag{10}$$

- and a unit $\eta_M : R \to M$, $r \mapsto r \cdot 1_M \equiv 1_M \cdot r$ for the multiplication μ_M which satisfies

$$1_M \triangleleft a = 1_M \cdot \varepsilon(a)\,, \qquad a \in A\,. \tag{11}$$

A left A-module algebroid over the left bialgebroid A is the same as the right A^{op}-module algebroid over the right bialgebroid A^{op}.

If ${}_AM$ is a left A-module algebroid over the left bialgebroid A then the opposite ring M^{op} yields a monoid in ${}_{R^{op}}M_{R^{op}}$ such that M^{op} becomes a left A^{cop}-module algebroid.

Similarly, a right A-module algebroid M_A gives rise to a right A^{cop}-module algebroid $M^{op}_{A^{cop}}$.

2.2. **The subring of invariants.** Let ${}_AM$ be a module algebroid over the left bialgebroid $\langle A, R, s, t, \Delta, \varepsilon\rangle$. The invariants of M is the subset

$$M^A := \{n \in M | a \triangleright m = s(\varepsilon(a)) \triangleright m, a \in A\}. \tag{12}$$

Notice that if $n \in M^A$ then

$$t(\varepsilon(a)) \triangleright n = s(\varepsilon(t(\varepsilon(a)))) \triangleright n = s(\varepsilon(a)) \triangleright n = a \triangleright n \tag{13}$$

for all $a \in A$. Therefore we obtain an equivalent definition if s is replaced by t in (12).

Lemma 2.2. *For $m \in M$, $n \in M^A$ and $a \in A$ we have*

$$a \triangleright (mn) = (a \triangleright m)n\,, \qquad a \triangleright (nm) = n(a \triangleright m)\,. \tag{14}$$

In particular, it follows that M^A is a subring of M.

Proof. In the next calculation, we use one of the two equivalent definitions of invariants, next the identity $m_1m_2 = (1_{(1)} \triangleright m_1)(1_{(2)} \triangleright m_2)$, then axiom (iii) of a left bialgebroid and finally one of the counit axioms.

$$\begin{aligned} a \triangleright (mn) &= (a_{(1)} \triangleright m)(s(\varepsilon(a_{(2)})) \triangleright n) = (1_{(1)}a_{(1)} \triangleright m)(1_{(2)}s(\varepsilon(a_{(2)})) \triangleright n) \\ &= (1_{(1)}t(\varepsilon(a_{(2)}))a_{(1)} \triangleright m)(1_{(2)} \triangleright n) = (a_{(1)} \cdot \varepsilon(a_{(2)}) \triangleright m)n \\ &= (a \triangleright m)n, \end{aligned}$$

and

$$\begin{aligned} a \triangleright (nm) &= t(\varepsilon(a_{(1)})) \triangleright n)(a_{(2)} \triangleright m) = (n \cdot \varepsilon(a_{(1)}))(a_{(2)} \triangleright m) \\ &= n(s(\varepsilon(a_{(1)}))a_{(2)} \triangleright m) = n(a \triangleright m). \quad \square \end{aligned}$$

In a similar way the invariants of a right bialgebroid M_A can be written in two ways

(15)

$$M^A = \{n \in M | m \triangleleft a = m \triangleleft s(\varepsilon(a)),\ a \in A\} = \{n \in M | m \triangleleft a = m \triangleleft t(\varepsilon(a)),\ a \in A\}$$

and form a subring $M^A \subset M$.

Another important subring in a module algebroid is the sub-A-module generated by the identity. For a left A-module algebroid M it is

$$M^j := \{a \triangleright 1_M | a \in A\}. \tag{16}$$

It is the image of the map

$$j_M \colon R \to M, \qquad r \mapsto s(r) \triangleright 1_M \tag{17}$$

which is a ring homomorphism. As a matter of fact, $t(r) \triangleright 1_M = s(\varepsilon(t(r))) \triangleright 1_M = s(r) \triangleright 1_M$ therefore

$$\begin{aligned} j_M(r)j_M(r') &= (s(r) \triangleright 1_M)(t(r') \triangleright 1_M) \\ &= ((r \cdot 1 \cdot r')_{(1)} \triangleright 1_M)((r \cdot 1 \cdot r')_{(2)} \triangleright 1_M) \\ &= s(r)t(r') \triangleright 1_M = s(r)s(r') \triangleright 1_M = s(rr') \triangleright 1_M \\ &= j_M(rr')\,. \end{aligned}$$

As a consequence of the lemma above, M^j commutes with the invariants,

$$(s(r) \triangleright 1_M)n = n(s(r) \triangleright 1_M) \quad r \in R,\ n \in M^A\,. \tag{18}$$

For the module algebroids we shall consider in Sections 3 and 4 the M^j is actually equal to the centralizer of M^A in M.

2.3. **The smash product.** If A is a left bialgebroid over R and M is a left A-module algebroid then M is a right R module via $m \cdot r := mj_M(r)$.

Definition 2.3. *The smash product $M \rtimes A$ of a left A-module algebroid ${}_AM$ with A is the ring with additive group $M \otimes_R A$ and multiplication defined by*

$$(m \rtimes a)(m' \rtimes a') := m(a_{(1)} \triangleright m') \rtimes a_{(2)}a' . \tag{19}$$

Analogously one defines $A \ltimes M$ for a right A-module algebroid M_A.

The multiplication is well-defined because of Eq. 9 and 2.iii. The maps $\imath_M : m \mapsto m \rtimes 1$ and $\imath_A : a \mapsto 1_M \rtimes a$ are ring homomorphisms of M, respectively of A, into $M \rtimes A$. One can check easily the relations

$$\imath_M(m)\imath_A(a) = m \rtimes a , \tag{20}$$

$$\imath_A(a)\imath_M(m) = (a_{(1)} \triangleright m) \rtimes a_{(2)} . \tag{21}$$

for $m \in M$, $a \in A$. The $\imath_M$ is always an embedding by the following argument. Lemma 2.2 allows us to map $M \rtimes A$ into $\mathrm{End}\, M_N$ where $N := M^A$, the subring of invariants, with A mapped into $\mathrm{End}\,({}_NM_N) \subset \mathrm{End}\, M_N$. As a matter of fact $m \rtimes a$ acts on M as $\lambda(m)(a \triangleright -)$. Composing the ring map $M \rtimes A \to \mathrm{End}\, M_N$ with $\imath_M$ one obtains λ_M, the left regular representation of M, which is faithful. Thus $\imath_M$ must be mono and the smash product is always a proper ring extension of M.

On the other hand, $\imath_A$ is not necessarily mono. If $\imath_A(a) = 0$ then using the above map into $\mathrm{End}\,{}_NM_N$ again we obtain that $a \triangleright m = 0$ for all $m \in M$. By Eq. 20, $m \rtimes a = 0$ for all $m \in M$. So if either ${}_AM$ is faithful or if M_R is faithfully flat, then A embeds into the smash product $M \rtimes A$ via $\imath_A$.

2.4. **Duals.** If A is a bialgebroid over R one may expect a bialgebroid structure on the dual bimodule A^* or ${}^*A$ provided A_R or ${}_RA$ is finitely projective. The fine point here is that in taking duals one really has to take into account that A is not only a bimodule over R but carries 4 actions of R: multiplying either from the left or right by either $s_A(r)$ or $t_A(r)$. Comultiplications of left and right bialgebroids are bimodule maps with respect to two different (and disjoint) pairs of R-actions. Multiplication, however, cannot be written as a bimodule map in either of these two categories but requires the use of "mixed" pairs of R-actions. This is why in defining duals of a bialgebroid we have to use new bimodule structures of A and not those appearing before in comultiplications.

2.4.1. *The right dual A^*.* Let A be a left bialgebroid over R and assume that A_R is finitely generated projective. Recall that the right action of $r \in R$ is $a \mapsto t_A(r)a$. We shall denote by $A^{(t)}$ the R-R-bimodule which is the additive group A on which $r \in R$ acts from the left via $a \mapsto at_A(r)$ and acts from the right via $a \mapsto t_A(r)a$. Thus the right R-action of $A^{(t)}$ coincides with the right R-action on A dictated by the left bialgebroid structure. But the left action is different. We define the right dual of A as the right dual bimodule of $A^{(t)}$, i.e., $A^* = \mathrm{Hom}\,(A_R, R_R)$ carrying the bimodule structure

$$\langle r \cdot b \cdot r', a\rangle := r\langle b, at_A(r')\rangle , \qquad b \in A^*, \ a \in A . \tag{22}$$

Here and below $\langle b, a\rangle$ denotes the canonical pairing, i.e., the evaluation of b on a. Now we make A^* into a ring by defining multiplication via the formula

$$\langle bb', a\rangle := \langle b', \langle b, a_{(1)}\rangle \cdot a_{(2)}\rangle . \tag{23}$$

which is associative due to coassociativity of Δ_A. Note with caution that $\cdot$ here denotes the ordinary R-bimodule structure on A: $r \cdot a = s(r)a$. The multiplication has a unit $1_{A^*} = \varepsilon_A$.

If A^* is going to be a right bialgebroid then the maps

$$s_{A^*} : R \to A^* , \qquad r \mapsto 1_{A^*} \cdot r = \varepsilon_A \circ \rho_A(t_A(r)) \tag{24}$$

$$t_{A^*} : R^{op} \to A^* , \qquad r \mapsto r \cdot 1_{A^*} = \varepsilon_A \circ \lambda_A(s_A(r)) \tag{25}$$

are ring homomorphisms. That this is indeed the case follows from previous identities such as Eq. (2):

$$\begin{aligned} \langle bs_{A^*}(r), a\rangle &= \langle s_{A^*}(r), \langle b, a_{(1)}\rangle \cdot a_{(2)}\rangle = \varepsilon_A(\langle b, a_{(1)}\rangle \cdot a_{(2)} t_A(r)) \\ &= \langle b, a_{(1)} \cdot \varepsilon_A(a_{(2)} t_A(r))\rangle = \langle b, at_A(r)\rangle \;=\; \langle b \cdot r, a\rangle \qquad (26) \\ \langle bt_{A^*}(r), a\rangle &= \langle t_{A^*}(r), \langle b, a_{(1)}\rangle \cdot a_{(2)}\rangle = r\varepsilon_A(\langle b, a_{(1)}\rangle \cdot a_{(2)}) \\ &= r\langle b, a_{(1)}\rangle \varepsilon_A(a_{(2)}) = r\langle b, a\rangle \;=\; \langle r \cdot b, a\rangle \qquad (27) \end{aligned}$$

For future convenience we list the five basic symmetry relations of the pairing, two of which have just been proved:

$$\begin{aligned} (28)\quad \langle b, t_A(r)a\rangle &= \langle b, a\rangle r \\ \langle b, s_A(r)a\rangle &= \langle b, \varepsilon_A(s_A(r)a_{(1)}) \cdot a_{(2)}\rangle = \langle b, \langle t_{A^*}(r), a_{(1)}\rangle \cdot a_{(2)}\rangle \\ (29)\quad &= \langle t_{A^*}(r)b, a\rangle \\ (30)\quad \langle b, at_A(r)\rangle &= \langle bs_{A^*}(r), a\rangle \\ \langle b, as_A(r)\rangle &= \langle b, \varepsilon_A(a_{(1)}) \cdot a_{(2)} s_A(r)\rangle = \langle b, \varepsilon_A(a_{(1)} t_A(r)) \cdot a_{(2)}\rangle \\ (31)\quad &= \langle s_{A^*}(r)b, a\rangle \\ (32)\quad \langle bt_{A^*}(r), a\rangle &= r\langle b, a\rangle \end{aligned}$$

In order to define comultiplication on A^* we have to utilize that A_R is finitely projective. A consequence of this is that the natural map

$$\begin{aligned} A^* \otimes_R A^* &\to \mathrm{Hom}\,((A^{(t)} \otimes_R A^{(t)})_R, R_R) \\ b \otimes b' &\mapsto \{a' \otimes a \mapsto \langle b \cdot \langle b', a'\rangle, a\rangle\} \end{aligned} \tag{33}$$

is an isomorphism. Its inverse can be given in terms of dual bases $\{a_i\}$ of A_R and $\{b_i\}$ of ${}_RA^*$ as $f \mapsto \sum_{i,j} f(a_j \otimes a_i) \cdot b_i \otimes b_j$.

