Classical dynamical r-matrices, Poisson homogeneous spaces, and Lagrangian subalgebras

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Abstract

In [18] Lu showed that any dynamical r-matrix for the pair $(\mathfrak{g}, \mathfrak{u})$ naturally induces a Poisson homogeneous structure on G/U. She also proved that if $\mathfrak{g}$ is complex simple, $\mathfrak{u}$ is its Cartan subalgebra and r is quasitriangular, then this correspondence is in fact 1-1. In the present paper we find some general conditions under which the Lu correspondence is 1-1. Then we apply this result to describe all triangular Poisson homogeneous structures on G/U for a simple complex group G and its reductive subgroup U containing a Cartan subgroup.

1 Introduction

The notion of a Poisson-Lie group was introduced almost 20 years ago by Drinfeld in [4]. Its infinitesimal counterpart, Lie bialgebras, were introduced in the same paper and later it was explained that these objects are in fact quasiclassical limits of quantum groups (see [5]). Lie bialgebra structures on a Lie algebra $\mathfrak{g}$ are in a natural 1-1 correspondence with Lie algebra structures on the vector space $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ with some compatibility conditions. $D(\mathfrak{g})$ with this Lie algebra structure is called the double of the Lie bialgebra $\mathfrak{g}$.

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The most popular and important class of Lie bialgebras is the class of quasitriangular Lie bialgebras (see [5]). They can be defined by an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ (called the classical r-matrix) such that

$$\Omega := r + r^{21}$$

is $\mathfrak{g}$-invariant, and the classical Yang-Baxter equation (CYBE)

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

is satisfied. If r is skew-symmetric, then one says that the corresponding Lie bialgebra is triangular. In general, $\Lambda := r - \frac{\Omega}{2}$ (i.e., the skew-symmetric part of r) satisfies the modified CYBE

$$[\Lambda^{12}, \Lambda^{13}] + [\Lambda^{12}, \Lambda^{23}] + [\Lambda^{13}, \Lambda^{23}] = \frac{1}{4}[\Omega^{12}, \Omega^{23}].$$

It is well known (and can be easily shown) that if $\mathfrak{g}$ is a complex simple finite-dimensional Lie algebra, then any Lie bialgebra structure on $\mathfrak{g}$ is quasitriangular. For the case $\Omega \neq 0$ ("quasitriangular case in the strict sense") they were classified by Belavin and Drinfeld, see [1, 2]. The triangular case was studied in [20, 21, 22].

In the paper [23] it was shown that for such $\mathfrak{g}$ there are only two possible structures of the $D(\mathfrak{g})$. In the triangular case $D(\mathfrak{g}) = \mathfrak{g}[\varepsilon] = \mathfrak{g} \oplus \mathfrak{g}\varepsilon$, where $\varepsilon^2 = 0$ and otherwise, $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$ (and $\mathfrak{g}$ is embedded diagonally into $\mathfrak{g} \times \mathfrak{g}$). Then it is clear that solutions of the CYBE (resp. the modified CYBE with $\Omega \neq 0$) are in a 1-1 correspondence with Lagrangian subalgebras $\mathfrak{l}$ in $\mathfrak{g}[\varepsilon]$ (resp. in $\mathfrak{g} \times \mathfrak{g}$) such that $\mathfrak{l} \cap \mathfrak{g} = 0$.

Along with the Poisson-Lie groups it is natural to study their Poisson actions, and in particular their Poisson homogeneous spaces. Drinfeld in [7] gave a general approach to the classification of Poisson homogeneous spaces. Namely, he showed that if G is a Poisson-Lie group, $\mathfrak{g}$ is the corresponding Lie bialgebra, then Poisson homogeneous G-spaces are essentially in a 1-1 correspondence with G-orbits on the set of all Lagrangian subalgebras in $D(\mathfrak{g})$. A classification of Lagrangian subalgebras in some important cases (including the case $\mathfrak{g}$ is complex simple, $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$) was obtained in [15, 16, 17].

At the same time an important generalization of the CYBE, the dynamical classical Yang-Baxter equation, was introduced in physics and mathematics. Notice that this equation is defined for a pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{u}$ is a subalgebra of $\mathfrak{g}$. From the mathematical point of view it was presented by Felder in [12, 13]. This equation and its quantum analogue were studied in many papers, see [10, 8, 19, 24]. First classification results for the solutions of the classical dynamical Yang-Baxter equation (dynamical r-matrices) were obtained by Etingof, Varchenko, and Schiffmann in [10, 19].

Later Lu ([18]) found a connection (which is essentially a 1-1 correspondence) between dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ (where $\mathfrak{u}$ is a Cartan subalgebra of the complex simple finite-dimensional algebra $\mathfrak{g}$), and Poisson homogeneous G-structures on G/U. Here $U \subset G$ are connected Lie groups corresponding to $\mathfrak{u} \subset \mathfrak{g}$, and G is equipped with the standard quasitriangular (with $\Omega \neq 0$) Poisson-Lie structure.

Lu also noticed that this connection can be generalized to the case $\mathfrak{u}$ is a subspace in a Cartan subalgebra (with some "regularity" condition). The dynamical r-matrices for the latter case were classified by Schiffmann in [19]. In this case connections between dynamical r-matrices and certain Lagrangian subalgebras can be derived directly from [19].

Now let G be a complex connected semisimple Lie group, and let U be its connected subgroup. Suppose $\mathfrak{u} \subset \mathfrak{g}$ be the corresponding Lie algebras. In the present paper we consider connections between Poisson homogeneous structures on G/U related to the triangular Poisson-Lie structures on G (i.e., with $\Omega = 0$), where U is a reductive subgroup containing a Cartan subgroup of G, and triangular dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$.

In fact, our results are based on a general result on relations between dynamical classical r-matrices and Poisson homogeneous structures (see Theorem 12), which is valid also in the quasitriangular case. Notice that the results of Sections 2 and 3 can be used to describe a 1-1 correspondence between dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{u} \subset \mathfrak{g}$ is a Cartan subalgebra, and Poisson homogeneous G-structures on G/U, where G is equipped with any quasitriangular (with $\Omega \neq 0$) Poisson-Lie structure (of course the latter result is due to Lu). Our approach is based on some strong classification results for dynamical r-matrices given recently by Etingof and Schiffmann in [9].

The paper is organized as follows. In Section 2 we describe a correspondence between the (moduli space of) dynamical r-matrices for a pair $(\mathfrak{g}, \mathfrak{u})$ and Poisson homogeneous G-structures on G/U proving that under certain assumptions it is a bijection. In Section 3 we consider a procedure of twisting for Lie bialgebras and examine its impact on the double $D(\mathfrak{g})$ and Poisson homogeneous spaces for corresponding Poisson-Lie groups. Then we use the twisting to weaken some restrictions needed in Section 2. In Section 4 we consider the basic example of our paper: $\mathfrak{g}$ is semisimple, $\mathfrak{u} \subset \mathfrak{g}$ is a reductive Lie subalgebra that contains some Cartan subalgebra of $\mathfrak{g}$, and the Lie bialgebra structure on $\mathfrak{g}$ is triangular (i.e., $D(\mathfrak{g}) = \mathfrak{g}[\varepsilon]$). Finally, in Appendix we present a general approach to the description of all Lagrangian subalgebras in $\mathfrak{g}[\varepsilon]$ and give a direct classification of the Lagrangian subalgebras $\mathfrak{l} \subset \mathfrak{g}[\varepsilon]$ such that $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{u}$.

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2 Classical dynamical r-matrices and Poisson homogeneous spaces

In this section we assume $\mathfrak{g}$ to be any finite-dimensional Lie algebra over $\mathbb{C}$. Let G be a connected Lie group such that $\mathrm{Lie}\, G = \mathfrak{g}$. Let $\mathfrak{u} \subset \mathfrak{g}$ be a Lie subalgebra (not necessary abelian). By U denote the connected subgroup in G such that $\mathrm{Lie}\, U = \mathfrak{u}$. We propose (under certain conditions) a connection between dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ and Poisson structures on G/U that make G/U a *Poisson* homogeneous G-space (for certain Poisson-Lie structures on G). Note that this connection was first introduced by Jiang-Hua Lu in [18] for the case $\mathfrak{g}$ is simple, $\mathfrak{u}$ is a Cartan subalgebra, and the dynamical r-matrix has non-zero coupling constant. Our result is inspired by [18].

