Which properties of a random sequence are dynamically sensitive?

Itai Benjamini Olle Häggström Yuval Peres Jeffrey E. Steif

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Abstract

Consider a sequence of i.i.d. random variables, where each variable is refreshed (i.e., replaced by an independent variable with the same law) independently, according to a Poisson clock. At any fixed time t, the resulting sequence has the same law as at time 0, but there can be exceptional random times at which certain almost sure properties of the time 0 sequence are violated. We prove that there are no such exceptional times for the law of large numbers and the law of the iterated logarithm, so these laws are *dynamically stable.* However, there are times at which run lengths are exceptionally long, i.e., run tests are *dynamically sensitive.* We obtain a multifractal analysis of exceptional times for run lengths and for prediction. In particular, starting from an i.i.d. sequence of unbiased random bits, the random set of times t where $\alpha \log_2(n)$ bits among the first n bits can be predicted from their predecessors, has Hausdorff dimension $1 - \alpha$ a.s. Finally, we consider simple random walk in the lattice $\mathbf{Z}^d$, and prove that transience is dynamically stable for $d \geq 5$, and dynamically sensitive for $d = 3, 4$. Moreover, for $d = 3, 4$, the nonempty random set of exceptional times t where the walk is recurrent, has Hausdorff dimension $(4 - d)/2$ a.s.

1 Introduction

Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables with common law ν. The strong law of large numbers (SLLN), the law of the iterated logarithm (LIL), Polya's theorem on recurrence of random walks, and other classical results in probability theory concern almost-sure (a.s.) properties of such sequences $\{X_n\}$. Our aim in this paper is to look at these properties from a dynamical perspective, and understand which of them are stable (respectively, sensitive) when the underlying sequence undergoes equilibrium dynamics. For each $n \in \mathbf{N}$, let $\{X_n(t)\}_{t \geq 0}$ be an independent process which at rate 1, replaces its current value by an independent sample from ν. More formally, given an array of i.i.d.

variables $\{X_n^{(j)} : j, n \in \mathbf{N}\}$ with law ν, and an independent Poisson process $\{\psi_n^{(j)}\}_{j\geq 0}$ of rate 1 for each n, define

$$X_n(t) := X_n^{(j)} \text{ for } \psi_n^{(j-1)} \leq t < \psi_n^{(j)},$$

where $\psi_n^{(0)} = 0$ for every n. Thus for different values of n, the processes $\{X_n(t)\}_{t\geq 0}$ are independent. The distribution of $X(t) := \{X_n(t)\}_{n\in\mathbf{N}}$ is $\mu := \nu^{\mathbf{N}}$ for every $t \geq 0$. Hence, any Borel event $\mathcal{A} \subseteq \{0,1\}^{\mathbf{N}}$ with $\mu(\mathcal{A}) = 1$, satisfies, for all $t \geq 0$,

$$\mathbf{P}(X(t) \in \mathcal{A}) = 1 \tag{1}$$

(here $\mathbf{P}$ is the probability measure on the underlying probability space on which the dynamical process is defined). By Fubini's Theorem, we immediately obtain

$$\mathbf{P}(X(t) \in \mathcal{A} \text{ for Lebesgue-a.e. } t) = 1\,. \tag{2}$$

Given an event $\mathcal{A}$ for which (1) holds, a natural question is whether (2) can be strengthened to

$$\mathbf{P}(X(t) \in \mathcal{A} \text{ for all } t) = 1\,. \tag{3}$$

In other words,

> *which almost sure properties of an i.i.d. sequence hold at all times under the above equilibrium dynamics?*

We classify almost sure properties of $\{X_n\}$ (events $\mathcal{A}$ with $\mu(\mathcal{A}) = 1$) as (dynamically) *stable* or *sensitive* according to whether or not (3) holds. We shall see below that the strong law of large numbers and the law of the iterated logarithm are dynamically stable, while other properties, involving run tests, prediction, and transience of random walks, are dynamically sensitive.

This type of problem was considered in [11] in the percolation context, where each X_n represents the status (open or closed, having probabilities p and $1-p$) of an edge in an infinite locally finite graph $G = (V, E)$, and $\mathcal{A}_G$ is the event that all open clusters are finite. Examples are given in [11] of graphs where $\mathcal{A}_G$ is stable for all $p \leq p_c$, and others where it is sensitive at the critical value $p = p_c$.

Here we shall consider the analogous problems for events $\mathcal{A}$ that are more central to classical probability theory. We denote the mean and variance of the law ν by m and σ^2, and begin with the strong law of large numbers (SLLN). Taking $\mathcal{A}$ to be the event that $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n X_k = m$, then $\mu(\mathcal{A}) = 1$ by the SLLN. Define $S_n(t) := \sum_{k=1}^n X_k(t)$.

Theorem 1.1 (dynamical SLLN) *Assume that m is finite. Then*

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{S_n(t)}{n} = m \text{ for all } t\right) = 1\,. \tag{4}$$

If the X_i's are bounded, this is easy; it is more interesting that it extends to the same generality as the SLLN itself, namely for random variables with finite mean. See Section 2 for the proof. We will prove there that a.s., the convergence in (4) is uniform in $t \in [0,1]$. This uniformity will be used in the proof of the dynamical LIL. Recall that the classical law of the iterated logarithm (LIL) asserts that

$$P\left(\limsup_{n\to\infty} \frac{S_n(0) - nm}{\sigma\sqrt{2n\log\log n}} = 1\right) = 1 \tag{5}$$

(See, e.g., [7], p. 437)

The dynamical extension of the LIL follows.

Theorem 1.2 (dynamical LIL) *Assume that $\sigma^2 < \infty$ and that the mean m is 0. Then*

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{S_n(t)}{\sigma\sqrt{2n\log\log n}} = 1 \text{ for all } t\right) = 1\,.$$

This is analogous to the fact that quasi-every Brownian path satisfies the LIL (see [9]). Theorem 1.2 is proved in Section 2.

Other natural properties turn out to be sensitive. In Section 3, we consider *run tests*. Let $\nu = p\delta_1 + (1-p)\delta_0$. Define

$$R_n := \sup\{j : X_k = 1 \text{ for } n \le k \le n+j-1\}\,,$$

i.e., R_n is the length of the run of 1's in X starting at bit n.

Theorem 1.3 (Erdős and Révész [8]) *Let $\{a_n\}_{n=0}^\infty$ be a sequence with $a_n \ge 1$ for all n. Then*

$$P(R_n \ge a_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty p^{a_n} < \infty \\ \\ 1 & \text{if } \sum_{n=1}^\infty p^{a_n} = \infty\,. \end{cases}$$

Next, we state the dynamical counterpart of 1.3, where $R_n(t)$ is defined analogously to R_n.

Theorem 1.4 *Let $\{a_n\}_{n=0}^\infty$ be a sequence with $a_n \ge 1$ for all n. Then*

$$\mathbf{P}(\exists t \ge 0 \text{ such that } \{R_n(t) \ge a_n \text{ i.o.}\}) = \begin{cases} 0 & \text{if } \sum_{n=1}^\infty a_n p^{a_n} < \infty \\ \\ 1 & \text{if } \sum_{n=1}^\infty a_n p^{a_n} = \infty\,. \end{cases}$$

In particular, if the sequence $\{a_n\}$ satisfies $\sum_{n=0}^\infty p^{a_n} < \infty$ and $\sum_{n=0}^\infty a_n p^{a_n} = \infty$, then the event $\{R_n(t) < a_n \text{ for all but finitely many } n\}$ holds for Lebesgue-a.e. t, but fails for an exceptional set of times. An example of such a sequence may be obtained by taking

$$a_n = \max\{1, \log_{1/p}(n) + r\log_{1/p}(\log_{1/p}(n))\} \tag{6}$$

for any $r \in (1, 2]$.

When an almost sure property is dynamically sensitive, the set of exceptional times where it fails is a.s. a nonempty set of Lebesgue measure 0, so it is natural to ask what is its Hausdorff dimension, and which deterministic sets it intersects with positive probability. For run tests with $\{a_n\}$ of the form (6), we answer these questions in Theorem 1.5 below. In that theorem, we calculate the Hausdorff dimension of the set of times t for which $R_n(t) \geq a_n$ i.o., and determine which fixed sets intersect this random exceptional set a.s. This yields a multifractal decomposition of $[0, 1]$. Analogous results of for dynamical percolation on certain trees appear in [11], [20]. We write $\dim_H$ for Hausdorff dimension and $\dim_P$ for packing dimension; see e.g. Mattila [18] for definitions.

Theorem 1.5 *Denote $\ell_p(x) = \log(x)/\log(1/p)$. For $r \in (1, 2]$, consider the random set of times*

$$W(r) = \{t \in [0,1] : R_n(t) \geq \ell_p(n) + r\ell_p(\ell_p(n)) \text{ i.o.}\}.$$

Then $\dim_H(W(r)) = 2 - r$ and $\dim_P(W(r)) = 1$ for all $1 < r \leq 2$, with probability 1. Furthermore, for any (deterministic) closed set $E \subseteq [0, 1]$ and any $r \in (1, 2]$, we have

$$\mathbf{P}(W(r) \cap E \neq \emptyset) = \begin{cases} 0 & \text{if } \dim_P(E) < r - 1 \\ 1 & \text{if } \dim_P(E) > r - 1. \end{cases}$$

In Section 4 we address prediction of random bits and algorithmic randomness. For these questions, we assume that $\nu = \delta_1/2 + \delta_0/2$. A function $g : \{0,1\}^{\mathbf{N}} \to \{0,1,*\}^{\mathbf{N}}$ is called a *predictor* if $(g(\xi))_n$ depends on ξ only via $(\xi_1, \ldots, \xi_{n-1})$ for each n. The idea is that a predictor is used to predict bits. Specifically, $(g(\xi))_n = i \in \{0, 1\}$ if based on $(\xi_1, \ldots, \xi_{n-1})$, g predicts that the nth bit has the value i, while $(g(\xi))_n = *$ if based on $(\xi_1, \ldots, \xi_{n-1})$, the function g does not predict the nth bit at all.