Noticing that for any $b \in A^*$ the map $a' \otimes a \mapsto \langle b, aa'\rangle$ belongs to the above hom-group, a comultiplication

$$\Delta_{A^*} : A^* \to A^* \otimes_R A^* , \qquad b \mapsto b_{(1)} \otimes b_{(2)} \tag{34}$$

can be defined by requiring

$$\langle b_{(1)} \cdot \langle b_{(2)}, a'\rangle, a\rangle \;=\; \langle b, aa'\rangle , \qquad b \in A^*,\; a, a' \in A . \tag{35}$$

In terms of the dual bases it can be written as

$$\Delta_{A^*}(b) \;=\; \sum_{i,j} \langle b, a_i a_j\rangle \cdot b_i \otimes b_j \; . \tag{36}$$

Now we turn to verifying the bialgebroid axioms for Δ_{A^*}.

Δ_{A^*} is a bimodule map by its very definition (a simple calculation with Eq. 35). In order to see that its image lies in $A^* \times_R A^*$ we compute

$$\begin{aligned}
\langle s_{A^*}(r)b_{(1)} \cdot \langle b_{(2)}, a' \rangle, a \rangle &= \langle b_{(1)} \cdot \langle b_{(2)}, a' \rangle, as_A(r) \rangle \\
&= \langle b, as_A(r)a' \rangle = \langle b_{(1)} \cdot \langle b_{(2)}, s_A(r)a' \rangle, a \rangle \\
&= \langle b_{(1)} \cdot \langle t_{A^*}(r)b_{(2)}, a' \rangle, a \rangle \qquad (37)
\end{aligned}$$

Now it is meaningful to ask whether the map $\Delta_{A^*} \colon A^* \to A^* \times_R A^*$ is a ring homomorphism. The proof of multiplicativity goes as follows (lines 6 to 7 below requires Eqs. 24, 23 and 2):

$$\begin{aligned}
\langle (bb')_{(1)} \cdot \langle (bb')_{(2)}, a' \rangle, a \rangle &= \langle bb', aa' \rangle = \langle b', \langle b, (aa')_{(1)} \rangle \cdot (aa')_{(2)} \rangle \\
&= \langle b', \langle b, a_{(1)}a'_{(1)} \rangle \cdot a_{(2)}a'_{(2)} \rangle \\
&= \left\langle b', \left(\langle b_{(1)} \cdot \langle b_{(2)}, a'_{(1)} \rangle, a_{(1)} \rangle \cdot a_{(2)} \right) a'_{(2)} \right\rangle \\
&= \left\langle b'_{(1)} \cdot \langle b'_{(2)}, a'_{(2)} \rangle, \langle b_{(1)} \cdot \langle b_{(2)}, a'_{(1)} \rangle, a_{(1)} \rangle \cdot a_{(2)} \right\rangle \\
&= \left\langle \left(b_{(1)} \cdot \langle b_{(2)}, a'_{(1)} \rangle \right) \left(b'_{(1)} \cdot \langle b'_{(2)}, a'_{(2)} \rangle \right), a \right\rangle \\
&= \left\langle b_{(1)} s_{A^*}(\langle b_{(2)}, a'_{(1)} \rangle) b'_{(1)} s_{A^*}(\langle b'_{(2)}, a'_{(2)} \rangle), a \right\rangle \\
&= \left\langle b_{(1)} b'_{(1)} s_{A^*} \left(\langle t_{A^*}(\langle b_{(2)}, a'_{(1)} \rangle) b'_{(2)}, a'_{(2)} \rangle \right), a \right\rangle \\
&= \langle b_{(1)} b'_{(1)} \cdot \langle b'_{(2)}, s_A(\langle b_{(2)}, a'_{(1)} \rangle) a'_{(2)} \rangle, a \rangle \\
&= \langle b_{(1)} b'_{(1)} \cdot \langle b'_{(2)}, \langle b_{(2)}, a'_{(1)} \rangle \cdot a'_{(2)} \rangle, a \rangle \\
&= \langle b_{(1)} b'_{(1)} \cdot \langle b_{(2)} b'_{(2)}, a' \rangle, a \rangle
\end{aligned}$$

Preservation of the unit, $\Delta_{A^*}(1) = 1 \otimes 1$, can be seen as

$$\begin{aligned}
\langle 1_{(1)} \cdot \langle 1_{(2)}, a' \rangle, a \rangle &= \langle 1, aa' \rangle = \varepsilon_A(aa') = \varepsilon_A(at_A(\varepsilon_A(a'))) \\
&= \langle 1, at_A(\langle 1, a' \rangle) \rangle = \langle 1 s_{A^*}(\langle 1, a' \rangle), a \rangle \\
&= \langle 1 \cdot \langle 1, a' \rangle, a \rangle
\end{aligned}$$

We are left with constructing the counit for A^*. Let

$$\varepsilon_{A^*} \colon A^* \to R\,, \qquad b \mapsto \langle b, 1_A \rangle\,. \qquad (38)$$

Then ε_{A^*} is an R-R-bimodule map,

$$\begin{aligned}
\varepsilon_{A^*}(r \cdot b \cdot r') &= \langle b s_{A^*}(r') t_{A^*}(r), 1_A \rangle = r \langle b, t_A(r') \rangle \\
&= r \varepsilon_{A^*}(b) r'\,,
\end{aligned}$$

it satisfies the counit properties because

$$\begin{aligned}
\langle b_{(1)} \cdot \varepsilon_{A^*}(b_{(2)}), a \rangle &= \langle b_{(1)} \cdot \langle b_{(2)}, 1_A \rangle, a \rangle = \langle b, a \rangle \\
\langle \varepsilon_{A^*}(b_{(1)}) \cdot b_{(2)}, a \rangle &= \langle b_{(1)}, 1_A \rangle \langle b_{(2)}, a \rangle = \langle b_{(1)}, t_A(\langle b_{(2)}, a \rangle) \rangle \\
&= \langle b_{(1)} \cdot \langle b_{(2)}, a \rangle, 1_A \rangle = \langle b, a \rangle\,,
\end{aligned}$$

it preserves the unit,

$$\varepsilon_{A^*}(1_{A^*}) \;=\; \varepsilon_A(1_A) \;=\; 1_R\,,$$

and finally it is compatible with multiplication of A^*,

$$\begin{aligned}\varepsilon_{A^*}(s_{A^*}(\varepsilon_{A^*}(b))b') &= \langle s_{A^*}(\langle b, 1_A\rangle)b', 1_A\rangle \\ &= \langle b', s_A(\langle b, 1_A\rangle)\rangle = \langle b', \langle b, 1_A\rangle \cdot 1_A\rangle = \langle bb', 1_A\rangle \\ &= \varepsilon_{A^*}(bb') \\ \varepsilon_{A^*}(t_{A^*}(\varepsilon_{A^*}(b))b') &= \langle b', s_A(\langle b, 1_A\rangle)\rangle \\ &= \varepsilon_{A^*}(bb')\,.\end{aligned}$$

What we have just proven is the following:

Proposition 2.4. *If A is a left bialgebroid over R such that A_R is finitely generated projective then $B := \mathrm{Hom}\,(A_R, R_R)$ has a unique right bialgebroid structure over R such that*

$$\langle bb', a\rangle = \langle b', \langle b, a_{(1)}\rangle \cdot a_{(2)}\rangle \tag{39}$$

$$\langle b, aa'\rangle = \langle b_{(1)} \cdot \langle b_{(2)}, a'\rangle, a\rangle \tag{40}$$

where $\langle\ ,\ \rangle\colon B \times A \to R$ denotes the canonical pairing.

2.4.2. *The left dual $^*A$.* Let A again be a left bialgebroid but now assume that $_RA$ is finitely generated projective. For $a \in A$ and $b \in {}^*A = \mathrm{Hom}\,({}_RA, {}_RR)$ we denote by $[a, b] \in R$ the evaluation of b on a. As a bimodule $^*A$ is considered to be the dual bimodule of $A^{(s)}$ where the latter is the additive group A on which $r \in R$ acts from the left by $a \mapsto s_A(r)a$ and from the right by $a \mapsto as_A(r)$. Then similarly as in the above Proposition we can construct a right bialgebroid structure on $^*A$. More precisely we have

Proposition 2.5. *If A is a left bialgebroid over R such that $_RA$ is finitely generated projective then $^*A := \mathrm{Hom}\,({}_RA, {}_RR)$ has a unique right bialgebroid structure over R such that*

$$[a, bb'] = [a_{(1)} \cdot [a_{(2)}, b], b'] \tag{41}$$

$$[aa', b] = [a, [a', b_{(1)}] \cdot b_{(2)}]\,. \tag{42}$$

The proof is very similar to the previous construction and therefore omitted. We only give here the symmetry properties of the $[\ ,\ ]$ pairing:

$$[s_A(r)a, b] = r[a, b] \tag{43}$$

$$[t_A(r)a, b] = [a, bt_{^*A}(r)] \tag{44}$$

$$[as_A(r), b] = [a, s_{^*A}(r)b] \tag{45}$$

$$[at_A(r), b] = [a, bs_{^*A}(r)] \tag{46}$$

$$[a, t_{^*A}(r)b] = [a, b]r \tag{47}$$

We note that both the $\langle\ ,\rangle$ and $[\ ,\ ]$ pairings are variations of Schauenburg's skew pairing τ of [33] with the caution that [33] uses only left bialgebroids in our language.

2.4.3. *Duals of right bialgebroids.* Left and right duals $^*B$ and B^* of a right bialgebroid B can be introduced directly using the above notions of duals of left bialgebroids. Let B be a right bialgebroid over R with B_R finitely generated projective. Then its right dual B^* is a left bialgebroid defined by

$$B^* := ((B^{\mathrm{op}})^*)^{\mathrm{op}} \tag{48}$$

This means that $B^* = \mathrm{Hom}(B_R, R_R)$ as an additive group and its bialgebroid structure is to be read from the canonical pairing

$$[a,b] := a(b) \text{ for } a \in B^*,\ b \in B$$

satisfying precisely the relations of the pairing $[\ ,\ ]$ of Proposition 2.5. Now it is easy to verify that for a left bialgebroid A such that ${}_RA$ is finitely generated projective the canonical isomorphism $A \cong ({}^*A)^*$ of Abelian groups is in fact an isomorphism of left bialgebroids. In other words, if B is the left dual of A then A is the right dual of B.

The same conclusion holds for a left bialgebroid A such that A_R is finite projective. Its left dual $B = {}^*A$ is a right bialgebroid such that ${}_RB$ is finite projective and for such a B a left dual can be introduced via

$$ {}^*B := ({}^*(B^{\mathrm{op}}))^{\mathrm{op}}\ . \tag{49}$$

Denoting the canonical pairing of $b \in B$ with $a \in^* B$ by $\langle b, a\rangle$ we obtain that $\langle\ ,\ \rangle$ satisfies the relations of Proposition 2.4. Thus again if B is the left dual of the left bialgebroid A then A is the right dual of B.