In order to recall the definition of the classical dynamical r-matrix we need some notation. Let $x_1, ..., x_r$ be a basis of $\mathfrak{u}$. By D denote the formal neighborhood of 0 in $\mathfrak{u}^*$. By functions from D to a vector space V we mean the elements of the space $V[[x_1, ..., x_r]]$, where x_i are regarded as coordinates on D. If $\omega \in \Omega^k(D, V)$ is a k-form with values in a vector space V, we denote by $\overline{\omega} : D \to \wedge^k \mathfrak{u} \otimes V$ the corresponding function. Finally, for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ we define the classical Yang-Baxter operator

$$\mathrm{CYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

Recall that a *classical dynamical r-matrix* for the pair $(\mathfrak{g}, \mathfrak{u})$ is an $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the classical dynamical Yang-Baxter equation (CDYBE):

$$\mathrm{Alt}(\overline{dr}) + \mathrm{CYB}(r) = 0, \tag{1}$$

where for $x \in \mathfrak{g}^{\otimes 3}$, we let $\mathrm{Alt}(x) = x^{123} + x^{231} + x^{312}$ (see [9, 10, 8]). Usually one requires also an additional *quasi-unitarity condition*:

$$r + r^{21} = \Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}.$$

Note that if r satisfies the CDYBE and the quasi-unitarity condition then Ω is a constant function.

Suppose $\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$. Let us denote by $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ the set of all classical dynamical r-matrices r for the pair $(\mathfrak{g}, \mathfrak{u})$ such that $r + r^{21} = \Omega$.

Denote by $\mathbf{Map}(D, G)^{\mathfrak{u}}$ the set of all regular $\mathfrak{u}$-equivariant maps from D to G. Suppose $g \in \mathbf{Map}(D, G)^{\mathfrak{u}}$. For any $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ set

$$r^g = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(r - \overline{\eta_g} + \overline{\eta_g}^{21} + \tau_g),$$

where $\eta_g = g^{-1} dg$, and $\tau_g(\lambda) = (\lambda \otimes 1 \otimes 1)([\overline{\eta_g}^{12}, \overline{\eta_g}^{13}](\lambda))$. Then r is a classical dynamical r-matrix iff r^g is (see [9]). The transformation $r \mapsto r^g$ is called a gauge transformation. It is indeed an action of the group $\mathbf{Map}(D,G)^{\mathfrak{u}}$ on $\mathbf{Dynr}(\mathfrak{g},\mathfrak{u},\Omega)$ (i.e., $(r^{g_1})^{g_2}=r^{g_2g_1}$). Following [9] denote by $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ the moduli space $\mathbf{Map}_0(D,G)^{\mathfrak{u}}\backslash\mathbf{Dynr}(\mathfrak{g},\mathfrak{u},\Omega)$, where $\mathbf{Map}_0(D,G)^{\mathfrak{u}}$ is the subgroup in $\mathbf{Map}(D,G)^{\mathfrak{u}}$ consisting of maps g satisfying $g(0)=e$.

In what follows we need the following notation. Suppose $a\in\mathfrak{g}^{\otimes k}$. By $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) denote the left (resp. right) invariant tensor field on G corresponding to a.

Suppose $\rho\in\mathfrak{g}\otimes\mathfrak{g}$ satisfies the classical Yang-Baxter equation (CYBE), i.e., $\mathrm{CYB}(\rho)=0$. Assume also that $\rho+\rho^{21}=\Omega$ (i.e., $\rho=\frac{\Omega}{2}+\Lambda$, where $\Lambda\in\wedge^2\mathfrak{g}$). Introduce a bivector field $\pi_\rho=\overrightarrow{\rho}-\overleftarrow{\rho}=\overrightarrow{\Lambda}-\overleftarrow{\Lambda}$ on G. It is well known that (G,π_ρ) is a Poisson-Lie group.

Now let $r\in\mathbf{Dynr}(\mathfrak{g},\mathfrak{u},\Omega)$. We have $r=\frac{\Omega}{2}+A$, where $A\in\wedge^2\mathfrak{g}$. Set $\tilde{\pi}_r:=\overrightarrow{r(0)}-\overleftarrow{\rho}=\overrightarrow{A(0)}-\overleftarrow{\Lambda}$. Consider a bivector field π_r on G/U defined by $\pi_r(\underline{g})=p_*\tilde{\pi}_r(g)$, where $p:G\to G/U$ is the natural projection, and $\underline{g}=p(g)$. Note that π_r is well defined since $r(0)\in(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{u}}$.

The following proposition belongs to Jiang-Hua Lu [18] (note that in [18] it is stated for the case $\mathfrak{g}$ is simple, $\mathfrak{u}$ is a Cartan subalgebra, but the proof fits the general case).

Proposition 1. *The bivector field π_r is Poisson, and $(G/U,\pi_r)$ is a Poisson homogeneous (G,π_ρ)-space.* □

Proposition 2. *Suppose $g\in\mathbf{Map}_0(D,G)^{\mathfrak{u}}$. Then $\pi_r=\pi_{r^g}$.*

Proof. Since $(G/U,\pi_r)$ is a Poisson homogeneous (G,π_ρ)-space, we see that π_r depends only on $\pi_r(\underline{e})=$ the image of $r(0)-\rho$ in $\wedge^2(\mathfrak{g}/\mathfrak{u})$. Thus it is enough to note that $r^g(0)-r(0)\in\mathfrak{u}\otimes\mathfrak{g}+\mathfrak{g}\otimes\mathfrak{u}$. □

Corollary 3. *The correspondence $r\mapsto\pi_r$ defines a map from $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ to the set of all Poisson (G,π_ρ)-homogeneous structures on G/U.* □

Suppose now that the following conditions hold:

(a) $\mathfrak{u}$ has an $\mathfrak{u}$-invariant complement $\mathfrak{m}$ in $\mathfrak{g}$ (we fix one).

(b) $\Omega\in(\mathfrak{u}\otimes\mathfrak{u})\oplus(\mathfrak{m}\otimes\mathfrak{m})$.

(c) $\rho\in\frac{\Omega}{2}+(\wedge^2\mathfrak{m})^{\mathfrak{u}}$.

Theorem 4. *Under the assumptions above the correspondence $r\mapsto\pi_r$ is a bijection between $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ and the set of all Poisson (G,π_ρ)-homogeneous structures on G/U.*

The rest of this section is devoted to the proof of Theorem 4. First we recall some results from [9]. Assume that (a) holds. Set

$$\mathcal{M}_\Omega=\left\{x\in\frac{\Omega}{2}+(\wedge^2\mathfrak{m})^{\mathfrak{u}}\;\middle|\;\mathrm{CYB}(x)=0\text{ in }\wedge^3(\mathfrak{g}/\mathfrak{u})\right\}.$$

Theorem 5 (Etingof, Schiffmann [9]). *1. Any class $\mathcal{C} \in \mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ has a representative $r \in \mathcal{C}$ such that $r(0) \in \mathcal{M}_\Omega$. Moreover, this defines an embedding $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$.*

2. Assume that (b) holds. Then the map $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$ defined above is a bijection. □

Now suppose $b \in (\wedge^2(\mathfrak{g}/\mathfrak{u}))^{\mathfrak{u}} = (\wedge^2\mathfrak{m})^{\mathfrak{u}}$. Set $\pi(\underline{g}) = (L_g)_* b + p_*\pi_\rho(g)$. Since ρ is $\mathfrak{u}$-invariant, we see that $\pi_\rho(g) = 0$ for $g \in U$; therefore π is a well-defined bivector field on G/U.