Definition 1.6 *Given an increasing mapping $r : \mathbf{N} \to \mathbf{N}$, we say that the predictor g* **has inverse rate** *r if for all $\xi \in \{0,1\}^{\mathbf{N}}$ and all $n \geq 1$,*

$$\sum_{k=1}^{r(n)} \mathbf{1}_{\{g(\xi))_k \neq *\}} \geq n.$$

In words, we require that, regardless of the input sequence, at least n bits are predicted by time $r(n)$.

Definition 1.7 *We say that a predictor is* **correct on the input** ξ *if* $(g(\xi))_n \in \{*, \xi_n\}$, *for all* n, i.e., *if all predictions made are correct.*

The next two theorems determine almost exactly the maximal prediction rate attainable at an exceptional random time t.

Theorem 1.8 *For any* $\varepsilon > 0$, *there exists a predictor* g *with inverse rate* $\lfloor 2^{(n+1)(1+\varepsilon)} \rfloor$ *such that* $\mathbf{P}(\exists t \geq 0 \text{ such that } g \text{ is correct on the input } X(t)) > 0$.

Theorem 1.9 *Let* g *be a predictor with inverse rate* $r(n)$ *that satisfies* $r(n) = O(2^n)$. *Then* $\mathbf{P}(\exists t \geq 0 \text{ such that } g \text{ is correct on the input } X(t)) = 0$.

In Section 4 we prove these theorems and establish a multifractal version, which, loosely speaking, states that the set of times t, where an inverse rate of $2^{n/\alpha}$ is attainable by a correct predictor, has Hausdorff dimension $1 - \alpha$ (provided that $0 < \alpha < 1$).

In Section 5 we study sensitivity of recurrence and transience of random walks. Note that if $\nu = \delta_1/2 + \delta_{-1}/2$, the random variables $\{S_n(t)\}_{n \in N}$, for fixed t, form a simple symmetric random walk on $\mathbf{Z}$. Stability of the LIL immediately yields the same for recurrence. Indeed, $S_n(t)$ only makes steps of size 1 (as n grows) and stability of the LIL implies that for all t, the process $\{S_n(t)\}_{n=1}^{\infty}$ takes both positive and negative values i.o. Hence,

Corollary 1.10 *If* $\nu = \delta_1/2 + \delta_{-1}/2$, *then*

$$\mathbf{P}\left(\forall t : S_n(t) = 0 \text{ i.o.}\right) = 1\,. \tag{7}$$

We will prove in Section 5 the following generalization of Corollary 1.10.

Theorem 1.11 *Let* ν *be concentrated on* $\mathbf{Z}$, *having finite support and mean 0. Then*

$$\mathbf{P}(\forall t :\ S_n(t) = 0 \text{ i.o.}) = 1\,.$$

Next, if $\{X_n\}_{n \geq 1}$ is a sequence of i.i.d. random vectors in $\mathbf{Z}^d$ with common law ν, the dynamic process $\{X_n(t)\}_{n \geq 1}$ is defined completely analogously as in the $d = 1$ case. We will see in Section 5 that the property of being transient for simple random walk on $\mathbf{Z}^d$ is stable for $d \geq 5$ but sensitive for $d = 3, 4$. This is analogous to the fact that quasi-every Brownian motion path in $\mathbf{R}^d$ is transient for $d \geq 5$, but not for $d = 3, 4$ (see [9], [16]).

Let $e_1, \ldots, e_d$ denote the unit vectors in the d coordinate directions in $\mathbf{R}^d$.

Theorem 1.12 *Fix $d \geq 1$ and let $\nu(\pm e_j) = 1/2d$ for each $j \in \{1, \ldots, d\}$. Then if $d \leq 4$, we have*

$$\mathbf{P}(\exists t : S_n(t) = 0 \text{ i.o.}) = 1 \tag{8}$$

while for $d \geq 5$, we have

$$\mathbf{P}(\exists t : S_n(t) = 0 \text{ i.o.}) = 0 .$$

This result is trivial for $d = 1, 2$. In Section 5, we will prove Theorem 1.12, and, more generally, characterize exactly those symmetric random walks on abelian groups for which transience is stable.

The next result describes the Hausdorff dimension of the set of times at which simple random walk is recurrent. This result will also be generalized to certain symmetric random walks on abelian groups.

Theorem 1.13 *The Hausdorff dimension of the set of times at which simple random walk is recurrent is 0 in 4 dimensions and 1/2 in 3 dimensions.*

Remark 1.14 Focusing on the first n bits, our dynamics produces a random number $N(n)$ of distinct finite sequences $\{X_j(t)\}_{j=1}^n$ as t ranges over $[0, 1]$. It is easy to see that (with probability 1) $C_1 n < N(n) < C_2 n$ for all large n, where C_1 and C_2 are positive constants. For some purposes (e.g., Theorem 1.4 on runs, Theorem 1.8 on prediction and Theorem 1.12 on transience), these sequences behave like n independently chosen random strings of n bits. However, the high correlations between these $N(n)$ strings are manifested in Theorems 1.2 and 1.11. These theorems should be contrasted with the fact that among $n^{1/2+\epsilon}$ random strings of ± 1's of length n, there is (with high probability) a string with all partial sums positive. Moreover, the stable recurrence exhibited in Theorem 1.11 indicates that the $N(n)$ strings obtained in our dynamical model by time 1 are more clustered than trajectories of a tree-indexed random walk: In [4, Theorem 1.1] it is shown that for a spherically symmetric tree with approximately $n^{1/2+\epsilon}$ vertices at level n for large n, the corresponding tree-indexed simple random walk on $\mathbf{Z}$ has, with positive probability, a ray with a trajectory that remains positive forever.

Notation: We will write $\{a_k\} \asymp \{b_k\}$ if $\sup_k\{\max(a_k/b_k, b_k/a_k,)\} \leq C$ for some constant C.

2 Two classical limit theorems

We begin this section with a strengthened version of Theorem 1.1.

Theorem 2.1 (uniform dynamical SLLN) *Assume that m is finite. Then a.s., for every $\varepsilon > 0$ there exists an $N = N(\varepsilon) < \infty$ such that*

$$\left|\frac{S_n(t)}{n} - m\right| < \varepsilon \text{ for all } t \in [0,1] \textit{ and all } n \geq N.$$

Proof: It clearly suffices to show that for every $\varepsilon > 0$, there is a.s. an N with the desired property. Fix $\varepsilon > 0$. Let N_k be the number of updates of the variable X_k during $[0, \varepsilon]$ and denote by $\{X_k^{(\ell)}\}_{\ell \geq 1}$ the successive values at these updates. Let $Y_k = X_k(0) I_{\{N_k = 0\}}$ and $M_k = \max_{1 \leq \ell \leq N_k} |X_k^{(\ell)}| I_{\{N_k \geq 1\}}$. Then

$$\mathbf{E}(Y_k) = \mathbf{E}(X_k(0)) e^{-\varepsilon}$$

and

$$\mathbf{E}(M_k) \leq \mathbf{E}\left(\sum_{\ell=1}^{\infty} |X_k^{(\ell)}| I_{\{N_k \geq \ell\}}\right) = \mathbf{E}(|X_k(0)|)\mathbf{E}(N_k) = \mathbf{E}(|X_1(0)|)\varepsilon.$$

Since $|X_k(t) - Y_k| \leq M_k$ for all $t \in [0, \varepsilon]$, we get

$$\sup_{t \in [0,\varepsilon]} \frac{1}{n} \sum_{k=1}^{n} |X_k(t) - Y_k| \leq \frac{1}{n} \sum_{k=1}^{n} M_k.$$

Hence

$$\sup_{t \in [0,\varepsilon]} \left|\frac{1}{n} \sum_{k=1}^{n} X_k(t) - m\right| \leq |m - \mathbf{E}(Y_1)| + \left|\frac{1}{n} \sum_{k=1}^{n} Y_k - \mathbf{E}(Y_1)\right| + \frac{1}{n} \sum_{k=1}^{n} M_k.$$

By the usual SLLN, we may choose N in such a way that $|\frac{1}{n}\sum_{k=1}^n Y_k - \mathbf{E}[Y_1]| < \varepsilon$ and $\frac{1}{n}\sum_{k=1}^n M_k < \varepsilon + \mathbf{E}(|X_1(0)|)\varepsilon$ for all $n \geq N$. We then have that for $n \geq N$,

$$\sup_{t \in [0,\varepsilon]} \left|\frac{1}{n} \sum_{k=1}^{n} X_k(t) - m\right| \leq |m|(1 - e^{-\varepsilon}) + \varepsilon + \varepsilon + \mathbf{E}(|X_1(0)|)\varepsilon.$$

Since we can cover $[0, 1]$ by finitely many intervals of length ε, we obtain the fact that there is an N such that for $n \geq N$,

$$\sup_{t \in [0,1]} \left|\frac{1}{n} \sum_{k=1}^{n} X_k(t) - m\right| \leq |m|(1 - e^{-\varepsilon}) + 2\varepsilon + \mathbf{E}(|X_1(0)|)\varepsilon.$$

□

To prove the dynamical LIL, we first establish the following lemma, which contains the key step of the proof. Suppose that the assumptions in Theorem 1.2 are in force, and let $b(n) := \sigma\sqrt{2n \log\log n}$.

