INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC CONES AND HERMITIAN SYMMETRIC SPACES

GENKAI ZHANG

ABSTRACT. We give a brief survey on the study of constructions of invariant differential operators on Riemannian symmetric spaces and of combinatorial and analytical properties of their eigenvalues, and pose some open questions.

1. INTRODUCTION

Let $D = G/K$ be a Riemannian symmetric space of rank r. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The algebra $\mathcal{D}^G(D)$ of G-invariant differential operators on $C^\infty(D)$ is a commutative algebra with r generators. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and let $\phi_{\underline{\lambda}}$ be the spherical function on D for $\underline{\lambda} \in \mathfrak{a}^*$. Then $\phi_{\underline{\lambda}}$ is an eigenfunction of all $L \in \mathcal{D}^G(D)$ and the eigenvalue $\widetilde{L}(\underline{\lambda})$ of L on $\phi_{\underline{\lambda}}$ is a W-invariant polynomials on $\mathfrak{a}^*$, where W is the Weyl group for the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Moreover the map $L \mapsto \widetilde{L}(\underline{\lambda})$ is an algebra isomorphism of $\mathcal{D}^G(D)$ onto algebra of W-invariant polynomials on $\mathfrak{a}^*$. It is therefore an interesting question to construct explicitly a system of generators and a linear basis of the algebra $\mathcal{D}^G(D)$ and find their eigenvalues. In particular, it is more interesting to construct geometrically those operators, since the invariance of such operators is naturally manifested. In certain cases it turns out that the kernels of these operators are closely related to some problems in representation theory and that their eigenvalues form some interesting symmetric functions and are closely related to the hypergeometric orthogonal polynomials of several variables. The purpose of the present article is to give a short survey of some known results. Basically all the results in this paper have been obtained earlier, or they are easy consequences of such, and we put these results in some (hopefully) interesting perspectives. The results as well as the references presented here are by no means exhaustive. We will mainly focus on constructions of invariant differential operators and their kernels as unitary representations of Lie groups, combinatorial and vanishing properties of their eigenvalues, and relation to orthogonal polynomials. Here is a brief overview and motivation of this survey.

Among Riemannian symmetric spaces the symmetric cones were studied in greater details due to their importance in many other areas in mathematics, notably wave equations, complex analysis, symmetric polynomials. We begin with a short account of the constructions of invariant differential operators on symmetric cones. So let X be an Euclidean

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Jordan algebra and Ω be the symmetric cone of positive elements. Let $V = X + iX$ be its complexification and $\mathcal{S} = X + i\Omega$ the tube domain over Ω (Siegel domain of type I in V). As Riemannian symmetric space, $\Omega = GL(\Omega)/O(\Omega)$ and $\mathcal{S} = G/K$. The Cayley transform maps $\mathcal{S}$ to a bounded symmetric domain D. The simplest case of a symmetric cone is the positive half line $\Omega = \mathbb{R}^+$ which is a group itself, the corresponding Siegel domain is the upper half space $\mathcal{S} = SL(2,\mathbb{R})/SO(2)$ and D is the unit disk. A linear basis of the invariant differential operators on $\mathbb{R}^+$ is $\{x^n(\frac{\partial}{\partial x})^n, n = 0, 1, \dots\}$. Each operator is associated to a symbol $x^n u^n$ defined on $\mathbb{R} \times \mathbb{R}$, and can be considered as the restriction of the $SO(2)$-invariant function $z^n \bar{w}^n$ on the $\mathbb{C}\times\mathbb{C}$; the later functions form a basis of the $SO(2)$-invariant polynomials on $\mathbb{C}\times\mathbb{C}$, holomorphic in the first variable and anti-holomorphic in the second variable. This construction can be generalized to any symmetric cone as follows. Consider the space $\mathcal{P}(V)$ of holomorphic polynomials on the complex space V. The compact group K acts, though the Cayley transform, linearly on V, and the corresponding action on $\mathcal{P}(V)$ is decomposed into irreducible representations $\mathcal{P}_{\underline{\mathbf{m}}}(V)$ with multiplicity one, of signatures $\underline{\mathbf{m}} = (m_1, m_2, \dots, m_r)$ where $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$ are integers. Let $K_{\underline{\mathbf{m}}}(z, w)$ be the reproducing kernel of $\mathcal{P}_{\underline{\mathbf{m}}}(V)$ with the Fock space norm. Then the operator $K_{\underline{\mathbf{m}}}(z, \partial_z)$ defines naturally K-invariant differential operator on $\mathcal{P}(V)$. Those polynomial differential operators can be naturally viewed as differential operators on Ω and are in fact invariant differential operators there. To understand the eigenvalues of these operators it is then sufficient to find their eigenvalues on each $\mathcal{P}_{\underline{\mathbf{n}}}(V)$. When $\underline{\mathbf{m}} = 1^r = (1, \dots, 1)$ the operator $K_{\underline{\mathbf{m}}}(z, \partial_z)$ is, up to a constant the Cayley type differential operator $\Delta(z)\Delta(\partial_z)$ where $\Delta(z)$ is the determinant function on the Jordan algebra X and its complexification V. The calculation of the eigenvalues of the Cayley type operators is then an easy consequence of the Gindikin Gamma function formula, which calculates the Laplace transform of the conical functions (see next section) on Ω. We can use this formula to determine the eigenvalues of the operators $K_{\underline{\mathbf{m}}}(z, \partial_z)$ when $\underline{\mathbf{m}}$ are the other smaller fundamental spherical signatures $\underline{\mathbf{m}} = 1^j$. This is done by Wallach [31] and Yan [32]. The combinatorial properties of the eigenvalue as a polynomial in $\underline{\mathbf{n}}$, indexed by $\underline{\mathbf{m}}$, have been taken up by Knop and Sahi, [21] and [12]. For general $\underline{\mathbf{m}}$ one can deduce some vanishing properties of the eigenvalues for $K_{\underline{\mathbf{m}}}(z, \partial_z)$ on $\mathcal{P}_{\underline{\mathbf{n}}}$. Some of those results will be presented in Section 3.

