BRANCHING COEFFICIENTS OF HOLOMORPHIC REPRESENTATIONS AND SEGAL-BARGMANN TRANSFORM

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ABSTRACT. Let $\mathbb{D} = G/K$ be a complex bounded symmetric domain of tube type in a Jordan algebra $V_{\mathbb{C}}$, and let $D = H/L = \mathbb{D} \cap V$ be its real form in a Jordan algebra $V \subset V_{\mathbb{C}}$. The analytic continuation of the holomorphic discrete series on $\mathbb{D}$ forms a family of interesting representations of G. We consider the restriction on D of the scalar holomorphic representations of G, as a representation of H. The unitary part of the restriction map gives then a generalization of the Segal-Bargmann transform. The group L is a spherical subgroup of K and we find a canonical basis of L-invariant polynomials in components of the Schmid decomposition and we express them in terms of the Jack symmetric polynomials. We prove that the Segal-Bargmann transform of those L-invariant polynomials are, under the spherical transform on D, multi-variable Wilson type polynomials and we give a simple alternative proof of their orthogonality relation. We find the expansion of the spherical functions on D, when extended to a neighborhood in $\mathbb{D}$, in terms of the L-spherical holomorphic polynomials on $\mathbb{D}$, the coefficients being the Wilson polynomials.

1. INTRODUCTION

Let $\mathbb{D} = G/K$ be a complex bounded symmetric domain of tube type. The weighted Bergman spaces $\mathcal{H}_\nu$ on $\mathbb{D}$ form unitary representations of G and are also called the scalar holomorphic discrete series. They have analytic continuation in terms of the weight ν and constitute some interesting and important family of unitary representations of G. It has turned out that it is very fruitful to study the restriction of the holomorphic representation to certain subgroups, both from the point of representation theory and harmonic analysis. In this paper we will pursue this by studying the restriction on some real forms of $\mathbb{D}$.

The domain $\mathbb{D}$ can be realized as a unit ball in a Jordan triple $V_{\mathbb{C}}$. Let V be a real form of $V_{\mathbb{C}}$, $V_{\mathbb{C}} = V + iV$. The real form $D = V \cap \mathbb{D}$ is called a real bounded symmetric domain if the complex conjugation τ with respect to V keeps $\mathbb{D}$ invariant. In this case $D = H/L$ is also a Riemannian symmetric space where H is a symmetric subgroup of G and L is a symmetric subgroup of K. Thus we have the following commutative diagram of subgroup inclusions

$$\begin{array}{ccc} G & \longleftarrow & K \\ \uparrow & & \uparrow \\ H & \longleftarrow & L \end{array}$$

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We consider the branching law of the holomorphic representation $\mathcal{H}_\nu$ on $\mathbb{D}$ along the diagram. The branching of $\mathcal{H}_\nu$ under K is given by the Schmid decomposition, whereas its restriction to H (the left vertical line) is given by the generalized Segal-Bargmann transform (see [26] and [39]), which gives the unitary equivalence between $\mathcal{H}_\nu$ as the L^2-space on the D. To continue this study, we consider the L-invariant elements in $\mathcal{H}_\nu$. The branching along the lower horizontal line is then given by the Helgason spherical transform; so to get diagram around we need to find all the L-invariant elements in the Schmid decomposition and calculate their Segal-Bargmann and spherical transform, this will be the main task of the present paper.

Let r be the rank of D, so that the rank of $\mathbb{D}$ is also r if the root system of H/L is of type A_{r-1} or D_r, it is $2r$ other wise; see Appendix 2. Let W be the Weyl group of the root system of the symmetric space H/L. The space $\mathcal{P} = \mathcal{P}(V_{\mathbb{C}})$ of holomorphic polynomials on $V_{\mathbb{C}}$ under the natural action of the compact group K is decomposed into irreducible subspaces $\mathcal{P}_{\underline{\mathbf{n}}}$ with signature $\underline{\mathbf{n}} = n_1\gamma_1 + \cdots + n_r\gamma_r$ for types A, D or B, and $\underline{\mathbf{n}} = n_1\gamma_1 + n_1'\gamma_1' + \cdots + n_r\gamma_r + n_r'\gamma_r'$ for other types, with multiplicity one. This decomposition can also be viewed as the diagonalization of a system of Cayley-Capelli type operators, namely $\Delta(z)^{-\alpha}\Delta(\partial)\Delta(z)^{\alpha+1}$, where Δ is the determinant polynomial of the Jordan triple $V_{\mathbb{C}}$ and α are nonnegative integers; moreover the eigenvalues of those operators separate the spaces $\mathcal{P}_{\underline{\mathbf{n}}}$. We consider its subspace $\underline{\mathbf{n}}\mathcal{P}^{\mathcal{L}}$ of L-invariant elements. By using the Cartan-Helgason theorem we find those signatures $\underline{\mathbf{n}}$, to be called spherical signature, for which $\mathcal{P}_{\underline{\mathbf{n}}}^L \neq 0$. Our first goal will be to find those polynomials.

For type A those polynomials are the well-known Jack symmetric polynomials and their norm has been calculated by Faraut-Koranyi [7]; there are also studied intensively by combinatorial method [23]. We shall study in [3] their Segal-Bargmann transform both in bounded and unbounded realization in relation to the Laplace-transform. However for other types those polynomials have not been determined in representation theory. We will use the Chevalley restriction theorem and the Dunkl-Cherednik operators to find them.

Let $\mathfrak{h} = \mathfrak{q} + \mathfrak{l}$ be the Cartan decomposition of the Lie algebra $\mathfrak{h}$ of H and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}$. Denote Σ the root system of $(\mathfrak{h}, \mathfrak{a})$ and W the corresponding Weyl group. The space $\mathfrak{q}$ can be identified with the Jordan triple V. By a well-known theorem of Chevalley the restriction map from $\mathcal{P}^L$ of L-invariant polynomials on $V_{\mathbb{C}}$ to $\mathfrak{a}$ is an isomorphism onto the subspace $\mathcal{P}(\mathfrak{a})^W$ of W-invariants polynomials in $\mathcal{P}(\mathfrak{a})$. Thus for each L-spherical signature $\underline{\mathbf{n}}$ there exists up to constants a unique polynomials in $\mathcal{P}_{\underline{\mathbf{n}}}$. Let D_ξ, $\xi \in \mathfrak{a}$, be the Dunkl operators. We can construct a family U_i of commuting operators acting on polynomials on $\mathfrak{a}$. For root system of type A, C those operators have been previously constructed by Dunkl [6]. We prove that when acting on L-invariant polynomials p

and restricted to $\mathfrak{a}$,

$$\Delta(z)^{-\alpha}\Delta(\partial)\Delta(z)^{\alpha+1}p(z) = \prod_{i=1}^{r}(U_i+\alpha)p(z).$$

Thus the problem of diagonalizing the Cayley-Capelli operator is reduced to that for the operators U_i. We find those polynomials $p_{\underline{\mathbf{n}}} \in \mathcal{P}_{\underline{\mathbf{n}}}^L$ in terms of the Jack polynomials and we calculated the Fock space norm of those polynomials; for type C those are done in [6] by some different method. We can then find the norm of those polynomials in the Bergman space $\mathcal{H}_\nu$, by using the result of Faraut-Koranyi.

We calculate then the Fourier and spherical transforms of the Segal-Bargmann transforms of those polynomials in the setting of Fock spaces and respectively Bergman spaces. They are, up to a factor of the square root of the symbol of the Berezin transform, Weyl group invariant orthogonal polynomials and will be called the *branching or coupling coefficients* as appeared in the title. In the former case we prove that they are of the form $e^{-\frac{1}{4\nu}||\underline{\lambda}||^2}\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$, where $\zeta_{\underline{\mathbf{n}},\nu}(\lambda)$ are Hermite type polynomials. Let $J_\nu(x,\lambda)$ be the Bessel function associated with the action of L on V, we prove that in the expansion of $e^{\frac{\nu}{2}||x||^2}J_\nu(x,\underline{\lambda})$ in terms of $p_{\underline{\mathbf{n}}}(x)$ the coefficients are exactly $\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$. In the later case (curved case) the spherical transform of the Segal-Bargmann transform of $p_{\underline{\mathbf{n}}}$ is of the form $b_\nu(\underline{\lambda})^{\frac{1}{2}}\xi_{\underline{\mathbf{n}}}(\underline{\lambda})$, where $b_\nu(\underline{\lambda})$ is the symbol of the Berezin transform; the symbol has been found independently in [34], [24] and [39]. (See also [32] for the case of complex bounded symmetric domains.) Thus the polynomials $\xi_{\underline{\mathbf{m}}}(\underline{\lambda})$ are W-invariant orthogonal polynomial with respect to the measure $b_\nu(\underline{\lambda})|c(\underline{\lambda})|^{-2}$ where $c(\underline{\lambda})$ is the Harish-Chandra c-function. They are some limiting cases of the multi-variable Wilson polynomials [33], so that the measure in the orthogonality relation has now an analytic significance. We find also an expansion of the product of the spherical function on D with the reproducing kernel in terms of the L-invariant polynomials, the coefficient being the spherical transform of their Segal-Bargmann transform (see Theorem 9.1). We give thus a unification of the two types of orthogonal polynomials associated to a root system, homogeneous symmetric polynomials of Jack type on one hand and the non-homogeneous Wilson polynomials on the other. For a general root system the Wilson polynomials have been studied by van-Diejen [33] under certain self-dual condition.

Along our way of the study we find an isometric version of the Chevalley restriction theorem (see Proposition 5.2) and an analogy of the Capelli identity expressing the product $\prod x_j$ and $\prod D_j$ in terms of the Cherednik operators $\prod U_j$, which we believe are also of independent interest; see [31] and [15] for the related study.

We remark that Theorem 6.9 can be deduced from [6], provided that one proves that they are eigenpolynomials of the (product of) the Cherednik operators and identify ours with that of Dunkl; in that paper Dunkl studied the a more general class of polynomials invariant under certain subgroups of the Weyl group and found the norm.

We refer also the reader to [14] for some algebraic consideration about finding polynomial invariants of linear groups and [17], [18] for some general results concerning the branching of unitary highest weight representations. We mention also that our results can also be interpreted as finding eigenfunctions of the Hamiltonians for the Calogero-Sutherland model for many body system, both with rational (flat case) and hyperbolic trigonometric (bounded case) first order differential operators in the Hamiltonian; see e.g. [6] for the flat case.

The paper is organized as follows. In Section 2 we give an abbreviated introduction of weighted Bergman spaces on bounded symmetric domains and fix some notation. In Section 3 we present the Segal-Bargmann transform and Berezin transform on real bounded symmetric domains, thus establish the abstract orthogonality relation for the spherical transforms of the Segal-Bargmann transforms of the L-invariant polynomials. We identify those Schmid components $\mathcal{P}_{\underline{\mathbf{n}}}$ which contain non-trivial L-invariant polynomials in Section 4. In Section 5 we express the radial part of the Cayley-Capelli type operator in terms of the Dunkl operator. Their eigenspace decomposition is done in Sections 6 and 7. In Sections 8 and 9 we study their Segal-Bargmann transforms and prove their orthogonality relation and find the expansion of the Bessel and spherical functions. We evaluate the constant in the Plancherel formula for the symmetric space H/L in our settings in Appendix 1, and we list all the real forms H/L of a general Hermitian symmetric space in Appendix 2.

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For the reader's convenience we list the main notation used in this paper:

- $\mathbb{D} = G/K$ a complex bounded symmetric domain of tube type and rank r' in a vector space $V_{\mathbb{C}}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition.
- V a real form in $V_{\mathbb{C}}$ and τ the conjugation of $V_{\mathbb{C}} = V + iV$ with respect to the real form V.
- $D = H/L$ a real bounded symmetric domain in V, $\mathfrak{h} = \mathfrak{l}+\mathfrak{q}$ the Cartan decomposition of $\mathfrak{h}$, $\mathfrak{q}$ is identified with V.
- r: rank of the real bounded symmetric domain D, so that the rank r' of the complex domain $\mathbb{D}$ is r if D is of type D_r, and it is $2r$ if D is of type C_r.
- $e_1, \dots, e_r$: a frame of the Jordan triple V.
- $\mathfrak{a} \subset \mathfrak{q}$ the maximal abelian subspace of $\mathfrak{q}$ spanned by the vectors $\xi_j = \xi_{e_j}$, Σ the root system of $\mathfrak{h}$ with respect to $\mathfrak{a}$;
- $\beta_1, \dots, \beta_r \in \mathfrak{a}^*$ the dual basis of $\frac{1}{2}\xi_1, \dots, \frac{1}{2}\xi_r$, identified also as linear functional on V.