Lemma 2.2 *Let $\varepsilon \in (0, 1)$. Then*

$$\lim_{N \to \infty} \mathbf{P}\left[\exists t \in \left[0, \frac{\varepsilon^2}{20}\right], n \geq N : |S_n(t) - S_n(0)| > \varepsilon b_n\right] = 0\,.$$

Proof: Let

$$A_{N,L} := \left\{ \exists t \in \left[0, \frac{\varepsilon^2}{20}\right], n \in [N, L] : |S_n(t) - S_n(0)| > \varepsilon b_n \right\}.$$

It suffices to show that $\lim_{N\to\infty} \sup_L \mathbf{P}(A_{N,L}) = 0$.

We introduce the following sequence of events.

$$\begin{aligned} B_N &:= \left\{ \exists t \in \left[0, \frac{\varepsilon^2}{20}\right], n \geq N : \sum_{k=1}^{n} X_k^2(t) \geq 2\sigma^2 n \right\}, \\ C_N &:= \left\{ \exists n \geq N : \left| S_n\left(\frac{\varepsilon^2}{20}\right) - S_n(0) \right| > \frac{\varepsilon}{3} b_n \right\}, \\ D_N &:= \left\{ \exists n \geq N : \left| S_n\left(\frac{\varepsilon^2}{20}\right) \right| > \left(1 + \frac{\varepsilon}{3}\right) b_n \right\}. \end{aligned}$$

We first claim that these three events have probabilities which go to 0 as N goes to ∞. For B_N, this follows from Theorem 2.1 applied to the random variables $\{X_k^2\}$ which have mean σ^2. For C_N, we observe that the distribution of $X_k(\frac{\varepsilon^2}{20}) - X_k(0)$ is $(1 - e^{-\frac{\varepsilon^2}{20}})\nu * \nu' + e^{-\frac{\varepsilon^2}{20}}\delta_0$ where $\nu'(A) := \nu(-A)$ for Borel sets A, and $*$ denotes convolution. This distribution clearly has mean 0 and variance $(1 - e^{-\frac{\varepsilon^2}{20}})2\sigma^2$ which is at most $\varepsilon^2\sigma^2/10$. The fact that $\lim_{N\to\infty} \mathbf{P}(C_N)=0$ now follows from the usual LIL applied to the sequence $\{X_k(\frac{\varepsilon^2}{20}) - X_k(0)\}_{k\geq 1}$. Finally, $\lim_{N\to\infty} \mathbf{P}(D_N) = 0$ also follows immediately from the usual LIL.

Since

$$\mathbf{P}(A_{N,L}) \leq \mathbf{P}(B_N) + \mathbf{P}(C_N) + \mathbf{P}(D_N) + \mathbf{P}(A_{N,L} \cap (B_N)^c \cap (C_N)^c \cap (D_N)^c),$$

we need to show that $\lim_{N\to\infty} \sup_{L\geq N} \mathbf{P}(A_{N,L} \cap (B_N)^c \cap (C_N)^c \cap (D_N)^c) = 0$.

Let $\mathcal{F}_t$ denote the σ-algebra generated by the process up until time t. We first observe the elementary identities

$$\mathbf{E}[X_k(t + \delta)|\mathcal{F}_t] = X_k(t)e^{-\delta} + \mathbf{E}(X_k(0))(1 - e^{-\delta}) = X_k(t)e^{-\delta}$$

and

$$\mathbf{E}[X_k^2(t + \delta)|\mathcal{F}_t] = X_k^2(t)e^{-\delta} + \sigma^2(1 - e^{-\delta}).$$

Hence

$$\mathbf{Var}[X_k(t + \delta)|\mathcal{F}_t] = X_k^2(t)(e^{-\delta} - e^{-2\delta}) + \sigma^2(1 - e^{-\delta}) \leq \delta[X_k^2(t) + \sigma^2],$$

from which it follows that

$$\mathbf{Var}[S_n(t + \delta)|\mathcal{F}_t] \leq \delta \left[\sum_{k=1}^{n} X_k^2(t) + n\sigma^2 \right].$$

On the event $A_{N,L}$, define t_* by

$$t_* := \inf\left\{t \in \left[0, \frac{\varepsilon^2}{20}\right] : |S_n(t) - S_n(0)| > \varepsilon b_n \text{ for some } n \in [N, L]\right\}$$

and n_* by

$$n_* := \inf\{n \in [N, L] : |S_n(t_*) - S_n(0)| > \varepsilon b_n\}.$$

On $(A_{N,L})^c$, take t_* to be $\varepsilon^2/20$ and n_* to be L. Let

$$B_N^* := \left\{\sum_{k=1}^{n_*} X_k^2(t_*) \geq 2\sigma^2 n_*\right\}$$

and note this is a subevent of B_N.

Observe that on the event $A_{N,L}$

$$|S_{n_*}(t_*) - S_{n_*}(0)| > \varepsilon b_{n_*}. \tag{9}$$

Let $Y := S_{n_*}(\frac{\varepsilon^2}{20}) - e^{-(\frac{\varepsilon^2}{20} - t_*)} S_{n_*}(t_*)$. Then, by the strong Markov property,

$$\mathbf{E}[Y|\mathcal{F}_{t_*}] = 0 \;\text{ and }\; \mathbf{Var}[Y|\mathcal{F}_{t_*}] \leq \frac{\varepsilon^2}{20}\left[\sum_{k=1}^{n_*} X_k^2(t_*) + \sigma^2 n_*\right]$$

and observe that on the event $(B_N^*)^c$, the latter is at most $3\varepsilon^2\sigma^2 n_*/20$.

We will now show that Y is large on the event $A_{N,L} \cap (C_N)^c \cap (D_N)^c$. We first note that on this event, we have that

$$\left|S_{n_*}\left(\frac{\varepsilon^2}{20}\right) - S_{n_*}(0)\right| \leq \frac{\varepsilon}{3} b_{n_*} \tag{10}$$

and

$$\left|S_{n_*}\left(\frac{\varepsilon^2}{20}\right)\right| \leq \left(1 + \frac{\varepsilon}{3}\right) b_{n_*} \tag{11}$$

By (11), we have

$$\left|e^{\frac{\varepsilon^2}{20} - t_*} S_{n_*}\left(\frac{\varepsilon^2}{20}\right) - S_{n_*}\left(\frac{\varepsilon^2}{20}\right)\right| \leq \frac{\varepsilon^2}{10} b_{n_*}$$

and with (10), we get

$$\left|e^{\frac{\varepsilon^2}{20} - t_*} S_{n_*}\left(\frac{\varepsilon^2}{20}\right) - S_{n_*}(0)\right| \leq \frac{\varepsilon}{2} b_{n_*}$$

Together with (9), we obtain the fact that

$$\left|e^{\frac{\varepsilon^2}{20} - t_*} S_{n_*}\left(\frac{\varepsilon^2}{20}\right) - S_{n_*}(t_*)\right| > \frac{\varepsilon}{2} b_{n_*}$$

which implies

$$|Y| > \frac{\varepsilon}{4} b_{n_*}.$$

We then have that

$$\begin{aligned}\mathbf{P}(A_{N,L} \cap (B_N)^c \cap (C_N)^c \cap (D_N)^c) &\leq \mathbf{P}\left(\left\{|Y| > \frac{\varepsilon}{4} b_{n_*}\right\} \cap (B_N^*)^c\right) \\ &= \mathbf{E}\left[\mathbf{P}\left[|Y| > \frac{\varepsilon}{4} b_{n_*} \middle| \mathcal{F}_{t_*}\right] I_{(B_N^*)^c}\right].\end{aligned}$$

By Chebyshev's inequality, the latter is at most $48 n_* \sigma^2 / 20 (b_{n_*})^2$ which goes to 0 as N goes to ∞ uniformly in L. □

Proof of Theorem 1.2: Let

$$B^\varepsilon := \{\exists t \in [0, \varepsilon^2/20] : S_n(t) \geq (1+\varepsilon) b_n \text{ i.o.}\}$$

and

$$A^\varepsilon := \{\exists t \in [0, \varepsilon^2/20] : S_n(t) \leq (1-\varepsilon) b_n \text{ for all sufficiently large } n\}.$$

It suffices to show that for any $\varepsilon > 0$, $\mathbf{P}(B^\varepsilon)$ and $\mathbf{P}(A^\varepsilon)$ are 0. Now, for all N,

$$\begin{aligned}B^\varepsilon &\subseteq \left\{\exists t \in [0, \varepsilon^2/20] : |S_n(t) - S_n(0)| \geq \frac{\varepsilon}{2} b_n \text{ for some } n \geq N\right\} \\ &\cup \left\{S_n(0) \geq \left(1 + \frac{\varepsilon}{2}\right) b_n \text{ for some } n \geq N\right\}.\end{aligned}$$

Lemma 2.2 implies that the probability of the first event goes to 0 with N and the usual LIL implies that the probability of the second event goes to 0 with N.