Proposition 6. *The bivector field π is Poisson iff* $\mathrm{CYB}(\rho + b) = 0$ *in* $\wedge^3(\mathfrak{g}/\mathfrak{u})$.

Proof. Set $a = \Lambda + b$. Define a bivector field $\tilde{\pi}$ on G by the formula $\tilde{\pi} = \overrightarrow{a} - \overleftarrow{\Lambda}$. Note that $\tilde{\pi} = \overrightarrow{b} + \pi_\rho$, therefore $\pi = p_*\tilde{\pi}$. Let us normalize the Schouten bracket of the bivector fields on G in a way that $[\overrightarrow{x}, \overrightarrow{x}] = \overrightarrow{\mathrm{CYB}(x)}$ for all $x \in \wedge^2\mathfrak{g}$. Then we have

$$[\tilde{\pi}, \tilde{\pi}] = [\overrightarrow{a}, \overrightarrow{a}] - 2[\overrightarrow{a}, \overleftarrow{\Lambda}] + [\overleftarrow{\Lambda}, \overleftarrow{\Lambda}] = \overrightarrow{\mathrm{CYB}(a)} - \overleftarrow{\mathrm{CYB}(\Lambda)}.$$

Since $\rho = \frac{\Omega}{2} + \Lambda$ satisfies the CYBE, we see that $\mathrm{CYB}(\Lambda) = \frac{1}{4}[\Omega^{12}, \Omega^{23}] \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. Thus

$$[\tilde{\pi}, \tilde{\pi}] = \overrightarrow{\mathrm{CYB}(a) - \frac{1}{4}[\Omega^{12}, \Omega^{23}]} = \overrightarrow{\mathrm{CYB}\left(\frac{\Omega}{2} + a\right)} = \overrightarrow{\mathrm{CYB}(\rho + b)}.$$

To finish the proof it is enough to note that $[\pi, \pi] = p_*[\tilde{\pi}, \tilde{\pi}]$. □

Proof of Theorem 4. Let us construct the inverse map. Suppose $(G/U, \pi)$ is a Poisson homogeneous (G, π_ρ)-space. Set $b = \pi(\underline{e}) \in \wedge^2(\mathfrak{g}/\mathfrak{u}) = \wedge^2\mathfrak{m}$. The condition (c) implies that in fact $b \in (\wedge^2(\mathfrak{g}/\mathfrak{u}))^{\mathfrak{u}} = (\wedge^2\mathfrak{m})^{\mathfrak{u}}$. Furthermore, (c) yields that $\rho + b \in \frac{\Omega}{2} + (\wedge^2\mathfrak{m})^{\mathfrak{u}}$. By Proposition 6, we have $\mathrm{CYB}(\rho + b) = 0$ in $\wedge^3(\mathfrak{g}/\mathfrak{u})$, i.e., $\rho + b \in \mathcal{M}_\Omega$. Then, by Theorem 5, there exists $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ such that $r(0) = \rho + b$, and the image of r in $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ is uniquely determined. It is now easy to verify that $\pi = \pi_r$. □

3 Twisting of Poisson homogeneous structures

Assume again that $\mathfrak{g}$ is an arbitrary finite-dimensional Lie algebra over $\mathbb{C}$. Recall that a Lie bialgebra structure on $\mathfrak{g}$ is a 1-cocycle $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ which satisfies the co-Jacobi identity. Denote by $\mathcal{D}(\mathfrak{g}, \delta)$ the classical double of $(\mathfrak{g}, \delta)$.

We say that two Lie bialgebra structures δ_1, δ_2 on $\mathfrak{g}$ are in the same class if there exists a Lie algebra isomorphism $f : \mathcal{D}(\mathfrak{g}, \delta_1) \to \mathcal{D}(\mathfrak{g}, \delta_2)$, which preserves the canonical forms Q_i on $\mathcal{D}(\mathfrak{g}, \delta_i)$, and such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathcal{D}(\mathfrak{g}, \delta_1) \\ & \searrow & \downarrow f \\ & & \mathcal{D}(\mathfrak{g}, \delta_2) \end{array}$$

Theorem 7. *Two Lie bialgebra structures δ, δ' on $\mathfrak{g}$ are in the same class if and only if $\delta' = \delta + ds$, where $s \in \wedge^2\mathfrak{g}$ and*

$$\mathrm{CYB}(s) = \mathrm{Alt}(\delta \otimes \mathrm{id})(s). \tag{2}$$

Proof. ($\Rightarrow$) Let us consider $\mathcal{D}(\mathfrak{g}, \delta)$. Then δ' is uniquely defined by a Lagrangian subalgebra $\mathfrak{l} \subset \mathcal{D}(\mathfrak{g}, \delta)$ such that $\mathfrak{l} \cap \mathfrak{g} = 0$. Clearly, $\mathfrak{l}$ is the graph of a linear map $S : \mathfrak{g}^* \to \mathfrak{g}$. Define an element $s = \sum_i s_i' \otimes s_i'' \in \mathfrak{g} \otimes \mathfrak{g}$ via

$$S(l) = \sum_i \langle l, s_i' \rangle s_i'' \tag{3}$$

for any $l \in \mathfrak{g}^*$. Since $\mathfrak{l}$ is Lagrangian, we see that s is skew-symmetric. Let us show that $\delta' = \delta + ds$.

Indeed, for any $a \in \mathfrak{g}$, $l_1, l_2 \in \mathfrak{g}^*$,

$$\begin{aligned} \langle \delta'(a), l_1 \otimes l_2 \rangle &= Q(\delta'(a), (S(l_1) + l_1) \otimes (S(l_2) + l_2)) = \\ &= Q(a, [S(l_1) + l_1, S(l_2) + l_2]) = \\ &= \langle a, [l_1, l_2] \rangle + Q(a, [S(l_1), l_2]) + Q(a, [l_1, S(l_2)]), \end{aligned}$$

and

$$\begin{aligned} \langle a, [l_1, l_2] \rangle &= \langle \delta(a), l_1 \otimes l_2 \rangle, \\ Q(a, [S(l_1), l_2]) &= \langle [a, S(l_1)], l_2 \rangle = \langle [1 \otimes a, s], l_1 \otimes l_2 \rangle, \\ Q(a, [l_1, S(l_2)]) &= -\langle [a, S(l_2)], l_1 \rangle = -\langle [1 \otimes a, s], l_2 \otimes l_1 \rangle = \\ &= \langle [1 \otimes a, s^{21}], l_2 \otimes l_1 \rangle = \langle [a \otimes 1, s], l_1 \otimes l_2 \rangle, \end{aligned}$$

where $\langle \cdot\,, \cdot \rangle$ is the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, and Q is the canonical bilinear form on $\mathcal{D}(\mathfrak{g}, \delta)$.

Now let $\{e_i\}$ be an arbitrary basis in $\mathfrak{g}$ and $\{f^i\}$ be its dual in $\mathfrak{g}^* \subset \mathcal{D}(\mathfrak{g}, \delta)$. Then the canonical element $r_\delta = \sum_i e_i \otimes f^i \in \mathcal{D}(\mathfrak{g}, \delta)^{\otimes 2}$ satisfies the CYBE and $r_{\delta'} = r_\delta + s$ satisfies the CYBE as well (since $r_\delta + s$ is the canonical element for the double $\mathcal{D}(\mathfrak{g}, \delta')$). It is easy to show that these two facts imply (2).