In [15] Nomura gives a construction of another system of generators on symmetric cone, generalizing earlier construction of Selberg in the case of the cone of positive symmetric matrices. The eigenvalues of those operators can be expressed in terms of the Cayley-Capelli type operators; however since the eigenvalues of the later are only explicitly known for certain signatures $\underline{\mathbf{m}}$, the result is less complete.

Now consider the symmetric Siegel domain $\mathcal{S} = X + i\Omega$. The Cayley differential operator $\Delta(\partial)$, and their higher powers $\Delta(\partial)^m$, considered as holomorphic differential operators on $\mathcal{S}$, have some remarkable intertwining properties: they intertwine two projective actions of the Mobius group G; in the case of the upper half plane this is the so called Bol's lemma.

The kernel of the operator $\Delta(\partial)$ on holomorphic polynomials forms a singular unitarizable representation of $(\mathfrak{g}, K)$; see Corollary 4.3. By conjugating with the power of the Bergman reproducing kernel one may get some more intertwining differential operators on $\mathcal{S}$. However, to understand those operators we need some geometric construction.

The domain D (or $\mathcal{S}$) with the Bergman metric is a Kähler manifold. On any Kähler manifold Englis and Peetre [5] have defined the invariant Cauchy-Riemann operator $\bar{D}$ by purely geometric data. Also, Shimura ([24], [26]) has earlier defined those operators on hermitian symmetric spaces. The power $\bar{D}^m$ of $\bar{D}$ maps a scalar valued function to a symmetric tensor valued function; by taking the projection onto certain one-dimensional subspaces we get essentially the Cayley operator $\Delta(\bar{\partial})$; see Proposition 4.4.

We may then form the higher order Laplacians by using the power of $\bar{D}$. It is proved by the author [38] these operators are the sum of the invariant differential operators defined by Shimura both in terms of the invariant Cauchy-Riemann operator and the universal enveloping algebras. Shimura [27] has found a system of generators for line bundles on Siegel domains. The eigenvalues of those operators have been found by the author [38]. These results are surveyed in Section 4. They are applied further in studying the tensor products of some minimal holomorphic representations [39]. Also in this case the eigenvalues have some interesting vanishing properties with respect to certain singular spherical unitary representations.

There is another natural way of constructing invariant differential operators on a bounded symmetric domain D by, roughly speaking, transporting the K-invariant differential operators $K_{\underline{\mathbf{m}}}(\bar{\partial}, \partial)$ at $z = 0$ to any point by a group transformation. See Definition 4.9. The eigenvalues of the these invariant differential operators are closely related to Berezin transform and some hypergeometric orthogonal polynomials. The precise statement is in Section 4.

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2. The symmetric cones and Hermitian symmetric spaces

2.1. **Symmetric cones.** In this section we recall some facts about the symmetric cones in a Jordan algebra and hermitian symmetric domains. To simplify the presentation we consider only tube domains. Most results in this section can be found, e.g. in [7], [6], [13], [30] and references therein, so that we will be very brief.

Let X be a d-dimensional simple, formally real Jordan algebra of rank r with product $x \circ y$ and unit element e.

$$\Omega = \{x^2, x \in X \text{ invertible}\}$$

is a convex cone. Let $GL(\Omega)$ be the identity component of the group

$$\{g \in GL(X) : g(\Omega) = \Omega\}.$$

The Lie algebra $\mathfrak{gl}(\Omega)$ has a Cartan decomposition $\mathfrak{gl}(\Omega) = \mathfrak{o}(\Omega) + \mathfrak{q}$, where $\mathfrak{o}(\Omega)$ is the Lie algebra of the "orthogonal" group

$$O(\Omega) = \{g \in GL(\Omega) : ge = e\}$$

and $\mathfrak{q} = \{M_x : x \in X\}$ consists of all multiplication operators

$$M_x y = x \circ y$$

on X. Moreover $\Omega = G(\Omega)/O(\Omega)$ is a Riemannian symmetric space.

Let $e_1, \dots, e_r$ be a maximal orthogonal system of idempotents in X, so that $e = e_1 + \dots + e_r$. Let

$$\mathfrak{a} = \mathbb{R}M_{e_1} + \dots + \mathbb{R}M_{e_r}$$

be the subspace of $\mathfrak{q}$ generated by $M_{e_1}, \dots, M_{e_r}$. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{q}$. We have thus a root space decomposition of $(\mathfrak{g}, \mathfrak{a})$. We let $\{\gamma_j\}_{j=1}^r$ be the dual basis in $\mathfrak{a}^*$ of $\{M_{e_j}\}_{j=1}^r$, i.e.

$$\gamma_j(M_{e_k}) = \delta_{jk}.$$

The corresponding root system consists of $\frac{\gamma_j - \gamma_k}{2}$, $j \neq k$ with common multiplicity a.