Next, let

$$A_N^\varepsilon := \{\exists t \in [0, \varepsilon^2/20] : S_n(t) \leq (1-\varepsilon) b_n \text{ for all } n \geq N\}$$

and note that $A^\varepsilon = \cup_N A_N^\varepsilon$. We next have that

$$\begin{aligned}A_N^\varepsilon &\subseteq \left\{\exists t \in [0, \varepsilon^2/20] : |S_n(t) - S_n(0)| \geq \frac{\varepsilon}{2} b_n \text{ for some } n \geq N\right\} \\ &\cup \left\{S_n(0) \leq \left(1 - \frac{\varepsilon}{2}\right) b_n \text{ for all } n \geq N\right\}.\end{aligned}$$

Lemma 2.2 implies that the probability of the first event goes to 0 with N and the usual LIL implies that the probability of the second event is 0. Since the events A_N^ε are increasing with N, we obtain $\mathbf{P}(A^\varepsilon) = 0$. □

Lemma 2.2 also yields a dynamical version of Strassen's invariance principle (see [7, p. 348]), in the same way that it led to the dynamical LIL. This will be needed in our proof of Theorem 1.11.

Corollary 2.3 (Dynamical Strassen invariance) *Assume that $m = 0$ and $\sigma^2 < \infty$. Let $f : [0,1] \to \mathbf{R}$ be such that $\int_0^1 |f'(x)|^2 dx \leq 1$. Let $Z_n(t)(\cdot)$ be the function from $[0,1]$ to $\mathbf{R}$ which is $S_j(t)/b_n$ at j/n for $j = 0, \ldots, n$ and defined at other points by linear interpolation. Then the event that for all t, there exists $n_j(t) \to \infty$ such that $Z_{n_j(t)}(t)(\cdot)$ approaches f uniformly, has probability 1.*

3 Run tests

In this section, we prove Theorems 1.4 and 1.5. For the proof of Theorem 1.4, it is convenient to define an auxiliary random variable τ, which is exponentially distributed with mean 1, and independent of $\{X_n(t)\}_{n\in\mathbf{N},t\geq 0}$. The idea is that considering the process up until the random time τ allows the exact calculation in Lemma 3.1 below. For a fixed sequence $a = \{a_n\}_{n=1}^{\infty}$ and $n \in \mathbf{N}$, define the random variable

$$U_n^a = \int_0^{\tau} \mathbf{1}_{\{R_n(t)\geq a_n\}} dt\,.$$

In words, U_n^a is the amount of time, up to time τ, that $R_n(t) \geq a_n$.

Lemma 3.1 *For any $a = \{a_n\}_{n=1}^{\infty}$, and any $n \in \mathbf{N}$, we have $\mathbf{E}[U_n^a] = p^{a_n}$ and $\mathbf{E}[U_n^a \,|\, U_n^a > 0] = \frac{1-p^{a_n+1}}{(a_n+1)(1-p)}$. Hence*

$$\mathbf{P}(U_n^a > 0) = \frac{\mathbf{E}[U_n^a]}{\mathbf{E}[U_n^a \,|\, U_n^a > 0]} = \frac{p^{a_n}(a_n+1)(1-p)}{1-p^{a_n+1}}\,.$$

Proof: This is a relatively easy computation left to the reader, or see [11, Lem. 5.2]. □

Proof of Theorem 1.4: Lemma 3.1 gives

$$\sum_{n=1}^{\infty} \mathbf{P}(U_n^a > 0) = \sum_{n=1}^{\infty} \frac{p^{a_n}(a_n+1)(1-p)}{1-p^{a_n+1}} \begin{cases} < \infty & \text{if } \sum_{n=1}^{\infty} a_n p^{a_n} < \infty \\ \\ = \infty & \text{if } \sum_{n=1}^{\infty} a_n p^{a_n} = \infty\,. \end{cases} \tag{12}$$

Hence, if $\sum_{n=1}^{\infty} a_n p^{a_n} < \infty$, we get

$$\mathbf{P}(\exists t \in [0,\tau] :\, \{R_n(t) \geq a_n \text{ i.o.}\}) \leq \mathbf{P}(U_n^a > 0 \text{ i.o.}) = 0$$

by Borel–Cantelli, and it follows easily that $\mathbf{P}(\exists t \geq 0 :\, \{R_n(t) \geq a_n \text{ i.o.}\}) = 0$.

Now assume that $\sum_{n=1}^{\infty} a_n p^{a_n} = \infty$. Define the event

$$\mathcal{C}_n = \{\exists t \in [0,1] :\, R_n(t) \geq a_n\}\,,$$

and note that

$$\begin{aligned} \mathbf{P}(U_n^a > 0) &= \mathbf{P}(\exists t \in [0,\tau] : R_n(t) \geq a_n) \\ &\leq \sum_{k=0}^{\infty} e^{-k}\mathbf{P}(\exists t \in [k,k+1] :\, R_n(t) \geq a_n) \\ &= \sum_{k=0}^{\infty} e^{-k}\mathbf{P}(\mathcal{C}_n) = \frac{\mathbf{P}(\mathcal{C}_n)}{1-e^{-1}} \end{aligned}$$

so that by (12) we have

$$\sum_{n=1}^{\infty} \mathbf{P}(\mathcal{C}_n) \geq (1-e^{-1}) \sum_{n=1}^{\infty} \mathbf{P}(U_n^a > 0) = \infty \,. \tag{13}$$

Now consider the event

$$\mathcal{A}_n = \{X_{n-1}(t) = 0 \text{ for all } t \in [0,1]\}.$$

Note that $\mathbf{P}(\mathcal{A}_n) = (1-p)e^{-p}$ and that $\mathcal{A}_n$ and $\mathcal{C}_n$ are independent for each n. Hence

$$\sum_n \mathbf{P}(\mathcal{A}_n \cap \mathcal{C}_n) = \infty.$$

Next, it is clear that the events $\{\mathcal{A}_n \cap \mathcal{C}_n\}_n$ are negatively correlated since for $m \neq n$, $\mathcal{A}_n \cap \mathcal{C}_n$ and $\mathcal{A}_m \cap \mathcal{C}_m$ are either disjoint or independent. It follows from the Kochen–Stone Theorem (see [7, p. 55]) that $\mathbf{P}(\mathcal{A}_n \cap \mathcal{C}_n \text{ i.o.}) = 1$ and so

$$\mathbf{P}(\exists \text{ infinitely many } n \in \mathbf{N} \text{ such that } \{\exists t \in [0,1] \text{ with } R_n(t) \geq a_n\}) = 1\,. \tag{14}$$

Clearly, (14) implies that for any rationals q, q' with $q < q'$,

$$\mathbf{P}(\exists \text{ infinitely many } n \in \mathbf{N} \text{ such that } \{\exists t \in (q,q') \text{ with } R_n(t) \geq a_n\}) = 1\,. \tag{15}$$

A time $t \in [0,1]$ for which $\{R_n(t) \geq a_n \text{ i.o.}\}$ can now be found (with probability 1) by an easy application of the Baire Category Theorem: Let

$$V_k = \cup_{n \geq k} \{t \in (0,1) : R_n(t) \geq a_n\}^o$$

where A^o denotes the interior of A. It follows from the above that each V_k is dense and open a.s. Hence, by the Baire Category Theorem, we have that $\cap_{k \geq 1} V_k$ is a.s. dense, completing the proof. $\square$

Our next task will be to prove Theorem 1.5. To this end, we need some notation, and a series of lemmas (Lemmas 3.2, 3.3 and 3.4 below). Let $p \in (0,1)$ be fixed. Theorem 1.5 will follow by sandwiching $W(r)$ between two discrete limsup random fractals in the sense of [14, Section 3] and [6, Section 3], that the reader needs to be familiar with in order to follow the proof. Throughout the rest of this section, we will often write x for $\lfloor x \rfloor$ when an integer is clearly intended.

We will now introduce a collection of so-called limsup random fractals $\{\underline{A}(r)\}_{1<r\leq 2}$ and $\{\overline{A}(r)\}_{1<r\leq 2}$. Let $\mathcal{D}_k$ be the collection of binary subintervals of $[0,1]$ of the form $[a/2^k, (a+1)/2^k]$ with a an integer. Let $J_m^{(r)} := [p^{-(m-1)}(m-1)^{-r}, p^{-m}m^{-r} - m - 1]$. For $I \in \mathcal{D}_k$, let $\underline{Z}_k^{(r)}(I)$ be the indicator function of

$$\{\exists m \in [2^k, 2^{k+1}] \text{ and } n \in J_m^{(r)} : R_n(t) \geq m \ \forall t \in I\}, \tag{16}$$

and let $\overline{Z}_k^{(r)}(I)$ be the indicator function of

$$\{\exists m \in [2^k, 2^{k+1}] \text{ and } n \le p^{-m}m^{-r} : R_n(t) \ge m \text{ for some } t \in I\}. \tag{17}$$

Next, let

$$\underline{A}(r) := \limsup_k \cup_{\{I \in \mathcal{D}_k : \underline{Z}_k^{(r)}(I)=1\}} I^o$$

and

$$\overline{A}(r) := \limsup_k \cup_{\{I \in \mathcal{D}_k : \overline{Z}_k^{(r)}(I)=1\}} I^o$$

where I^o denotes the interior of I.

The following three lemmas will do all the work necessary to apply the limsup random fractal theory developed in [14] and in [6].