- a the root multiplicity of $\frac{\beta_j-\beta_k}{2}$ and are independent of j, k and choice of the frame. Observe that $\dim A_{jj} = 1$ and $\dim B_{jj} = \iota - 1$; $\dim B_{jk} = \dim A_{jk} = a$ for type C and D_r ($r \geq 3$).
- $\gamma_1, \dots, \gamma_r$ (if type D) or $\gamma_1, \gamma_1' \dots, \gamma_r, \gamma_r'$ (if type C) the Harish-Chandra strongly orthogonal roots; a' the non-compact root multiplicity of $\frac{\gamma_j-\gamma_k}{2}$.
- $\mathfrak{k} = \mathfrak{k}_\tau^+ + \mathfrak{k}_\tau^-$ induced Cartan decomposition of $\mathfrak{k}$ by τ with $\mathfrak{k}_\tau^+ = \mathfrak{l} = \mathfrak{k} \cap q$
- $\mathfrak{t} = \mathfrak{t}_\tau^- + \mathfrak{t}_\tau^+$ the induced decomposition of a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$.
- c_ν the normalization constant for the Berezin transform, c_ν' the normalization constant for the weighted Bergman measure, C_0 the one for the Plancherel formula on $D = H/L$, C_1 the one for the integral of L-invariant functions on V in terms of polar coordinates on $\mathfrak{a}$.

2. Complex bounded symmetric domains

We recall very briefly in this and next sections some preliminary results on bounded symmetric domains and fix notation; see [21] and [7] and reference therein.

Let $\mathbb{D} = G/K$ be an irreducible bounded symmetric domain of tube type in a d-dimensional complex vector space $V_{\mathbb{C}} = \mathbb{C}^d$ of rank r'. (The symbol r will be reserved for the rank of the real bounded symmetric domain D in Section 3.) Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ be the Harish-Chandra decomposition of its complexification. The space $V_{\mathbb{C}} = \mathfrak{p}^+$ has then a Jordan triple structure so that the subspace $\mathfrak{p}$ is of the form

$$\mathfrak{p} = \{v - Q(z)\bar{v}, v \in V_{\mathbb{C}}\}, \tag{2.1}$$

when the elements are realized as holomorphic vector fields, where $Q(z) : \bar{V_{\mathbb{C}}} \mapsto V_{\mathbb{C}}$ is quadratic in z. We denote $\{x\bar{y}z\}$ the Jordan triple product $\{x\bar{y}z\} = (Q(x+z) - Q(x) - Q(z))\bar{y}$. We fix a K-invariant Hermitian inner product $(\cdot, \cdot)$ on $V_{\mathbb{C}}$ so that a minimal tripotent has norm 1. We let $dm(z)$ be the corresponding Lebesgue measure. The Bergman reproducing kernel up to a positive constant is of the form $h(z, \bar{w})^{-p}$ where p is the genus of $\mathbb{D}$, defined by $p = \frac{2d}{r'}$ (for tube domains) and $h(z, \bar{w})$ is an irreducible polynomial holomorphic in z and anti-holomorphic in w.

Let $\nu > p - 1$ and consider the probability measure $d\mu_\nu(z) = c_\nu' h(z, \bar{z})^{\nu - p} dm(z)$ where c_ν' is the normalization constant, and the corresponding weighted Bergman space $\mathcal{H}_\nu = \mathcal{H}_\nu(\mathbb{D})$ of holomorphic functions f so that

$$\|f\|_\nu^2 = \int_{\mathbb{D}} |f(z)|^2 d\mu_\nu(z) < \infty.$$

It has reproducing kernel $h(z, \bar{w})^{-\nu}$. The group G acts unitarily on $\mathcal{H}_\nu$ via the following

$$\pi_\nu f(z) = J_{g^{-1}}(z)^{\frac{\nu}{p}} f(g^{-1}z), \tag{2.2}$$

and it forms a unitary projective representation of G.

Let $\mathcal{F}_\nu = \mathcal{F}_\nu(V_\mathbb{C})$, for $\nu > 0$, be the Fock space of entire function on $V_\mathbb{C}$ with the norm defined by

$$(p, q)_{\mathcal{F}_\nu} = \frac{\nu^d}{\pi^d} \int_{V_\mathbb{C}} p(z)\overline{q(z)} e^{-\nu(z,z)} dm(z).$$

Thus it has reproducing kernel $e^{\nu(z,w)}$. When $\nu = 1$ we write for simplicity $\mathcal{F} = \mathcal{F}_1$. The norm in $\mathcal{F}$ can alternatively defined by

$$(f, g)_{\mathcal{F}_\nu} = \nu^{-\deg(p)} f(\partial_z) g^*(z)\big|_{z=0} \tag{2.3}$$

for polynomials f and g, where g^* is obtained from g by taking formerly the complex conjugate of the coefficients of the monomials (in terms of an orthonormal basis in $V_\mathbb{C}$).

The first part of the next result is due to Hua [16] for classical domains and Schmid [30] for general domains, the second and third part is due to Faraut and Koranyi [7], Theorem 3.8; this result will be of fundamental importance for our work here. Fix a Cartan subalgebra of $\mathfrak{k}$ and let $\gamma_1 > \cdots > \gamma_{r'}$ be the Harish-Chandra strongly orthogonal roots. Let a' be the root multiplicity of $\frac{\gamma_j - \gamma_k}{2}$ in $\mathfrak{p}^+$.

Theorem 2.1. *The space $\mathcal{P}$ of holomorphic polynomials on $V_\mathbb{C}$ decomposes into irreducible subspaces under $Ad(K)$, with multiplicity one as:*

$$\mathcal{P} \cong \sum_{\underline{\mathbf{n}} \geq 0} \mathcal{P}_{\underline{\mathbf{n}}}. \tag{2.4}$$

Each $\mathcal{P}_{\underline{\mathbf{n}}}$ is of lowest weight $-\underline{\mathbf{n}} = -(n_1\gamma_1 + \cdots + n_r\gamma_r)$ with $n_1 \geq \cdots \geq n_r \geq 0$. The quotient of the norms of a polynomial $f \in \mathcal{P}_{\underline{\mathbf{n}}}$ in the Fock space $\mathcal{F}$ and in the weighted Bergman space $\mathcal{H}_\nu$ is given by

$$\frac{\|f\|_{\mathcal{F}}}{\|f\|_{\mathcal{H}_\nu}} = \sqrt{(\nu)_{\underline{\mathbf{n}}}}$$

where

$$(\nu)_{\underline{\mathbf{n}}} = \prod_{j=1}^{r'} (v - \frac{a'}{2}(j-1))_{n_j} = \prod_{j=1}^{r} \prod_{k=1}^{n_j} (v - \frac{a'}{2}(j-1) + k - 1).$$

is the generalized Pochhammer symbol.

Let $K_{\underline{\mathbf{n}}}$ be the reproducing kernel of $\mathcal{P}_{\underline{\mathbf{n}}}$ with in the the Fock space $\mathcal{F}_\nu$ for $\nu = 1$. Then as a consequence we have, writing $|\underline{\mathbf{n}}| = n_1 + \cdots + n_{r'}$,

$$e^{\nu(z,w)} = \sum_{\underline{\mathbf{n}}} v^{|\underline{\mathbf{n}}|} K_{\underline{\mathbf{n}}}(z, w) \tag{2.5}$$

and

$$h(z, \bar{w})^{-\nu} = \sum_{\underline{\mathbf{n}}} (\nu)_{\underline{\mathbf{n}}} K_{\underline{\mathbf{n}}}(z, \bar{w}). \tag{2.6}$$

3. Berezin and Bargmann transform on real bounded symmetric domains

Let V be a real form of $V_{\mathbb{C}}$ and τ the complex conjugation with respect to V. Suppose $\tau(\mathbb{D}) = \mathbb{D}$, namely, τ fixes the bounded symmetric domain. Then the real form $D = \mathbb{D} \cap V$ is called a *real bounded symmetric domain*. In this case the triple product $D(x, \bar{y})z = \{x\bar{y}z\}$ restricted on V defines also a triple product on V.

A complete list of real bounded symmetric domains D is given in [21]. As a Riemannian symmetric space, $D = H/L$, where H is the connected component of the subgroup of G of biholomorphic transformations of $\mathbb{D}$ which keep D invariant. The coset space G/H is called a causal symmetric space, a complete list of the pairs (G, H) can be found in e.g. [25] and [13], see also Appendix 2.

The H-invariant measure on D is

$$h(z, \bar{z})^{-\frac{p}{2}} dm(z), \tag{3.1}$$

and H acts unitarily on $L^2(D, h^{-\frac{p}{2}} dm)$ via change of variables,

$$\pi_0(g)f(x) = f(g^{-1}x), \quad g \in H. \tag{3.2}$$

We describe briefly some algebraic and geometric structures of the domain D.

Let $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{l}$ be the Cartan decomposition of the Lie algebra $\mathfrak{h}$ of H. Similar to (2.1) we have

$$\mathfrak{q} = \{v - Q(z)\bar{v}, v \in V\}. \tag{3.3}$$

We thus identify $\mathfrak{q}$ with the underlying space V via the mapping $v \mapsto \xi_v(z)$.

Let r the rank of $D = H/L$, so that the rank r' of $\mathbb{D}$ will be r or $2r$; see below. Let $\{e_j, j = 1, \dots, r\}$ be a frame of minimal tripotents in V. The elements $\xi_j = \xi_{e_j} = e_j - Q(z)\bar{e_j}$, $j = 1, \dots, r$, span a maximal subspace $\mathfrak{a}$ of $\mathfrak{q}$ of dimension r. Let $\{\beta_j, j = 1, \dots, r\}$ in $\mathfrak{a}^*$ be the dual basis of $\frac{1}{2}\xi_j$,

$$\beta_j(\xi_k) = 2\delta_{j,k},$$

where $\delta_{j,k}$ is the Kronecker symbol. Then the root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of types A_r, C_r or D_r. Type A_r corresponds to the case that V is a formal real Jordan algebra; we will be only concerned with type C_r and type $D_r (r \geq 3)$:

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\pm\beta_j, \frac{\beta_j \pm \beta_k}{2}\};$$

and respectively

$$\Sigma(\mathfrak{g}, \mathfrak{a}) = \{\frac{\beta_j \pm \beta_k}{2}\}.$$

We let, as in [39], a be the root multiplicity of $\frac{\beta_j \pm \beta_k}{2}$ and $\iota - 1$ that of β_j. Arrange an ordering of the roots so that

$$\beta_1 < \beta_2 < \cdots < \beta_r.$$

The ranks r' and r and the multiplicities a', a and $\iota - 1$ are related to each other:

$$r' = r, a' = 2a \tag{3.4}$$

if it is type $D_r (r \geq 3)$, and

$$r' = 2r, a = 2a', \iota = 2 + a'; \tag{3.5}$$

if it is type C_r for $r \geq 2$; $a' = 0$ and $\iota = 2 + a'$ if $r = 1$. Those relation can easily be obtained by calculations of the dimensions of the root spaces or by a case by case check of the table in the Appendix 2.

We note that with the above normalization of the inner product on $V_{\mathbb{C}}$ a minimal tripotent of V has norm $\sqrt{2}$ if the root system is of Type C, otherwise it is of norm 1; see below. (The inner product in [39] is normalized so that a minimal tripotent of V always has norm 1.)

To consider the branching law of $(\mathcal{H}_\nu, \pi_\nu)$ of G under H we let R be the restriction map ([26]) $R : H_\nu \to C^\infty(D)$ by

$$Rf(x) = f(x)h(x, \bar{x})^{\frac{\nu}{2}}, \quad x \in D. \tag{3.6}$$

Then R is an H-intertwining map, as one can easily checks from the transformation properties of $h(x, \bar{x})$. Consider its formal conjugate operator R^* from $L^2(D, d\mu_0)$ to H^ν and form the operator R^*R on $L^2(D, \omega)$. It is

$$RR^* = \frac{1}{c_\nu} B_\nu, \tag{3.7}$$

where

$$B_\nu f(z) = c_\nu \int_D f(w) \frac{\overline{h(z,)^{\frac{\nu}{2}}} h(w, \bar{w})^{\frac{\nu}{2}}}{h(z, \bar{w})^\nu} \frac{dm(w)}{h(w, \bar{w})^{\frac{p}{2}}}. \tag{3.8}$$

is the (normalized) Berezin transform and the constant c_ν is such that $B_\nu 1 = 1$. The constant c_ν is evaluated in [39]. Let

$$R = |R|U \tag{3.9}$$

be the polar decomposition of R. Thus $|R|^2 = RR^*$.

It is proved in [39] that when $\nu > p - 1$ for type C and suppose $\nu > \frac{p}{2} - 1$ the Berezin transform RR^* is bounded, and that the multiplier $h^{\frac{\nu}{2}}$ is in $L^1(D, h^{-\frac{p}{2}} dm(z))$. Since for those values of ν the space $\mathcal{H}_\nu$ contains all polynomials, thus the range of R contains the functions of the form $h(z, \bar{z})^{\frac{\nu}{2}} p(z, \bar{z})$ where $p(z, \bar{z})$ are polynomials of z and it clearly forms a dense subspace in $L^2(D, h^{-\frac{p}{2}} dm(z))$. This proves

Proposition 3.1. *Suppose $\nu > p - 1$ for type C and suppose $\nu > \frac{p}{2} - 1$ for type D. The operator U is unitary and intertwines H-actions (2.2) on $\mathcal{H}_\nu$ and (3.2) onto $L^2(D)$.*

Definition 3.2. The unitary operator U is called a (generalized) Segal-Bargmann transform.