We fix an ordering of $\mathfrak{a}^*$ by letting

$$\gamma_r > \gamma_{r-1} > \dots > \gamma_1.$$

Let $\underline{\delta}$ be the half sum of positive roots

$$\underline{\delta} = \frac{a}{2}\sum_{j>k}\frac{\gamma_j - \gamma_k}{2} = \sum_{j=1}^r \delta_j\gamma_j = \frac{a}{4}\sum_{j=1}^r(2j - (r+1))\gamma_j,$$

with

$$\delta_j = \frac{a}{4}(2j - (r+1)) = \frac{a}{2}(j-1) - \frac{a}{4}(r-1))$$

We will now describe the spherical functions on Ω. Let

$$X = \sum_{1 \le i \le j \le r} X_{ij} \tag{2.1}$$

be the Peirce decomposition of X induced by $\{e_, \dots, e_r\}$. Denote the determinant polynomial on X by Δ. We let

$$\Delta_j(x) = \Delta_{X_j}(P_j x),$$

where Δ_{X_j} is the corresponding determinant in the Jordan algebra

$$X_j = \sum_{1 \le k \le l \le j} X_{kl}$$

with identity $e_1 + \dots + e_j$, and P_j the orthogonal projection of X onto X_j.

For any $\underline{\mathbf{s}} = (s_1, \dots, s_r) \in \mathbb{C}^r$ we define the conical function by

$$\Delta_{\underline{s}}(x) = \prod_{j=1}^{r} \Delta_j^{s_j - s_{j+1}}(x), \tag{2.2}$$

with $s_{r+1} = 0$. For $\lambda = \sum_{j=1}^{r} \lambda_j \gamma_j \in \mathfrak{a}^*_{\mathbb{C}}$, the complexification of $\mathfrak{a}^*$, we let $\underline{s} = (s_1, s_2, \dots, s_r)$ be defined by

$$s_j = \lambda_j + \delta_j;$$

the spherical functions $\phi^{\Omega}_{\underline{\lambda}}$ is then

$$\phi^{\Omega}_{\underline{\lambda}}(x) = \int_{O(\Omega)} \Delta_{\underline{s}}(kx)dk = \int_{O(\Omega)} \prod_{j=1}^{r} \Delta_j^{s_j - s_{j+1}}(kx)dk.$$

If

$$\boxed{\underline{\mathbf{m}} = \underline{\lambda} + \underline{\delta}}, \tag{2.3}$$

with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$ being integers, we define spherical polynomials $\Omega_{\underline{\mathbf{m}}}$ by

$$\Omega_{\underline{\mathbf{m}}}(x) = \phi^{\Omega}_{\underline{\lambda}}(x) = \int_{O(\Omega)} \Delta_{\underline{\mathbf{m}}}(kx)dk.$$

We recall the Gindikin formula for the Laplace transform of the conical function on the symmetric cone Ω:

$$\Delta_{\underline{\mathbf{s}}}(z) = \frac{1}{\Gamma_{\Omega}(-\underline{\mathbf{s}}^*)} \int_{\Omega} e^{-\langle z, \eta \rangle} \Delta^*_{-\underline{\mathbf{s}}^*}(\eta) \Delta(\eta)^{-\frac{d}{r}} d\eta \tag{2.4}$$

where $\underline{\mathbf{s}}^* = (s_r, s_{r-1}, \dots, s_1)$, $\Delta^*_{-\underline{\mathbf{s}}^*}(\eta)$ is the conical function defined with the opposite ordering $(e_r, e_{r-1}, \dots, e_1)$ of the idempotents, and

$$\Gamma_{\Omega}(\underline{\mathbf{t}}) = \frac{1}{\sqrt{2\pi}^{d-r}} \prod_{j=1}^{r} \Gamma(t_j - \frac{a}{2}(j-1)),$$

is the Gindikin Gamma function.

2.2. **Siegel domains and bounded symmetric domains.** Consider the complexification $V = X + iX$ of the Jordan algebra X. The domain

$$\mathcal{S} = \mathcal{S}(\Omega) := \{w \in V = X + iX; \Im(w) \in \Omega\} = X + i\Omega$$

is a symmetric Siegel tube domain. Let c be the Cayley transform

$$c(w) = \frac{w - ie}{w + ie}.$$

c maps $\mathcal{S}$ onto a bounded symmetric domain D in V. As a hermitian symmetric space $D = G/K$ where K is acting on V by linear transformations. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G. The Cayley transform c maps $\mathfrak{a}$ into a maximal abelian subspace, to be also denoted by $\mathfrak{a}$, of $\mathfrak{p}$. (See e.g. [7], [13] for the precise formulation). Let β_j the Cayley transform of γ_j. The root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is of type C. Let W be the Weyl group. We order the roots according to

$$\beta_r > \cdots > \beta_1 > 0.$$

Let $\underline{\rho}$ be the half sum of the positive roots. It is

$$\underline{\rho} = \sum_{j=1}^{r} \rho_j \beta_j = \sum_{j=1}^{r} \frac{1 + a(j-1)}{2} \beta_j. \tag{2.5}$$