Lemma 3.2 *Let $p \in (0,1)$, $r \in (1,2]$. Let*

$$W_k^{(r)} := \sum_{m=2^k}^{2^{k+1}} \sum_{n \in J_m^{(r)}} I_{\{R_n(t) \ge m \forall t \in [0, \frac{1}{2^k}]\}},$$

and

$$U_k^{(r)} := \sum_{m=2^k}^{2^{k+1}} \sum_{n=1}^{p^{-m}m^{-r}} I_{\{R_n(t) \ge m \text{ for some } t \in [0, \frac{1}{2^k}]\}}.$$

Then there exists a constant $C_1 = C_1(r,p)$ such that for any k, the ratio between any of $\underline{Z}_k^{(r)}([0, \frac{1}{2^k}])$, $\overline{Z}_k^{(r)}([0, \frac{1}{2^k}])$, $\mathbf{E}[W_k^{(r)}]$, $\mathbf{E}[U_k^{(r)}]$ and $2^{k(1-r)}$ is at most C_1.

Proof: Throughout this proof, C will denote an arbitrary constant (depending on p and r) whose value might change from line to line.

Clearly there exists $C > 0$ such that for any k, any set of at most 2^{k+1} locations has the property that during a time interval of length $1/2^k$, none of the locations will flip with probability at least C. From this, it is easy to see that $\mathbf{E}\left(\underline{Z}_k^{(r)}([0, \frac{1}{2^k}])\right)$, $\mathbf{E}\left(\overline{Z}_k^{(r)}([0, \frac{1}{2^k}])\right)$, $\mathbf{E}(W_k^{(r)})$ and $\mathbf{E}(U_k^{(r)})$ all change by at most a multiplicative constant if all the events in question are modified by replacing "$R_n(t) \ge m\ \forall t \in I$" or "$R_n(t) \ge m$ for some $t \in I$" by "$R_n(0) \ge m$" in (16) and (17). For the rest of this proof, we will work with these modified events, which we denote by $\widehat{\underline{Z}_k^{(r)}}$, $\widehat{\overline{Z}_k^{(r)}}$, $\widehat{W_k^{(r)}}$ and $\widehat{U_k^{(r)}}$.

First, note that

$$\mathbf{E}[\widehat{U_k^{(r)}}] = \sum_{m=2^k}^{2^{k+1}} \sum_{n=1}^{p^{-m}m^{-r}} p^m \le C \int_{2^k}^{2^{k+1}} \frac{1}{x^r} dx \le C 2^{k(1-r)}.$$

It is also easy to see from the above that $\mathbf{E}[\widehat{U_k^{(r)}}] \ge C 2^{k(1-r)}$. Since

$$\mathbf{E}\left[\widehat{\underline{Z}_k^{(r)}}\left(\left[0, \frac{1}{2^k}\right]\right)\right] \le \mathbf{E}\left[\widehat{\overline{Z}_k^{(r)}}\left(\left[0, \frac{1}{2^k}\right]\right)\right]$$

and

$$\mathbf{E}\left[\widehat{W_k^{(r)}}\right] \leq \mathbf{E}\left[\widehat{U_k^{(r)}}\right],$$

we need only show that $\mathbf{E}\left[\widehat{\underline{Z}_k^{(r)}}([0,\frac{1}{2^k}])\right] \geq C\mathbf{E}\left[\widehat{U_k^{(r)}}\right]$. The reader can easily check that $\mathbf{E}\left[\widehat{W_k^{(r)}}\right] \geq C\mathbf{E}\left[\widehat{U_k^{(r)}}\right]$, and so we need to show that $\mathbf{E}\left[\widehat{\underline{Z}_k^{(r)}}([0,\frac{1}{2^k}])\right] \geq C\mathbf{E}\left[\widehat{W_k^{(r)}}\right]$.

We first show that

$$\mathbf{E}\left[\left(\widehat{W_k^{(r)}}\right)^2\right] \leq C\mathbf{E}\left[\widehat{W_k^{(r)}}\right]. \tag{18}$$

Since $\mathbf{E}\left[\widehat{W_k^{(r)}}\right] \leq 1$ for large k (because we have already seen it is $\asymp 2^{k(1-r)}$), all we need to do to prove (18), is to show that $\mathbf{Var}\left[\widehat{W_k^{(r)}}\right] \leq C\mathbf{E}\left[\widehat{W_k^{(r)}}\right]$.

Note now that if $m_1, m_2 \in [2^k, 2^{k+1}]$ with $m_1 \neq m_2$ and $n_1 \in J_{m_1}$ and $n_2 \in J_{m_2}$, then $\{R_{n_1} \geq m_1\}$ and $\{R_{n_2} \geq m_2\}$ are independent. Hence

$$\mathbf{Var}\left[\widehat{W_k^{(r)}}\right] = \sum_{m=2^k}^{2^{k+1}} \sum_{n_1,n_2 \in J_m^{(r)}} \mathbf{Cov}[I_{\{R_{n_1}(0)\geq m\}}, I_{\{R_{n_2}(0)\geq m\}}].$$

Note that

$$\mathbf{Cov}[I_{\{R_{n_1}(0)\geq m\}}, I_{\{R_{n_2}(0)\geq m\}}] \leq p^m p^{|n_1-n_2|}$$

since the covariance is 0 if $|n_1 - n_2| \geq m$ and is at most $\mathbf{E}[I_{\{R_{n_1}(0)\geq m\}} I_{\{R_{n_2}(0)\geq m\}}]$ otherwise. It follows that for any $m \in [2^k, 2^{k+1}]$ and any $n_1 \in J_m^{(r)}$,

$$\sum_{n_2 \in J_m^{(r)}} \mathbf{Cov}[I_{\{R_{n_1}(0)\geq m\}}, I_{\{R_{n_2}(0)\geq m\}}] \leq Cp^m$$

and from this, it follows that $\mathbf{Var}\left[\widehat{W_k^{(r)}}\right] \leq C\mathbf{E}\left[\widehat{W_k^{(r)}}\right]$. From (18) and the Cauchy–Schwartz inequality, we obtain

$$\mathbf{E}\left(\widehat{\underline{Z}_k^{(r)}}\left(\left[0,\frac{1}{2^k}\right]\right)\right) = \mathbf{P}\left(\widehat{W_k^{(r)}} > 0\right) \geq \frac{\mathbf{E}\left[\widehat{W_k^{(r)}}\right]^2}{\mathbf{E}\left[\left(\widehat{W_k^{(r)}}\right)^2\right]} \geq C\mathbf{E}\left[\widehat{W_k^{(r)}}\right].$$

□

Lemma 3.3 *Let $I = [\frac{a}{2^k}, \frac{a+1}{2^k}]$ and $J = [\frac{b}{2^k}, \frac{b+1}{2^k}]$ where $b = a+1+j$ with $0 \leq j \leq 2^k$. Let $W_k^{(r)}(I), W_k^{(r)}(J)$ be defined as $W_k^{(r)}$ in Lemma 3.2 but with $[0, 1/2^k]$ replaced by I and J respectively. Then for any $p \in (0,1)$ and $r \in (1,2]$, there exist constants $C_2 = C_2(r,p)$ $C_3 = C_3(r,p)$ such that*

$$\mathbf{Cov}[W_k^{(r)}(I), W_k^{(r)}(J)] \leq C_2 e^{-C_3 j}\mathbf{E}[W_k^{(r)}(I)].$$

Proof: Again, $C = C(r,p)$ will denote an arbitrary constant whose value might change from appearance to appearance. Note that for $m_1, m_2 \in [2^k, 2^{k+1}]$ with $m_1 \neq m_2$ and $n_1 \in J_{m_1}$ and $n_2 \in J_{m_2}$, $\{R_{n_1}(t) \geq m_1 \forall t \in I\}$ and $\{R_{n_2}(t) \geq m_2 \forall t \in J\}$ are independent. Hence the covariance above is

$$\sum_{m=2^k}^{2^{k+1}} \sum_{n_1,n_2 \in J_m^{(r)}} \mathbf{Cov}[I_{\{R_{n_1}(t) \geq m \forall t \in I\}}, I_{\{R_{n_2}(t) \geq m \forall t \in J\}}]. \tag{19}$$

Next,

$$\begin{aligned}
&\mathbf{Cov}[I_{\{R_{n_1}(t) \geq m \forall t \in I\}}, I_{\{R_{n_2}(t) \geq m \forall t \in J\}}] \\
\leq\ & \mathbf{P}\left(R_{n_1}\left(\frac{a+1}{2^k}\right) \geq m, R_{n_2}\left(\frac{b}{2^k}\right) \geq m\right) \\
=\ & p^{2|n_1-n_2|}(pe^{-\frac{j}{2^k}} + p^2(1-e^{-\frac{j}{2^k}}))^{m-|n_1-n_2|} \\
=\ & \left(pe^{-\frac{j}{2^k}} + p^2(1-e^{-\frac{j}{2^k}})\right)^m \left(\frac{1}{\frac{e^{-\frac{j}{2^k}}}{p} + (1-e^{-\frac{j}{2^k}})}\right)^{|n_1-n_2|}.
\end{aligned}$$

Now, $p < 1$ implies that

$$\inf_{0 \leq j \leq 2^k} \frac{e^{-\frac{j}{2^k}}}{p} + (1-e^{-\frac{j}{2^k}}) > 1.$$

Summing over n_2, we are dominated by a geometric series, whence (19) is at most

$$C \sum_{m=2^k}^{2^{k+1}} \sum_{n_1 \in J_m^{(r)}} p^m [e^{-\frac{j}{2^k}} + p(1-e^{-\frac{j}{2^k}})]^m. \tag{20}$$