In Sections 4 and 5 we shall meet a situation when the left bialgebroid A has both a left and a right dual and they are isomorphic to a right bialgebroid B. In this case it is fair to say that A and B are dual pairs of bialgebroids.

3. Depth 2 Ring Extensions

The tensor-square $M \otimes_N M$, left and right endomorphism rings, End ${}_NM$ and End M_N, of a ring extension $M|N$ have the natural M-M bimodule structures given by $m \cdot x \otimes y \cdot m' = mx \otimes ym'$, $(m\eta m')(x) = \eta(xm)m'$
and $(mfm')(x) = mf(m'x)$ for $m, m', x, y \in M$, $\eta \in$ End ${}_NM$, and $f \in$ End M_N, respectively.

Definition 3.1. *A ring extension $M|N$ is called left depth two or*
left d2 *if ${}_NM \otimes_N M_M \oplus * \cong \oplus^n\, {}_NM_M$ for some integer n.*
right d2 *if ${}_MM \otimes_N M_N \oplus * \cong \oplus^m\, {}_MM_N$ for some integer m.*
$M|N$ is called **d2** *if it is d2 both from the left and from the right.*

In particular, the natural modules ${}_MM \otimes M$ and $M \otimes M_M$ are f.g. projective for a d2 extension $M|N$.

Remark 3.2. If left d2, M and $M \otimes_N M$ are in fact H-equivalent as $M \otimes N^{\mathrm{op}}$-modules, since the multiplication mapping $\mu_N : M \otimes_N M \to M$ is a split N-M-epi for any ring extension $M|N$. A similar statement is equivalent to the right d2 condition.

Example 3.3. A classical depth two subfactor $N \subseteq M$ of finite index is of depth two by Proposition 6.2.

Example 3.4. A centrally projective ring extension $M|N$ is d2, since ${}_NM_N \oplus * \cong \oplus^n\, {}_NN_N$ and we may arrive at the definition above by tensoring from the left or right by ${}_MM_N$ or ${}_NM_M$. In particular, if $N \subset Z(M)$, the center of M, and M is finitely generated and projective as a N-module, then M is a d2 extension of N, a "d2 algebra" over N.

Example 3.5. An H-separable extension [14] $M|N$ is d2, since its defining property is

$$ {}_M(M \otimes_N M)_M \oplus * \cong \oplus^n\, {}_MM_M.$$

Let $A := \mathrm{End}\,{}_N M_N$ and $B := (M \otimes_N M)^N$.

Lemma 3.6. *$N \subset M$ is left d2 iff there exist $b_i \in B$ and $\beta_i \in A$ (called a left d2 quasibasis) such that*

$$\sum_i b_i^1 \otimes b_i^2 \beta_i(m) \;=\; m \otimes 1\,, \qquad m \in M\,.$$

$N \subset M$ is right d2 iff there exist $c_i \in B$ and $\gamma_i \in A$ such that

$$\sum_i \gamma_i(m) c_i^1 \otimes c_i^2 \;=\; 1 \otimes m\,, \qquad m \in M\,.$$

Proof. (Left d2 $\Rightarrow$ existence of quasibasis.) Let $\pi : \oplus^n {}_N M_M \to {}_N M \otimes M_M$ and $\sigma : {}_N M \otimes M_M \to \oplus^n {}_N M_M$ denote the split epi and its section implied by the definition. Furthermore, let $\{e_i\}_{i=1}^n$ be the standard basis of the free module $\oplus^n M_M$, and $p_i : \oplus^n {}_N M_M \to {}_N M_M$ be the standard projections. Then we let $b_i := \pi(e_i)$ where clearly $b_i \in B$. If $\iota_1 : {}_N M_N \hookrightarrow {}_N M \otimes_N M_N$ denotes the map $m \mapsto m \otimes 1$, then we let $\beta_i := p_i \circ \sigma \circ \iota_1 \in A$. Then

$$m \otimes 1 = \pi(\sigma(\iota_1(m))) = \sum_i \pi(e_i)\beta_i(m) = \sum_i b_i^1 \otimes b_i^2 \beta_i(m)$$

The rest of the proof is similar. □

Remark 3.7. If M is a finite dimensional k-algebra with $N = k1$, with dual bases $e_i \in M$ and $\pi_i \in M^* = \mathrm{Hom}_k(M, k)$, then a left and right d2 quasibasis is given by $\beta_i = \gamma_i = \iota\pi_i$, $b_i = e_i \otimes 1_M$ and $c_i = 1_M \otimes e_i$, where $\iota : k \hookrightarrow M$ is the unit map.

For every $m \in M$, we let $\lambda(m) \in \mathrm{End}\, M_N$ denote $\lambda(m)(x) = mx$ and $\rho(m) \in \mathrm{End}\,{}_N M$ denote $\rho(m)(x) = xm$. If $r \in R := C_M(N)$, we note that $\lambda(r), \rho(r) \in \mathrm{End}\,{}_N M_N = (\mathrm{End}\,{}_N M)^N = (\mathrm{End}\, M_N)^N$. In the sequel the R-bimodule structure on A is understood to be $r \cdot \alpha \cdot r' = \lambda(r)\rho(r')\alpha$.

Proposition 3.8. *If $M|N$ is a right d2 extension, then* $\mathrm{End}\,{}_N M \cong A \otimes_R M$ *as N-M-bimodules via $\alpha \otimes m \mapsto \rho(m)\alpha$. If $M|N$ is a left d2 extension, then* $\mathrm{End}\, M_N \cong M \otimes_R A$ *as M-N-bimodules via $m \otimes \alpha \mapsto \lambda(m) \circ \alpha$.*

Proof. We claim that $f \mapsto \sum_i \gamma_i \otimes c_i^1 f(c_i^2)$ for $f \in \mathcal{E}' := \mathrm{End}\,{}_N M$ defines an inverse. Since $\sum_i \gamma_i(m) c_i^1 f(c_i^2) = f(m)$, we see that $f = \sum_i \rho(c_i^1 f(c_i^2))\gamma_i \in \rho(M)A$.

Similarly an inverse to the second statement is given by

$$f \mapsto \sum_i f(b_i^1) b_i^2 \otimes \beta_i$$

for each $f \in \mathcal{E} := \mathrm{End}\, M_N$. □

Proposition 3.9. *If $M|N$ is left or right d2, then*

$$A \otimes_R A \cong \mathrm{Hom}_{N-N}(M \otimes_N M, M) \tag{50}$$

via $\alpha \otimes \beta \longmapsto (m \otimes m' \mapsto \alpha(m)\beta(m'))$.

Proof. The inverse mapping is given by

$$\mathrm{Hom}_{N-N}(M \otimes_N M, M) \to A \otimes_R A, \quad f \longmapsto \sum_i f(- \otimes b_i^1) b_i^2 \otimes_R \beta_i, \tag{51}$$

since

$$\sum_i \alpha(-)\beta(b_i^1) b_i^2 \otimes \beta_i = \sum_i \alpha \otimes \beta(b_i^1) b_i^2 \beta_i(-) = \alpha \otimes \beta$$

and $\sum_i f(m \otimes b_i^1) b_i^2 \beta_i(m') = f(m \otimes m')$ for each $m, m' \in M$. We can carry out a similar proof with a right d2 quasibasis. □

Next is a main theorem for depth two extensions. We make use of the "step one" centralizer R, and "step two centralizers" A and B defined above; in addition, a "step three" centralizer $C := \mathrm{End}\,_N(M \otimes_N M)_M$ (cf. [12], [19]).

Theorem 3.10. *If $M|N$ is left d2, then C and R are Morita equivalent rings with invertible bimodules $_CB_R$ and $_RA_C$ in a Morita context. In particular, B_R and $_RA$ are f.g. projective generators with the following isomorphisms:*

1. $\mu_R : B \otimes_R A \xrightarrow{\cong} C$ *via* $b \otimes \alpha \mapsto (m \otimes m' \mapsto b\alpha(m)m')$.
2. $\psi : B_R \xrightarrow{\cong} \mathrm{Hom}\,(\,_RA,\,_RR)_R$ *via* $b \mapsto (\alpha \mapsto \alpha(b^1)b^2)$.
3. $\tau : B \otimes_R M \xrightarrow{\cong} M \otimes_N M$ *defined by* $\tau(b \otimes m) = bm$.
4. $\iota : M \otimes_N M \xrightarrow{\cong} \mathrm{Hom}\,(\,_RA,\,_RM)$ *via* $\iota(m \otimes m')(\alpha) = \alpha(m)m'$.
5. $C \cong \mathrm{End}\, B_R$ *via* $c \mapsto (b \mapsto c(b))$.
6. $C \cong \mathrm{End}\,_RA$ *via*

$$c \longmapsto (\alpha \mapsto \mu(\alpha \otimes \mathrm{id}_M)c\iota_1) \tag{52}$$

where $\iota_1 : M \to M \otimes M$ *by* $m \mapsto m \otimes 1$.

Proof. First we note that $R \cong \mathrm{End}\,_{N-M}(M)$ via $r \mapsto \lambda(r)$ with inverse $f \mapsto f(1)$. Next we note that $\mathrm{Hom}\,_{N-M}(M, M \otimes_N M) \cong B$ via $f \mapsto f(1)$ with inverse $b \longmapsto (m \mapsto bm)$. The bimodule structure on $_CB_R$ is given by

$$c \cdot b \cdot r = c(b^1 \otimes b^2 r).$$

We next note that $A \cong \mathrm{Hom}\,_{N-M}(M \otimes_N M, M)$ via $\alpha \mapsto (m \otimes m' \mapsto \alpha(m)m')$ with inverse $f \mapsto f \circ \iota_1$. The bimodule structure on $_RA_C$ is given by

$$r \cdot \alpha \cdot c = \lambda(r)\mu_N(\alpha \otimes \mathrm{id}_M)c\iota_1\,.$$

The rest follows strictly from the Lemma and Proposition in the introduction; however, we note some useful inverses to some of the isomorphisms above.

$$\tau^{-1}(m \otimes m') = \sum_i b_i \otimes \beta_i(m)m' \tag{53}$$

$$\iota^{-1}(f) = \sum_i b_i^1 \otimes b_i^2 f(\beta_i) \tag{54}$$

$$\psi^{-1}(\phi) = \sum_i b_i \phi(\beta_i) \tag{55}$$

Dual bases for $_RA$ are given by $\{\psi(b_i)\}$, $\{\beta_i\}$. □

By yet another application of Lemma 1.1 we prove in a similar way (but writing arguments to the left of a function) that if $M|N$ is *right* d2, then the natural module $_RB$ and A_R, where $\alpha \cdot r = \rho(r) \circ \alpha$, are progenerators with corresponding isomorphisms, such as $M \otimes_R B \cong M \otimes M$ via $m \otimes b \mapsto mb$,

$$B \cong \mathrm{Hom}\,(A_R, R_R), \quad b \longmapsto (\alpha \mapsto b^1\alpha(b^2)) \tag{56}$$

and

$$C \cong \mathrm{End}\, A_R. \tag{57}$$

From Prop. 3.8 and the theorem we easily establish

Corollary 3.11. *If $M|N$ is d2, then*

$$ {}_N\mathcal{E}'_M \oplus * \cong \oplus^s {}_N M_M, \tag{58}$$

$$ {}_M\mathcal{E}_N \oplus * \cong \oplus^t {}_M M_N. \tag{59}$$

We obtain a type of converse to the theorem by noting that if μ_R is epi, then $M|N$ is left d2. Equivalently, ${}_RA$ f.g. projective, ψ an isomorphism with $C \cong \operatorname{End}{}_RA$ implies $M|N$ is left d2. This shows that a classical depth two pair of semisimple algebras is left d2 and similarly right d2 [12].