Let $\phi_{\underline{\lambda}}$ be the spherical function on $D = H/L$. Let

$$\widehat{f}(\underline{\lambda}) = \int_D f(z)\phi_{\underline{\lambda}}(z)\frac{dm(z)}{h(z,\bar{z})^{\frac{p}{2}}}$$

be the spherical transform on D, where f is a L-invariant C^∞-function on D with compact support. It is well known ([11], Chapter IV) that the map extends to a unitary operator from $L^2(D)^L$ onto $L^2(\mathfrak{a}^*, C_o|c(\underline{\lambda})|^{-2}d\underline{\lambda})^W$ of W-invariant functions in the L^2 space on $\mathfrak{a}^*$. Here $c(\underline{\lambda})$ is the Harish-Chandra c-function, with the same normalization as in [11], Chapter IV, $d\underline{\lambda}$ is the regularly normalized measure and the constant C_0 can be evaluated by using the evaluation of the constant c_ν in [39] and the evaluation of the Selberg integral, see Appendix 1.

Corollary 3.3. *The map $f \mapsto \widehat{Uf}$ is a unitary operator from $\mathcal{H}_\nu^L$ onto $L^2(\mathfrak{a}^*, C_o|c(\underline{\lambda})|^{-2}d\underline{\lambda})^W$.*

In particular if we find a canonical orthogonal basis $\mathcal{H}_\nu^L$, their image under the above unitary operator then gives an orthogonal basis $L^2(\mathfrak{a}^*, C_o|c(\underline{\lambda})|^{-2}d\underline{\lambda})^W$.

4. Existence of the L-invariant holomorphic polynomials

We identify now those polynomial spaces $\mathcal{P}_{\underline{\mathbf{n}}}$ that contain L-invariant vectors.

Lemma 4.1. *In the decomposition (2.4) the component $(\mathcal{P}_{\underline{\mathbf{n}}})^L \neq 0$ if and only if*

$$\underline{\mathbf{n}} = (m_1, m_1, m_2, m_2, \ldots, m_r, m_r) = \sum_{j=1}^r m_j(\gamma_{2j-1} + \gamma_{2j}) \tag{4.1}$$

if Σ is type C_r, and

$$\underline{\mathbf{n}} = (2m_1, 2m_2, \ldots, 2m_r) + m(1, 1, \ldots 1) = \sum_{j=1}^r (2m_j + m)\gamma_j \tag{4.2}$$

if Σ is of type D_r, where in all cases m_j and m are nonnegative integers and $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$.

Proof. The involution τ on $V_{\mathbb{C}}$ induced an involution on $\mathfrak{k}$ and its fixed point set is $\mathfrak{l}$, thus $(\mathfrak{k}, \mathfrak{l})$ is a symmetric pair. Let $\mathfrak{k} = \mathfrak{l} + \mathfrak{k}_\tau^-$ be the corresponding Cartan decomposition. Let $\mathfrak{t}_\tau^- \subset \mathfrak{k}_\tau^-$ be a maximal abelian subspace of $\mathfrak{k}_\tau^-$, and $\mathfrak{t} = \mathfrak{k}_\tau^+ + \mathfrak{k}_\tau^-$ a Cartan subalgebra of $\mathfrak{k}$ with $\mathfrak{t}_\tau^+ \subset \mathfrak{l}$. We apply the Cartan-Helgason theorem [12], Chapter V, Theorem 4.1, to identify the L-spherical representations, which asserts in our case that the representation $\mathcal{P}_{\underline{\mathbf{n}}}$ contains a L-fixed vector if and only if

$$\underline{\mathbf{n}}\big|_{(\mathfrak{k}_\tau^+)^{\mathbb{C}}} = 0 \tag{4.3}$$

and

$$\frac{(\underline{\mathbf{n}}, \alpha)}{(\alpha, \alpha)} \in \mathbb{N}, \tag{4.4}$$

for all roots α in the root system $\Sigma(\mathfrak{k}^{\mathbb{C}}, (\mathfrak{t}_\tau^-)^{\mathbb{C}})$ of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{t}_\tau^-$, in which case $\mathcal{P}_{\underline{\mathbf{n}}}^L$ is one-dimensional. Here $\mathbb{N}$ is the set of nonnegative integers.

Consider the type D_r first. By (3.1) the frame $\{e_1, e_2, \dots, e_r\}$ of minimal tripotents in V is also a frame in $V_{\mathbb{C}}$. Recall the involution τ induced by the complex conjugation on $V_{\mathbb{C}}$ with respect to V. We have $\tau(\xi_{e_j}) = \xi_{e_j}$ $\tau(\xi_{ie_j}) = -\xi_{e_j}$. Thus

$$\tau(iD(e_j, e_j)) = -iD(e_j, e_j),$$

since

$$iD(e_j, e_j) = \frac{1}{2}[\xi_{e_j}, \xi_{ie_j}],$$

so that $iD(e_j, e_j) \in \mathfrak{t}_\tau^-$, $j = 1, \dots, r$, and they span an r-dimensional abelian subspace $i(\mathbb{R}D(e_1, e_1) + \cdots + \mathbb{R}D(e_1, e_1))$ of $\mathfrak{t}_\tau^-$, whereas the dimension of $\mathfrak{t}_\tau^-$ is $\dim(\mathfrak{t}_\tau^-) = \operatorname{rank}(K/L)$. We claim that $\operatorname{rank}(K/L) > r$. Indeed the symmetric pair $(\mathfrak{k}, \mathfrak{l})$ is $(s(\mathfrak{u}(r) \oplus \mathfrak{u}(r)), \mathfrak{so}(r) \oplus \mathfrak{so}(r))$, $(\mathfrak{u}(2r), \mathfrak{so}(2r))$ or $(\mathfrak{so}(10) + \mathbb{R}, \mathfrak{sp}(4))$ (with $r = 3$), the rank of the first pair being $2r - 1 > r$, the second $2r > r$ and the third $7 > 3$ (see [11], Table V, p. 518). This means that the subspace $i(\mathbb{R}D(e_1, e_1) + \cdots + \mathbb{R}D(e_1, e_1))$ is a nontrivial subspace of $\mathfrak{k}_\tau^-$. So the vanishing condition (4.3) is already satisfied since $\underline{\mathbf{n}}$ is vanishing on the orthogonal complement of $i(\mathbb{R}D(e_1, e_1) + \cdots + \mathbb{R}D(e_1, e_1))$ in $\mathfrak{t}$. To check the second condition, recall (see [30], formulas (16) and (17)) that all roots of in $\Sigma(\mathfrak{k}^{\mathbb{C}}, (\mathfrak{t}_\tau^-)^{\mathbb{C}})$ are of the form $\frac{\gamma_j - \gamma_k}{2} + \alpha$, with α orthogonal to $\frac{\gamma_j - \gamma_k}{2}$ and if nonzero, $\|\alpha\|^2 = \|\frac{\gamma_j - \gamma_k}{2}\|^2$. Notice that since $\frac{\gamma_j - \gamma_k}{2}$, viewed as linear functional on $\mathfrak{t}_\tau^-$, only span a $r - 1$-dimensional subspace, thus there exist roots in $\Sigma(\mathfrak{k}^{\mathbb{C}}, (\mathfrak{t}_\tau^-)^{\mathbb{C}})$ of the form $\frac{\gamma_j - \gamma_k}{2} + \alpha$ with nonzero α for all $j, k = 1, \dots, r$, $j \neq k$. Thus by the Cartan-Helgason theorem, the polynomial representation $\underline{\mathbf{n}}$ is L-spherical if and only if

$$\frac{\langle \underline{\mathbf{n}}, \frac{\gamma_j - \gamma_k}{2} + \alpha \rangle}{\langle \frac{\gamma_j - \gamma_k}{2} + \alpha, \frac{\gamma_j - \gamma_k}{2} + \alpha \rangle} = \frac{1}{2}(n_j - n_k) \in \mathbb{Z}_{\geq 0},$$

for all $j < k$, and

$$\frac{\langle \underline{\mathbf{n}}, \frac{\gamma_j - \gamma_k}{2} \rangle}{\langle \frac{\gamma_j - \gamma_k}{2}, \frac{\gamma_j - \gamma_k}{2} \rangle} = (n_j - n_k) \in \mathbb{Z}_{\geq 0},$$

when $\alpha = 0$. The first condition clearly implies the second and it is just our stated condition.

Now consider type C_r. There are only three cases and we study them case by case. Consider the case $(\mathfrak{k}, \mathfrak{l}) = (s(\mathfrak{u}(2r) \oplus \mathfrak{u}(2r)), \mathfrak{sp}(r) \oplus \mathfrak{sp}(r))$. The highest weight $\underline{\mathbf{n}} = (m_1\varepsilon_1 + m_2\varepsilon_2 + \cdots + m_{2r}\varepsilon_{2r}) \otimes (m_1\varepsilon_1 + m_2\varepsilon_2 + \cdots + m_{2r}\varepsilon_{2r})^*$ where ε_j is the dual of diagonal matrix with jth entry being 1 and rest 0, in the standard matrix representation of $\mathfrak{u}(2r)$, and $(m_1\varepsilon_1 + m_2\varepsilon_2 + \cdots + m_{2r}\varepsilon_{2r})^*$ is the contra-gradient representation. The representation $\underline{\mathbf{m}}$ has a $\mathfrak{l} = \mathfrak{sp}(r) \oplus \mathfrak{sp}(r)$-fixed vector if and only if the representation $m_1\varepsilon_1 + m_2\varepsilon_2 + \cdots + m_{2r}\varepsilon_{2r}$ has a $\mathfrak{sp}(r)$-fixed vector, and the later happens precisely when $m_1 = m_2, m_3 = m_4, \dots, m_{2r-1} = m_{2r}$, namely our condition, again by the Cartan-Helgason theorem, since $(\mathfrak{su}(2r), \mathfrak{sp}(r))$ is a symmetric pair. Now let $(\mathfrak{k}, \mathfrak{l}) = (\mathfrak{u}(2r), \mathfrak{sp}(r))$. The Harish-Chandra roots are $\gamma_j = 2\varepsilon_{2j-1}, \gamma_j' = 2\varepsilon_{2j}$. Our result immediately follows from the above argument. Finally consider $(\mathfrak{k}, \mathfrak{l}) = (\mathfrak{so}(2) \oplus \mathfrak{so}(p), \mathfrak{so}(p))$. The highest weight $\underline{\mathbf{n}} = m_2(\gamma_1 + \gamma_2) +$

$(m_1 - m_2)(\gamma_1)$, and as representations $\underline{\mathbf{n}} = m_2(\gamma_1 + \gamma_2) \otimes (m_1 - m_2)\gamma_1$ with $m_2(\gamma_1 + \gamma_2)$ being the trivial representation of $\mathfrak{so}(p)$ and $(m_1 - m_2)\gamma_1$ the representation of $\mathfrak{so}(p)$ on the spherical harmonics of degree $m_1 - m_2$, the later has an $\mathfrak{so}(p)$-invariant vector if and only if $m_1 - m_2$ is even. This completes the proof. $\square$

Remark 4.2. In [19] Krämer has given a classification of all spherical connected subgroups L of a simple compact Lie group K (Tabelle 1, loc. cit) and listed all the fundamental spherical highest weights. Our result can also be deduced from that list. One may also prove the above result somewhat by slightly general argument by following the classification as in [25] and [21]. Moreover the above result for type C holds also for type BC and that for type D holds also for type B (with $m = 0$).

Let $\Delta(z)$ be the determinant polynomial of the Jordan triple $V_{\mathbb{C}}$ and $\Delta(\partial)$ the corresponding differential operator which we call the Cayley type operator. The next result gives the eigevalue of the operator $\Delta(z)^{-\alpha}\Delta(\partial)\Delta(z)^{\alpha+1}$, sometimes also named as the Cayley-Capelli operator, under the Schmid decomposition (2.4), and its follows easily by using Theorem 2.1; see e.g. [36], [35].

Lemma 4.3. *Let r' be the rank of $\mathbb{D}$. The differential operators $\Delta(z)^{-\alpha}\Delta(\partial)\Delta(z)^{\alpha+1}$ for r' different nonnegative integers α form a system of generators of K-invariant differential operators on $\mathcal{P}(V_{\mathbb{C}})$. A holomorphic polynomial $f \in \mathcal{P}(V_{\mathbb{C}})$ is in the space $\mathcal{P}_{\underline{\mathbf{n}}}$ if and only if it is a solution of the system of differential equations*

$$\Delta(z)^{-\alpha}\Delta(\partial)\Delta(z)^{\alpha+1}p(z) = \prod_{j=1}^{r'}(\frac{a'}{2}(r' - j) + 1 + \alpha + n_j)p(z), \tag{4.5}$$

for r' different nonnegative integers α.