($\Leftarrow$) $s \in \wedge^2\mathfrak{g}$ defines $S : \mathfrak{g}^* \to \mathfrak{g}$ via (3) and the graph of S is $\mathfrak{l} \subset \mathcal{D}(\mathfrak{g}, \delta)$, a Lagrangian subspace because s is skew-symmetric. Let us prove that for any $l_1, l_2, l_3 \in \mathfrak{g}^*$,

$$\langle l_1 \otimes l_2 \otimes l_3, \mathrm{CYB}(s) - \mathrm{Alt}(\delta \otimes \mathrm{id})(s) \rangle = Q([l_1 + S(l_1), l_2 + S(l_2)], l_3 + S(l_3)).$$

Let us verify that, for instance,

$$\langle l_1 \otimes l_2 \otimes l_3, [s^{12}, s^{13}] \rangle = Q([l_1, S(l_2)], S(l_3)).$$

Indeed, if $s = \sum_i s'_i \otimes s''_i$, then we have

$$[s^{12}, s^{13}] = \sum_{i,j} [s'_i, s'_j] \otimes s''_i \otimes s''_j$$

and

$$\begin{aligned}\langle l_1 \otimes l_2 \otimes l_3, [s^{12}, s^{13}]\rangle &= \langle l_1, \sum_{i,j} [\langle s''_i, l_2\rangle s'_i, \langle s''_j, l_3\rangle s'_j]\rangle = \\ &= \langle l_1, \sum_{i,j} [\langle s'_i, l_2\rangle s''_i, \langle s'_j, l_3\rangle s''_j]\rangle = \langle l_1, [S(l_2), S(l_3)]\rangle = \\ &= Q(l_1, [S(l_2), S(l_3)]) = Q([l_1, S(l_2)], S(l_3)).\end{aligned}$$

Similarly,

$$-\langle l_1 \otimes l_2 \otimes l_3, (\delta \otimes \mathrm{id})(s)\rangle = -\langle [l_1, l_2] \otimes l_3, s\rangle = \langle [l_1, l_2], S(l_3)\rangle = Q([l_1, l_2], S(l_3)),$$

and so on. Since $Q([l_1, l_2], l_3)$ and $Q([S(l_1), S(l_2)], S(l_3))$ vanish, the identity is proved.

Now it follows that $Q([l_1 + S(l_1), l_2 + S(l_2)], l_3 + S(l_3)) = 0$ for any $l_1, l_2, l_3 \in \mathfrak{g}^*$. Since $\mathfrak{l}$ is Lagrangian, we conclude that $[l_1 + S(l_1), l_2 + S(l_2)] \in \mathfrak{l}$ and hence $\mathfrak{l}$ is a subalgebra. Clearly, $\mathfrak{l}$ defines $\delta' := \delta + ds$, and this completes the proof of the theorem. □

Remark 8. If we consider our Lie bialgebra $(\mathfrak{g}, \delta)$ as a Lie quasibialgebra, then $(\mathfrak{g}, \delta + ds)$ is called "twisting via s". The notions of Lie quasibialgebra and twisting via s was introduced by Drinfeld in [6]. The theorem above can be also deduced from results of [6].

Further, we are going to examine the effect of the twisting on Poisson homogeneous spaces. First we recall some definitions and rather well-known results.

Let G be a connected complex Poisson-Lie group, $(\mathfrak{g}, \delta)$ its Lie bialgebra, and $\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}, \delta)$ the corresponding classical double of $\mathfrak{g}$ with the canonical invariant form Q.

Recall that an action of G on a Poisson manifold M is called Poisson if the defining map $G \times M \to M$ is a Poisson map, where $G \times M$ is equipped with the product Poisson structure. If the action is transitive, we say that M is a *Poisson homogeneous G-space.*

Let M be a homogeneous G-space, and let π be any bivector field on M. For any $x \in M$ let us consider the map

$$\pi_x : T_x^* M \to T_x M, \ \pi_x(l) = (l \otimes \mathrm{id})(\pi(x)).$$

On the other hand, $M \cong G/H_x$ and $T_x M = \mathfrak{g}/\mathfrak{h}_x$, $T_x^* M = (\mathfrak{g}/\mathfrak{h}_x)^* = \mathfrak{h}_x^\perp \subset \mathfrak{g}^*$, where $\mathfrak{h}_x = \mathrm{Lie}\, H_x$. Therefore we can consider π_x as a map $\pi_x : \mathfrak{h}_x^\perp \to \mathfrak{g}/\mathfrak{h}_x$.

Now let us consider the following set of subspaces in $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$:

$$\mathfrak{l}_x = \{a + l \mid a \in \mathfrak{g},\, l \in \mathfrak{h}_x^{\perp},\, \pi_x(l) = \overline{a}\}, \tag{4}$$

where $\overline{a}$ is the image of a in $\mathfrak{g}/\mathfrak{h}_x$. Observe that $\mathfrak{l}_x$ are Lagrangian (i.e., maximal isotropic) subspaces, and $\mathfrak{l}_x \cap \mathfrak{g} = \mathfrak{h}_x$. The following result was obtained in [7].

Theorem 9 (Drinfeld [7]). *(M, π) is a Poisson homogeneous G-space if and only if for any $x \in M$ $\mathfrak{l}_x$ is a subalgebra of $\mathcal{D}(\mathfrak{g})$, and $\mathfrak{l}_{gx} = \mathrm{Ad}_g \mathfrak{l}_x$ for all $g \in G$.* □

Now set $\delta' = \delta + ds$, where $s \in \wedge^2 \mathfrak{g}$ satisfies (2). Then we have two Poisson-Lie groups, (G, π_δ) and $(G, \pi_{\delta'})$, whose Lie bialgebras are $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, \delta')$ respectively. Let (M, π) be a Poisson homogeneous (G, π_δ)-space. Consider a bivector field ξ on M defined by the formula $\xi(x) =$ the image of s in $\wedge^2(\mathfrak{g}/\mathfrak{h}_x) = \wedge^2 T_x M$. Set $\pi' = \pi - \xi$.

Proposition 10. *(M, π') is a Poisson homogeneous $(G, \pi_{\delta'})$-space, and thus one obtains a bijection between the sets of all Poisson (G, π_δ)- and $(G, \pi_{\delta'})$-homogeneous structures on M.*

Proof. Theorem 7 allows one to identify $\mathcal{D}(\mathfrak{g}, \delta)$ and $\mathcal{D}(\mathfrak{g}, \delta')$. It is easy to verify that under this identification the sets of Lagrangian subspaces that correspond to (M, π) and (M, π') are the same. This completes the proof, according to Theorem 9. □

Finally, we are going to generalize the main result of the previous section to the twisted case. Assume that $(\mathfrak{g}, \delta)$ is a quasitriangular Lie bialgebra, i.e., $\delta = d\rho$, where $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ and $\mathrm{CYB}(\rho) = 0$. It is easy to verify that the condition (2) for an element $s \in \wedge^2 \mathfrak{g}$ is equivalent to

$$\mathrm{CYB}(s) + [\rho, s] + [s, \rho] = 0, \tag{5}$$

where for $a, b \in \mathfrak{g}^{\otimes 2}$ we set $[a, b] = [a^{12}, b^{13}] + [a^{12}, b^{23}] + [a^{13}, b^{23}] \in \mathfrak{g}^{\otimes 3}$ (i.e., $\mathrm{CYB}(a) = [a, a]$).

Fix $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and assume that $\rho \in \frac{\Omega}{2} + \wedge^2 \mathfrak{g}$. As before, consider the Poisson-Lie group (G, π_δ), where $\pi_\delta = \pi_\rho = \overrightarrow{\rho} - \overleftarrow{\rho}$. Suppose $s \in \wedge^2 \mathfrak{g}$ satisfies (5). Set $\delta' = \delta + ds = d(\rho + s)$; then $\pi_{\delta'} = \pi_{\rho,s} := \overrightarrow{\rho + s} - \overleftarrow{\rho + s}$, and $(G, \pi_{\rho,s})$ is a Poisson-Lie group.

Let U be a connected Lie subgroup in G, and $\mathfrak{u} = \mathrm{Lie}\, U$. Consider $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$. As usually, set $\tilde{\pi}_r = \overrightarrow{r(0)} - \overleftarrow{\rho}$ and denote by π_r the natural projection of $\tilde{\pi}_r$ on G/U. By Proposition 1, $(G/U, \pi_r)$ is a Poisson homogeneous (G, π_ρ)-space. Set also $\tilde{\pi}_{r,s} = \tilde{\pi}_r - \overleftarrow{s} = \overrightarrow{r(0)} - \overleftarrow{\rho + s}$ and denote by $\pi_{r,s}$ its projection on G/U. According to Proposition 10, $(G/U, \pi_{r,s})$ is a Poisson homogeneous $(G, \pi_{\rho,s})$-space.