4. The Left Bialgebroid A and its Action

We now turn to our main topic, the construction of a bialgebroid associated to any given d2 ring extension $M|N$. Its underlying ring is the step 2 centralizer $A := \operatorname{End}{}_N M_N$ over $N \subset M$. For the next theorem, we recall that a right R-module V is *balanced* if the natural left $\mathcal{E} := \operatorname{End} V_R$-module on V has left endomorphism ring naturally anti-isomorphic to R: $R \stackrel{\cong}{\to} \operatorname{End}_{\mathcal{E}} V$.

Theorem 4.1. *Let $N \subset M$ be a depth two extension of rings. Then A is a left bialgebroid over R with left action of A on M. If M_N is moreover balanced, then the subring of invariants under this action is N.*

More explicitly, the bialgebroid is $\langle A, R, s_A, t_A, \Delta_A, \varepsilon_A \rangle$ where

$$ A = \operatorname{End}{}_N M_N \tag{60}$$

$$ R = C_M(N) \tag{61}$$

$$ s_A(r) = \lambda(r)\colon m \mapsto rm \tag{62}$$

$$ t_A(r) = \rho(r)\colon m \mapsto mr \tag{63}$$

$$ r \cdot \alpha \cdot r' = \lambda(r)\rho(r')\alpha\colon m \mapsto r\alpha(m)r' \tag{64}$$

$$ \Delta_A(\alpha) = \sum_i \gamma_i \otimes_R c_i^1 \alpha(c_i^2 -) \tag{65}$$

$$ \varepsilon_A(\alpha) = \alpha(1_M) \tag{66}$$

The A-module action on M is simply the action of endomorphisms, $\alpha \triangleright m = \alpha(m)$.

Proof. At first we check the left bialgebroid axioms.

Δ_A is an R-R-bimodule map:

$$\begin{aligned}
\Delta_A(r \cdot \alpha \cdot r') &= \gamma_j \otimes_R c_j^1 r \alpha(c_j^2 -) r' \\
&= \gamma_j \otimes_R c_j^1 r \alpha(c_j^2 b_i^1) b_i^2 \beta_i(-) r' \\
&= \gamma_j(-) c_j^1 r \alpha(c_j^2 b_i^1) b_i^2 \otimes_R \beta_i(-) r' \\
&= r \cdot \big(\alpha(-b_i^1) b_i^2 \otimes_R \beta_i(-)\big) \cdot r'
\end{aligned}$$

Putting $r = r' = 1$ yields an alternative formula for the coproduct,

$$ \Delta_A(\alpha) = \alpha(-b_i^1) b_i^2 \otimes_R \beta_i \tag{67}$$

which, when plugged back, gives

$$ \Delta_A(r \cdot \alpha \cdot r') = r \cdot \Delta_A(\alpha) \cdot r' $$

Coassociativity:

$$\begin{aligned}
(\Delta_A \otimes \mathrm{id}_A) \circ \Delta_A(\alpha) &= \gamma_j \otimes_R c_j^1 \alpha(c_j^2 - b_i^1) b_i^2 \otimes_R \beta_i \\
(\mathrm{id}_A \otimes \Delta_A) \circ \Delta_A(\alpha) &= \gamma_j \otimes_R c_j^1 \alpha(c_j^2 - b_i^1) b_i^2 \otimes_R \beta_i
\end{aligned}$$

The property $\Delta_A(\alpha)(\rho(r)\otimes 1) = \Delta_A(\alpha)(1\otimes\lambda(r))$:

$$\begin{aligned} LHS &= \alpha(-rb_i^1)b_i^2 \otimes_R \beta_i = \\ &= \alpha(-b_j^1)b_j^2\beta_j(rb_i^1)b_i^2 \otimes_R \beta_i \\ &= \alpha(-b_i^1)b_i^2 \otimes_R \beta_i(r-) \\ &= RHS \end{aligned}$$

Multiplicativity of Δ_A:

$$\begin{aligned} \Delta_A(\alpha)\Delta_A(\alpha') &= \alpha(\alpha'(-b_j^1)b_j^2b_i^1)b_i^2 \otimes_R \beta_i(\beta_j(-)) \\ &= \gamma_k \otimes_R c_k^1\alpha(\alpha'(c_k^2b_j^1)b_j^2b_i^1)b_i^2 \cdot \beta_i(\beta_j(-)) \\ &= \gamma_k \otimes_R c_k^1\alpha(\alpha'(c_k^2-)) \\ &= \Delta_A(\alpha\circ\alpha') \end{aligned}$$

Unitalness: $\Delta_A(1) = 1\otimes_R 1$ and $\varepsilon_A(1) = 1_R$ are obvious.
The compatibility of ε_A with multiplication:

$$\varepsilon_A(\alpha\circ\lambda(\varepsilon_A(\alpha'))) = \alpha(\alpha'(1_M)) = \varepsilon_A(\alpha\alpha')$$

and the same for ρ instead of λ.

This completes the proof that A is a bialgebroid.

Module algebra properties: A acts on M by the simple formula $\alpha\triangleright m := \alpha(m)$. The induced R-R-bimodule structure on M is also the obvious one arising from R being a subring of M.

$$\alpha(mm') = \alpha_{(1)}(m)\alpha_{(2)}(m') \tag{68}$$

therefore multiplication $\mu_M\colon M\otimes_R M\to M$ is a left A-module map.

$$\alpha\triangleright 1_M = \varepsilon_A(\alpha)1_M \tag{69}$$

therefore the unit ${}_RR_R \to {}_RM_R$ is a left A-module map, too. This means precisely that M together with its ring structure, written as maps in ${}_R\mathcal{M}_R$, is a monoid in ${}_A\mathcal{M}$, i.e., M is a left A-module algebroid.

The invariants are determined as follows. First of all $N\subset M^A$ is obvious. On the other hand if $m\in M^A$, then $\beta_i(m) = \varepsilon_A(\beta_i)m = \beta_i(1)m$ for each i, so for every $\psi\in\mathcal{E} := \mathrm{End}\, M_N$,

$$\psi(m) = \psi(b_i^1)b_i^2\beta_i(m) = \psi(1)m,$$

whence also $\psi\circ\lambda(m')(m) = \psi(m'm) = \psi(m')m$ for each $m'\in M$. Thus, $\rho(m)\in \mathrm{End}_{\mathcal{E}} M = \rho(N)$, and $m\in N$. Only here in the last step have we used that M_N is balanced. □

We set down some equivalent formulae for the invariant subring, the proof of which are left to the reader.

$$M^A := \{n\in M \mid \alpha\triangleright n = \varepsilon_A(\alpha)n,\ \forall\alpha\in A\} \tag{70}$$

$$= \{n\in M \mid \alpha\circ\rho(n) = \rho(n)\circ\alpha,\ \forall\alpha\in A\} \tag{71}$$

$$= \{n\in M \mid \alpha\circ\lambda(n) = \lambda(n)\circ\alpha,\ \forall\alpha\in A\} \tag{72}$$

Example 4.2. If $M|N$ is an algebra extension with $N = k1$ trivial and M finite dimensional, we recover Lu's bialgebroid $A = \mathrm{End}_k M$ [23, 3.4] since $R = M$. In case R is not semisimple, A is a bialgebroid over R which is not a weak bialgebra [9]. This provides a wealth of examples of action by bialgebroids.

Remark 4.3. For each $\alpha \in A$, its coproduct $\Delta_A(\alpha)$ may be considered a map in $\mathrm{Hom}_{N-N}(M \otimes_N M, M)$ via Prop. 3.9. We compute the simple form it takes:

$$\Delta_A(\alpha)(m \otimes m') = \sum_i \gamma_i(m) c_i^1 \alpha(c_i^2 m') = \alpha(mm'). \tag{73}$$

Example 4.4. That M_N should be balanced in the theorem is a necessary condition, for consider M to be the algebra of 2-by-2 matrices over a field k with N the upper triangular matrices. It is left as an exercise to show that R is trivial ($k1_M$), $M|N$ is H-separable (therefore d2) since $1 \otimes 1 \in (M \otimes_N M)^M$, and that

$$\mathcal{E} := \mathrm{End}\, M_N \cong M (\cong \mathrm{End}\,_N M \cong M \otimes_N M).$$

Consequently $A \cong R$ is trivial, so $M^A = M \neq N$. But M_N is not balanced, since $_{\mathcal{E}}M = {}_M M$ and $N \ncong \mathrm{End}\,_M M = \rho(M)$.

We next note that the endomorphism ring of a left d2 extension M/N is isomorphic to a smash product of M with the bialgebroid A.

Corollary 4.5. *$\mathrm{End}\, M_N$ is isomorphic to a smash product ring $M \rtimes A$ via $m \rtimes \alpha \mapsto \lambda(m)\alpha$.*

Proof. Define $\pi : M \rtimes A \to \mathrm{End}\, M_N$ by the mapping just given. By Proposition 3.8, π is a linear isomorphism of $M \otimes_R A$ with $\mathrm{End}\, M_N$. We compute using Eq. (67) for $x \in M$:

$$\begin{aligned}
\pi((m \rtimes \alpha)(m' \rtimes \beta))(x) &= \pi(m\alpha_{(1)}(m') \rtimes \alpha_{(2)}\beta)(x) \\
&= m\alpha_{(1)}(m')\alpha_{(2)}(\beta(x)) \\
&= m\alpha(m'\beta(x)) \\
&= \pi(m \rtimes \alpha) \circ \pi(m' \rtimes \beta)(x)
\end{aligned}$$

Hence, π is a ring isomorphism. □

Remark 4.6. Taking into account the finite projectiveness of A over the step 1 centralizer R established in the previous section, we propose that a d2 extension M is an B-Galois extension of N provided M is a balanced N-module. The justification for this terminology requires further investigation in a future paper.

5. The Right Bialgebroid B

Let $B = (M \otimes_N M)^N$ the elements of which are denoted $b = b^1 \otimes b^2$ suppressing a possible summation. B is a ring with multiplication $bb' = b'^1 b^1 \otimes b^2 b'^2$ and unit $\mathbb{1} = 1 \otimes 1$. This multiplication does not extend to $M \otimes_N M$ but $M \otimes_N M$ is a left B-module via

$$b \cdot (m \otimes m') = mb^1 \otimes b^2 m'. \tag{74}$$

The so defined ring homomorphism $B \to \mathrm{End}\,_{M-M}(M \otimes_N M)$ is in fact an isomorphism. The inverse is provided by $f \mapsto f(1 \otimes 1)$.