In the next sections we will find the L-invariant polynomials in $\mathcal{P}_{\underline{\mathbf{n}}}$.

5. Determination of the L-invariant holomorphic polynomials: Some general results

We consider the Chevalley restriction map Res from $\mathcal{P}(V_{\mathbb{C}})^L$ onto the space $\mathcal{P}(\mathfrak{a})^W$ of W-invariant polynomials on $\mathfrak{a}$. By using the Dunkl operator we define an inner product on $\mathcal{P}(\mathfrak{a})^W$ and we prove that the restriction map Res is an unitary map from $\mathcal{P}(V_{\mathbb{C}})^L$ with the Fock space norm onto $\mathcal{P}(\mathfrak{a})^W$. This reduces the problem of finding and calculating of L-invariant polynomials to the corresponding one of W-invariant polynomials on $\mathfrak{a}$.

Some consideration that follows will be true for some general root system, we shall, however, only consider the root systems Σ as in Section 3.

We fix a frame $\{e_j, j = 1, \dots, r\}$ of minimal tripotent in V, enumerated so that $e_j \in V \subset V^{\mathbb{C}} = \mathfrak{p}^+$ is a root vector of the Harish-Chandra orthogonal root γ_j, $j = 1, \dots r$, if Σ is of type C, and e_j is a sum of two root vectors with roots γ_{2j-1} and γ_{2j} for type D.

For any root $\alpha \in \Sigma$ let $r_\alpha \in W$ be the reflection defined by α. We recall the Dunkl operator [4],

$$D_j = \partial_j + \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \frac{\alpha(\xi_j)}{\alpha(x)}(1 - r_\alpha)$$

acting on polynomials $f(x)$ on $\mathfrak{a}$, where the r_α acts $f(x)$ via

$$(r_\alpha f)(x) = f(r_\alpha^{-1}x).$$

The operators $D_j, j = 1, \ldots, r$ are pairwise commuting, and thus define an isomorphism between the ring of polynomials on $\mathfrak{a}$ and the ring of difference-differential operators generated by them. We can thus define, for any polynomial $x = x_1e_1 + \cdots + x_re_r \to f(x) = f(x_1, \ldots, x_r)$ the operator $f(D)$ by assigning D_j to the polynomials x_j, $j = 1, \ldots, r$.

Definition 5.1. (Dunkl [5]) Let Σ be the root system of type C or type D. The Σ-inner product on $(\mathcal{P}(\mathfrak{a}))^W$ is defined by

$$(f, g)_\Sigma = f(D_x)g^*(x)\big|_{x=0},$$

if Σ is of type D, and

$$(f, g)_\Sigma = f(\frac{1}{2}D_x)g^*(x)\big|_{x=0}$$

if Σ is of type C.

The discrepancy for type C here is due to the unmatched norms of minimal tripotents in $V_{\mathbb{C}}$ and V.

Proposition 5.2. *The restriction map* Res *is an isometric mapping from* $(\mathcal{P}(V_{\mathbb{C}})^L, \|\cdot\|_{\mathcal{F}})$ *onto* $(\mathcal{P}(\mathfrak{a})^W, \|\cdot\|_\Sigma)$.

Proof. We consider the operators

$$E = \frac{1}{2}(z_1^2 + \cdots + z_d^2), \tag{5.1}$$

$$F = -\frac{1}{2}(\partial_1^2 + \cdots + \partial_d^2), \tag{5.2}$$

and

$$H = (z_1\partial_1 + \cdots + z_d\partial_d) + \frac{d}{2} \tag{5.3}$$

acting on the space $\mathcal{P}(V_{\mathbb{C}})^L$. Then it is clear that they form the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = 2F.$$

Let us, following Heckman [10], define similarly

$$E_0 = \frac{1}{2}(x_1^2 + \cdots + x_r^2), \tag{5.4}$$

$$F_0 = -\frac{1}{2}(D_1^2 + \cdots + D_r^2), \tag{5.5}$$

for type D, and

$$E_0 = x_1^2 + \cdots + x_r^2, \tag{5.6}$$

$$F_0 = -\frac{1}{4}(D_1^2 + \cdots + D_r^2), \tag{5.7}$$

for type C (again due to the ill-matching of the norms of minimal tripotents in $V_{\mathbb{C}}$ and V), and

$$H_0 = (x_1\partial_1 + \cdots + x_r\partial_d) + \frac{1}{2}(r + \sum_{\alpha\in R} m_\alpha), \tag{5.8}$$

for all types. They form a copy of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We claim that

$$\operatorname{Res} E = E_0, \quad \operatorname{Res} F = F_0, \quad \operatorname{Res} H = H_0.$$

The second is trivial, the third follows since that the dimension $d = \dim_{\mathbb{C}} V_{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{q} = \frac{r}{2} + \frac{1}{2}\sum_{\alpha\in R} m_\alpha$. The first is just the formula for the radial part of the Laplace operator on $\mathfrak{q}$; see [12], Proposition 3.13 for the formula and [4] (or [10]) for the calculation of F_0. It is proved in [10] Proposition 3.4 that for any polynomial p of degree m on $\mathfrak{a}$, viewed as multiplication operator on $\mathcal{P}(\mathfrak{a})$, namely in $Aut(\mathcal{P}(\mathfrak{a}))$,

$$p(D) = (-1)^m \frac{1}{m!} \operatorname{ad}(F_0)^m(p), \tag{5.9}$$

where the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is acting on $Aut(\mathcal{P}(\mathfrak{a}))$ via the adjoint action. The essentially same (even easier) calculation shows that, for polynomial P on $\mathcal{P}(V_{\mathbb{C}})$ we have

$$P(\partial) = (-1)^m \frac{1}{m!} \operatorname{ad}(F)^m(P) \tag{5.10}$$

For any $P, Q \in \mathcal{P}(V_{\mathbb{C}})^L$, let $p = \operatorname{Res} P$, $q = \operatorname{Res} Q$; if $x \in \mathfrak{a}$,

$$\begin{aligned} P(\partial)Q^*(x) &= (\operatorname{Res} P(\partial)Q^*)(x) \\ &= \left(\operatorname{Res}((-1)^m \frac{1}{m!} \operatorname{ad}(F)^m(P)Q^*)\right)(x) \\ &= \left((-1)^m \frac{1}{m!} \operatorname{ad}(F_0)^m(\operatorname{Res} P) \operatorname{Res} Q^*\right)(x) \\ &= p(\partial)q^*(x) \end{aligned}$$

and

$$(P, Q)_{\mathcal{F}} = P(\partial)Q^*(0) = p(\partial)q^*(0) = (p, q)_\Sigma = (\operatorname{Res} P, \operatorname{Res} Q)_\Sigma,$$

completing the proof. □

It is noted in [10] that the above idea of reducing computation to second order operators goes back to Harish-Chandra.

Remark 5.3. The above result clarifies the significance of the Dunkl operator and the inner product $(\cdot, \cdot)_\Sigma$. Moreover it gives an isometric version of the Chevalley restriction theorem ([12], Chapter 2, Corollary 5.12). It seems to the author that this was not been known before.

As a consequence we get

Corollary 5.4. *The Cayley operator* $\operatorname{Res}\Delta(\partial)$ *on* L*-invariant polynomials is given by*

$$(\operatorname{Res}\Delta(\partial))f(x) = \prod_{j=1}^{r} D_j f(x)$$

for type D *and*

$$(\operatorname{Res}\Delta(\partial))f(x) = 2^{-2r}\prod_{j=1}^{r} D_j^2 f(x)$$

for type C.

Proof. We consider only the type D_r, the type C_r-case can be proved similarly. Recall that the determinant function Δ on $V_{\mathbb{C}}$ is of degree r and when restricted on $\mathfrak{a}$ it is

$$\operatorname{Res}\Delta(x) = \Delta(x_1e_1 + \cdots + x_re_r) = \prod_{j=1}^{r} x_j = (\prod_{j=1}^{r}\frac{\beta_j}{2})(x).$$

We calculate now the adjoint operator of Δ and $\operatorname{Res}\Delta$. The adjoint operator of multiplication by $\Delta(z)$ on $\mathcal{P}(V_{\mathbb{C}})$ with respect to the Fock norm is $\Delta(\partial)$, $\Delta^* = \Delta(\partial)$ and respectively the multiplication by $\prod_{j=1}^{r}\frac{\beta_j}{2}$ is $(\prod_{j=1}^{r}\frac{\beta_j}{2})^* = \prod_{j=1}^{r} D_j$ with respect to the Σ-norm by the definition of the inner product. Thus for $P, Q \in \mathcal{P}(V_{\mathbb{C}})^L$ we have, by the preceding proposition,

$$\begin{aligned}
(\operatorname{Res}(\Delta(\partial)P), \operatorname{Res}Q)_R &= (\Delta(\partial)P, Q)_{\mathcal{F}} = (P, \Delta Q)_{\mathcal{F}}\\
&= (\operatorname{Res}P, \operatorname{Res}\Delta Q)_\Sigma = (\operatorname{Res}P, (\prod_{j=1}^{r}\frac{\beta_j}{2})\operatorname{Res}\Delta Q)_\Sigma\\
&= ((\prod_{j=1}^{r}\frac{\beta_j}{2})^*\operatorname{Res}P, \operatorname{Res}Q)_\Sigma = ((\prod_{j=1}^{r}D_j)\operatorname{Res}P, \operatorname{Res}Q)_\Sigma
\end{aligned}$$

proving the result. ∎

6. L-invariant holomorphic polynomials: Type C_r

In this section we find the L-invariant polynomials in $\mathcal{P}_{\underline{\mathbf{n}}}^L$ and calculate their norm in the Fock space. We will express them in terms of the Jack symmetric polynomials. For that purpose we recall some basic facts.

Let

$$D_j = D_j^A = \partial_j + \frac{a}{2}\sum_{i\neq j}\frac{1}{y_j - y_i)}(1 - s_{ij}) \tag{6.1}$$

be the Dunkl operator ([4], [10]) acting on functions on the r-dimensional vector space $\mathbb{R}^r$, where the superscript indicates that the underlying root system is of type A. Let

$$U_j = U_j^A = D_jy_j - \frac{a}{2}\sum_{i<j}s_{ij} \tag{6.2}$$

be the Cherednik operator for type A ([6] and [2]). Then U_j^A, $j = 1, \dots, r$ are commuting operators. The relation between U_j^A and the Cayley type operator $\prod_{j=1}^r D_j \prod_{j=1}^r y_j$ is the following. (It can be viewed as a symmetric invariant analogue of the Cayley-Capelli identity, which express the product $\Delta(\partial)\Delta(z)$ as another determinant.)

Lemma 6.1. *The following identity hold*

$$(\prod_{j=1}^r D_j)(\prod_{j=1}^r y_j) = \prod_{j=1}^r U_j$$

Proof. The proof relies on the commutation relations $D_i s_{ij} = s_{ij} D_j$ and $[D_i, x_j] = -\frac{a}{2} s_{ij}$. From this we deduce that

$$\prod_{j=1}^r D_j \prod_{j=1}^r y_j = D_1 \dots D_{r-1} D_r y_1 y_2 \dots y_r$$

$$= D_1 \dots D_{r-1} y_1 D_r y_2 \dots y_r - \frac{a}{2} (\prod_{j=1}^{r-1} D_j)(\prod_{j=1}^{r-1} y_j) s_{1r};$$

repeating the argument (moving D_r until it reaches y_r) we get

$$\prod_{j=1}^r D_j \prod_{j=1}^r y_j = D_1 \dots D_{r-1} y_1 y_2 \dots y_{r-1} U_r. \tag{6.3}$$

Performing the above computation with D_{r-1} and so on proves the formula. □

Essentially the same computation gives

$$(\prod_{j=1}^r y_j)^{-\alpha} \prod_{j=1}^r D_j (\prod_{j=1}^r y_j)^{1+\alpha} = \prod_{j=1}^r (U_j + \alpha). \tag{6.4}$$

Remark 6.2. The Cherednik operators U_j^A are introduced before (see [6] for root systems of type B and reference therein) in order to study the non-symmetric Jack polynomials. The above lemma shows that it can also be obtained, though less systematically, in trying to write the Cayley-type operator $(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)$ as a product of r operators.