Moreover, if we combine Theorem 4 and Proposition 10, we get the following

Theorem 11. *Assume that* $\mathfrak{u}$, Ω, *and* ρ *satisfy the conditions (a), (b), and (c) from the previous section. Then the correspondence* $r \mapsto \pi_{r,s}$ *is a bijection between* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *and the set of all Poisson* $(G, \pi_{\rho,s})$*-homogeneous structures on* G/U. □

Clearly, this can be reformulated as follows:

Theorem 12. *Assume that* $\mathfrak{u}$ *and* Ω *satisfy the conditions (a) and (b) from the previous section. Suppose also that there exists* $s \in \wedge^2\mathfrak{g}$ *such that (5) holds, and* $\rho + s \in \frac{\Omega}{2} + (\wedge^2\mathfrak{m})^{\mathfrak{u}}$. *Then the correspondence* $r \mapsto \pi_r$ *is a bijection between* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *and the set of all Poisson* (G, π_ρ)*-homogeneous structures on* G/U. □

Let us apply our previous results to the triangular case.

Corollary 13. *Assume that* $\mathfrak{u}$ *satisfies the condition (a) from the previous section. Set* $\Omega = 0$. *Consider any* $\rho \in \wedge^2\mathfrak{g}$ *that satisfies the CYBE. Then the correspondence* $r \mapsto \pi_r$ *is a bijection between* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *and the set of all Poisson* (G, π_ρ)*-homogeneous structures on* G/U.

Proof. Set $s = -\rho$ and apply Theorem 12. □

4 Poisson homogeneous structures in triangular case

Now assume that $\mathfrak{g}$ is semisimple. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\mathbf{R}$ the corresponding root system. In this section we apply the results of the previous sections to the case $\mathfrak{u}$ is reductive Lie subalgebra in $\mathfrak{g}$ containing $\mathfrak{h}$, $\Omega = 0$, and $\rho \in \wedge^2\mathfrak{g}$ such that $\mathrm{CYB}(\rho) = 0$.

To be more precise, consider $\mathbf{U} \subset \mathbf{R}$, and suppose $\mathfrak{u} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\mathbf{U}} \mathfrak{g}_\alpha)$ is a reductive Lie subalgebra in $\mathfrak{g}$. If this is the case, then we say that a subset $\mathbf{U} \subset \mathbf{R}$ is *reductive* (i.e., $(\mathbf{U} + \mathbf{U}) \cap \mathbf{R} \subset \mathbf{U}$ and $-\mathbf{U} = \mathbf{U}$; see [14, Ch. 6, §1.2]). Condition (a) is satisfied since $\mathfrak{m} = \bigoplus_{\alpha\in\mathbf{R}\setminus\mathbf{U}} \mathfrak{g}_\alpha$ is an $\mathfrak{u}$-invariant complement to $\mathfrak{u}$ in $\mathfrak{g}$.

Applying Corollary 13 (and results of Etingof and Schiffmann cited in Section 2), we get:

1. Any structure of a Poisson homogeneous (G, π_ρ)-space on G/U is of the form $p_*(\overrightarrow{x} - \overleftarrow{\rho})$, where $x \in \mathcal{M}_\Omega$.
2. If $x \in \mathcal{M}_\Omega$, then there exists (a unique up to the $\mathbf{Map}_0(D, G)^{\mathfrak{u}}$-action) $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ such that $r(0) = x$.

Let us now describe $\mathcal{M}_\Omega$ and thus get an explicit description of all G-invariant Poisson structures on G/U. Recall that in our case by definition

$$\mathcal{M}_\Omega = \left\{ x \in (\wedge^2\mathfrak{m})^{\mathfrak{u}} \;\middle|\; \mathrm{CYB}(x) = 0 \text{ in } \wedge^3(\mathfrak{g}/\mathfrak{u}) \right\}.$$

We need to fix some notation. Fix a nondegenerate invariant bilinear form (*invariant scalar product*) $\langle \cdot , \cdot \rangle$ on $\mathfrak{g}$. For any $\alpha \in \mathbf{R}$ choose $E_\alpha \in \mathfrak{g}_\alpha$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$. Further, suppose $\mathbf{N}$ is a reductive subset which contains $\mathbf{U}$. We say that $h \in \mathfrak{h}$ is $(\mathbf{N}, \mathbf{U})$-*regular* if $\alpha(h) = 0$ for all $\alpha \in \mathbf{U}$, and $\alpha(h) \neq 0$ for all $\alpha \in \mathbf{N} \setminus \mathbf{U}$.

Proposition 14. *$x \in \mathcal{M}_\Omega$ iff*

$$x = x_{\mathbf{N},h} = \sum_{\alpha \in \mathbf{N} \setminus \mathbf{U}} \frac{1}{\alpha(h)} E_\alpha \otimes E_{-\alpha}, \tag{6}$$

where $\mathbf{N}$ is a reductive subset in $\mathbf{R}$ containing $\mathbf{U}$, and $h \in \mathfrak{h}$ is $(\mathbf{N}, \mathbf{U})$-regular.

Proof. First we calculate $(\wedge^2 \mathfrak{m})^{\mathfrak{u}}$. It is easy to see that $x \in \wedge^2 \mathfrak{m}$ is $\mathfrak{h}$-invariant iff it is of the form

$$x = \sum_{\alpha \in \mathbf{R} \setminus \mathbf{U}} x_\alpha \cdot E_\alpha \otimes E_{-\alpha},$$

where $x_{-\alpha} = -x_\alpha$.

Define $c_{\alpha\beta}$ by the formula $[E_\alpha, E_\beta] = c_{\alpha\beta} E_{\alpha+\beta}$ for $\alpha, \beta, \alpha + \beta \in \mathbf{R}$.

Furthermore, suppose $\gamma \in \mathbf{U}$. One can easily verify that the condition $\mathrm{ad}_{E_\gamma}(x) = 0$ is equivalent to the following statement: for all $\alpha, \beta \in \mathbf{R} \setminus \mathbf{U}$ such that $\alpha + \beta + \gamma = 0$ we have $c_{\alpha\gamma} x_\alpha = c_{\beta\gamma} x_\beta$.

Lemma 15. *Suppose $\alpha, \beta, \gamma \in \mathbf{R}$, $\alpha + \beta + \gamma = 0$. Then $c_{\alpha\gamma} + c_{\beta\gamma} = 0$.*

Proof. $c_{\alpha\gamma} = c_{\alpha\gamma} \langle E_{-\beta}, E_\beta \rangle = \langle [E_\alpha, E_\gamma], E_\beta \rangle = \langle E_\alpha, [E_\gamma, E_\beta] \rangle = -c_{\beta\gamma} \langle E_\alpha, E_{-\alpha} \rangle = -c_{\beta\gamma}$. □

Therefore we obtain

Lemma 16. *$x \in (\wedge^2 \mathfrak{m})^{\mathfrak{u}}$ iff*

$$x = \sum_{\alpha \in \mathbf{R} \setminus \mathbf{U}} x_\alpha \cdot E_\alpha \otimes E_{-\alpha},$$

where $x_{-\alpha} = -x_\alpha$, and for all $\alpha, \beta \in \mathbf{R} \setminus \mathbf{U}$, $\gamma \in \mathbf{U}$, $\alpha + \beta + \gamma = 0$, we have $x_\alpha + x_\beta = 0$. □

Lemma 17. *Suppose*

$$x = \sum_{\alpha \in \mathbf{R} \setminus \mathbf{U}} x_\alpha \cdot E_\alpha \otimes E_{-\alpha} \in (\wedge^2 \mathfrak{m})^{\mathfrak{u}}.$$

Then $x \in \mathcal{M}_\Omega$ iff the following condition holds: for all $\alpha, \beta, \gamma \in \mathbf{R} \setminus \mathbf{U}$, $\alpha + \beta + \gamma = 0$, we have $x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha = 0$.