Let R be the centralizer of N in M, $R = C_M(N)$. Define the ring homomorphisms

$$s_B: \quad R \to B, \qquad s_B(r) = 1 \otimes r\,, \tag{75}$$

$$t_B: \quad R^{op} \to B, \qquad t_B(r) = r \otimes 1\,. \tag{76}$$

Since we are going to make B ino a right bialgebroid over R we define its R-R-bimodule via the actions

$$r \cdot b \cdot r' \;=\; bt_B(r)s_B(r') \;=\; rb^1 \otimes b^2 r' \,. \tag{77}$$

Lemma 5.1. *Let $N \subset M$ be a left d2 extension of rings. Then the tensor product bimodule $B \otimes_R B$ is isomorphic, as an R-R-bimodule, to $(M \otimes_N M \otimes_N M)^N$ where the bimodule structure of the latter is defined by $r \cdot (m \otimes m' \otimes m'') \cdot r' = rm \otimes m' \otimes m'' r'$. An isomorphism is given by*

$$\imath \colon B \otimes_R B \to (M \otimes_N M \otimes_N M)^N \,, \quad b \otimes b' \mapsto b^1 \otimes b^2 b'^1 \otimes b'^2 \,. \tag{78}$$

Proof. That $\imath$ is a bimodule map is clear. To show that it is an isomorphism we write down its inverse using the left d2 quasibasis $\{b_i, \beta_i\}$ of Lemma 3.6.

$$\imath^{-1}(t) = \sum_i b_i \otimes_R (\beta_i(t^1)t^2 \otimes_N t^3) \,, \quad t \in (M \otimes_N M \otimes_N M)^N \quad \square$$

Now the right bialgebroid structure on the ring and bimodule B is defined by the following coproduct and counit

$$\Delta_B(b) \;=\; \sum_i (b_i^1 \otimes_N b_i^2) \;\otimes_R\; (\beta_i(b^1) \otimes_N b^2) \tag{79}$$

$$\varepsilon_B(b) \;=\; b^1 b^2 \tag{80}$$

By the lemma, $\Delta_B(b) = \imath^{-1}(b^1 \otimes 1 \otimes b^2)$.

Theorem 5.2. *Let $N \subset M$ be a d2 extension of rings. Then $\langle B, R, s_B, t_B, \Delta_B, \varepsilon_B \rangle$ is a right bialgebroid and* End ${}_N M$ *is a right B-module algebroid w.r.t. the action $\xi \triangleleft b := b^1 \xi(b^2 -)$. The subring of invariants is $\rho(M)$, the right multiplications with elements of M.*

Proof. At first we check the bialgebroid axioms:

Coassociativity: Apply $\imath_3 \colon B \otimes_R B \otimes_R B \to (M \otimes_N M \otimes_N M \otimes_N M)^N$, $b \otimes b' \otimes b'' \mapsto b^1 \otimes b^2 b'^1 \otimes b'^2 b''^1 \otimes b''^2$ to both hand sides of $(\Delta_B \otimes \mathrm{id}_B) \circ \Delta_B = (\mathrm{id}_B \otimes \Delta_B) \circ \Delta_B$ and check that the result on a $b \in B$ is $b^1 \otimes 1 \otimes 1 \otimes b^2$ in both cases. (The inverse $\imath_3^{-1}$ sends $t \in (M \otimes_N M \otimes_N M \otimes_N M)^N$ into

$$\sum_{i,j} b_i \otimes_R (\beta_i(t^1)t^2 \otimes_N t^3 \gamma_j(t^4)) \otimes_R c_j \in B \otimes_R B \otimes_R B.)$$

Counit properties: Obvious

$\Delta_B(B) \subset B \times_R B$

$$\begin{aligned}(s_B(r) \otimes 1)\Delta_B(b) &= (b_i^1 \otimes r b_i^2) \otimes_R (\beta_i(b^1) \otimes b^2) = \\ &= \imath^{-1}\left(b_i^1 \otimes r b_i^2 \beta_i(b^1) \otimes b^2\right) = \imath^{-1}(b^1 \otimes r \otimes b^2)\end{aligned}$$

and similarly

$$(1 \otimes t_B(r))\Delta_B(b) \;= \imath^{-1}(b^1 \otimes r \otimes b^2)$$

Δ_B is multiplicative:

$$\begin{aligned}\Delta_B(b)\Delta_B(b') &= (b_j^1 b_i^1 \otimes b_i^2 b_j^2) \otimes_R (\beta_j(b'^1)\beta_i(b^1) \otimes b^2 b'^2) \\ &= \imath^{-1}\left(b_j^1 b_i^1 \otimes b_i^2 b_j^2 \beta_j(b'^1)\beta_i(b^1) \otimes b^2 b'^2\right) \\ &= \imath^{-1}\left(b'^1 b_i^1 \otimes b_i^2 \beta_i(b^1) \otimes b^2 b'^2\right) = \imath^{-1}(b'^1 b^1 \otimes 1 \otimes b^2 b'^2) \\ &= \Delta_B(bb')\end{aligned}$$

$\Delta_B(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$, $\varepsilon_B(\mathbb{1}) = 1$ Obvious
$\varepsilon_B(t_B(\varepsilon_B(b))b') = \varepsilon_B(bb') = \varepsilon_B(s_B(\varepsilon_B(b))b')$

$$\varepsilon_B(t_B(\varepsilon_B(b))b') = \varepsilon_B((b^1b^2 \otimes 1)(b'^1 \otimes b'^2)) = b'^1b^1b^2b'^2 = \varepsilon_B(bb')$$

and the same for t_B replaced by s_B.

This finishes the proof that B is a bialgebroid.

Module algebroid properties:

$$(\xi \circ \xi') \triangleleft b = b_i^1\xi(b_i^2\beta_i(b^1)\xi'(b^2-)) = (\xi \triangleleft b_{(1)}) \circ (\xi' \triangleleft b_{(2)})$$

The induced bimodule structure on End $_NM$ is $r \cdot \xi \cdot r' = \lambda(r) \circ \xi \circ \lambda(r')$

$$\mathrm{id}_M \triangleleft b = \lambda(\varepsilon_B(b))$$

The invariants:

$$(\mathrm{End}\ _NM)^B := \{\xi \mid \xi \triangleleft b = \lambda(\varepsilon_B(b)) \circ \xi\}$$

Clearly ξ is an invariant iff

$$b^1 \otimes \xi(b^2m) = b_i^1 \otimes b_i^2\beta_i(b^1)\xi(b^2m) = b_i^1 \otimes b_i^2\beta_i(b^1)b^2\xi(m) = b^1 \otimes b^2\xi(m)$$

for all $m \in M$, $b \in B$. Thus

$$1 \otimes \xi(m) = \gamma_j(m)c_j^1 \otimes \xi(c_j^2) = \gamma_j(m)c_j^1 \otimes c_j^2\xi(1) = 1 \otimes m\xi(1)$$

Applying multiplication $\xi(m) = m\xi(1)$ follows. Thus an invariant $\xi = \rho(\xi(1))$ and belongs to $\rho(M)$. The opposite inclusion is trivial. This proves

$$(\mathrm{End}\ _NM)^B \;=\; \rho(M) \quad \square$$

□

Recalling the theory of the dual of a left bialgebroid in Section 2.6, we have:

Corollary 5.3. *B is isomorphic as bialgebroids over R to the right bialgebroid dual of A via the isomorphism in Eq. 56. Similarly, B is isomorphic to the left bialgebroid dual $^*A$ via ψ in Theorem 3.10.*

Proof. We prove the first statement and leave the second as an exercise. Recall the nondegenerate pairing $\langle b, a\rangle = b^1a(b^2)$. Let A^* denote the right bialgebroid dual of A with $\eta : B \to A^*$ the linear isomorphism given by $\eta(b) = \langle b, -\rangle$. We note that η is an R-R-bimodule homomorphism, since

$$\langle r \cdot b \cdot r', a\rangle = rb^1a(b^2r') = r\langle b, at(r')\rangle.$$

η is a ring homomorphism since

$$\langle bb', a\rangle = b'^1b^1a(b^2b'^2)$$

while

$$\langle b', \langle b, a_{(1)}\rangle \cdot a_{(2)}\rangle = b'^1b^1a_{(1)}(b^2)a_{(2)}(b'^2) = b'^1b^1\alpha(b^2b'^2).$$

η is a homomorphism of corings since

$$\begin{aligned}\langle b, aa'\rangle &= b^1aa'(b^2) \\ &= \sum_i b_i^1a(b_i^2\beta_i(b^1)a'(b^2)) \\ &= \sum_i \langle b_i\beta_i(b^1)a'(b^2), a\rangle = \langle b_{(1)} \cdot \langle b_{(2)}, a'\rangle, a\rangle. \quad \square\end{aligned}$$

Remark 5.4. There is also a right action of B on $\mathrm{End}\, M_N$ given by $\xi \triangleleft b := \xi(-b^1)b^2$. It however satisfies

$$(\xi \circ \xi') \triangleleft b = (\xi \triangleleft b_{(2)}) \circ (\xi' \triangleleft b_{(1)})$$

Its invariants are also the right multiplications with elements of M.

Remark 5.5. The coring $\langle B, R, \Delta_B, \varepsilon \rangle$ is a restriction of the Sweedler coring [35]

$$\langle M \otimes_N M,\, M,\, \Delta : M \otimes_N M \to M \otimes_N M \otimes_N M,\, \varepsilon : M \otimes_N M \to M \rangle.$$

If $N = k1$ for a ground field k with M finite dimensional, we recover Lu's bialgebroid $B = M^{\mathrm{op}} \otimes_k M$ [23, 3.1] up to a twist S. In this case, B is a Hopf algebroid, with antipode S.

6. The Frobenius Case

Recall that a ring extension $M|N$ is *Frobenius* if there is (a Frobenius homomorphism) $E \in \mathrm{Hom}_{N-N}(M, N)$ and (dual bases) $x_i, y_i \in M$ such that $\sum_i \lambda(x_i)E\lambda(y_i) = \mathrm{id}_M = \sum_i \rho(y_i)E\rho(x_i)$. Throughout this section and part of the next, we assume $M|N$ is Frobenius with this data. We recall several facts about $M|N$.

Proposition 6.1. *We have* $\mathrm{End}\, M_N \cong M \otimes_N M$*, which is a Frobenius extension over* $\lambda(M) \cong M$*, with Frobenius homomorphism* $E_M = \mu$ *and dual bases* $\{x_i \otimes 1\}$, $\{1 \otimes y_i\}$. *Moreover,* $\mathrm{End}\,{}_N M$ *and* $\mathrm{End}\, M_N$ *are anti-isomorphic.*

Proof. The isomorphism $\mathcal{F} : \mathrm{End}\, M_N \to M \otimes_N M$ is given by $f \mapsto \sum_i f(x_i) \otimes y_i$ with inverse $m \otimes m' \mapsto \lambda(m)E\lambda(m')$. A multiplication on $M \otimes_N M$ is induced from composition of endomorphisms, the E-multiplication given by $(m \otimes m')(m'' \otimes m''') = mE(m'm'') \otimes m'''$ and unity $1_1 = \sum_i x_i \otimes y_i$. An anti-isomorphism is then given by

$$\phi : \; \mathrm{End}\,{}_N M \stackrel{\cong}{\to} M \otimes_N M, \quad f \mapsto \sum_i x_i \otimes f(y_i) \tag{81}$$

with inverse $m \otimes m' \mapsto \rho(m')E\rho(m)$. The rest of the proof is somewhat standard [17]. □

We set $e_1 = 1 \otimes 1$ and $M_1 := Me_1M = M \otimes_N M$. Note that $M_1 \cong \mathrm{End}\, M_N$ via the M-M map induced by $e_1 \mapsto E$ and M identified with $\lambda(M)$. Note too the key identities

$$e_1 m e_1 = e_1 E(m) = E(m) e_1, \quad E_M(m e_1 m') = mm'. \tag{82}$$

In this notation $\{x_i e_1\}$, $\{e_1 y_i\}$ are dual bases for $E_M \in \mathrm{Hom}_{M-M}(M_1, M)$.

We note then that $A = \mathrm{End}\,{}_N M_N \cong (M \otimes_N M)^N$ via $\alpha \mapsto \sum_i \alpha(x_i) \otimes y_i$.