The Jack symmetric polynomial are then a sum of joint (nonsymmetric) eigenfunctions of U_j. Let $\Omega_{\underline{\mathbf{m}}}$ be the normalized Jack symmetric polynomial, so that it is a symmetric eigenpolynomial of the operators $\prod_{j=1}^r (U_j + \alpha)$

$$\prod_{j=1}^r (U_j^A + \alpha) \Omega_{\underline{\mathbf{m}}} = (\prod_{j=1}^r (\frac{a}{2}(r-k) + 1 + \alpha + m_k)) \Omega_{\underline{\mathbf{m}}}(y_1, \dots, y_r) \tag{6.5}$$

for all nonnegative integers α, normalized so that

$$\Omega_{\underline{\mathbf{m}}}(1, \dots, 1) = 1.$$

Let D_j be the Dunkl operator on $\mathfrak{a} = \mathbb{R}e_1 + \dots \mathbb{R}e_r$ for the root system Σ of type C,

$$D_j = \partial_j + \frac{\iota - 1}{2} \frac{1}{x_j}(1 - s_j) + \frac{a}{2} \sum_{i \neq j} (\frac{1}{x_j - x_i}(1 - s_{ij}) + \frac{1}{x_j + x_i}(1 - \sigma_{ij}))$$

where σ_j, s_{ij} and σ_{ij} are the reflections in Weyl group corresponding to the roots γ_j, $\frac{1}{2}(\gamma_i - \gamma_j)$ and respectively $\frac{1}{2}(\gamma_i + \gamma_j)$.

Recall that the Hermitian form on V is normalized so that e_j has norm $\sqrt{2}$.

Lemma 6.3. *The restriction of the operator* $\mathrm{Res}(\Delta(\partial_x)\Delta(x))$ *on* $\mathfrak{a}$ *is given by*

$$
\begin{aligned}
&\mathrm{Res}(\Delta(\partial_x)\Delta(x))\\
&= 2^{-2r}\prod_{j=1}^{r} D_j^2 \prod_{j=1}^{r} x_j^2 = 2^{-2r}\prod_{j=1}^{r} D_j \prod_{j=1}^{r}(D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}))\prod_{j=1}^{r} x_j\\
&= 2^{-2r}\prod_{j=1}^{r}\left(D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}) + 1 - (\iota-1)\sigma_j\right)\prod_{j=1}^{r}\left(D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij})\right)
\end{aligned}
$$

Proof. The first equality follows by Corollary 5.4, for the restriction of $\Delta(x)$ on $\mathfrak{a}$ is $\prod_{j=1}^r x_j^2$. The rest of the proof is similar to that of Lemma 6.1. We have the product formula

$$
(\prod_{j=1}^{r} D_j)(\prod_{j=1}^{r} x_j) = \prod_{j=1}^{r}(D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}));
$$

and shift formula

$$
(\prod_{k=1}^{r} D_k)(D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij})) = (D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}) + 1 - (\iota-1)\sigma_j)(\prod_{k=1}^{r} D_k),
$$

which can be obtained by repeatedly using the commutation relation

$$
[D_j, x_k] = \frac{a}{2}(\sigma_{jk} - s_{jk}), \quad (j \neq k),
$$

$$
[D_j, x_j] = 1 + (\iota - 1)s_j + \frac{a}{2}\sum_{i\neq j}(\sigma_{ij}+s_{ij}),
$$

and

$$
\sigma_j D_j = -D_j\sigma_j, \quad s_{ij}D_j = D_i s_{ij}, \quad \sigma_{ij}D_j = -D_i\sigma_{ij}.
$$

Consequently

$$\begin{aligned}
&(\prod_{j=1}^r D_j^2)(\prod_{j=1}^r x_j^2)\\
&= (\prod_{j=1}^r D_j)(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)(\prod_{j=1}^r x_j)\\
&= (\prod_{j=1}^r D_j)\prod_{j=1}^r (D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}))(\prod_{j=1}^r x_j)\\
&= \prod_{j=1}^r (D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}) + 1 - (\iota-1)\sigma_j)(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)\\
&= \prod_{j=1}^r (D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}) + 1 - (\iota-1)\sigma_j)\prod_{j=1}^r (D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij}))
\end{aligned}$$

□

Lemma 6.4. *The operator* $\mathrm{Res}(\Delta(\partial_x)\Delta(x))$, *when acting on W-invariant polynomials of the form $f(x_1^2,\dots,x_r^2)$ with $f(y_1,\dots,y_r)$ being a symmetric polynomials in r variables, is given in terms of the coordinates $y_j = x_j^2$ by*

$$\mathrm{Res}(\Delta(\partial_x)\Delta(x))f(x) = \prod_{j=1}^r (U_j^A + \frac{1}{2}(\iota-1) - \frac{1}{2})\prod_{j=1}^r U_j^A$$

where U_j^A is the Cherednik operator (6.2) of type A acting on functions of $y_1,\dots,y_r$.

Proof. We use the previous lemma and consider the operator $D_j x_j - \frac{a}{2}\sum_{i<j}(\sigma_{ij}+s_{ij})$ acting on the functions of the form $f(y_1,\dots,y_r) = f(x_1^2,\dots,x_r^2)$; this operator maps f into functions of the same form, and in terms of the variables $y_j = x_j^2$, it is

$$2(D_j^{(y)} - \frac{a}{2}\sum_{i<j}s_{ij} + \frac{\iota-1}{2} - \frac{1}{2}) = 2(U_j^A + \frac{\iota-1}{2} - \frac{1}{2}) \tag{6.6}$$

by direct calculation, proving our Lemma. □

The above lemma can also be proved by using [6], Proposition 5.2, where the operator $(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)$ is expressed in terms of the Cherednik operators. (Note that our operator U_j^A differs by a constant with the operator $U_{A,j}$ there.) Together with (6.5) it implies then

Lemma 6.5. *The Jack symmetric polynomials $\Omega_{\underline{\mathbf{m}}}(x_1^2,\dots,x_r^2)$ are eigenfunction of the operator* $\mathrm{Res}(\Delta^{-\alpha}\Delta(\partial)\Delta^{1+\alpha})$ *with eigenvalue*

$$\prod_{j=1}^r (\frac{a}{2}(r-j) + \frac{\iota}{2} + m_j + \alpha)(\frac{a}{2}(r-j) + m_j + \alpha). \tag{6.7}$$

Proposition 6.6. *For each* $\underline{\mathbf{m}} = (m_1, \dots, m_r)$ *there exists a unique polynomial* $p_{\underline{\mathbf{n}}}$ *in the space* $\mathcal{P}^L_{\underline{\mathbf{n}}}$ *with* $\underline{\mathbf{n}}$ *given by* $\underline{\mathbf{m}}$ *as in (4.1) such that*

$$\operatorname{Res} p_{\underline{\mathbf{n}}}(x_1, \dots, x_r) = \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2). \tag{6.8}$$

Proof. The existence of such polynomial $p_{\underline{\mathbf{n}}}$ in $\mathcal{P}(V_{\mathbb{C}})^L$ is by the Chevalley restriction theorem. Lemma 4.3 implies that $p_{\underline{\mathbf{n}}}$ is in the space $\mathcal{P}_{\underline{\mathbf{n}}}$, with $\underline{\mathbf{n}}$ as given, if and only if it is an eigenfunction of the operator $\Delta^{-\alpha}\Delta(\partial)\Delta^{1+\alpha}$ with eigenvalue

$$\begin{aligned}
&\prod_{j=1}^{2r}(\frac{a'}{2}(2r-j)+1+n_j+\alpha)\\
=&\prod_{j=2i-1,i=1}^{r}(\frac{a'}{2}(2r-2i+1)+1+n_{2i-1}+\alpha)\prod_{j=2i,i=1}^{r}(\frac{a'}{2}(2r-2i)+1+n_{2i}+\alpha)\\
=&\prod_{i=1}^{r}(a'(r-i+\frac{1}{2})+1+m_i+\alpha)\prod_{i=1}^{r}(a'(r-i)+1+m_i+\alpha).
\end{aligned}$$

Using (3.5) we see that the factors in the product (6.7) are

$$\frac{a}{2}(r-j)+\frac{\iota}{2}+m_j+\alpha = a'(r-j)+1+\frac{a'}{2}+m_j+\alpha = a'(r-j+\frac{1}{2})+1+m_j+\alpha$$

and

$$\frac{a}{2}(r-j)+1+m_j+\alpha = a'(r-j)+1+m_j+\alpha,$$

so that (6.7) coincides with the previous formula. □

We calculate now the Fock space norm of $p_{\underline{\mathbf{n}}}$ by a direct calculation using certain recurrence formula of Macdonald.

Lemma 6.7. *In terms of the coordinates* $y_j = x_j^2, j = 1, \dots, r$, *the operator* F_0 *when acting on* W*-invariant polynomials has the following form,*

$$F_0 = -(-2F_0^A + (\frac{a}{2}(r-1)+\frac{1}{2}+\frac{\iota-1}{2})\sum_{j=1}^{r}\partial_j^{(y)}) \tag{6.9}$$

where F_0^A *is the corresponding operator for the root system of type A,*

$$F_0^A = -\frac{1}{2}(\sum_{j=1}^{r}(D_j^A)^2) = -\frac{1}{2}\left(\sum_{j=1}^{r} y_j(\partial_j^{(y)})^2 + a\sum_{i\neq j}\frac{y_i}{y_i-y_j}\partial_i^{(y)}\right) \tag{6.10}$$

With some abuse of notation we denote $\Omega^{(x)}_{\underline{\mathbf{m}}}$ the function $\Omega^{(x)}_{\underline{\mathbf{m}}} = \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)$.

To state our next result we let $\binom{\underline{\mathbf{m}}}{\underline{\mathbf{m}}'}$ be the generalized binomial coefficients [20] and let $\underline{\mathbf{m}}_j$ respectively $\underline{\mathbf{m}}^j$ stand for the signature $\underline{\mathbf{m}}_j = (m_1, \cdots, m_r) - (0, \dots, 0, 1, 0, \dots, 0)$ and $\underline{\mathbf{m}}^j = (m_1, \cdots, m_r) + (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in the j'th position). The binomial coefficients $\binom{\mathbf{m}}{\mathbf{m}'}$ is then (loc. cit., Section 14)

$$\binom{\underline{\mathbf{m}}}{\underline{\mathbf{m}}_j} = (m_j + \frac{a}{2}(r-j)) \prod_{i\neq j} \frac{m_j - m_i + \frac{a}{2}(i-j-1)}{m_j - m_i + \frac{a}{2}(i-j)} \tag{6.11}$$

Lemma 6.8. *The operators E_0 and F_0 have following upper respectively lower triangular form when acting on the polynomials $\Omega^{(x)}_{\underline{\mathbf{m}}}$*

$$E_0 \Omega^{(x)}_{\underline{\mathbf{m}}} = \sum_{j=1}^{r} c_{\underline{\mathbf{m}}}(j) \Omega^{(x)}_{\underline{\mathbf{m}}^j}, \tag{6.12}$$

(6.13)

$$F_0 \Omega^{(x)}_{\underline{\mathbf{m}}} = -\frac{1}{4} \sum_{j=1}^{r} \Big(4(m_j - 1 - \frac{a}{2}(j-1)) + 2a(r-1) + 2(\iota - 1) + 2\Big) \binom{\underline{\mathbf{m}}}{\underline{\mathbf{m}}_j} \Omega^{(x)}_{\underline{\mathbf{m}}_j},$$

where

$$c_{\underline{\mathbf{m}}}(j) = \prod_{i\neq j} \frac{m_j - m_i + \frac{a}{2}(1+i-j)}{m_j - m_i + \frac{a}{2}(i-j)}.$$

Proof. This follows from our formula (6.9) and the result of Macdonold [22], Section D. Following temporarily the notation there, let

$$\square_1 = \sum_{j=1}^{r} y_j (\partial^{(y)}_j)^2 + a \sum_{i\neq j} \frac{y_i}{y_i - y_j} \partial^{(y)}_i,$$

and

$$\varepsilon_1 = \sum_{j=1}^{r} \partial^{(y)}_j;$$

it is proved by Macdonald that

$$\square_1 \Omega_{\underline{\mathbf{m}}} = \sum_{j=1}^{r} (m_j - 1 - \frac{a}{2}(j-1)) \binom{\underline{\mathbf{m}}}{\underline{\mathbf{m}}_j} \Omega_{\underline{\mathbf{m}}_j},$$

$$\varepsilon_1 \Omega_{\underline{\mathbf{m}}} = \sum_{j=1}^{r} \binom{\underline{\mathbf{m}}}{\underline{\mathbf{m}}_j} \Omega_{\underline{\mathbf{m}}_j}$$

and

$$(\sum_{j=1}^{r} y_j) \Omega_{\underline{\mathbf{m}}} = \sum_{j=1}^{r} c_{\underline{\mathbf{m}}}(j) \Omega_{\underline{\mathbf{m}}_j},$$

which then imply our result. □

Theorem 6.9. *The norm square $\|p_{\underline{\mathbf{n}}}\|^2_{\mathcal{F}}$ of the L-invariant polynomial $p_{\underline{\mathbf{n}}}$, with $\underline{\mathbf{n}}$ determined by $\underline{\mathbf{m}}$ as in (4.1), is given by*

$$\prod_{1\le i<j\le r}\frac{\Gamma(\frac{a}{2}(j+1-i))(\frac{a}{2}(j-i))}{\Gamma(1+\frac{a}{2}(j-1-i))}\prod_{j=1}^{r}(\frac{\iota-1}{2}+\frac{1}{2}+\frac{a}{2}(r-j))_{m_j}\prod_{j=1}^{r}(1+\frac{a}{2}(r-j))_{m_j}$$
$$\times\prod_{1\le i<j\le r}\frac{\Gamma(m_i+1-m_j+\frac{a}{2}(j-1-i))}{\Gamma(m_i-m_j+\frac{a}{2}(j+1-i))(m_i-m_j+\frac{a}{2}(j-i))}$$