Proof. One can check directly (using Lemma 15) that the image of $\mathrm{CYB}(x)$ in $\wedge^3(\mathfrak{g}/\mathfrak{u}) = \wedge^3\mathfrak{m}$ is equal to

$$\sum_{\substack{\alpha,\beta,\gamma\in \mathbf{R}\setminus\mathbf{U},\\ \alpha+\beta+\gamma=0}} c_{-\alpha,-\beta}\,(x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha)\, E_\alpha \otimes E_\beta \otimes E_\gamma.$$

This immediately proves the lemma. □

Now consider the following properties of the function $\mathbf{R}\setminus\mathbf{U}\to\mathbb{C}$, $\alpha\mapsto x_\alpha$:

(d) $x_{-\alpha} = -x_\alpha$ for all $\alpha\in\mathbf{R}\setminus\mathbf{U}$.

(e) If $\alpha,\beta\in\mathbf{R}\setminus\mathbf{U}$, $\gamma\in\mathbf{U}$, $\alpha+\beta+\gamma=0$, then $x_\alpha+x_\beta=0$.

(f) If $\alpha,\beta,\gamma\in\mathbf{R}\setminus\mathbf{U}$, $\alpha+\beta+\gamma=0$, then $x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha = 0$.

Lemma 18. *x_α satisfies (d)–(f) iff*

$$x_\alpha = \begin{cases} 1/\alpha(h), & \text{if } \alpha\in\mathbf{N}\setminus\mathbf{U}\\ 0, & \text{if } \alpha\in\mathbf{R}\setminus\mathbf{N}, \end{cases} \tag{7}$$

for a certain reductive subset $\mathbf{N}\subset\mathbf{R}$ such that $\mathbf{N}\supset\mathbf{U}$, and $(\mathbf{N},\mathbf{U})$-regular element $h\in\mathfrak{h}$.

Proof. Suppose x_α satisfies (d)–(f). Set $\mathbf{N}=\mathbf{U}\cup\{\alpha\in\mathbf{R}\setminus\mathbf{U}\,|\,x_\alpha\neq 0\}$. Let us prove that $\mathbf{N}$ is reductive. Using (d), we see that $-\mathbf{N}=\mathbf{N}$. Further, suppose $\alpha,\beta\in\mathbf{N}$, $\gamma\in\mathbf{R}$, $\alpha+\beta+\gamma=0$. We have to verify that $\gamma\in\mathbf{N}$. If $\alpha,\beta\in\mathbf{U}$, then also $\gamma\in\mathbf{U}\subset\mathbf{N}$ (since $\mathbf{U}$ is reductive). If $\alpha\in\mathbf{U}$, $\beta\in\mathbf{N}\setminus\mathbf{U}$, then $\gamma\in\mathbf{R}\setminus\mathbf{U}$. Applying (e), we see that $x_\beta+x_\gamma=0$. Since $x_\beta\neq 0$, we have $x_\gamma\neq 0$, i.e., $\gamma\in\mathbf{N}\setminus\mathbf{U}$. Finally, let $\alpha,\beta\in\mathbf{N}\setminus\mathbf{U}$. Assume also that $\gamma\in\mathbf{R}\setminus\mathbf{U}$ (we have nothing to prove in the case $\gamma\in\mathbf{U}$). Using (f), we see that $x_\alpha\neq 0$, $x_\beta\neq 0$ imply that $x_\gamma\neq 0$, i.e., $\gamma\in\mathbf{N}\setminus\mathbf{U}$.

Furthermore, set $y_\alpha = 1/x_\alpha$ for $\alpha\in\mathbf{N}\setminus\mathbf{U}$. Suppose $\alpha,\beta,\gamma\in\mathbf{N}\setminus\mathbf{U}$, $\alpha+\beta+\gamma=0$. Then $y_\alpha+y_\beta+y_\gamma=0$ according to (f). This means that $y_\alpha=\alpha(h)$ for some $h\in\mathfrak{h}$.

Finally, we prove that h is $(\mathbf{N},\mathbf{U})$-regular. By construction, $\alpha(h)\neq 0$ for all $\alpha\in\mathbf{N}\setminus\mathbf{U}$. Now assume that $\gamma\in\mathbf{U}$. Take any $\alpha\in\mathbf{N}\setminus\mathbf{U}$ (note that if $\mathbf{N}=\mathbf{U}$, then we have nothing to prove here), and set $\beta=-(\alpha+\gamma)$. Obviously, $\beta\in\mathbf{N}\setminus\mathbf{U}$. By (e), we have $0=x_\alpha+x_\beta=1/\alpha(h)+1/\beta(h)$, i.e., $\gamma(h)=0$.

Conversely, if x_α is of the form (7), then the conditions (e)–(f) can be verified without difficulties. □

The last lemma proves the proposition. □

Remark 19. We note that Lemmas 16, 17, and 18 are essentially contained in [3].

In [3], among other results, the *symplectic* G-invariant structures on G/U are classified if U is a Levi subgroup of G. Actually, in this case there exists a G-equivariant symplectomorphism from G/U to a semisimple coadjoint G-orbit equipped with the Kirillov-Kostant-Souriau bracket.

Moreover, it is easy to show that if G/U has a G-invariant symplectic structure, then U is a Levi subgroup. Indeed, let $p_*\overrightarrow{x_{\mathbf{N},h}}$ (where $x_{\mathbf{N},h}$ is defined by (6)) be a G-invariant Poisson structure on G/U. Obviously, it is symplectic iff $\mathbf{N}=\mathbf{R}$. Since h is $(\mathbf{R},\mathbf{U})$-regular, i.e., $\alpha(h)=0$ for all $\alpha\in\operatorname{Span}\mathbf{U}$ and $\alpha(h)\neq 0$ for all $\alpha\in\mathbf{R}\setminus\mathbf{U}$, we see that $(\operatorname{Span}\mathbf{U})\cap\mathbf{R}=\mathbf{U}$. It is well known that the latter condition is equivalent to the fact that U is a Levi subgroup.

Let us also remark that the existence of reductive non-Levi subgroups is the main difference between the triangular and the strictly quasitriangular cases. Indeed, suppose U is a Cartan subgroup. Then in the triangular case the Poisson homogeneous structures on G/U relate to *all* reductive subgroups of G, while in the strictly quasitriangular case they relate to the Levi subgroups only (see [17, 18]).

Now we are going to describe the Lagrangian subalgebras corresponding to the Poisson (G,π_ρ)-homogeneous structures on G/U. Since the Lie bialgebras corresponding to (G,π_ρ) are all in the same class, we may assume without loss of generality that $\rho=0$. It is clear that the double of our Lie bialgebras is $\mathfrak{g}[\varepsilon]=\mathfrak{g}\oplus\mathfrak{g}\varepsilon$, where $\varepsilon^2=0$ (see Appendix for details).

Suppose $\rho=0$. Assume that $\mathbf{N}$ and h are as in Proposition 14. Set $\pi_{\mathbf{N},h}=p_*\overrightarrow{x_{\mathbf{N},h}}$, where $x_{\mathbf{N},h}$ is defined by (6). By $\mathfrak{l}_{\mathbf{N},h}$ denote the Lagrangian subalgebra corresponding to $(G/U,\pi_{\mathbf{N},h})$ at the base point $\underline{e}$.