If we iterate this (basic) construction, we construct $M_2 = M_1 e_2 M_1$ with $e_2 m_1 e_2 = e_2 E_M(m_1) = E_M(m_1) e_2$ and $E_{M_1}(m_1^1 e_2 m_1^2) = m_1^1 m_1^2$ for each $m_1 \in M_1$. Note that $M_2 = M_1 \otimes_M M_1 \cong M \otimes_N M \otimes_N M$. We arrive at a generalized Jones tower,

$$N \to M \hookrightarrow M_1 \hookrightarrow M_2 \hookrightarrow \cdots$$

with *Temperley-Lieb generators* $e_i \in M_i$ such that

$$e_i e_{i+1} e_i = e_i 1_{M_{i+1}}, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \quad e_i e_j = e_j e_i \tag{83}$$

if $|i - j| > 1$. Note that the e_i are not the Jones projections even if they exist, $e_i^2 \neq e_i$. For example if $M|N$ is not a split extension then there is no unit preserving

Frobenius homomorphism E. However, the Temperley-Lieb generators exist for any Frobenius extension as shown above.

We also have the Pimsner-Popa relations:

$$m_i e_i = E_{M_{i-1}}(m_i e_i) e_i, \quad e_i m_i = e_i E_{M_{i-1}}(e_i m_i) \tag{84}$$

for $m_i \in M_i$, $i = 1, 2$ and $M_0 := M$.

Introduce the notation $\hat{A} := M_1^N$ for the centralizer $C_{M_1}(N)$ in $N \to M \hookrightarrow M_1$. We introduce the canonical isomorphism ψ_A of $A = \mathrm{End}\,_N M_N$ with $\hat{A}$ given by the restriction of $\mathcal{F}$ above to A:

$$\alpha \longmapsto \sum_i \alpha(x_i) e_1 y_i,$$

with inverse $a^1 e_1 a^2 \mapsto \lambda(a^1) \circ E \circ \lambda(a^2)$. Similarly, let $\hat{C} := M_2^N$, which is isomorphic as rings to the step three centralizer C introduced in Section 3.

We first show that classical depth two extensions are depth two in the sense of this paper. It is known that a semisimple pair $N \subset M$ over a field, where M has faithful trace T that restricts to a faithful trace on N, is a (split, separable) Frobenius extension; cf. [12, Prop. 2.6.2]. Also subfactors of finite index are Frobenius extensions by the Pimsner-Popa orthonormal basis result [12, 17]; these have semisimple centralizers. Of course, a module over a semisimple ring is always projective.

Proposition 6.2. *Suppose $M|N$ is Frobenius extension, $\hat{A}_R$ and $_R\hat{A}$ are f.g. projective, and $\hat{C} = \hat{A} e_2 \hat{A}$. Then $M|N$ is a depth two ring extension.*

Proof. We first show that $E_M : M_1 \to M$ has dual bases in $\hat{A}$. By the classical d2 hypothesis on $\hat{C}$, $1_{M_2} = \sum_k a_k e_2 b_k$ for some $a_k, b_k \in \hat{A}$. Let $m_1 \in M_1$, then:

$$e_2 m_1 = \sum_k e_2 m_1 a_k e_2 b_k = \sum_k e_2 E_M(m_1 a_k) b_k.$$

By applying E_{M_1} we arrive at $m_1 = \sum_k E_M(m_1 a_k) b_k$; similarly,
$m_1 = \sum_k a_k E_M(b_k m_1)$, so $\{a_k\}$, $\{b_k\}$ are indeed dual bases for E_M.

It follows that $M_1 \cong M \otimes_R \hat{A}$ as M-N-bimodules via $m_1 \mapsto \sum_k E_M(m_1 a_k) \otimes b_k$ with inverse mapping given simply by $m \otimes a \mapsto ma$: note that $E_M(\hat{A}) \subseteq R$. Since $M \otimes_N M \cong M_1$ and $_R\hat{A}$ is f.g. projective, it follows that $M|N$ is right d2. Similarly, $M_1 \cong \hat{A} \otimes_R M$ as N-M-bimodules and $M|N$ is left d2. □

Proposition 6.3. *If $M|N$ is a left or right d2 Frobenius extension, then E_M : $M_1 \to M$ has dual bases in $\hat{A}$.*

Proof. Let $b_i \in B, \beta_i \in A$ be a left d2 quasibasis, then $\{b_i\}$, $\{\sum_j \beta_i(x_j) e_1 y_j\}$ are dual bases, obviously in M_1^N, for E_M. As a matter of fact

$$\begin{aligned} \sum_{i,j} E_M(m e_1 m' b_i^1 e_1 b_i^2) \beta_i(x_j) e_1 y_j &= \sum_{i,j} m E(m' b_i^1) b_i^2 \beta_i(x_j) e_1 y_j \\ &= m E(m' x_j) e_1 y_j = m e_1 m' \end{aligned}$$

for $m, m' \in M$, and

$$\sum_{i,j} b_i^1 e_1 b_i^2 E_M(\beta_i(x_j) e_1 y_j m e_1 m') = \sum_i b_i^1 e_1 b_i^2 \beta_i(m) m' = m e_1 m'.$$

The proof starting with a right d2 quasibasis is similar. □

Corollary 6.4. *We have* $M_1 \cong M \otimes_R \hat{A}$ *via* $m \otimes a \mapsto ma$ *for each* $m \in M, a \in \hat{A}$.

Proof. An inverse is given by $m_1 \mapsto \sum_{i,j} E_M(m_1 b_i) \otimes_R \beta_i(x_j) e_1 y_j \in M \otimes \hat{A}$ by Proposition 3.8. □

Similarly we show $M_1 \cong \hat{A} \otimes_R M$ via $a \otimes m \mapsto am$. If R is a field coincident with centralizers of M and N as in [19], it follows from the proposition that M_1 is f.g. free as a left or right natural M-module.

Corollary 6.5. *M_1 is isomorphic to a smash product algebra:* $M_1 \cong M \rtimes A$.

Proof. Define $\Pi : M \rtimes A \to M_1$ by $\Pi(m \rtimes \alpha) = \sum_i m\alpha(x_i) e_1 y_i$ for all $\alpha \in A, m \in M$. We see then that Π is a composition of two algebra isomorphisms, π in Corollary 4.5 and $\mathcal{F}$ above (cf. Proposition 6.1). □

From Section 3 we recall the step 3 centralizer $C = \mathrm{End}_{N-M}(M \otimes_N M)$.

Corollary 6.6. *If $M|N$ is d2 Frobenius, then the ring extensions $R \hookrightarrow A$, $r \mapsto \lambda(r)$ and $C|A$ (given by Eq. 52) are Frobenius extensions.*

Proof. If $M|N$ is Frobenius, we arrive at $A|R$ Frobenius from the proposition by restriction of E_M to A (identified with $\{\alpha(x_i)e_1 y_i | \, \alpha \in A\}$) and noting that $E_M(A) \subseteq R$. Since $C \cong \mathrm{End}_R A$, we conclude from the (left) endomorphism ring theorem for Frobenius extensions [17] that $C|A$ via right regular representation is Frobenius. □

Conversely, $A|R$ Frobenius implies $\mathcal{E}|M$ is Frobenius by Prop. 3.8. If M_N is a progenerator, then a endomorphism ring theorem-and-converse assures us that $M|N$ is also Frobenius (cf. [17]). By the same token, $A|R$ is Frobenius iff $C|A$ since ${}_RA$ is a progenerator (Theorem 3.10).

The next result is an endomorphism ring theorem for Frobenius d2 extensions, and answers a question posed at the end of [19].

Theorem 6.7. *If $M|N$ is a left d2 extension, then $M_1|M$ is a right d2 extension. Similarly, if $M|N$ is right d2, then $M_1|M$ is left d2.*

Proof. We note the bimodule ${}_{M_1}M_N$ given by

$$(me_1m') \cdot m'' \cdot n = mE(m'm'')n, \tag{85}$$

which is of course isomorphic to the natural bimodule ${}_{\mathcal{E}}M_N$ where $\mathcal{E} = \mathrm{End}\, M_N$. Now tensor from the left the first isomorphism in Def. 3.1 by this bimodule:

$${}_{M_1}M \otimes_N M \otimes_N M_M \oplus * \cong \oplus^n {}_{M_1}M \otimes_N M_M$$

which is isomorphic to

$${}_{M_1}M_1 \otimes_M M_{1M} \oplus * \cong \oplus^n {}_{M_1}M_{1M},$$

the condition for $M_1|M$ to be right d2. The second statement is proven similarly. □

Let $\hat{B} := M_2^M$, and note the canonical algebra isomorphism $\psi_B : B \cong \hat{B}$ given by

$$\psi_B(b) = \sum_i x_i b^1 e_1 b^2 e_2 e_1 y_i$$

with inverse

$$b^1 e_2 b^2 \mapsto b^1 E_M(b^2 e_1)$$

($b^1, b^2 \in M_1$) obtained by following the ring isomorphisms

$$M_2^M \cong \mathrm{Hom}_{M-M}(M_1, M_1) \cong (M \otimes_N M)^N \tag{86}$$

via first the Frobenius map $\Psi : x \otimes y \mapsto \lambda(x) E_M \lambda(y)$, for each $x, y \in M_1$, composed with the general map $\Phi : f \mapsto f(1_M \otimes 1_M)$ for each $f \in \mathrm{End}_M(M_1)_M$. The E_M-multiplication on M_2^M is therefore identifiable with composition as well as the multiplication on B from Section 5.

Now the endomorphism ring theorem for d2 Frobenius extensions may be used to show, in a similar way to the earlier propositions in this section, that E_{M_1} has dual bases in $\hat{B}$, $B|R^{\mathrm{op}}$ is a Frobenius extension and M_2 is a smash product of M_1 and $\hat{B}$.

7. The Biseparable Case

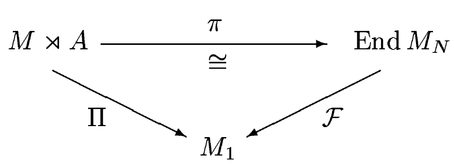
Suppose $M|N$ is a Frobenius d2 R-algebra extension where the centralizer R is *trivial*, i.e., coincides with the centers of M and N. In this case, A and B are bialgebras which are finitely generated projective over R by Cor. 6.6 and its analog for B. In this section, under the additional constraints that R coincides with a ground field and $M|N$ is a biseparable algebra extension, we show that A and B are dual semisimple Hopf algebras with Galois actions on M and M_1^{op}, respectively.

The next proposition shows that A acts on M via the same formula as the Hopf algebra action of $\mathrm{End}_{M-M} M_1$ on M_1 in [19, Eq. (21)]. The proof does not make use of the triviality assumption on R.

Proposition 7.1. *If $M|N$ is Frobenius, then the action of A on M defined in Section 4 is the Ocneanu-Szymański action,*

$$a \triangleright m = E_M(ame_1).$$

The algebra homomorphism $\pi : M \rtimes A \to \mathrm{End}\, M_N$ given by $m \rtimes a \mapsto \lambda(m)(a \triangleright -)$ is an isomorphism which fits into a commutative triangle with the isomorphisms given in Cor. 6.5 and Prop. 6.1.



Proof. We let $a \in (M \otimes_N M)^N$ be the image of $\alpha \in A$ under the isomorphism above. Then:

$$E_M(ame_1) = E_M(\alpha(x_i)e_1 y_i m e_1) = E_M(\alpha(x_i)E(y_i m)e_1) = \alpha(m).$$

The commutativity of the diagram is immediate from the definitions. □

The bimodule action on A induced by $R \subset M \hookrightarrow M_1$ is given by the somewhat different formula $r \cdot \alpha \cdot r' = \lambda(r)\alpha\lambda(r')$ (cf. Eq. 63). However, the two bimodule structures coincide when R is trivial.