Proof. For a fixed j write $\underline{\mathbf{m}}'=\underline{\mathbf{m}}_j=(m_1,\dots,m_{j-1},m_j-1,m_{j+1},\dots,m_r)$ and let $\underline{\mathbf{n}}'$ be the corresponding $\underline{\mathbf{n}}$. $p_{\underline{\mathbf{n}}}$ and $p_{\underline{\mathbf{n}}'}$ are orthogonal in $\mathcal{F}$ as they are in the different Schmid components, so are the polynomials $\Omega^{(x)}_{\underline{\mathbf{m}}}$ and $\Omega^{(x)}_{\underline{\mathbf{m}}'}$ with the Σ-inner product by Proposition 5.2. Now, on the space of all W-invariant polynomials, the adjoint $(E_0)^*=-F_0$ with respect to the Σ-inner product, from this we obtain

$$(E_0\Omega^{(x)}_{\underline{\mathbf{m}}'},\Omega^{(x)}_{\underline{\mathbf{m}}})_\Sigma-(\Omega^{(x)}_{\underline{\mathbf{m}}'},F_0\Omega^{(x)}_{\underline{\mathbf{m}}})_\Sigma,$$

this gives

$$\frac{\|\Omega^{(x)}_{\underline{\mathbf{m}}}\|^2_\Sigma}{\|\Omega^{(x)}_{\underline{\mathbf{m}}'}\|^2_\Sigma}=\binom{\underline{\mathbf{m}}}{\underline{\mathbf{m}}'}\frac{m_j-\frac{1}{2}+\frac{\iota-1}{2}+\frac{a}{2}(r-j)}{c_{\underline{\mathbf{m}}'}(j)}.$$

This recursion formula, together with the fact that $(p_0,p_0)_{\mathcal{F}}=(\Omega_0,\Omega_0)_\Sigma=1$ uniquely determine the norm. Carrying out the calculation gives our result. □

7. L-invariant holomorphic polynomials: Type D_r ($r\ge 3$)

The Weyl group W in this case consists of the signed permutation of vectors $(x_1,\dots,x_r)$ keeping the product $x_1\dots x_r$ invariant. Thus any W-invariant polynomial is of the form $(x_1\dots x_r)^m f(x_1^2,\dots,x_r^2)$, where f is a symmetric polynomial in r variables. The Dunkl operators are

$$D^D_j=\partial_j+\frac{a}{2}\sum_{i\ne j}(\frac{1}{x_j-x_i}(1-s_{ij})+\frac{1}{x_j-x_i}(1-\sigma_{ij})).$$

We are interested in the operator $(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)$. Similar to the proof of Lemma 6.1 we have

$$D_r(\prod_{i=1}^r x_i)=(\prod_{i=1}^{r-1}x_i)(D_rx_r-\frac{a}{2}\sum_{i<r}(s_{ir}+\sigma_{ir})),\tag{7.1}$$

and generally

$$D_j(\prod_{i\le j}x_i)=(\prod_{i\le j-1}x_i)(D_jx_j-\frac{a}{2}\sum_{i<j}(s_{ij}+\sigma_{ij}).\tag{7.2}$$

We define therefore Cherednik operator D_j, $j=1,\dots,r$ for type D,

$$U^D_j=D_jx_j-\frac{a}{2}\sum_{i<j}(s_{ij}+\sigma_{ij}).$$

Thus we have

$$(\prod_{j=1}^{r} D_j)(\prod_{j=1}^{r} x_j) = \prod_{j=1}^{r} U_j \tag{7.3}$$

Moreover, it is easy to prove that

$$U_j^D(\prod_{j=1}^{r} x_j) = (\prod_{j=1}^{r} x_j)(U_j^D + 1), \tag{7.4}$$

Therefore we have

Lemma 7.1. *We have the following formula*

$$(\prod_{j=1}^{r} D_j)^m(\prod_{j=1}^{r} x_j)^m = \prod_{j=1}^{r}(U_j^D)_m$$

where $(T)_m = T(T+1)\dots(T+m-1)$ *is the Pochhammer symbol of any operator* T*.*

Consider the Cherednik operator U_j^D acting on even functions $f(x_1^2, \dots, x_r^2)$. Performing the change of variables $y_j = x_j^2$, we have

$$U_j^D = 2(U_j^A - \frac{1}{2}) \tag{7.5}$$

where U_j^A is the operator (6.2). This proves consequently that the Cherednik operators U_j^D are also commuting operators when acting on even functions.

The next result follows then immediately from Corollary 5.4, the formulas (7.3) and (7.4).

Lemma 7.2. *The operator* $\mathrm{Res}(\Delta(\partial_x)\Delta(x))$*, when acting on even polynomials of the form* $f(x_1^2, \dots, x_r^2)$*, is, after the change of variables* $y_j = x_j^2$*, given by*

$$\mathrm{Res}(\Delta(x)^{-\alpha}\Delta(\partial_x)\Delta(x)^{1+\alpha}) = 2^r\prod_{j=1}^{r}(U_j^A - \frac{1}{2} + \frac{\alpha}{2})$$

where $U_j^A = D_j^{(y)}y_j - \frac{a}{2}\sum_{i<j}\sigma_{ij}$ *is the Cherednik operator in variables* y_j*'s.*

Proposition 7.3. *The polynomials* $(\prod_{j=1}^{r} x_j)^m\Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)$ *are eigenfunction of the operator* $\mathrm{Res}(\Delta(x)^{-\alpha}\Delta(\partial_x)\Delta(x)^{1+\alpha})$ *with eigenvalue*

$$\prod_{j=1}^{r}(a(r-k) + 2 + \alpha + m + 2m_j).$$

Moreover, it is the restriction of a unique L*-invariant polynomials* $p_{\underline{\mathbf{n}}}$ *in the space* $\mathcal{P}_{\underline{\mathbf{n}}}$*,*

$$\mathrm{Res}\, p_{\underline{\mathbf{n}}}(x_1, \dots, x_r) = (\prod_{j=1}^{r} x_j)^m\Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2), \tag{7.6}$$

with $\underline{\mathbf{n}}$ *given by* $\underline{\mathbf{m}}$ *as in (4.2).*

Proof. The operator $\mathrm{Res}(\Delta(x)^{-\alpha}\Delta(\partial_x)\Delta(x)^{1+\alpha})$ has the form

$$(\prod_{j=1}^r x_j)^{-\alpha}(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)^{1+\alpha}$$

by Corollary 5.4. We calculate its action on $(\prod_{j=1}^r x_j)^m \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)$. It is

$$(\prod_{j=1}^r x_j)^m \left((\prod_{j=1}^r x_j)^{-\alpha-m}(\prod_{j=1}^r D_j)(\prod_{j=1}^r x_j)^{1+\alpha+m} \right) \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2).$$

Applying Lemma 7.2 for the operator in the parenthesises and using (6.4) we see that our polynomials is indeed an eigenfunction with eigenvalue

$$2^r \prod_{j=1}^r (\frac{a}{2}(r-k) + \frac{1}{2} + \frac{\alpha}{2} + \frac{m}{2} + m_j)$$

as claimed. The rest of the proof is similar to that of Proposition 6.6. $\square$

Theorem 7.4. *The norm square $\|p_{\underline{\mathbf{n}}}\|^2_{\mathcal{F}}$ of the L-invariant polynomial $p_{\underline{\mathbf{n}}}$ is given by*

$$\prod_{1\le i<j\le r} \frac{\Gamma(\frac{a}{2}(j+1-i))(\frac{a}{2}(j-i))}{\Gamma(1+\frac{a}{2}(j-1-i))} 2^{2(m_1+\cdots+m_r)} \prod_{j=1}^r (\frac{1}{2} + \frac{a}{2}(r-j))_{m_j} \prod_{j=1}^r (1+\frac{a}{2}(r-j))_{m_j}$$
$$\times \prod_{j=1}^r (\frac{a}{2}(r-j) + 1 + m_j - \frac{1}{2})_m \prod_{1\le i<j\le r} \frac{\Gamma(m_i + 1 - m_j + \frac{a}{2}(j-1-i))}{\Gamma(m_i - m_j + \frac{a}{2}(j+1-i))(m_i - m_j + \frac{a}{2}(j-i))}$$

Proof. Proposition 5.2 implies that

$$\|p_{\underline{\mathbf{n}}}\|^2_{\mathcal{F}} = \|\,\mathrm{Res}\, p_{\underline{\mathbf{n}}}\|^2_{\Sigma} = \Omega_{\underline{\mathbf{m}}}(D_1^2, \dots, D_r^2)(\prod_{j=1}^r D_j)^m (\prod_{j=1}^r x_j)^m \Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)\big|_{x=0}.$$

However, by Lemma 7.1

$$(\prod_{j=1}^r D_j)^m (\prod_{j=1}^r x_j)^m = \prod_{j=1}^r (U_j)_m = 2^r \prod_{j=1}^r (U_j^A - \frac{1}{2})_m,$$

which has $\Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)$ as an eigenfunction with eigenvalue

$$\prod_{j=1}^r (\frac{a}{2}(r-j) + 1 + m_j - \frac{1}{2})_m$$

by (6.5). Thus

$$\|p_{\underline{\mathbf{n}}}\|^2_{\mathcal{F}} = \prod_{j=1}^r (\frac{a}{2}(r-j) + 1 + m_j - \frac{1}{2})_m \Omega_{\underline{\mathbf{m}}}(D_1^2, \dots, D_r^2)\Omega_{\underline{\mathbf{m}}}(x_1^2, \dots, x_r^2)\big|_{x=0}$$
$$= \prod_{j=1}^r (\frac{a}{2}(r-j) + 1 + m_j - \frac{1}{2})_m (\Omega^x_{\underline{\mathbf{m}}}, \Omega^x_{\underline{\mathbf{m}}})_{\Sigma}.$$

The inner product above can be calculated by the same recursion formula as for type C. Indeed the Laplace operator F_0 in this case, after changing of variables is of the form

$$F_0 = -2\left(-2F_0^A + (\frac{a}{2}(r-1) + \frac{1}{2})\sum_{j=1}^{r}\partial_j^{(y)}\right)$$

which is of the same form as (6.9) except the term $\frac{\iota-1}{2}$ is missing and that the coefficient -1 in front is replace by -2. □

8. Bargmann transform of L-invariant polynomials: Flat case

Consider again the restriction map

$$R = R_\nu : \mathcal{F}_\nu(V_{\mathbb{C}}) \mapsto C^\omega(V), \quad Rf(x) = f(x)e^{-\frac{\nu}{2}||x||^2} \tag{8.1}$$

It defines a bounded operator and the isometric part U in the polar decomposition $R = |R|U$ of R is a unitary operator, and is the Segal-Bargmann transform. The Berezin transform in this case is

$$|R|^2 f(x) = RR^* f(x) = \int_V e^{-\frac{\nu}{2}||x-y||^2} f(y)dy$$

Recall our identification of V with $\mathfrak{q}$. In this identification $\mathfrak{a} = \mathbb{R}e_1 + \cdots + \mathbb{R}e_r$ is a subspace of V.

Define the Bessel function on V by

$$J_{\underline{\lambda}}(x) = \int_L e^{-i\underline{\lambda}((lx))} dl \tag{8.2}$$

for $\underline{\lambda} \in \mathfrak{a}^*$ where we extend the linear functional $\underline{\lambda}$ on $\mathfrak{a}$ to V via the orthogonal projection onto $\mathfrak{a}$ and where dl is the normalized Haar measure on L, so that $J_{\underline{\lambda}}(0) = 1$.

Definition 8.1. The Hermite polynomial $\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$ related to the root system Σ of type C or D on $\mathfrak{a}$ is defined by the Rodrigues type formula:

$$\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda}) = \frac{1}{||p_{\underline{\mathbf{n}}}||^2_{\mathcal{F}}} p_{\underline{\mathbf{n}}}(\partial_x)\left(e^{\frac{\nu}{2}||x||^2} J_{\underline{\lambda}}(x)\right)|_{x=0}.$$

for those $\underline{\mathbf{n}}$ as determined in (4.2). Here $p_{\underline{\mathbf{n}}}(\partial_x)$ is the differential operator on V obtained from $p_{\underline{\mathbf{n}}}(x)$ by the same convention as in (2.3).