Proposition 20. $\mathfrak{l}_{\mathbf{N},h} = \mathfrak{u}\oplus\left(\bigoplus_{\alpha\in\mathbf{R}\setminus\mathbf{N}}\varepsilon\mathfrak{g}_\alpha\right)\oplus\left(\bigoplus_{\alpha\in\mathbf{N}\setminus\mathbf{U}}(1-\alpha(h)\varepsilon)\mathfrak{g}_\alpha\right)$ *(cf. Proposition 26 below).*

Proof. By definition (see (4)),

$$\mathfrak{l}_{\mathbf{N},h}=\{a+b\varepsilon\mid a\in\mathfrak{g}, b\in\mathfrak{u}^\perp=\mathfrak{m}, (b\otimes 1)(x_{\mathbf{N},h})=\overline{a}\},$$

where $\overline{a}$ is the image of a in $\mathfrak{g}/\mathfrak{u}=\mathfrak{m}$. Suppose $b=E_\alpha$, where $\alpha\in\mathbf{R}\setminus\mathbf{U}$. Then

$$(b\otimes 1)(x_{\mathbf{N},h})=\begin{cases}-\frac{1}{\alpha(h)}E_\alpha, & \text{if } \alpha\in\mathbf{N}\setminus\mathbf{U}\\ 0, & \text{if } \alpha\in\mathbf{R}\setminus\mathbf{N}.\end{cases}$$

This completes the proof. □

5 Appendix: Lagrangian subalgebras in $\mathfrak{g}[\varepsilon]$

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, G a connected Lie group such that $\operatorname{Lie} G = \mathfrak{g}$. Fix an invariant scalar product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$. Consider the complex Lie algebra $\mathfrak{g}[\varepsilon] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon] = \mathfrak{g} \oplus \mathfrak{g}\varepsilon$, where $\mathbb{C}[\varepsilon] = \mathbb{C}[x]/(x^2)$ is the algebra of dual numbers. We identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 \subset \mathfrak{g}[\varepsilon]$. Equip $\mathfrak{g}[\varepsilon]$ with the invariant scalar product defined by

$$\langle a + b\varepsilon, c + d\varepsilon \rangle = \langle a, d \rangle + \langle b, c \rangle.$$

Then the pair $(\mathfrak{g}[\varepsilon], \mathfrak{g})$ is a Manin pair.

Recall that a Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}[\varepsilon]$ is called *Lagrangian* if it is a maximal isotropic subspace in $\mathfrak{g}[\varepsilon]$.

Let $\mathfrak{n} \subset \mathfrak{g}$ be a Lie subalgebra, B be a $\mathbb{C}$-valued 2-cocycle on $\mathfrak{n}$. By $\mathbf{Pairs}(\mathfrak{g})$ denote the set of all such pairs $(\mathfrak{n}, B)$. Define $f : \mathfrak{n} \to \mathfrak{n}^*$ by

$$\langle f(x), y \rangle = B(x, y) \tag{8}$$

(here we identify $\mathfrak{n}^*$ with $\mathfrak{g}/\mathfrak{n}^\perp$ via $\langle \cdot\,, \cdot \rangle$). One can easily see that f is a skew-symmetric 1-cocycle (with respect to the coadjoint action of $\mathfrak{n}$ on $\mathfrak{n}^*$). Set

$$\mathfrak{l}(\mathfrak{n}, B) = \{a + b\varepsilon \mid a \in \mathfrak{n}, b \in \mathfrak{g}, f(a) = \bar{b}\} \subset \mathfrak{g}[\varepsilon],$$

where $\bar{b}$ is the image of b in $\mathfrak{n}^* = \mathfrak{g}/\mathfrak{n}^\perp$. Obviously, $\mathfrak{u} := \operatorname{Ker} B = \operatorname{Ker} f$ is a Lie subalgebra in $\mathfrak{n}$.

We denote by $\mathbf{Lagr}(\mathfrak{g})$ the set of all Lagrangian subalgebras in $\mathfrak{g}$. Note that G acts naturally on $\mathbf{Pairs}(\mathfrak{g})$ and on $\mathbf{Lagr}(\mathfrak{g})$.

Theorem 21. *1.* $\mathfrak{l}(\mathfrak{n}, B)$ *is a Lagrangian subalgebra in* $\mathfrak{g}[\varepsilon]$ *and* $\mathfrak{l}(\mathfrak{n}, B) \cap \mathfrak{g} = \mathfrak{u}$.

2. The mapping $(\mathfrak{n}, B) \mapsto \mathfrak{l}(\mathfrak{n}, B)$ *is a* G*-equivariant bijection between* $\mathbf{Pairs}(\mathfrak{g})$ *and* $\mathbf{Lagr}(\mathfrak{g})$.

Proof. Suppose $\mathfrak{l} \in \mathbf{Lagr}(\mathfrak{g})$. Denote by $\mathfrak{n}$ the projection of $\mathfrak{l}$ onto $\mathfrak{g}$ along $\mathfrak{g}\varepsilon$. Since $\mathfrak{l} \subset \mathfrak{n} \oplus \mathfrak{g}\varepsilon$, we have $\mathfrak{l} = \mathfrak{l}^\perp \supset (\mathfrak{n} \oplus \mathfrak{g}\varepsilon)^\perp = \mathfrak{n}^\perp\varepsilon$. Consider

$$\bar{\mathfrak{l}} := \mathfrak{l}/(\mathfrak{n}^\perp\varepsilon) \subset (\mathfrak{n} \oplus \mathfrak{g}\varepsilon)/(\mathfrak{n}^\perp\varepsilon) = \mathfrak{n} \oplus \mathfrak{n}^*\varepsilon.$$

Since $\dim \bar{\mathfrak{l}} = \dim \mathfrak{l} - \dim \mathfrak{n}^\perp = \dim \mathfrak{g} - (\dim \mathfrak{g} - \dim \mathfrak{n}) = \dim \mathfrak{n}$, we see that $\bar{\mathfrak{l}}$ is the graph of a linear map $f : \mathfrak{n} \to \mathfrak{n}^*$, i.e.,

$$\bar{\mathfrak{l}} = \{a + f(a)\varepsilon \mid a \in \mathfrak{n}\}.$$

This yields that

$$\mathfrak{l} = \{a + b\varepsilon \mid a \in \mathfrak{n}, b \in \mathfrak{g}, f(a) = \bar{b}\} \subset \mathfrak{g}[\varepsilon],$$

where $\bar{b}$ is the image of b in $\mathfrak{n}^* = \mathfrak{g}/\mathfrak{n}^\perp$.

Now let $a+b\varepsilon, c+d\varepsilon \in \mathfrak{l}$ (i.e., $a, c \in \mathfrak{n}$, $f(a)=\bar{b}$, $f(c)=\bar{d}$). Since $\mathfrak{l}$ is a Lie subalgebra, we have

$$\mathfrak{l} \ni [a+b\varepsilon, c+d\varepsilon] = [a,c] + ([a,d]+[b,c])\varepsilon.$$

Therefore

$$f([a,c]) = \overline{[a,d]+[b,c]} = [a, f(c)] + [f(a), c],$$

i.e., f is a 1-cocycle. Since $\mathfrak{l}$ is isotropic, we have

$$0 = \langle a+b\varepsilon, c+d\varepsilon\rangle = \langle a,d\rangle + \langle b,c\rangle = \langle a, f(c)\rangle + \langle f(a), c\rangle,$$

i.e., f is skew-symmetric. Finally, define B by (8). It is easy to check that B is a 2-cocycle.

Conversely, $\mathfrak{l}(\mathfrak{n}, B)$ is a Lie subalgebra since $\mathfrak{n}$ is a Lie subalgebra and f is a 1-cocycle (recall that f and B are connected via (8)); $\mathfrak{l}(\mathfrak{n}, B)$ is isotropic since f is skew-symmetric; finally, $\mathfrak{l}(\mathfrak{n}, B)$ is Lagrangian since $\dim \mathfrak{l}(\mathfrak{n}, B) = \dim \mathfrak{n} + \dim \mathfrak{n}^* = \dim \mathfrak{g}$.