2.3. **The Schmid decomposition.** The space $\mathcal{P}(V)$ is decomposed under K into irreducible subspaces of multiplicity one as

$$\mathcal{P}(V) = \sum_{\underline{\mathbf{m}}} \mathcal{P}_{\underline{\mathbf{m}}}(V), \tag{2.6}$$

where each $\mathcal{P}_{\underline{\mathbf{m}}}(V)$ is of highest weight $-(m_1\gamma_1 + \cdots + m_r\gamma_r)$ with $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$ being integers, and the summation is over all $\underline{\mathbf{m}}$. The group $O(\Omega)$ is a subgroup of K and the polynomial $\Omega_{\underline{\mathbf{m}}}$ is the unique $O(\Omega)$-invariant polynomial in $\mathcal{P}(V)$. See [23] and [6].

2.4. **The Faraut-Koranyi expansion.** We fix a K-invariant hermitian inner product $\langle z, w\rangle$ on V so that a minimal tripotent has norm 1. We consider the Fock space $\mathcal{F} = \mathcal{F}(V)$ on V with respect to the normalized Gaussian measure $\frac{1}{\pi^d} e^{-\|z\|^2} dm(z)$. Its norm can be also calculated, for polynomials f and g on V, by

$$(f, g)_{\mathcal{F}} = f(\partial) g^*(z)\big|_{z=0}$$

where g^* is obtained by g by taking the complex conjugate of its coefficients in its expansion in terms of the monomials. It has reproducing kernel $e^{\langle z, w\rangle}$. Similarly we can define the Bergman space of holomorphic functions on D. Its reproducing kernel is, up to a positive constant, $h(z, w)^{-p}$, where $h(z, w)$ is an irreducible K-invariant polynomial on $V \times V$ so that $h(\sum_{j=1}^r t_j e_j, \sum_{j=1}^r t_j e_j) = \prod_{j=1}^r (1 - t_j^2)$ and $p = \frac{2d}{r}$ is called the genus of D. Let $\mathcal{K}_{\underline{\mathbf{m}}}(z, w)$ be the reproducing kernel of $\mathcal{P}_{\underline{\mathbf{m}}}$ in $\mathcal{F}$. It follows immediately from (2.6) that

$$e^{\langle z, w\rangle} = \sum_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, w).$$

In [6] Faraut and Koranyi has given an expansion of the kernel $h(z, w)^{-\nu}$.

Theorem 2.1. *The following expansion holds*

$$h(z, w)^{-\nu} = \sum_{\underline{\mathbf{m}}} (\nu)_{\underline{\mathbf{m}}} K_{\underline{\mathbf{m}}}(z, w), \tag{2.7}$$

where $(\nu)_{\underline{\mathbf{m}}} = \prod_{j=1}^r (\nu - \frac{a}{2}(j-1))_{m_j} = \prod_{j=1}^r \prod_{k=1}^{m_j} (\nu - \frac{a}{2}(j-1) + k - 1)$ *is the generalized Pochhammer symbol.*

In particular the expansion implies that $h(z,w)^{-\nu}$ is positive definite exactly for all ν in the Wallach set

$$W(D) = \{0, \frac{a}{2}, \dots, \frac{a}{2}(r-1)\} \cup (\frac{a}{2}(r-1), \infty).$$

For each such ν there is a Hilbert space H_ν of holomorphic functions on D with reproducing kernel $h(z,w)^{-\nu}$; H_ν forms a unitary irreducible (projective) representation of G via the action

$$\pi_\nu(g)f(z) = (J_{g^{-1}}(z))^{\frac{\nu}{p}} f(g^{-1}z) \tag{2.8}$$

of $g \in G$, where $J_{g^{-1}}(z)$ is the complex Jacobian. For $\nu > p-1 = \frac{2d}{r} - 1$ it is the weighted Bergman space with the weighted probability measure $c_\nu h(z,z)^{\nu-p} dm(z)$.

The discrete points $\nu_j = \frac{a}{2}(j-1)$, $j = 1, \dots, r$, correspond to some singular unitary representations of G. The spaces are then

$$H_{\frac{a}{2}(j-1)} = \sum_{m_j=0}^{\oplus} \mathcal{P}_{\underline{\mathbf{m}}}. \tag{2.9}$$

The annihilating operator for the space H_{ν_j} is Cayley-Capelli type operators in the next section and its invariance is in the Section 4.2; see Lemma 4.2 and Remark 4.5. (See further [2] for integral formulas for those spaces.)

3. Invariant differential operators on symmetric cones

In this section we present several constructions of invariant differential operators on the symmetric cone Ω and some results on the properties of their eigenvalues. By the formula (2.3) we see that the eigenvalues of any invariant differential operator on Ω are uniquely determined by its eigenvalues on the spherical polynomials, and this remark will be applied to all operators in the following calculations.