We introduce two canonical anti-isomorphisms of $\phi_A : A \to \hat{A} = M_1^N$ and $\phi_B : B \to \hat{B} = M_2^M$ given by

$$\phi_A(\alpha) = \sum_i x_i e_1 \alpha(y_i), \quad \phi_B(b) = \sum_i x_i e_1 e_2 b^1 e_1 b^2 y_i.$$

The Ocneanu-Szymański action $\triangleright'$ of $\hat{B}$ on M_1 in [19, Eq. (21)] is related to our action of B on $\mathcal{E}' = \mathrm{End}\ {}_N M$ via the anti-isomorphism ϕ in Eq. 81 as we see next. Again, we do not need triviality of R.

Proposition 7.2. *If $M|N$ is a depth two Frobenius extension, then for every $b \in B$, $f \in \mathcal{E}'$:*

$$\phi_B(b) \triangleright' \phi(f) := E_{M_1}(\phi_B(b)\phi(f)e_2) = \phi(f \triangleleft b).$$

Proof.

$$\begin{aligned} E_{M_1}(\phi_B(b)\phi(f)e_2) &= \sum_{i,j} E_{M_1}(x_i e_1 e_2 b^1 e_1 b^2 y_i x_j e_1 f(y_j) e_2) \\ &= \sum_i E_{M_1}(x_i e_1 E_M(b^1 e_1 E(b^2 y_i x_j) f(y_j)) e_2) \\ &= \sum_i x_i e_1 b^1 f(b^2 y_i) = \phi(f \triangleleft b). \quad \square \end{aligned}$$

A ring extension $M|N$ is said to be *biseparable* if M_N and ${}_N M$ are f.g. projective while $M|N$ is a separable extension (i.e., $\mu : M \otimes_N M \to M$ is split M-M-epimorphism) and a split extension (i.e., there is N-bimodule V such that $M \cong N \oplus V$ as N-bimodules). If R is trivial, a biseparable Frobenius extension of R-algebras coincides with the notion of strongly separable extension [17, 19].

Theorem 7.3. *If R is a field and $M|N$ is a biseparable Frobenius R-algebra extension of depth two, then A and B are dual semisimple Hopf algebras isomorphic to $\hat{A}$ and $\hat{B}$, respectively.*

Proof. Since R is trivial, A and B are dual bialgebroids over R, it follows easily that A and B are dual bialgebras. We next note that the nondegenerate pairing $\langle b, a\rangle = b^1 a(b^2)$ is equal to the nondegenerate pairing

$$\langle \phi_A(a), \psi_B(b)\rangle' := E_M E_{M_1}(\psi_B(b) e_1 e_2 \phi_A(a))$$

analyzed in [19, 4.4], since:

$$\begin{aligned} E_M E_{M_1}(\psi_B(b) e_1 e_2 \phi_A(a)) &= \sum_{i,j} E_M E_{M_1}(x_i b^1 e_1 b^2 e_2 e_1 y_i e_1 e_2 x_j e_1 a(y_j)) \\ &= \sum_{i,j} E_M E_{M_1}(x_i b^1 e_1 b^2 e_2 E_M(e_1 E(y_i)) x_j e_1 a(y_j))) \\ &= \sum_{i,j} E_M(x_i b^1 e_1 b^2 E(y_i) x_j e_1 a(y_j)) \\ &= \sum_j b^1 E(b^2 x_j) a(y_j) = \langle b, a\rangle. \end{aligned}$$

In [19, Section 4], it is shown that for $a' \in \hat{A}, b' \in \hat{B}$

$$\langle a', b'\rangle' = E_M E_{M_1}(a' e_2 e_1 S(b')) := \langle a', S(b')\rangle''$$

for antipode $S : \hat{B} \to \hat{B}$. Now let $\psi_B(b) = b'$ and $\alpha, \alpha' \in A$. We compute that ψ_B is a coalgebra homomorphism:

$$\begin{aligned}
\langle \phi_A(\alpha'), \psi_B(b_{(1)})\rangle' \langle \phi_A(\alpha), \psi_B(b_{(2)})\rangle' &= \langle b_{(1)}, \alpha'\rangle \langle b_{(2)}, \alpha\rangle \\
&= \langle b, \alpha'\alpha\rangle \\
&= \langle \phi_A(\alpha)\phi_A(\alpha'), S(b')\rangle'' \\
&= \langle \phi_A(\alpha), S(b'_{(2)})\rangle'' \langle \phi_A(\alpha'), S(b'_{(1)})\rangle'' \\
&= \langle \phi_A(\alpha'), b'_{(1)}\rangle' \langle \phi_A(\alpha), b'_{(2)}\rangle',
\end{aligned}$$

since S is coalgebra anti-isomorphism and by definition of $\Delta(b')$ in [19]. Finally, ψ_B preserves the counit:

$$\begin{aligned}
\varepsilon_{\hat{B}}(\psi_B(b)) &= \sum_i E_{M_1}(e_2 x_i b^1 e_1 b^2 e_2 e_1 y_i) \\
&= \sum_i E_{M_1}(e_2 E_M(x_i b^1 e_1 b^2) e_1 y_i) \\
&= \sum_i x_i b^1 b^2 e_1 y_i = \varepsilon_B(b) 1_{M_1}
\end{aligned}$$

by triviality of R and [19, 3.13, 4.3]. It follows that ψ_B is a bialgebra isomorphism, whence B has an antipode. Semisimplicity of A and B follow from [19]. A is then also a Hopf algebra since it is the dual of B. □

Moreover, the antipode is involutive, $S^2 = \mathrm{id}$, by a powerful theorem of Etingof and Gelaki [8].

7.1. **D2 biseparable extensions are QF.** In this subsection, we no longer assume $M|N$ is Frobenius: in fact, we will be interested in *when* d2 biseparable extensions are Frobenius. A *depth one* ring extension $M|N$ is a centrally projective ring extension defined in Example 3.4: compare [19, 3.1]. It follows from Example 3.4 that a depth one extension is automatically d2.

The following theorem answers [6, Problem 3.8] for depth two extensions. A ring extension $M|N$ is left QF (quasi-Frobenius) if M_N and ${}_N M$ are f.g. projective and ${}_N M^*_M \oplus * \cong \oplus^n {}_N M_M$ [25]. Similarly there is a notion of right QF extension with two-sided QF extensions being denoted by "QF." Of course, a QF extension is a weakening of the notion of Frobenius extension.

It is already well-known and easily derived that a depth one (bi)separable extension is QF (e.g., see [6]: in fact, it is Frobenius [34]). The same is true of depth two extensions:

Theorem 7.4. *A depth two biseparable extension is QF.*

Proof. Since $M|N$ is separable, it follows easily that $\mathcal{E}|M$ (identified with $\lambda(M)$) is split with bimodule projection given by $f \mapsto \sum_i f(x_i)y_i$ where $\sum_i x_i \otimes y_i$ is a separability element. Let ${}_M W_M$ be the complementary bimodule satisfying $\mathcal{E} \cong M \oplus W$ as M-bimodules. By Corollary 3.11 for left d2 extensions we note that ${}_M \mathcal{E}_N \oplus * \cong \oplus^n {}_M M_N$. Then also ${}_M W_N \oplus * \cong \oplus^n {}_M M_N$. Since the canonical map $W \otimes_N M \to W$ is a split right M-epimorphism by separability, it follows from ${}_N W \otimes_N M_M \oplus * \cong \oplus^n {}_N M \otimes_N M_M$ and the definition of left d2 extension that

$${}_N W_M \oplus * \cong \oplus^m {}_N M_M.$$

On the other hand, since $M|N$ is split, we have $M \cong N \oplus V$ for N-bimodule V, so $\mathcal{E} \cong \mathrm{Hom}\,(M_N, N_N \oplus V_N)$; whence the isomorphism of N-M-bimodules,

$$M \oplus W \cong \mathrm{Hom}\,(M_N, N_N) \oplus \mathrm{Hom}\,(M_N, V_N).$$

From the two displayed equations, we conclude that $M^* \oplus * \cong \oplus^{m+1} M$ as N-M-bimodules. Hence $M|N$ is left QF. We prove similarly that a right d2 biseparable extension is right QF. □

By comparing with the definition of depth three in [19, 3.1], we propose that a ring extension $M|N$ be *right depth three* if ${}_{\mathcal{E}}M \otimes_N M \otimes_N M_N$ and ${}_{\mathcal{E}}M \otimes_N M_N$ are H-equivalent modules; and *left depth three* if ${}_N M \otimes_N M \otimes_N M_{\mathcal{E}'}$ and ${}_N M \otimes_N M_{\mathcal{E}'}$ are H-equivalent. We recall our notations $\mathcal{E} = \mathrm{End}\, M_N$ and $\mathcal{E}' = \mathrm{End}\,{}_N M$. The following is proved in the same way as Theorem 6.7.

Proposition 7.5. *A depth two extension is depth three.*

We propose the following problem in extension of Theorem 7.4: is a biseparable depth three extension QF or Frobenius? Yet another problem is to determine a reasonable definition of finite depth ring extensions.

8. The Irreducible Case

From [19] we recall (and slightly extend the notion) that a K-algebra extension $N \to M$ is *irreducible* if the centralizer is trivial: $R = K1_M$ for K a commutative ground ring. In this section, we show that a depth two irreducible Frobenius extension M/N has *Hopf algebras* A and B with bialgebra structure defined as before. Because of the results in Sections 6 and 7, this extends by entirely different means the main theorem 1.1 in [19]. To be precise, we obtain the same results without the hypotheses of proper extension, biseparability and K a field; however, the result that the fixed points of M under the action of A be exactly N requires the new condition of M_N being balanced, as shown in Section 4.

Theorem 8.1. *Suppose M/N is a depth two irreducible Frobenius extension. Then $A := \mathrm{End}\,{}_N M_N$ and $B := (M \otimes_N M)^N$ are dual Hopf algebras acting on M and $\mathrm{End}\,{}_N M$ respectively, with $M_1 \cong M \rtimes A$.*

Proof. From Section 4 recall that A is a left bialgebroid over $R = K$; whence a K-bialgebra which by Theorem 3.10 is a progenerator K-module. We also recall that A acts on M with M_1 isomorphic as rings to $M \rtimes A$. From Proposition 2.4 and Corollary 5.3, B is the bialgebra dual of A, since

$$\langle bb', a\rangle = \langle b, a_{(1)}\rangle\langle b', a_{(2)}\rangle, \quad \langle b, aa'\rangle = \langle b_{(1)}, a\rangle\langle b_{(2)}, a'\rangle.$$

It suffices then to show that A has an antipode.

Let $E : M \to N$ be a Frobenius homomorphism with dual bases $\{x_i\}$, $\{y_i\}$, and b_i, β_i be a left d2 quasibasis for M/N. We now claim that $\psi : A \to K$ defined by $\psi(\alpha) = \sum_j \alpha(x_j)y_j$ is a Frobenius homomorphism satisfying

$$a_{(1)}\psi(a_{(2)}) = \psi(a)1_A \tag{87}$$

for every $a \in A$. ψ is shown to be a Frobenius homomorphism by either noting that it corresponds to E_M restricted to $\hat{A}$ via the isomorphism $A \cong \hat{A}$ given in

Corollary 6.6, or computing that $\{b_i^1 E(b_i^2 -)\}$, $\{\beta_i\}$, are dual bases for ψ. Now we compute:

$$\begin{aligned} a_{(1)}\psi(a_{(2)}) &= a(-b_i^1)b_i^2\beta_i(x_j)y_j \\ &= \alpha(-x_j)y_j \\ &= \alpha(x_j)y_j \mathrm{id}_M = \psi(a)1_A, \end{aligned}$$

since $\sum_i x_i \otimes y_i \in (M \otimes_N M)^M$, so $a(mx_j)y_j = \psi(a)m$ for $m \in M$.