Next, it is an elementary calculus exercise to check that for $p \in (0,1)$, there exists $C_3 > 0$ such that

$$e^{-x} + p(1-e^{-x}) \leq e^{-C_3 x} \quad \text{for } x \in [0,1].$$

Hence (20) is at most

$$\begin{aligned}
C \sum_{m=2^k}^{2^{k+1}} \sum_{n_1 \in J_m^{(r)}} p^m e^{\frac{-C_3 jm}{2^k}} &\leq Ce^{-C_3 j} \sum_{m=2^k}^{2^{k+1}} \sum_{n_1 \in J_m^{(r)}} p^m \\
&\leq Ce^{-C_3 j} \mathbf{E}[W_k^{(r)}(I)].
\end{aligned}$$

Here, the last inequality (recall C might change each time it appears) uses the observation at the beginning of the proof of Lemma 3.2, that the probabilities of $W_k^{(r)}$ and $\widehat{W_k^{(r)}}$ are of the same order of magnitude. □

Lemma 3.4 *There exists a constant L such that if*

$$f(k) := \max_{I\in\mathcal{D}_k} |J \in \mathcal{D}_k : \mathbf{E}[\underline{Z}_k^{(r)}(I)\underline{Z}_k^{(r)}(J)] \geq L\mathbf{E}[\underline{Z}_k^{(r)}(I)]\mathbf{E}[\underline{Z}_k^{(r)}(J)]|,$$

then

$$\limsup_{k\to\infty} \frac{\log_2 f(k)}{k} = 0.$$

Proof: Let $L := (C_1C_2+1)C_1^2$ where C_1 and C_2 come from Lemmas 3.2 and 3.3. If $\mathbf{E}[\underline{Z}_k^{(r)}(I)\underline{Z}_k^{(r)}(J)] \geq L\mathbf{E}[\underline{Z}_k^{(r)}(I)]\mathbf{E}[\underline{Z}_k^{(r)}(J)]$, then by Lemma 3.2

$$\mathbf{E}[W_k^{(r)}(I)W_k^{(r)}(J)] \geq LC_1^{-2}\mathbf{E}[W_k^{(r)}(I)]\mathbf{E}[W_k^{(r)}(J)]$$

which implies that

$$\mathbf{Cov}[W_k^{(r)}(I)W_k^{(r)}(J)] \geq (LC_1^{-2}-1)\mathbf{E}[W_k^{(r)}(I)]\mathbf{E}[W_k^{(r)}(J)].$$

Lemma 3.3 now implies that

$$C_2e^{-C_3j} \geq (LC_1^{-2}-1)\mathbf{E}[W_k^{(r)}(J)] \geq (LC_1^{-2}-1)C_1^{-1}2^{k(1-r)}$$

where $j/2^k$ is the distance between I and J. By definition of L, we get $e^{-C_3j} \geq 2^{k(1-r)}$ or $j \leq (r-1)k\log(2)/C_3$. This implies the conclusion of the lemma. □

Proof of Theorem 1.5: Note that Lemma 3.2 implies that

$$\lim_{k\to\infty} \frac{\log_2(\mathbf{E}[\underline{Z}_k^{(r)}(I)])}{k} = 1-r = \lim_{k\to\infty} \frac{\log_2(\mathbf{E}[\overline{Z}_k^{(r)}(I)])}{k}. \tag{21}$$

This, together with [14, Theorem 3.1], implies that if E is any closed set in $[0,1]$ with $\dim_P(E) < r-1$, then $\mathbf{P}(\overline{A}(r)\cap E \neq \emptyset) = 0$. Also, [14, Corollary 3.3] implies that $\dim_H(\overline{A}(r)) \leq 2-r$. We next show that if E is any closed set in $[0,1]$ with $\dim_P(E) > r-1$, then

$$\mathbf{P}(\underline{A}(r)\cap E \neq \emptyset) = 1. \tag{22}$$

To this end, we will apply [6, Theorem 3.1]. Condition I in that theorem holds with $\gamma = r-1$, by (21). Condition II holds since it is easy to check that the random variables $\{\underline{Z}_k^{(r)}(I)\}$ are quasi-localized as defined in that paper; in fact the $\mathcal{F}_i$'s as defined there are trivial σ-algebras. Condition III follows from Lemma 3.4 above. Hence, [6, Theorem 3.1] implies (22). Furthermore, [6, Corollary 3.2] implies that $\dim_H(\underline{A}(r)) = 2-r$ a.s., and by the remark following that same corollary, we have $\dim_P(\underline{A}(r)) = 1$ a.s.

Finally, an easy computation shows that

$$W(r) \subseteq \cap_{r'<r}\overline{A}(r')$$

and

$$\cup_{r'>r}\underline{A}(r') \subseteq W(r).$$

The results proved for $\underline{A}(r)$ and $\overline{A}(r)$ now immediately imply the statements about $W(r)$ in the theorem. □

4 Prediction and von Mises–Church randomness

We begin this section with a proof of Theorem 1.8. The proof is based on parity tests. Define

$$B_{(m,n)}(t) = \Big(\sum_{k=m}^{n} X_k(t)\Big) \bmod 2\,.$$

The crucial lemma is as follows.

Lemma 4.1 *let $\{m_k\}_{k\geq 1}$ be a sequence of positive integers which is lacunary in the sense that* $\inf_k m_{k+1}/m_k > 1$. *Let $a_0 = 0$ and $a_{i+1} = a_i + m_{i+1}$ for $i \geq 0$. Then*

$$\mathbf{P}(\exists t \in [0,1] : \{B_{(a_{n-1}+1,a_n)}(t) = 0 \textit{ for all } n \in \mathbf{N}\}) > 0$$

if and only if

$$\sum_{\ell=1}^{\infty} \frac{2^\ell}{m_\ell} < \infty.$$

Proof: Let τ be an independent exponential time and as usual, it suffices to show that

$$\mathbf{P}(\exists t \in [0,\tau] : \{B_{(a_{n-1}+1,a_n)}(t) = 0 \text{ for all } n \in \mathbf{N}\}) > 0$$

if and only if

$$\sum_{\ell=1}^{\infty} \frac{2^\ell}{m_\ell} < \infty.$$

For $n \geq 1$, let $U_n(t) := \{B_{(a_{k-1}+1,a_k)}(t) = 0 \;\forall k \in \{1,\dots,n\}\}$ and $Z_n := \int_0^\tau \mathbf{1}_{\{U_n(t)\}} dt$. It is easy to see that $\mathbf{E}[Z_n] = (1/2)^n$. We will now compute $\mathbf{E}[Z_n \,|\, Z_n > 0]$. It is easy to check that for $t > s$,

$$\mathbf{P}(U_n(t)|U_n(s)) = \prod_{k=1}^{n} \left(\frac{1}{2} + \frac{e^{-m_k(t-s)}}{2}\right).$$

Using the strong Markov property for the stopping time $\inf\{t \geq 0 : U_n(t) \text{ occurs}\}$ and the memoryless property of the exponential distribution, we obtain

$$\begin{aligned}
\mathbf{E}[Z_n|\, Z_n > 0] &= \int_0^\infty \mathbf{P}(U_n(t)|U_n(0)) e^{-t} dt \int_0^\infty \prod_{k=1}^{n} \left(\frac{1}{2} + \frac{e^{-m_k t}}{2}\right) e^{-t} dt \\
&= \frac{1}{2^n} \int_0^\infty \prod_{k=1}^{n} \left(1 + e^{-m_k t}\right) e^{-t} dt \\
&= \frac{1}{2^n} \int_0^\infty \sum_{S \subseteq \{1,\dots,n\}} e^{-(\sum_{k\in S} m_k)t} e^{-t} dt \\
&= \frac{1}{2^n} \sum_{S \subseteq \{1,\dots,n\}} \frac{1}{1 + \sum_{k\in S} m_k} \leq \frac{1}{2^n} \sum_{S \subseteq \{1,\dots,n\}} \frac{1}{1 + m_S} \\
&= \frac{1}{2^n} \left(1 + \sum_{k=1}^{n} \frac{2^{k-1}}{1 + m_k}\right)
\end{aligned}$$

where $m_S := \max\{m_i : i \in S\}$ with the usual convention $m_\emptyset := 0$. Note, importantly, that by the lacunary assumption, the inequality can be reversed up to a uniform multiplicative constant. Since $\mathbf{P}(Z_n > 0) = \mathbf{E}[Z_n]/\mathbf{E}[Z_n \mid Z_n > 0]$, it follows that

$$\liminf_{n\to\infty} \mathbf{P}(Z_n > 0) = \begin{cases} 0 & \text{if } \sum_{\ell=1}^{\infty} \frac{2^\ell}{m_\ell} = \infty \\ L > 0 & \text{if } \sum_{\ell=1}^{\infty} \frac{2^\ell}{m_\ell} < \infty \,. \end{cases}$$

Note that the events $\{Z_n > 0\}$ are decreasing, so that if $\sum_{\ell=1}^{\infty} \frac{2^\ell}{m_\ell} = \infty$, then

$$\mathbf{P}(\exists t \in [0,\tau] : \{B_{(a_{n-1}+1,a_n)}(t) = 0 \text{ for all } n \in \mathbf{N}\}) \le \mathbf{P}(\cap_n \{Z_n > 0\}) = 0.$$