3.1. **Cayley type differential operators.** The determinant polynomial $\Delta(x)$ transforms under the group $GL(\Omega)$ by a scalar character, from which one can deduce that the operators $\Delta^{-\alpha}\Delta(\partial)\Delta^{\alpha+1}$ are $GL(\Omega)$-invariant differential operators on Ω. The eigenvalues of those operators can be obtained by using the Gindikin Gamma function formula (2.4). This is the method of Shimura [25], which generalizes the Cayley identity studied earlier by Cayley and Gårding. Moreover by taking r-different non-negative integers α one gets a system of generators of $\mathcal{D}^{GL(\Omega)}(\Omega)$. This is realized by Yan [32].

Theorem 3.1. ([25], [32]) *The differential operators $\Delta(x)^{-\alpha}\Delta(\partial)\Delta(x)^{\alpha+1}$ for r different nonnegative integers α form a system of generators of $\mathcal{D}^{G(\Omega)}(\Omega)$. Their eigenvalue on the spherical polynomial $\Omega_{\underline{\mathbf{m}}}$ is given by*

$$\prod_{j=1}^{r}(\frac{a}{2}(r-j) + 1 + \alpha + m_j) \tag{3.1}$$

3.2. **Cayley-Capelli type differential operators.** Define the differential operator $K_{\underline{\mathbf{m}}}(x, \partial)$ by

$$K_{\underline{\mathbf{m}}}(x, \partial_x)e^{\langle x,y\rangle} = K_{\underline{\mathbf{m}}}(x, y)e^{\langle x,y\rangle} \tag{3.2}$$

Clearly $K_{\underline{\mathbf{m}}}(x, \partial)$ is K-invariant and thus is a diagonal operator under the Schmid decomposition. We denote $k_{\underline{\mathbf{m}}}(\underline{\mathbf{n}})$ its eigenvalues on the component $\mathcal{P}_{\underline{\mathbf{n}}}$. Let $\preceq$ be the natural (partial) ordering (see [14], Chapter I) on all signatures. The following vanishing and interpolation property of $k_{\underline{\mathbf{m}}}(\underline{\mathbf{n}})$ follows easily from the definition and from that fact that $K_{\underline{\mathbf{m}}}K_{\underline{\mathbf{n}}}$ is a linear combination of $K_{\underline{\mathbf{l}}}$ so that $\underline{\mathbf{m}} \preceq \underline{\mathbf{l}}$. See further [21].

Proposition 3.2. *The eigenvalue $k_{\underline{\mathbf{m}}}(\underline{\mathbf{n}})$ of $K_{\underline{\mathbf{m}}}(x, \partial)$ on $\mathcal{P}_{\underline{\mathbf{n}}}$ satisfies*

$$k_{\underline{\mathbf{m}}}(\underline{\mathbf{n}}) = 0, \quad \textit{if} \quad \underline{\mathbf{m}} \not\preceq \underline{\mathbf{n}}$$

and

$$k_{\underline{\mathbf{m}}}(\underline{\mathbf{m}}) = 1.$$

When $\underline{\mathbf{m}}$ are the fundamental signature

$$\underline{\mathbf{m}} = \underline{\mathbf{1}^j} = \sum_{k=1}^{j} \gamma_k$$

one may further find explicit formulas for their eigenvalues $k_{\underline{\mathbf{m}}}$. The following result is proved in [31] and [32], Theorem 2.3. See also [12] for a combinatorial study of the eigenvalues and [9], [10], [11] for related results.

Theorem 3.3. *The differential operators $K_{\underline{\mathbf{m}}}(x, \partial)$ for $\underline{\mathbf{m}} = 1^j$, $j = 1, \dots, r$ form a system of generators of $G(\Omega)$-invariant differential operators. Their eigenvalue on the spherical function $\phi_{\underline{\lambda}}^{\Omega}$ is given by*

$$\Big(\prod_{l=1}^{j}(1 + \frac{a}{2}(j-l))^{-1} \sum_{1\le i_1<\cdots<i_j\le r} \prod_{l=1}^{j}(\lambda_{i_l} + \delta_{i_l} + \frac{a}{2}(j-l)\Big). \tag{3.3}$$

Remark 3.4. In [12] a different vector of $\underline{\delta}$ is chosen so that the formula appears a bit differently.

3.3. **Selberg differential operators.** When Ω is the symmetric cone of positive symmetric matrices Selberg defined earlier a system of operators; they are formally $\mathrm{Tr}(X\partial_X)^j$ with the convention that in calculating the power $(X\partial_X)^j$ the differentiation ∂_X does not operate on X (namely ∂_X and X are viewed as commuting operators); see [29], Chapter IV. In [15] Nomura gives a generalization of the Selberg differential operators for all symmetric cones. Let $P(x)$ be the quadratic operator of the Jordan algebra X. Consider $p_j(x, y) = \mathrm{Tr}(P(x^{\frac{1}{2}})y)^j$ defined on $\Omega \times X$, where Tr is the trace function on the Jordan algebra. It is proved in [15] that $p_j(x, y)$ can be extended to a polynomial function on $X \times X$ such that