Conceptually it is better to write this as

$$\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda}) = \frac{1}{||p_{\underline{\mathbf{n}}}||^2_{\mathcal{F}}} p_{\underline{\mathbf{n}}}(\partial_z)(R^{-1}J_{\underline{\lambda}})(0) = \frac{1}{||p_{\underline{\mathbf{n}}}||^2_{\mathcal{F}}} p_{\underline{\mathbf{n}}}(\partial_z)(e^{\frac{\nu}{2}(z,\bar{z})} J_{\underline{\lambda}}(z))(0)$$

where we extend the function $e^{\frac{\nu}{2}||x||^2}$ on V to a *holomorphic* function $e^{\frac{\nu}{2}(z,\bar{z})}$ on the whole space $V_{\mathbb{C}}$.

The following expansion is a direct consequence of the definition.

Lemma 8.2. *The L-invariant function $e^{\frac{\nu}{2}||x||^2}J_{\underline{\lambda}}(x)$ on V has the following expansion in terms of $p_{\underline{\mathbf{n}}}(x)$*

$$e^{\frac{\nu}{2}||x||^2}J_{\underline{\lambda}}(x) = \sum_{\underline{\mathbf{n}}} p_{\underline{\mathbf{n}}}(x)\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda}), \tag{8.3}$$

where the summation is over all $\underline{\mathbf{n}}$ as determined in (4.1) and (4.2).

The next result computes the Fourier transform of the Segal-Bargmann transform $Up_{\underline{\mathbf{n}},\nu}$ of $p_{\underline{\mathbf{n}}}(\underline{\lambda})$. Denote $f(x)\mapsto \widetilde{f}(\underline{\lambda})$ the Fourier transform on V evaluated on $\mathfrak{a}^*$,

$$\widetilde{f}(\underline{\lambda}) = \int_V f(x)e^{i\underline{\lambda}(x)}dx, \qquad \underline{\lambda}\in\mathfrak{a}^*. \tag{8.4}$$

Proposition 8.3. *The Fourier transform $\widetilde{Up_{\underline{\mathbf{n}}}}(\underline{\lambda})$ of $Up_{\underline{\mathbf{n}}}$ is*

$$\widetilde{Up_{\underline{\mathbf{n}}}}(\underline{\lambda}) = (-\nu)^{-|\underline{\mathbf{n}}|}(\frac{2\pi}{\nu})^{\frac{d}{4}}\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})e^{-\frac{1}{4\nu}||\underline{\lambda}||^2} \tag{8.5}$$

Proof. The Berezin transform is the convolution operator with Gaussian kernel, whose Fourier transform is also a Gaussian. Namely $|R|^2$ has $e^{i\underline{\lambda}(x)}$ and therefore the Bessel function as its eigenfunction. More precisely, we have

$$|R|^2J_{\underline{\lambda}}(x) = (\frac{2\pi}{\nu})^{\frac{d}{2}}e^{-\frac{1}{2\nu}||\underline{\lambda}||^2}J_{\underline{\lambda}}(x). \tag{8.6}$$

We may rewrite it as

$$\int_V e^{\nu(x,y)}e^{-\frac{\nu}{2}||y||^2}J_{\underline{\lambda}}(x)dy = (\frac{2\pi}{\nu})^{\frac{d}{2}}(J_{\underline{\lambda}}(x)e^{\frac{\nu}{2}||x||^2})e^{-\frac{1}{2\nu}||\underline{\lambda}||^2}.$$

We differentiate both side by the differential operator $p_{\underline{\mathbf{n}}}(\partial_x)$ and evaluate at $x=0$. To do this, we observe that

$$p_{\underline{\mathbf{n}}}(\partial_x)e^{-\nu(x,y)}(0) = (-\nu)^{|\underline{\mathbf{n}}|}p_{\underline{\mathbf{n}}}(y),$$

so that the resulting formula is

$$(-\nu)^{|\underline{\mathbf{n}}|}\int_V p_{\underline{\mathbf{n}}}(y)e^{-\frac{\nu}{2}||y||^2}J_{\underline{\lambda}}(y)dy = (\frac{2\pi}{\nu})^{\frac{d}{2}}||p_{\underline{\mathbf{n}}}||^2_{\mathcal{F}}\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})e^{-\frac{1}{2\nu}||\underline{\lambda}||^2}. \tag{8.7}$$

The left hand side is actually

$$(-\nu)^{|\underline{\mathbf{n}}|}\widetilde{Rp_{\underline{\mathbf{n}}}}(\underline{\lambda}) = (-\nu)^{|\underline{\mathbf{n}}|}\widetilde{|R|Up_{\underline{\mathbf{n}}}}(\underline{\lambda}). \tag{8.8}$$

On the other hand formula the formula (8.6) implies that

$$\widetilde{|R|^2f}(\underline{\lambda}) = (\frac{2\pi}{\nu})^{\frac{d}{2}}e^{-\frac{1}{2\nu}||\underline{\lambda}||^2}\widetilde{f}(\underline{\lambda})$$

for any $f\in L^2(V)$. Thus

$$\widetilde{|R|f}(\underline{\lambda}) = (\frac{2\pi}{\nu})^{\frac{d}{4}}e^{-\frac{1}{4\nu}||\underline{\lambda}||^2}\widetilde{f}(\underline{\lambda}).$$

Substituting this into (8.8) for $f=Up_{\underline{\mathbf{n}}}$ we get

$$(-\nu)^{|\underline{\mathbf{n}}|}\widetilde{Rp_{\underline{\mathbf{n}}}}(\underline{\lambda}) = (-\nu)^{|\underline{\mathbf{n}}|}(\frac{2\pi}{\nu})^{\frac{d}{4}}e^{-\frac{1}{4\nu}||\underline{\lambda}||^2}\widetilde{Up_{\underline{\mathbf{n}}}}(\underline{\lambda}).$$

The equality (8.7) becomes now

$$(-\nu)^{|\underline{\mathbf{n}}|}(\frac{2\pi}{\nu})^{\frac{d}{4}}e^{-\frac{1}{4\nu}||\underline{\lambda}||^2}\widetilde{Up_{\underline{\mathbf{n}}}}(\underline{\lambda}) = (\frac{2\pi}{\nu})^{\frac{d}{2}}||p_{\underline{\mathbf{n}}}||^2_{\mathcal{F}}\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})e^{-\frac{1}{2\nu}||\underline{\lambda}||^2},$$

namely

$$\widetilde{Up_{\underline{\mathbf{n}}}}(\underline{\lambda}) = (\frac{2\pi}{\nu})^{\frac{d}{4}}(-\nu)^{-|\underline{\mathbf{n}}|}||p_{\underline{\mathbf{n}}}||^2_{\mathcal{F}}\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})e^{-\frac{1}{4\nu}||\underline{\lambda}||^2}, \tag{8.9}$$

completing the proof. □

The unitarity of U and the Fourier transform implies then the orthogonality relation of $\zeta_{\underline{\mathbf{n}},\nu}$. Let C_1 be the normalization constant so that the following measure is a probability measure on $\mathfrak{a}$:

$$C_1(\frac{2\pi}{\nu})^{\frac{d}{2}}e^{-\frac{1}{2\nu}||\underline{\lambda}||^2}\prod_{j=1}^{r}|\lambda_j|^{\iota-1}\prod_{1\le i<j\le r}^{r}|\lambda_i-\lambda_j|^a \tag{8.10}$$

for root system of type C, and

$$C_1(\frac{2\pi}{\nu})^{\frac{d}{2}}e^{-\frac{1}{2\nu}||\underline{\lambda}||^2}\prod_{1\le i<j\le r}^{r}|\lambda_i^2-\lambda_j^2|^a \tag{8.11}$$

for type D_r. (Note C_1 is independent to the parameter ν.) C_1 can be evaluated by the Selberg-Macdonald formula (proved by Opdam [27]).

Corollary 8.4. *The W-invariant polynomials $||p_{\underline{\mathbf{n}}}||_{\mathcal{F}_\nu}\zeta_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$ for an orthonormal basis for the Hilbert space of W-invariant L^2-functions with the above probability measure.*

The expansion (8.3) takes now the following form

Corollary 8.5. *The L-invariant function $e^{\frac{\nu}{2}||x||^2}J_{\underline{\lambda}}(x)$ on V has the following expansion in terms of $p_{\underline{\mathbf{n}}}(x)$, writing $q_{\underline{\mathbf{n}},\nu}=\frac{p_{\underline{\mathbf{n}}}(x)}{||p_{\underline{\mathbf{n}}}||_{\mathcal{F}_\nu}}$ the normalized L-invariant polynomial,*

$$e^{-\frac{1}{4\nu}||\underline{\lambda}||^2}e^{\frac{\nu}{2}||x||^2}J_{\underline{\lambda}}(x) = (\frac{2\pi}{\nu})^{\frac{d}{2}}\sum_{\underline{\mathbf{n}}}(-1)^{|\underline{\mathbf{n}}|}q_{\underline{\mathbf{n}},\nu}(x)\widetilde{Uq_{\underline{\mathbf{n}},\nu}}(\underline{\lambda}). \tag{8.12}$$

9. Segal-Bargmann transform of L-invariant polynomials: bounded case

The result is this section is parallel to that of the previous section and is an explicit realization of the Corollary 3.3. We will be brief; see also [29] and [28].

Recall the Berezin transform B_ν in (3.8). It is proved in [39] that B_ν defines a H-invariant bounded operator on $L^2(D)$ with the invariant measure (3.1). Let $b_\nu(\underline{\lambda})$ be its spectral symbol, namely

$$B_\nu\phi_{\underline{\lambda}} = b_\nu(\underline{\lambda})\phi_{\underline{\lambda}}$$

in the sense of spectral decomposition, where $\phi_{\underline{\lambda}}$ on D is the spherical function; the function $b_\nu(\underline{\lambda})$ is explicitly calculated in [39].

Define the polynomials $\xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$ by the Rodrigues type formula

$$\xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda}) = \frac{1}{\|p_{\underline{\mathbf{n}}}\|^2_{\mathcal{F}}} p_{\underline{\mathbf{n}}}(\partial_x)(h^{-\frac{\nu}{2}}(x)\phi_{\underline{\lambda}}(x))\big|_{x=0}.$$

Thus $\xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$ is W-invariant.

Theorem 9.1. *The L-invariant analytic function $h(x,\bar{x})^{-\frac{\nu}{2}}\phi_{\underline{\lambda}}(x)$, when extended to a holomorphic function $h(z,z)^{-\frac{\nu}{2}}\phi_{\underline{\lambda}}(z)$ in a neighborhood of D in $\mathcal{D}$, has the following expansion in terms of the L-invariant polynomials $p_{\underline{\mathbf{n}}}(z)$,*

$$h(z,z)^{-\frac{\nu}{2}}\phi_{\underline{\lambda}}(z) = \sum_{\underline{\mathbf{n}}} \xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda}) p_{\underline{\mathbf{n}}}(z) \tag{9.1}$$

near $z = 0$ in $\mathbb{D}$. Moreover the spherical transform of the Segal-Bargmann transform of $p_{\underline{\mathbf{n}}}$ is

$$\widehat{Up_{\underline{\mathbf{n}}}}(\underline{\lambda}) = \frac{1}{c_\nu^{\frac{1}{2}}} b_\nu(\underline{\lambda})^{\frac{1}{2}} \xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda}) \|p_{\underline{\mathbf{n}}}\|^2_\nu. \tag{9.2}$$

Thus $\|p_{\underline{\mathbf{n}}}\|_\nu \xi_{\underline{\mathbf{n}}}(\underline{\lambda})$ for all $\underline{\mathbf{n}}$ form an orthonormal basis for the space $L^2(\mathfrak{a}^*, C_0\frac{b_\nu(\underline{\lambda})}{c_\nu}|c(\underline{\lambda})|^{-2})^W$.

Conceptually it is better to write (9.1) in the form

$$(R^{-1}\phi_{\underline{\lambda}})(z) = h(z,z)^{-\frac{\nu}{2}}\phi_{\underline{\lambda}}(z) = \sum_{\underline{\mathbf{n}}} (\|p_{\underline{\mathbf{n}}}\|_\nu \xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda})) \otimes \frac{p_{\underline{\mathbf{n}}}(z)}{\|p_{\underline{\mathbf{n}}}\|_\nu}. \tag{9.3}$$

Note that we have extended the real analytic function $h(x,\bar{x})^{-\frac{\nu}{2}}$ to a *holomorphic* function, formally written as $h(z,z)^{-\frac{\nu}{2}}$, on $\mathbb{D}$. Notice also that both $\frac{p_{\underline{\mathbf{n}}}(z)}{\|p_{\underline{\mathbf{n}}}\|_\nu}$ and $\|p_{\underline{\mathbf{n}}}\|_\nu \xi_{\underline{\mathbf{n}}}(\underline{\lambda})$ are orthonormal basis in the respective Hilbert spaces. Thus $(R^{-1}\phi_{\underline{\lambda}})(z)$ is the Schwarz kernel for the unitary operator from $\mathcal{H}_\nu^L$ onto $L^2(\mathfrak{a}^*, C_0\frac{b_\nu(\underline{\lambda})}{c_\nu}|c(\underline{\lambda})|^{-2})^W$ obtained by taking the composition of the spherical and Segal-Bargmann transforms.