We note next that E is left norm for the augmented Frobenius algebra (A, ψ, ε), since for each $a \in A$:

$$\psi(aE) = \sum_j a(E(x_j))y_j = a(1)\sum_j E(x_j)y_j = \varepsilon(a).$$

Now it follows from Eq. (67) and a standard lemma (due to Pareigis) that

$$S : A \to A, \quad S(a) = E_{(1)}\psi(aE_{(2)}) = \sum_{i,j} E(-b_i^1)b_i^2 a\beta_i(x_j)y_j \tag{88}$$

is an antipode for A, since $\mathcal{A} := \mathrm{Hom}_K(A, A)$ is f.g. projective algebra with respect to the convolution product $*$ induced from A, clearly $1_A * S = 1_{\mathcal{A}}$, whence $S * 1_A = 1_{\mathcal{A}}$. □

The theorem provides the key to computing formulas for the Hopf algebra structures on $\hat{A}$ and $\hat{B}$ in [19](cf. Section 6) and its extension to the nonbiseparable case with commutative ground ring. For example, we show that the action of $\hat{A}$ on M in [19], expressed by a conjugation formula (Eq. 25), is indeed given by the Ocneanu-Szymański action. Let $\hat{S}$ be the antipode induced from S on $\hat{A}$ by the Hopf algebra isomorphism $\psi_A(\alpha) = \sum_j \alpha(x_j)e_1 y_j$ (cf. Theorem 7.3).

Proposition 8.2. *For $a \in \hat{A}$ and $m \in M$, we have*

$$E_M(ame_1) = a_{(1)}m\hat{S}(a_{(2)}) \tag{89}$$

Proof. Let $\psi_A(\alpha) = a$ for $\alpha \in A$. By Proposition 7.1, it will suffice to compute that $a_{(1)}m\hat{S}(a_{(2)}) = \alpha(m)1_{M_1}$. We compute using Eq. 88:

$$\begin{aligned} a_{(1)}m\hat{S}(a_{(2)}) &= \sum_{i,j,k,r,s} \alpha(x_j b_i^1)b_i^2 e_1 y_j m E(x_k b_r^1)b_r^2\beta_i\beta_r(x_s)y_s e_1 y_k \\ &= \sum_{i,j,k,r,s} \alpha(x_j b_i^1)b_i^2 e_1 E(y_j m E(x_k b_r^1)b_r^2\beta_i\beta_r(x_s)y_s)y_k \\ &= \sum_{k,r,i,s} \alpha(mE(x_k b_r^1)b_r^2 b_i^1)b_i^2\beta_i\beta_r(x_s)y_s e_1 y_k \\ &= \sum_{k,r,s} \alpha(mE(x_k b_r^1)b_r^2\beta_r(x_s))y_s e_1 y_k \\ &= \sum_{k,s} \alpha(mE(x_k x_s))y_s e_1 y_k \\ &= \alpha(m)\sum_k x_k e_1 y_k = \alpha(m)1_{M_1}, \end{aligned}$$

since $\sum_s \beta_i\beta_r(x_s)y_s \in R = K$. □

We next note a criterion for when an irreducible d2 Frobenius extension M/N is split (i.e., $N \oplus * \cong M$ as N-bimodules).

Proposition 8.3. *M/N is split $\Leftrightarrow$ A is K-separable.*

Proof. Since M/N is Frobenius with Frobenius homomorphism $E : M \to N$, it is split iff there is $d \in R$ such that $E(d) = 1$. Since R is trivial, $d \in K$, so $E(1)d = 1$ and $\varepsilon(E) = E(1)$ is invertible. Since E is a left norm in A, it is a left integral, or by direct computation for $m \in M$ and $\alpha \in A$:

$$\alpha E(m) = \alpha(1)E(m) = \varepsilon(\alpha)E(m).$$

But then A is K-separable iff $\varepsilon(E)$ is invertible (e.g., [21, 5.2]). □

Similarly M/N is separable iff there is $d \in R$ such that $\sum_i x_i d y_i = 1$. Then $E_M(1_{M_1}) = \sum_i x_i y_i$ is invertible in K. Now the multiplication in B yields

$$1_{M_1} b = b^1 x_j e_1 y_j b^2 = 1_{M_1} b^1 b^2 = 1_{M_1} \varepsilon(b),$$

whence 1_{M_1} is a right integral for the Hopf algebra B. We then similarly complete the proof of the next proposition:

Proposition 8.4. *M/N is separable $\Leftrightarrow$ B is K-separable.*

Next we conclude from Proposition 7.1 and [40, 1.1] (cf. [19, Fig. 2]) that M/N is a B-Galois extension of K-algebras, a generalization of [19, Theorem 6.5]. We assume that M_N is balanced as used in Section 4.

Corollary 8.5. *An irreducible Frobenius (ring) extension of depth two is a Hopf-Galois extension.*

If K is a field of characteristic zero, it follows from the Larson-Radford theorem [22] that A is a semisimple Hopf algebra over K iff its dual B is semisimple. From the propositions above we deduce:

Corollary 8.6. *Suppose K is a field of characteristic zero and M/N is an irreducible d2 Frobenius K-algebra extension. Then M is a split extension of N if and only if M is a separable extension of N.*

References

[1] O. Babelon, Universal exchange algebra for Bloch waves and Liouville theory, *Comm. Math. Phys.* **139** (1991), 619–643.

[2] G. Böhm, K. Szlachányi, A coassociative C^*-quantum group with nonintegral dimensions, *Lett. Math. Phys.* **35** (1996), 437–456

[3] G. Böhm, F. Nill and K. Szlachányi, Weak Hopf algebras, I. Integral theory and C^*-structure, *J. Algebra* **221** (1999), 385-438.

[4] G. Böhm, K. Szlachányi, Weak Hopf Algebras, II. Representation theory, dimensions, and the Markov trace, *J. Algebra* **233** (2000) 156–212

[5] T. Brzezinski and G. Militaru, Bialgebroids, $\times_A$-bialgeras and duality, preprint QA/0012164.

[6] S. Caenepeel and L. Kadison, Are biseparable extensions Frobenius? *J. K-Theory*, to appear, RA/0104019.

[7] S. Doplicher, J.E. Roberts, A new duality theory for compact groups, *Invent. Math.* **98** (1989), 157–218

[8] P. Etingof and S. Gelaki, On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, *Int. Math. Res. Not.* **16** (1998), 851–864.

[9] P. Etingof and D. Nikshych, Dynamical quantum groups at roots of 1, *Duke Math. J.*, to appear, QA/0003221.

[10] P. Etingof and A. Varchenko, Exchange dynamical quantum groups, *Comm. Math. Phys.* **205** (1999), 19–52.
[11] D. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Univ. Press, New York, 1998.
[12] F. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, M.S.R.I. Publ. **14**, Springer, Heidelberg, 1989.
[13] R. Haag, *Local Quantum Physics, Fields, Particles, Algebras* (2nd edition) Springer-Verlag, Berlin-Heidelberg, 1996
[14] K. Hirata, Some types of separable extensions of rings, *Nagoya Math. J.* **33** (1968), 107–115.
[15] V.F.R. Jones, Index for subfactors, *Inventiones Math.* **72** (1983), 1–25.
[16] V.F.R. Jones and V.S. Sunder, *Introduction to Subfactors*, LMS **234**, Cambridge Univ. Press, 1997.
[17] L. Kadison, *New examples of Frobenius extensions*, University Lecture Series **14**, Amer. Math. Soc., Providence, 1999.
[18] L. Kadison and D. Nikshych, Outer actions of Hopf algebra centralizers on separable extensions, *Comm. Alg.* **29**, no. 9 (2001).
[19] L. Kadison and D. Nikshych, Hopf algebra actions on strongly separable extensions of depth two, *Adv. Math.*, to appear, `QA/0107064`.
[20] L. Kadison and D. Nikshych, Frobenius extensions and weak Hopf algebras, *J. Algebra*, to appear, `RA/0102010`.
[21] L. Kadison and A. Stolin, Separability and Hopf algebras, in: Proc. Conf. Rings & Modules (Athens, Ohio, March 1999), eds. Huynh, Jain and Lopez-Permouth, Contemporary Math., vol. 259, A.M.S., Providence, 2000; 279–298.
[22] R.G. Larson and D.E. Radford, Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, *J. Algebra* **117** (1988), 267–289.
[23] J.-H. Lu, Hopf algebroids and quantum groupoids, *Int. J. Math.* **7** (1996), 47–70.
[24] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conf. Series in Math. Vol. 82, AMS, Providence, 1993.
[25] B. Müller, Quasi-Frobenius Erweiterungen I, *Math. Zeit.* **85** (1964), 345–368.
[26] D. Nikshych and L. Vainerman, A characterization of depth 2 subfactors of II_1 factors, *J. Func. Analysis* **171** (2000), 278–307.
[27] D. Nikshych and L. Vainerman, A Galois correspondence for actions of quantum groupoids on II_1-factors, *J. Func. Analysis*, **178** (2000), 113-142.
[28] D. Nikshych and L. Vainerman, Finite dimensional quantum groupoids and their applications, in: Proceedings of "Hopf Algebras" workshop, MSRI Publications, 2000; `QA/0006057`.
[29] F. Nill, Axioms for weak bialgebras, preprint, 1998, `QA/9805104`.
[30] S. Popa, *Classification of Subfactors and their Endomorphisms*, CBMS **86**, Amer. Math. Soc., Providence, 1995.
[31] D. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, Pure and Appl. Math. Series, Academic Press, 1986.
[32] A.C. da Silva and A. Weinstein, Geometric Models for Noncommutative Algebras, Berkeley Math. Lect. Notes, vol. 10, AMS, Providence, 1999.
[33] P. Schauenburg, Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras), *Contemp. Math.* to appear
[34] K. Sugano, Separable extensions and Frobenius extensions, *Osaka J. Math.* **7** (1970), 291–299.
[35] M.E. Sweedler, The predual theorem to the Jacobson-Bourbaki theorem, *Trans. A.M.S.* **213** (1975), 391–406.
[36] M.E. Sweedler, Groups of simple algebras, *Publ. Math. I.H.E.S.* **44** (1974), 79–189.

[37] K. Szlachányi, Finite quantum groupoids and inclusions of finite type, *Fields Inst. Comm. Ser.*, to appear, `math.QA/0011036`.

[38] W. Szymański, Finite index subfactors and Hopf algebra crossed products, *Proc. Amer. Math. Soc.* **120** (1994), no. 2, 519–528.

[39] M. Takeuchi, Groups of algebras over $A \otimes \overline{A}$, *J. Math. Soc. Japan* **29** (1977), 459–492.

[40] K.H. Ulbrich, Galois erweiterungen von nicht-kommutativen ringen, *Comm. Alg.* **10** (1982), 655–672.

[41] A. Weinstein, Symplectic groupoids and Poisson manifold, *Bull. Amer. Math. Soc.* **16** (1987), 101–104.

[42] A. Weinstein, Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan* **40** (4) (1988), 705–727.

[43] P. Xu, Quantum groupoids and deformation quantization, *C. R. Acad. Sci. Paris*, I, **326** (1998), 289-294.

[44] P. Xu, Quantum groupoids, preprint, 1999, `QA/9905192`.

Göteborg University, S-412 96 Göteborg, Sweden
E-mail address: `kadison@math.chalmers.se`

Research Institute for Particle and Nuclear Physics, Budapest, H-1525 Budapest, P. O. Box 49, Hungary
E-mail address: `szlach@rmki.kfki.hu`