$p_j(x,y) = p_j(y,x)$. One may define the operator S_j on Ω with symbol $p_j(x,y)$ using the method of (3.2),

$$S_j e^{\langle x,y\rangle} = p_j(x,y)e^{\langle x,y\rangle}.$$

Observe that the symbol $p_j(x,y)$ of S_j when $y = e$ and when $x = x_1e_1 + \cdots + x_re_r$, is the power sum,

$$p_j(x,e) = x_1^j + \cdots + x_r^j;$$

whereas the symbol $K_{1^j}(x,y)$ of the operators $K_{1^j}(x,\partial)$ is the elementary symmetric function

$$K_{1^j}(x,e) = \frac{1}{\prod_{k=1}^{j}(1+\frac{a}{2}(k-1))} \sum_{1\le i_i<i_2<\cdots<i_j\le r} x_{i_1}x_{i_2}\dots x_{i_j}.$$

Both the power sums and the elementary symmetric functions form systems of generators for the ring of symmetric polynomials. Parallel to this and Theorem 3.3 we have

Proposition 3.5. ([15]) *The operators S_j for $j = 1,\dots r$ form a system of generators for $\mathcal{D}^{GL(\Omega)}(\Omega)$.*

We may find some formula expressing the eigenvalues of the Selberg operators in terms of $k_{\underline{\mathbf{m}}}$. Indeed, let $J_{\underline{\mathbf{m}}}(x)$ be the Jack symmetric polynomials as defined in [14], Chapter VI, the symbol $K_{\underline{\mathbf{m}}}(x,y)$ of the operator $K_{\underline{\mathbf{m}}}(x,\partial)$ at the point $y = e$ is the a constant multiple of the Jack symmetric polynomials $J_{\underline{\mathbf{m}}}(x)$; one has $K_{\underline{\mathbf{m}}}(x,e) = K_{\underline{\mathbf{m}}}(e,x) = (\frac{2}{a})^{|\mathbf{m}|}\frac{J_{\underline{\mathbf{m}}}(x)}{j_{\underline{\mathbf{m}}}}$ in the notation in [22] (see Lemma 5.2 there). By using the formulas (10.31-32) in [14], Chapter VI, one may express the power sum symmetric polynomial $p_j(x)$ in terms of the Jack symmetric polynomials and thus express the Selberg operator S_j in terms of the operators $K_{\underline{\mathbf{m}}}(x,\partial)$. This involves, however, $K_{\underline{\mathbf{m}}}$ for other $\underline{\mathbf{m}}$ than just the fundamental signatures, for which we have no explicit formulas for the eigenvalues $k_{\underline{\mathbf{m}}}$. So we may ask

Question 1. Find explicit formulas for the eigenvalues of S_j, $j = 1,\dots,r$.

4. Invariant Cauchy-Riemann operator and Shimura operators on Hermitian symmetric spaces

4.1. **Invariant Cauchy-Riemann operator.** Let $B(z,z)$ be as in [13] the Bergman operator, so that the Bergman metric at $z \in D$ is given by $B(z,z)^{-1}$. The invariant Cauchy-Riemann operator $\bar{D}$ is defined by

$$\bar{D}f(z) = B(z,z)\bar{\partial}f(z). \tag{4.1}$$

The power $\bar{D}^m$ of $\bar{D}$ maps a scalar-valued function to one with value in the symmetric tensor $S_m(V)$ of V ([19]), which is further decomposed under K into irreducible subspaces $S_{\underline{\mathbf{m}}}(V)$ with signature $\underline{\mathbf{m}}$. Denote the corresponding projection onto $S_{\underline{\mathbf{m}}}(V)$ by $P_{\underline{\mathbf{m}}}$. We form the higher Laplacians

$$\mathcal{L}_m = (\bar{D}^m)^*\bar{D}^m$$

and the Shimura operators

$$\mathcal{L}_{\underline{\mathbf{m}}} = (\bar{D}^m)^* P_{\underline{\mathbf{m}}} \bar{D}^m,$$

where $(\bar{D}^m)^*$ is taken in the sense of operators on Hilbert spaces, since there is [38] natural G-invariant inner product on the $S_{\underline{\mathbf{m}}}(V)$-valued functions on D. Thus $\mathcal{L}_m$ is the sum of $\mathcal{L}_{\underline{\mathbf{m}}}$ with $m_1 + \cdots + m_r = m$.

We may naturally ask

Question 2. (Englis-Peetre [5]) Do the operators $\{\mathcal{L}_m, m = 0, 1, \dots\}$ generate $\mathcal{D}^G(D)$?

For rank two and three domain the above question has been answered to be positive; see [4] and [36].

However if we take the Shimura operators $\mathcal{L}_{\underline{\mathbf{m}}}$ for the fundamental spherical signature $\underline{\mathbf{m}} = 1^j$ then they do generate $\mathcal{D}^G(D)$.

Theorem 4.1. *(*[27]*) Let $D = G/K$ be a bounded symmetric domain. The operator $\mathcal{L}_{\underline{\mathbf{1}^j}}$ for $j = 1, 2, \cdots, r$, form a system of algebraically independent generators of $\mathcal{D}^G(D)$.*

4.2. **Relation between invariant Cauchy-Riemann operator and Cayley type differential operators.** Let π_ν be the action (2.8) of G on all smooth functions on D. We recall the following generalization of the Bol's lemma; see [24], Lemma 7.1 and [1].