The fact that $\mathfrak{l}(\mathfrak{n}, B) \cap \mathfrak{g} = \mathfrak{u}$ is obvious. □

Now fix a Lie subalgebra $\mathfrak{u} \subset \mathfrak{g}$. Set

$$\mathbf{Pairs}(\mathfrak{g}, \mathfrak{u}) = \{(\mathfrak{n}, B) \in \mathbf{Pairs}(\mathfrak{g}) \mid \mathfrak{n} \supset \mathfrak{u}, \operatorname{Ker} B = \mathfrak{u}\},$$

$$\mathbf{Lagr}(\mathfrak{g}, \mathfrak{u}) = \{\mathfrak{l} \in \mathbf{Lagr}(\mathfrak{g}) \mid \mathfrak{l} \cap \mathfrak{g} = \mathfrak{u}\}.$$

Denote by $N(\mathfrak{u})$ the normalizer of $\mathfrak{u}$ in G. Clearly, $N(\mathfrak{u})$ acts on $\mathbf{Pairs}(\mathfrak{g}, \mathfrak{u})$ and $\mathbf{Lagr}(\mathfrak{g}, \mathfrak{u})$.

Corollary 22. *The mapping* $(\mathfrak{n}, B) \mapsto \mathfrak{l}(\mathfrak{n}, B)$ *is a* $N(\mathfrak{u})$*-equivariant bijection between* $\mathbf{Pairs}(\mathfrak{g}, \mathfrak{u})$ *and* $\mathbf{Lagr}(\mathfrak{g}, \mathfrak{u})$. □

As before, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and denote by $\mathbf{R}$ the corresponding root system. Consider a reductive subset $\mathbf{U} \subset \mathbf{R}$ and set $\mathfrak{u} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \mathbf{U}} \mathfrak{g}_\alpha)$. We would like to describe more explicitly the set $\mathbf{Lagr}(\mathfrak{g}, \mathfrak{u})$ or in other words Lagrangian subalgebras $\mathfrak{l} \subset \mathfrak{g}[\varepsilon]$ such that $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{u}$. By Corollary 22, it is sufficient to describe the set $\mathbf{Pairs}(\mathfrak{g}, \mathfrak{u})$.

Theorem 23. *Suppose* $(\mathfrak{n}, B) \in \mathbf{Pairs}(\mathfrak{g})$. *Then* $(\mathfrak{n}, B) \in \mathbf{Pairs}(\mathfrak{g}, \mathfrak{u})$ *if and only if* $\mathfrak{n} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \mathbf{N}} \mathfrak{g}_\alpha)$ *is a reductive Lie subalgebra in* $\mathfrak{g}$ *that contains* $\mathfrak{u}$, *and* $B(x,y) = \langle h, [x,y]\rangle$ *(i.e., B is a 2-coboundary), where* $h \in \mathfrak{h}$ *is* $(\mathbf{N}, \mathbf{U})$*-regular (for definition see the previous section).*

Remark 24. Suppose $h \in \mathfrak{h}$. It is clear that $B(x,y) = \langle h, [x,y]\rangle$ depends only on the image of h in $\mathfrak{h}/\mathfrak{z}(\mathfrak{n})$, where $\mathfrak{z}(\mathfrak{n})$ is the center of $\mathfrak{n}$. Note also that $\mathfrak{z}(\mathfrak{n}) = \{h \in \mathfrak{h} \mid \alpha(h) = 0 \text{ for all } \alpha \in \mathbf{N}\}$.

Proof of Theorem 23. Suppose $(\mathfrak{n}, B) \in \mathbf{Pairs}(\mathfrak{g}, \mathfrak{u})$, i.e., $\mathfrak{n} \supset \mathfrak{u}$, $\operatorname{Ker} B = \mathfrak{u}$. Since $\mathfrak{n} \supset \mathfrak{h}$, we see that $\mathfrak{n} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \mathbf{N}} \mathfrak{g}_\alpha)$ for some $\mathbf{N} \subset \mathbf{R}$. Clearly, $\mathbf{U} \subset \mathbf{N}$.

Lemma 25. *If $\alpha, \beta \in \mathbf{N}$, $\alpha + \beta \neq 0$, then $B(x, y) = 0$ for all $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$.*

Proof. If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $h \in \mathfrak{h}$, then, since B is a 2-cocycle, we have $B([x, y], h) + B([y, h], x) + B([h, x], y) = 0$, i.e., $B(h, [x, y]) = (\alpha + \beta)(h) \cdot B(x, y)$. Since $\operatorname{Ker} B \supset \mathfrak{h}$ and $\alpha + \beta \neq 0$, we see that $B(x, y) = 0$. □

Now we continue the proof of the theorem. If $\alpha \in \mathbf{N}$, but $-\alpha \notin \mathbf{N}$, then, by Lemma 25, we see that $\mathfrak{g}_\alpha \subset \operatorname{Ker} B = \mathfrak{u}$. Then $\pm\alpha \in \mathbf{U} \subset \mathbf{N}$ because $\mathbf{U}$ is reductive. This contradiction proves that $-\mathbf{N} = \mathbf{N}$, i.e., $\mathfrak{n}$ is reductive.

Let us prove that B is a 2-coboundary. Recall that $H^2(\mathfrak{n}, \mathbb{C}) = \wedge^2 \mathfrak{z}(\mathfrak{n})$ (see [11]). To be more precise, any 2-cocycle B can be presented uniquely in the form $B' + B''$, where B' is a 2-coboundary, and $B''(x, y) = \langle u, x \otimes y \rangle$ for $u \in \wedge^2 \mathfrak{z}(\mathfrak{n})$. Assume that $B'' \neq 0$. Then there exists $a \in \mathfrak{z}(\mathfrak{n}) \subset \mathfrak{h}$ such that $a \notin \operatorname{Ker} B''$. Since B' is a 2-coboundary, we see that $a \in \operatorname{Ker} B'$. Therefore $a \notin \operatorname{Ker} B$, and we get a contradiction. This means that $B(x, y) = \langle h, [x, y] \rangle$, where $h \in \mathfrak{n}$.

It remains to prove that h is a $(\mathbf{N}, \mathbf{U})$-regular element of $\mathfrak{h}$. Suppose $\alpha \in \mathbf{N}$, $x \in \mathfrak{g}_\alpha$, $h' \in \mathfrak{h}$. Since $\operatorname{Ker} B \supset \mathfrak{h}$, we have $0 = B(h', x) = \langle h, [h', x] \rangle = \alpha(h') \cdot \langle h, x \rangle$. Therefore h is orthogonal to $\mathfrak{g}_\alpha$ for all $\alpha \in \mathbf{N}$. This implies that $h \in \mathfrak{h}$.

If $\alpha \in \mathbf{N}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{n}$, then $B(x, y) = \langle [h, x], y \rangle = \alpha(h) \cdot \langle x, y \rangle$. This shows that $\mathfrak{g}_\alpha \subset \operatorname{Ker} B$ iff $\alpha(h) = 0$. Therefore $\operatorname{Ker} B = \mathfrak{u}$ iff h is $(\mathbf{N}, \mathbf{U})$-regular.

The converse statement of the theorem can be verified directly. □

Suppose $\mathbf{N}$ is a reductive subset in $\mathbf{R}$ containing $\mathbf{U}$. By $\mathfrak{n}$ denote the reductive Lie subalgebra in $\mathfrak{g}$ that corresponds to $\mathbf{N}$. Consider a $(\mathbf{N}, \mathbf{U})$-regular element $h \in \mathfrak{h}$. Denote by B the 2-coboundary which corresponds to h (see Theorem 23).

Proposition 26. $\mathfrak{l}(\mathfrak{n}, B) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \mathbf{R} \setminus \mathbf{N}} \varepsilon \mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha \in \mathbf{N}} (1 + \alpha(h)\varepsilon) \mathfrak{g}_\alpha\right) = \mathfrak{u} \oplus \left(\bigoplus_{\alpha \in \mathbf{R} \setminus \mathbf{N}} \varepsilon \mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha \in \mathbf{N} \setminus \mathbf{U}} (1 + \alpha(h)\varepsilon) \mathfrak{g}_\alpha\right)$.

Proof. Direct calculations. □

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