Conversely, if $\sum_{\ell=1}^{\infty} \frac{2^\ell}{m_\ell} < \infty$, then $\mathbf{P}(\cap_n \{Z_n > 0\}) > 0$, and a simple compactness argument implies that $\cap_n \{Z_n > 0\} \subseteq \{\exists t \in [0,\tau] : \{B_{(a_{n-1}+1,a_n)}(t) = 0 \text{ for all } n \in \mathbf{N}\}\}$, completing the proof. $\square$

Proof of Theorem 1.8: Let $m_k := \lfloor 2^{k(1+\varepsilon)} \rfloor$, $a_0 = 0$ and $a_{i+1} = a_i + m_{i+1}$ for $i \ge 0$ as in Lemma 4.1. Define $g : \{0,1\}^{\mathbf{N}} \to \{*,0,1\}^{\mathbf{N}}$ by setting

$$(g(\xi))_n = \begin{cases} \left(\sum_{k=a_{i-1}+1}^{n-1} \xi_k\right) \bmod 2 & \text{if } n = a_i \text{ for some } i \in \mathbf{N}, \text{ and} \\ * & \text{otherwise.} \end{cases}$$

A simple computation shows that this g has inverse rate $2^{(n+1)(1+\varepsilon)}$. By Lemma 4.1, $\mathbf{P}(\exists t \in [0,1] : \{B_{(a_{n-1}+1,a_n)}(t) = 0 \text{ for all } n \in \mathbf{N}\}) > 0$. However, one may simply observe that if $B_{(a_{n-1}+1,a_n)}(t) = 0$ for all $n \in \mathbf{N}$, then it follows immediately from the definition of g that g is correct on the input $X(t)$. This completes the proof. $\square$

Before giving the proof of Theorem 1.9, we prove a slightly weaker version, which has a more elementary proof in the sense that it does not appeal to a general result from Markov process theory.

Theorem 4.2 *Let g be a predictor with inverse rate $r(n)$ such that $r(n) = o(2^n)$. Then* $\mathbf{P}(\exists t \ge 0$ such that g is correct on the input $X(t)) = 0$.

Proof: Let g be a fixed predictor with the given rate assumption. Let C be such that $r(n) \le 2^n \le C$ for all n. Let A_n be the event that for some $t \in [0,1]$, the first n bits of $X(t)$ which are predicted by g are predicted correctly. Let V_n be the number of different sequences of the first $r(n)$ bits that arise during the time interval $[0,1]$. Clearly, $V_n \le 1 + Y_n$ where Y_n has a Poisson distribution with mean $r(n)$. Note that given a single sequence of $r(n)$ random unbiased bits, the first n bits which are predicted by g are predicted correctly with probability 2^{-n}. It follows that

$$\mathbf{P}(A_n) \le \sum_{k=1}^{\infty} \mathbf{P}(V_n = k)\ k 2^{-n} = \frac{1 + r(n)}{2^n}$$

which goes to 0 as $n \to \infty$. $\square$

Proof of Theorem 1.9: Fix $n \geq 1$, $i \in \{1, \ldots, n\}$, and a predictor g with the given rate assumption. Let A_n^i be the event that for some $t \in [(i-1)/n, i/n]$, the first n bits of $X(t)$ which are predicted by g are predicted correctly. Next, let $T_n := \sum_{i=1}^n \mathbf{1}_{\{A_n^i\}}$. A similar computation to the proof of Theorem 4.2 shows that $\mathbf{E}[T_n] \leq r(n)(1/2)^n + n(1/2)^n$ and hence $\liminf_n \mathbf{E}[T_n] \leq C$. If

$$T := \{t \in [0,1] \text{ such that } g \text{ is correct on the input } X(t)\},$$

we easily have that $T \leq \liminf_{n\to\infty} T_n$ and so $\mathbf{E}[T] \leq C$ by Fatou's lemma. Hence $T < \infty$ a.s. It follows from general Markov process theory (see [20, Lemma 2.3]) that $T = \emptyset$ a.s., as desired. $\square$

Let $\mathcal{H}^\beta$ denote β–dimensional Hausdorff measure and recall that $\dim_H(S)$ denotes the Hausdorff dimension of the set S. We say that a closed set A has positive β-capacity if there exists a probability measure ρ on A such that

$$\int_A \int_A |t-s|^{-\beta} d\rho(t) d\rho(s) < \infty.$$

Recall Frostman's Theorem (see [13]) which says that for any closed set $A \subseteq \mathbf{R}^n$, $\dim_H(A) = \sup\{\beta : A \text{ has positive } \beta\text{-capacity}\}$. Our next result is a multi-fractal result for prediction.

Theorem 4.3 *Given a predictor g, let*

$$T_g := \{t \in [0,1] : g \text{ predicts } X(t) \text{ correctly}\}.$$

Given $\alpha \in (0,1)$, let g_α be the predictor which is defined as in Lemma 4.1 with $m_k = 2^{k/\alpha}$. If A is a closed set which has positive α-capacity, then $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) > 0$. If $\mathcal{H}^\alpha(A) = 0$, then $\mathbf{P}(T_g \cap A \neq \emptyset) = 0$ for any predictor g which has inverse rate $2^{n/\alpha}$. Furthermore $\dim_H(T_{g_\alpha}) = 1 - \alpha$.

Remark 4.4 By Frostman's Theorem, we have that the above is stronger than the statements that $\dim_H(A) > \alpha$ implies $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) > 0$, and that $\dim_H(A) < \alpha$ implies $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) = 0$.

Remark 4.5 Theorem 4.3 above comes close to, but does not succeed in, obtaining a sufficient and necessary condition for which sets A satisfy $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) > 0$. The gap here is the usual gap between zero Hausdorff measure and positive capacity when studying the hitting probabilities of random sets.

Proof of Theorem 4.3: If A has positive α-capacity, choose a probability measure ρ on A such that $\int_A \int_A |t-s|^{-\alpha} d\rho(t) d\rho(s) < \infty$. For $n \geq 1$, let $U_n(t)$ be as in Lemma 4.1 and $Z_n := \int_0^1 \mathbf{1}_{\{U_n(t)\}} d\rho(t)$. Then $\mathbf{E}[Z_n] = (1/2)^n$ and the same computation as in Lemma 4.1 yields

$$\mathbf{E}[Z_n^2] = \frac{1}{4^n}\left(1 + \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} \int_0^1 \int_0^1 e^{-(\sum_{k\in S} m_k)|t-s|} d\rho(t) d\rho(s)\right).$$

Replacing $\sum_{k\in S} m_k$ by $m_S := \max\{m_i : i \in S\}$ and proceeding as in Lemma 4.1, the latter is at most

$$\frac{1}{4^n}\int_0^1 \int_0^1 \left(1 + \sum_{k=1}^{n} 2^{k-1} e^{-m_k|t-s|}\right) d\rho(t) d\rho(s)\,.$$

We claim that for any $t \geq 0$ and any n,

$$\sum_{k=1}^{n} 2^{k-1} e^{-m_k t} \leq C t^{-\alpha} \tag{23}$$

where $C := \int_0^\infty u^{\alpha-1} e^{-u} du$. Once established, (23) immediately implies that

$$\mathbf{E}[Z_n^2] \leq \frac{1}{4^n}\left[1 + C \int_A \int_A |t-s|^{-\alpha}\right] d\rho(t) d\rho(s)\,.$$

By the Cauchy–Schwartz inequality, we have that $\mathbf{P}(Z_n > 0) \geq \mathbf{E}[Z_n]^2/\mathbf{E}[Z_n^2]$, which is larger than some constant $C' > 0$ for all n. Then $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) > 0$ follows exactly as in Lemma 4.1 and Theorem 1.8.

To prove (23), it is elementary to check that for any $k \geq 1$

$$\int_{2^{\frac{k-1}{\alpha}}}^{2^{\frac{k}{\alpha}}} x^{\alpha-1} e^{-tx} dx \geq 2^{k-1} e^{-m_k t}$$

and hence that

$$\sum_{k=1}^{n} 2^{k-1} e^{-m_k t} \leq \int_0^\infty x^{\alpha-1} e^{-tx} dx$$

which by a change of variables is $\left(\int_0^\infty u^{\alpha-1} e^{-u} du\right) t^{-\alpha}$, and (23) is established.

We now assume that $\mathcal{H}^\alpha(A) = 0$ and g is any predictor which has inverse rate $2^{n/\alpha}$. Let $\delta > 0$ be arbitrary. Choose intervals $\{I_i\}_{i\in J}$ such that $A \subseteq \cup_{i\in J} I_i$ and $\sum_{i\in J} |I_i|^\alpha < \delta$. We claim that

$$\mathbf{P}(\exists t \in I_i \text{ such that } g \text{ is correct on the input } X(t)) \leq 4|I_i|^\alpha. \tag{24}$$

Once this claim is established, it follows that

$$\mathbf{P}(\exists t \in A \text{ such that } g \text{ is correct on the input } X(t)) \leq 4 \sum_{i\in J} |I_i|^\alpha < 4\delta.$$

As $\delta > 0$ is arbitrary, we may then conclude that

$$\mathbf{P}(\exists t \in A \text{ such that } g \text{ is correct on the input } X(t)) = 0\,.$$

It only remains to prove (24). Consider the first $1/|I_i|$ bits. ($1/|I_i|$ need of course not be an integer but we leave this easy correction to the reader.) The number of different sequences within the first $1/|I_i|$ bits that we see during the time interval I_i has distribution which is $\delta_1 * \text{Poisson}(1)$, where $*$ denotes convolution. The number of bits predicted with the first $1/|I_i|$ bits is at least $-\alpha \log_2 |I_i| - 1$ and the probability that g_α predicts this many of the first bits correctly in a random sequence is at most $2|I_i|^\alpha$. Hence, as in the proof of Theorem 4.2, (24) follows.