Lemma 4.2. *(Bol's lemma) We have*

$$\Delta(\partial)^l \pi_{\frac{d}{r}-l} = \pi_{\frac{d}{r}+l} \Delta(\partial)^l.$$

An immediate application is that the kernel of $\Delta(\partial)$ on the holomorphic functions is invariant under the action $\pi_{\frac{d}{r}-l}$. Using (2.9) we have

Corollary 4.3. *The kernel* $\operatorname{Ker}\Delta(\partial)$ *on the space $\mathcal{P}(V)$ of holomorphic polynomials is the algebraic sum of polynomials in $H_{\frac{a}{2}(r-1)}$ and thus forms an irreducible unitarizable $(\mathfrak{g}, K)$-module.*

The next result explains then the relation between the Cayley type operator and the invariant Cauchy-Riemann operator $\bar{D}$.

Proposition 4.4. *(*[28]*) Let l be a positive integer. Up to a positive constant the operator $P_{l1^r}\bar{D}^{lr}$ is $h(z, z)^{-l+\frac{d}{r}} \Delta(\bar{\partial})^l h(z, z)^{l-\frac{d}{r}}$.*

The exact constant is calculated in [38].

Remark 4.5. The operator $\Delta(\partial)$ in the Bol's lemma can be viewed as the irreducible component with signature $\underline{\mathbf{m}} = 1^r$ in the decomposition of the differentiation $\partial^r = \partial \otimes \cdots \otimes \partial$ under K. One may consider the irreducible component with signature $\underline{\mathbf{m}} = 1^j$ in the lower order differentiation ∂^j. Indeed Shimura [28] proves that these operators are also intertwining operators, thus generalizing the Bol's lemma. They constitute the annihilating operators for the other reducible point in the Wallach set. In [3] the authors studied the annihilating operators for general unitarizable highest weight modules.

Remark 4.6. It is proved in [34] that the relative discrete series of a line bundle on D are in the kernel of the power $\bar{D}^{m+1}$ of $\bar{D}$ and that $\bar{D}^m$ is an intertwining operator realizing the relative discrete series as holomorphic discrete series.

4.3. **Eigenvalues of the Shimura invariant differential operators.** Let $\phi_{\underline{\lambda}}$ be the spherical function on $D = G/K$ in the usual convention [8]. The following result is proved in [38] for line bundles on all bounded symmetric domains.

Theorem 4.7. *Let $D = G/K$ be a bounded symmetric domain of tube type. The eigenvalue $\widetilde{\mathcal{L}_{\underline{1^j}}}(\underline{\lambda})$ of the operator $\mathcal{L}_{\underline{1^j}}$ for $j = 1, 2, \cdots, r$, on the spherical function $\phi_{\underline{\lambda}}$ is*

$$C_j \sum_{k=0}^{j} h_{j-k}\left(\rho_2^2 - \frac{1}{4}, \cdots, (\rho_{r-j+1})^2 - \frac{1}{4}\right) m_k \left(\lambda_1^2 + \frac{1}{4}, \lambda_2^2 + \frac{1}{4}, \cdots, \lambda_r^2 + \frac{1}{4}\right),$$

where the constant C_j is given by

$$C_j = \frac{j!}{\prod_{k=1}^{j}(\frac{a}{2}(j-k)+1)},$$

h_k is the complete symmetric function of degree k and m_k the kth elementary symmetric function.

Moreover there is some similar result as that of Corollary 4.3. Consider the tensor product $H_\nu \otimes \overline{H_\nu}$ of the holomorphic representation H_ν with its conjugate for the singular point of ν, $\nu = \frac{a}{2}(j-1)$. Consider the intertwining operator R, $RF(z) = h(z,z)^\nu F(z,z)$, from $H_\nu \otimes \overline{H_\nu}$ into the space $C^\omega(D)$ of real analytical functions on D with the regular action of G. The following result is proved in [39]

Proposition 4.8. *Let $1 \leq j \leq r$. For $\nu = \frac{a}{2}(j-1)$ the image $R(H_\nu \otimes \overline{H_\nu})$ of the intertwining map R is annihilated by the Shimura invariant differential operators $\mathcal{L}_k$, for $k \geq j$.*

One can further prove [39] that any unitary representation appearing in the decomposition of the tensor product is spherical, and thus is determined uniquely by a spherical function $\phi_{\underline{\lambda}}$. Any such $\phi_{\underline{\lambda}}$ functions satisfies $L_k \phi_{\underline{\lambda}} = 0$ for $k \geq j$. It suggests the following

Question 3. Study systematically the combinatorial and vanishing properties of the eigenvalues of the Shimura invariant differential operators.

In view of Remark 4.6 it is even more interesting to study the question for the Shimura operators on line or vector bundles over D.

Following Shimura [27], and motivated by the above results, we define

$$\Lambda = \{\underline{\lambda} \in \mathfrak{a}_{\mathbb{C}}^*/W; \widetilde{\mathcal{L}_{\underline{\mathbf{m}}}}(\underline{\lambda}) \geq 0 \quad \text{for all} \quad \underline{\mathbf{m}}\}.$$

It is not difficult to prove, by using the Plancherel formula for $L^2(D)$, that $\mathfrak{a}^* \subset \Lambda$, namely the spherical functions $\phi_{\underline{\lambda}}$ appearing in the Plancherel have nonnegative eigenvalues under

$\mathcal{L}_{\underline{\mathbf{m}}}$. All those $\phi_{\underline{\lambda}}$ are positive definite. So we define

$$\Sigma = \{\underline{\lambda} \in \mathfrak{a}_{\mathbb{C}}^*/W; \phi_{\underline{\lambda}} \quad \text{is positive definite}\}$$

and ask

Question 4. What is the relation between the sets Λ and Σ?