Remark 9.2. Associated to each root system there are the multi-variable Askey-Wilson polynomials determined by the root multiplicities (which are positive real numbers) and two more extra parameters [33]. Our polynomial $\xi_{\underline{\mathbf{n}},\nu}(\underline{\lambda})$ depend on extra parameter and can be reviewed as some limiting cases of the Askey-Wilson polynomials polynomials. We have thus found an alternative simple proof of the their orthogonality relation. Moreover the (known) expansion of $h^{-\frac{\nu}{2}}$ in terms of the Jack symmetric polynomials will also give some evaluation formula for the polynomials; we will however not pursue it here.

Remark 9.3. Consider the spherical function on the Hermitian symmetric space $\mathbb{D}$ or on its compact dual. There are Jacobi type functions. There have been attempts (see [1]) in expanding the Jacobi functions in terms of the Jack symmetric polynomials and study the combinatorial properties of the coefficients. It turned out (see [37], [28] and [38]) however that the expansion of the spherical transform multiplied with the reproducing kernel has a much better analytical significance. This is also the case here. Further more by considering analytic continuation in the parameter ν we may recover the expansion of the spherical functions itself.

Appendix 1: Evaluation of the constant C_0

We consider type D_r ($r \geq 3$) first, in this case $\Delta(x) = \delta(x)$ and the rank $r(\mathbb{D})$ is r.

It is proved in [12], Chapter IV, Exercise C4, that when the H-invariant measure $d\iota(z)$ on $D = H/K$ is (regularly) normalized so that

$$\int_D f(z) d\iota(z) = \int_{\mathfrak{a}^+} f(\exp H) \prod_{\alpha \in \Sigma^+} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha} d_n H$$

with $d_n H$ the regular normalization, then the Plancherel formula reads

$$|W| \int_D |f(z)|^2 d\iota(z) = \int_{\mathfrak{a}^*} |\widehat{f}(\underline{\lambda})||c(\underline{\lambda})|^{-2} d_n \underline{\lambda}.$$

By regular normalizations $d_n H$ on $\mathfrak{a}$ it is meant that the $d_n H$ is the Euclidean measure on $\mathfrak{a}$ induced by the restriction on it of the Killing form on $\mathfrak{g}$, multiplied with the factor $\frac{1}{\sqrt{2\pi}^r}$; by the Riesz lemma we get an identification of $\mathfrak{a}^*$ with $\mathfrak{a}$ and thus similarly get a regular measure $d_n \underline{\lambda}$ on $\mathfrak{a}^*$. In our case they are

$$d_n H = \frac{1}{\sqrt{2\pi}^r} \sqrt{2a(r-1)}^r dx_1 \cdots dx_r, \quad d_n \underline{\lambda} = \frac{1}{\sqrt{2\pi}^r} \left(\frac{2}{\sqrt{2a(r-1)}} \right)^r d\lambda_1 \cdots d\lambda_r,$$

if $H = x_1 \xi_1 + \ldots x_r \xi_r \in \mathfrak{a}^*$ and $\underline{\lambda} = \lambda_1 \beta_1 + \cdots + \lambda_r \beta_r \in \mathfrak{a}^*$ and the order $|W|$ of the Weyl group is $2^{r-1} r!$. There exists now a constant C_0 so that

$$\int_D f(z) \frac{dm(z)}{h^{\frac{p}{2}}(z)} = C_0 \int_D f(z) d\iota(z) = C_0 \int_{\mathfrak{a}^+} f(\exp H) \prod_{\alpha \in \Sigma^+} (e^{\alpha(H)} - e^{-\alpha(H)})^{m_\alpha} d_n H.$$

Take $f(z) = h(z, \bar{z})^\sigma$ for sufficiently large σ. The left hand side can, by performing Cayley transform $z \mapsto w = \frac{e+z}{e-z}$, $z = \frac{w-e}{w+e}$ mapping D to its the Siegel domain $\mathcal{S}$ (see [21], [39]), be evaluated eventually by the Gindikin Gamma function, while the right hand side is a kind of Selberg type integral and is also known. Let us carry out this calculation. Write I_1 for the left hand side,

$$I_1 = \int_D h^{\sigma - \frac{p}{2}}(z) dm(z) = 4^{r(\sigma - \frac{p}{2})} 2^d \int_{\mathcal{S}} \frac{\Delta(w)^{\sigma - \frac{p}{2}}}{\Delta(e+w)^{2\sigma - p}} \frac{dw}{\Delta(w+e)^{\frac{d}{r}}}$$

since the determinant of the differential of the Cayley transform $z = \gamma_{-1}(w)$ is

$$J_{\gamma_{-1}}(w) = 2^d \Delta(e+w)^{-2\frac{d}{r}},$$

(see [8], Chapter X, Proposition 2.4, for the calculation of the complex Jacobian of the Cayley transform), and that

$$h(z, \bar{z}) = 4^r \frac{\Delta(w)}{\Delta(e+w)^2}.$$

This integral over $\mathcal{S}$ is evaluated in [39], Proposition 4.2, the result is

$$I_1 = 4^{r(\sigma - \frac{p}{2})} 2^d \sqrt{\pi}^d 4^{n_A - r(\sigma + \frac{1}{2})} \frac{\Gamma_\Omega(\nu - \frac{d_B}{r})}{\Gamma_\Omega(\sigma + \frac{1}{2})} = \sqrt{\pi}^d \frac{\Gamma_\Omega(\sigma - \frac{d_B}{r})}{\Gamma_\Omega(\sigma + \frac{1}{2})};$$

here, adopting the notation there, $\Gamma_\Omega(x)$ is the Gindikin Gamma function. The right hand side is $\frac{1}{\sqrt{2\pi}^r}\sqrt{2a(r-1)}^r I_0$ where

$$I_0=\int_{x_1>x_2>\cdots>x_r>0}\prod_{j=1}^r(1-\tanh^2 x_j)^\sigma\prod_{i<j}(e^{x_i-x_j}-e^{x_j-x_i})^a(e^{x_i+x_j}-e^{-x_i-x_j})^a dx_1\cdots dx_r.$$

Perform change of variables $t_j=\tanh x_j$ then $dx_j=(1-t_j^2)^{-1}dt_j$ and

$$(e^{x_i-x_j}-e^{x_j-x_i})(e^{x_i+x_j}-e^{-x_i-x_j})=\frac{4(t_i^2-t_j^2)}{(1-t_i^2)(1-t_j^2)}.$$

So that

$$\begin{aligned}I_0&=4^{\frac{a}{2}r(r-1)}\int_{1>t_1>t_2>\cdots>t_r>0}\prod_{j=1}^r(1-t_j^2)^{\sigma-a(r-1)-1}\prod_{i<j}(t_i^2-t_j^2)^a dt_1\cdots dt_r\\ &=4^{\frac{a}{2}r(r-1)}\frac{1}{r!}\int_{[0,1]^r}\prod_{j=1}^r(1-t_j^2)^{\sigma-a(r-1)-1}\prod_{i<j}(t_i^2-t_j^2)^a dt_1\cdots dt_r.\end{aligned}\tag{9.4}$$

Changing again variables, letting $s_j=t_j^2$, the integral becomes

$$I_0=4^{\frac{a}{2}r(r-1)}2^{-r}\frac{1}{r!}\int_{[0,1]^r}\prod_{j=1}^r(1-s_j)^{\sigma-a(r-1)-1}\prod_{j=1}^r s_j^{-\frac{1}{2}}\prod_{i<j}(s_i-s_j)^a ds_1\cdots ds_r;\tag{9.5}$$

its value is

(9.6)

$$\frac{2^{ar(r-1)-r}}{r!}\prod_{i=1}^r\frac{\Gamma(\sigma-\frac{a}{2}(r-1)-\frac{a}{2}(i-1))\Gamma(\frac{1}{2}+\frac{a}{2}(r-1)-\frac{a}{2}(i-1))}{\Gamma(\sigma+\frac{1}{2}-\frac{a}{2}(i-1))}\frac{\prod_{j=1}^r\Gamma(\frac{a}{2}(r-j+1))}{\Gamma(\frac{a}{2})^r};$$

see [23], Chapter VI, example 7, pp. 385-386. One of the fraction in the above product is the following, and can also be written as

$$\prod_{i=1}^r\frac{\Gamma(\sigma-\frac{a}{2}(r-1)-\frac{a}{2}(i-1))}{\Gamma(\sigma+\frac{1}{2}-\frac{a}{2}(i-1))}=\frac{\Gamma_\Omega(\sigma-\frac{a}{2}(r-1))}{\Gamma_\Omega(\sigma+\frac{1}{2})}=\frac{\Gamma_\Omega(\sigma-\frac{d_B}{r})}{\Gamma_\Omega(\sigma+\frac{1}{2})};$$

this fraction appears also in I_1. Finally the constant C_0 is determined by the formula

$$I_1=C_0\frac{\sqrt{2ar}^r}{\sqrt{2\pi}^r}I_0$$

and we find that

$$C_0=\frac{\sqrt{2\pi}^r}{\sqrt{2ar}^r}\frac{I_1}{I_0}=\frac{\sqrt{2\pi}^r}{\sqrt{2ar}^r}\sqrt{\pi}^d\frac{r!}{2^{ar(r-1)-r}}\frac{\Gamma(\frac{a}{2})^r}{\prod_{j=1}^r\Gamma(\frac{a}{2}(r-j+1))\Gamma(\frac{1}{2}+\frac{a}{2}(r-1)-\frac{a}{2}(i-1))}$$

The constant for type C_r can be evaluated similarly and we leave it to the interested reader.

Appendix 2: Irreducible real bounded symmetric domains

We list here the the associated Lie algebras $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{h}, \mathfrak{l})$ of non-compact irreducible bounded symmetric domains $\mathbb{D} = G/K \subset V_{\mathbb{C}}$ and respectively their real bounded symmetric subdomain $D = H/L \subset V$, when V is not an Euclidean Jordan algebra; see [21]. They were also classified by Olafsson ([25], [9]) through Lie theoretic methods. The restricted root systems Σ of $(\mathfrak{h}, \mathfrak{l})$ are also indicated, the name of the types (Types B, BC, etc) is in consistence with Loos [21], Proposition 11.18.

Table 1. Irreducible real bounded symmetric subdomains

$\mathfrak{g}, \mathfrak{k}$	$\mathfrak{h}, \mathfrak{l}$	Σ
$\mathfrak{su}(r,r), s(\mathfrak{u}(r) \oplus \mathfrak{u}(r))$	$\mathfrak{so}(r,r), \mathfrak{so}(r) \oplus \mathfrak{so}(r)$	D_r
$\mathfrak{su}(r,r+b), s(\mathfrak{u}(r) \oplus \mathfrak{u}(r+b))$	$\mathfrak{so}(r,r+b), \mathfrak{so}(r) \oplus \mathfrak{so}(r+b)$	B_r
$\mathfrak{su}(2r,2r), s(\mathfrak{u}(2r) \oplus \mathfrak{u}(2r))$	$\mathfrak{sp}(r,r), \mathfrak{sp}(r) \oplus \mathfrak{sp}(r)$	C_r
$\mathfrak{su}(2r,2r+2b), s(\mathfrak{u}(2r) \oplus \mathfrak{u}(2r+2b))$	$\mathfrak{sp}(r,r+b), \mathfrak{sp}(r) \oplus \mathfrak{sp}(r+b)$	BC_r
$\mathfrak{so}^*(4r), \mathfrak{u}(2r)$	$\mathfrak{so}(2r,\mathbb{C}), \mathfrak{so}(2r)$	D_{2r}
$\mathfrak{so}^*(2r), \mathfrak{u}(r)$	$\mathfrak{so}(r,\mathbb{C}), \mathfrak{so}(r)$	B_r (r odd)
$\mathfrak{so}(2,p+q), \mathfrak{so}(2) \oplus \mathfrak{so}(p+q))$	$\mathfrak{so}(1,p) \oplus \mathfrak{so}(1,q), \mathfrak{so}(p) \oplus \mathfrak{so}(q)$	$D_2 (q \neq 0)$
$\mathfrak{so}(2,p), \mathfrak{so}(2) \oplus \mathfrak{so}(p))$	$\mathfrak{so}(1,p), \mathfrak{so}(p)$	C_1
$\mathfrak{sp}(2r,\mathbb{R}), \mathfrak{u}(2r)$	$\mathfrak{sp}(r,\mathbb{C}), \mathfrak{sp}(r)$	$C_r (b=0)$
$\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R}$	$\mathfrak{sp}(2,2), \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$	B_2
$\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R}$	$\mathfrak{f}_{4(-20)}, \mathfrak{so}(9)$	BC_1
$\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 + \mathbb{R}$	$\mathfrak{su}^*(8), \mathfrak{sp}(4)$	D_3

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