4.4. **Differential operators defined by their symbols.** There is another natural way of constructing invariant differential operators by a simple idea of change of variables. Any invariant differential operator is uniquely determined by its symbol at a fixed point say $z = 0$, which is then invariant under the action of the isotropic group K. Now the polynomials $K_{\underline{\mathbf{m}}}(z, z)$ form a basis of all K-invariants. This leads the following

Definition 4.9. For any smooth function f on D let $\mathcal{K}_{\underline{\mathbf{m}}}f(z) = K_{\underline{\mathbf{m}}}(\overline{\partial_w}, \partial_w)f(g_z(w))\big|_{w=0}$ where $g_z \in G$ is such that $g_z(0) = z$.

For general consideration see Helgason [8]; see also Rudin [20] for the case of the unit ball in $\mathbb{C}^n$.

One can prove by definition that $\mathcal{K}_{\underline{\mathbf{m}}}$ are G-invariant on D and they form a linear basis for $\mathcal{D}^G(D)$. The following result can be deduced by the calculations in [17] and [36].

Proposition 4.10. *The eigenvalues of $\mathcal{K}_{\underline{\mathbf{m}}}$ and of $\mathcal{L}_{\underline{\mathbf{m}}}$ on the spherical function $\phi_{\underline{\lambda}}$ as polynomials of $\underline{\lambda}$ have the same leading homogeneous term. In particular, $\mathcal{K}_{\underline{\mathbf{m}}}$ for $\underline{\mathbf{m}} = 1^j$, $j = 1, \dots, r$, form also a system of the generators of $\mathcal{D}^G(D)$.*

Question 5. Find explicit formulas for the eigenvalues of operators $\mathcal{K}_{\underline{\mathbf{m}}}$ for $\underline{\mathbf{m}} = 1^j$, $j = 1, \dots, r$.

For rank 2 domains the eigenvalue of $\mathcal{K}_{1^2}$ has been found in [36] by using Berezin transform and the result of Unterberger and Upmeier [30], which is related to the results in the next subsection.

4.5. **Eigenvalues as special evaluation of orthogonal polynomials.** It is realized in [17] and [36] that the eigenvalues of the operators $\mathcal{K}_{\underline{\mathbf{m}}}$ are closely related to certain hypergeometric orthogonal polynomials.

Definition 4.11. We define the polynomials $p_{\underline{\mathbf{m}},\nu}(\underline{\lambda})$ by

$$p_{\underline{\mathbf{m}},\nu}(\underline{\lambda}) = \mathcal{K}_{\underline{\mathbf{m}}}(\phi_{\underline{\lambda}}(z)h(z,z)^{-\nu})(0) = K_{\underline{\mathbf{m}}}(\bar{\partial}, \partial)(\phi_{\underline{\lambda}}(z)h(z,z)^{-\nu})(0).$$

Note that, a priori, $p_{\underline{\mathbf{m}},\nu}(\underline{\lambda})$ is only a function of $\underline{\lambda}$ and however it is easy to prove [17] that it is a W-invariant polynomial in $\underline{\lambda}$ and in ν.

The following orthogonality relation is observed earlier in [35] and [18] for the unit disk and is proved in [36]. Let $b_\nu(\underline{\lambda})$ be the symbol of the Berezin transform [30], and $c(\underline{\lambda})$ the Harish-Chandra c-function.

Proposition 4.12. *Suppose $\nu > p - 1 = \frac{2d}{r} - 1$. The polynomials $p_{\underline{\mathbf{m}},\nu}(\underline{\lambda})$ form an orthogonal basis for the L^2-space of W-invariant functions on $\mathfrak{a}^*$ with the measure $|c(\underline{\lambda})|^{-2} b_\nu(\underline{\lambda})$.*

In the representation theoretic terms, the polynomials $p_{\underline{\mathbf{m}},\nu}(\underline{\lambda})$ are the Clebsch-Gordan coefficients of the K-invariant elements $K_{\underline{\mathbf{m}}}(z, w)$ in the tensor product $H_\nu \otimes \overline{H_\nu}$ of the weighted Bergman space H_ν with its conjugate.

The following result follows immediately from the definitions, which establishes some interesting relation.

Proposition 4.13. *The polynomials $p_{\underline{\mathbf{m}},\nu}$ for $\nu = 0$ are the eigenvalues of the invariant differential operators $\mathcal{K}_{\underline{\mathbf{m}}}$.*

Finally we remark that so far we have treated only the cases of symmetric cones and hermitian symmetric spaces. Another closely related class of Riemannian symmetric spaces is that of the real bounded symmetric domains, where some analysis can be studied by using the so-called generalized Segal-Bargmann transform (see [16], [37]). Some results in this direction have been proved in [37] and [33]. Naturally, we may ask similar questions in the context of general Riemannian symmetric spaces.

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DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY AND GÖTEBORG UNIVERSITY, S-412 96 GÖTEBORG, SWEDEN

E-mail address: genkai@math.chalmers.se