Finally, as mentioned in the above remark, the first part of the theorem says that $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) > 0$ if $\dim_H(A) > \alpha$ and $\mathbf{P}(T_{g_\alpha} \cap A \neq \emptyset) = 0$ if $\dim_H(A) < \alpha$. It follows from a standard codimension argument originally due to the works of Taylor and Hawkes (see [14, Lemma 3.4] and [19, Proposition 2.1]) that $\dim_H(T_{g_\alpha}) = 1 - \alpha$ as desired. $\square$

Next, we discuss notions of "algorithmic randomness", i.e., criteria for an individual sequence $\xi \in \{0,1\}^{\mathbf{N}}$ to be a "typical" sample from the measure $m := \prod_i (\delta_1/2 + \delta_0/2)$. (See Kolmogorov and Uspenskii [15] and Li and Vitányi [17].)

Today, the most widely used notion of algorithmic randomness is *Martin-Löf randomness*, which can be described informally as follows. Let $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ be the collection of Borel sets in $\{0,1\}^{\mathbf{N}}$ that

(i) have m-measure 0, and

(ii) are computable in the sense of the Church–Turing thesis (see [15] or [17] for details).

Let $\mathcal{A} = \bigcup_{\alpha \in I} \mathcal{A}_\alpha$. Since there are only countably many sets satisfying (ii), we infer that $m(\mathcal{A}) = 0$. A sequence $\xi \in \{0,1\}^{\mathbf{N}}$ is said to be *Martin-Löf random* if $\xi \notin \mathcal{A}$.

The existence of any sensitive computable a.s. property of Bernoulli sequences implies that Martin-Löf randomness is a sensitive. Such properties can easily be extracted from, e.g., Theorem 1.4 or 1.8. (Alternatively, sensitivity of Martin-Löf randomness follows immediately from Corollary 4.6 below.)

We shall show that even the less restrictive (and older) notion of *von Mises–Church randomness* is sensitive. This notion of randomness is defined as follows. First, a function $g:\ \{0,1\}^{\mathbf{N}} \to \{s,d\}^{\mathbf{N}}$ is a *selector* if $(g(\xi))_n$ depends on ξ only via $(\xi_1, \dots, \xi_{n-1})$ for each n. The idea is that a selector is used to choose which bits we will use in forming a subsequence. Specifically, $(g(\xi))_n = s$ if based on $(\xi_1, \dots, \xi_{n-1})$, the selector g selects to use the n:th bit, while $(g(\xi))_n = d$ if based on $(\xi_1, \dots, \xi_{n-1})$, the selector g does not use the n:th bit ("s" for select and "d" for decline). A sequence $\xi \in \{0,1\}^{\mathbf{N}}$ is said to be *von Mises–Church random* if for all computable selectors $g : \{0,1\}^{\mathbf{N}} \to \{s,d\}^{\mathbf{N}}$ such that $\sum_{k=1}^{\infty} \mathbf{1}_{\{(g(\xi))_k = s\}} = \infty$, we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbf{1}_{\{(g(\xi))_k=s\}}\xi_k}{\sum_{k=1}^{n} \mathbf{1}_{\{(g(\xi))_k=s\}}} = 1/2\,.$$

In other words, ξ is von Mises–Church random if the limiting fraction of 1's is 1/2 along all infinite subsequences obtained algorithmically without peeking at $\{\xi_n, \xi_{n+1}, \ldots\}$ when deciding whether ξ_n should be included in the subsequence. Since there are only countably many algorithms,

$$m(\xi : \xi \text{ is von Mises–Church random}) = 1\,.$$

Corollary 4.6 *von Mises–Church randomness is dynamically sensitive, i.e.*

$$\mathbf{P}(\exists t \geq 0 : \ X(t) \textit{ is not von Mises–Church random}) = 1\,. \tag{25}$$

Proof: As in the proof of Theorem 1.8, let $\varepsilon > 0$ be arbitrary, let $m_k := \lfloor 2^{k(1+\varepsilon)} \rfloor$, $a_0 = 0$ and $a_{i+1} = a_i + m_{i+1}$ for $i \geq 0$. Define $g : \{0,1\}^{\mathbf{N}} \to \{s,d\}^{\mathbf{N}}$ by setting

$$(g(\xi))_n = \begin{cases} s & \text{if } n = a_i \text{ for some } i \in \mathbf{N}, \text{ and } \left(\sum_{k=a_{i-1}+1}^{n-1} \xi_k\right) \bmod 2 = 1 \\ d & \text{otherwise.} \end{cases}$$

Define the event

$$\mathcal{A} = \{\exists \text{ infinitely many } i \in \mathbf{N} \text{ such that } \{X_{a_i}(t) = 1 \text{ for all } t \in [0,1]\}\}\,,$$

and note that $\mathbf{P}(\mathcal{A}) = 1$ by the second Borel–Cantelli lemma. Also define the event

$$\mathcal{B} = \{\exists t \in [0,1] : \ \{B_{a_{n-1},a_n}(t) = 0 \text{ for all } n \in \mathbf{N}\}\}\,.$$

By Lemma 4.1, we have that $\mathbf{P}(\mathcal{B}) > 0$, so that

$$\mathbf{P}(\mathcal{A} \cap \mathcal{B}) > 0\,.$$

On the event $\mathcal{B}$, we have, for the times t such that $\{B_{a_{n-1}+1,a_n}(t) = 0 \text{ for all } n \in \mathbf{N}\}$, that $X_n(t) = 1$ for all n such that $(g(X(t)))_n = s$. On the event $\mathcal{A} \cap \mathcal{B}$, for such t, we furthermore have that $(g(X(t)))_n = s$ for infinitely many n. However, then

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbf{1}_{\{(g(X(t)))_k=s\}}X_k(t)}{\sum_{k=1}^{n} \mathbf{1}_{\{(g(X(t)))_k=s\}}} = 1 \tag{26}$$

so that von Mises–Church randomness fails at time t. Hence,

$$\mathbf{P}(\exists t \in [0,1] \text{ such that } X(t) \text{ is not von Mises–Church random}) > 0$$

and (25) follows by Kolmogorov's 0-1-law. $\square$

5 Recurrence and transience of random walks

In this section, we prove Theorems 1.11, 1.12 and 1.13. We need the following lemma, which is proved in [21, p. 382].

Lemma 5.1 *Let $\{S_n\}$ be an irreducible 1-dimensional integer-valued random walk with steps which have mean 0 and have support in $\{-S, \ldots, S\}$. Then there exists a constant C such that*

$$P_x(S_k \neq 0 \textit{ for } k = 1, 2, \ldots, n) \leq Cn^{-1/2}$$

for all $x \in \{-S, \ldots, S\}$ and $n \geq 1$.

We continue with three more lemmas needed to prove Theorem 1.11. As in Section 3, we run our process up until a random time τ which has an exponential distribution with mean 1. Theorem 1.11 will follow if we can show that for all $u \in \mathbf{N}$,

$$\mathbf{P}(\forall t \in [0, \tau] : S_n(t) = 0 \text{ for some } n \geq u) = 1. \tag{27}$$

We now fix such a u.

For $n \geq 20$, let $I_1^n = [n/20, n/10]$ and for $i = 2, 3, 4, 5$, let $I_i^n = [(2i-2)n/10, (2i-1)n/10]$ (where if these fractions are not integers, we use the greatest integer function instead.). Assume that n is sufficiently large so that $\lfloor n/20 \rfloor \geq u$. Let E_n be the event that

$$\{\{S_k\}_{k\geq 0} \text{ takes both strictly positive and negative values in each of } I_1^n, I_2^n, I_3^n, I_4^n \text{ and } I_5^n\}$$

and F_n be the event that

$$\{\{S_k\}_{k\geq 0} \text{ does not return to 0 in } [u, n]\}.$$

Lemma 5.2 *For all n such that $\lfloor n/20 \rfloor \geq u$, we have that*

$$\mathbf{P}(E_n \cap F_n) \leq C^5 n^{-5/2}$$

where C comes from Lemma 5.1.

Proof: For $i = 1, 2, 3, 4$ and 5, let f_i^n be the smallest element in I_i^n and let

$$c_i^n := \inf\{\ell \in I_i^n \backslash \{f_i^n\} : S_{\ell-1} S_\ell < 0\},$$

where we take c_i^n to be ∞ if $S_{\ell-1} S_\ell \geq 0$ for all $\ell \in I_i^n \backslash \{f_i^n\}$.

For $i = 1, 2, 3, 4$ and 5, let $A_i^n := \{c_i^n < \infty, S_k \neq 0 \text{ for } k \in \{c_i^n + 1, \ldots, c_i^n + n/10\}\}$. Then

$$\mathbf{P}(E_n \cap F_n) \leq \mathbf{P}(\cap_{i=1}^5 A_i^n).$$

By the strong Markov property and Lemma 5.1, for all $i \in \{1,2,3,4,5\}$,

$$\mathbf{P}\Big(A_i^n \Big| \bigcap_{j=1}^{i-1} A_j^n\Big) \leq Cn^{-1/2}$$

and the statement of the lemma follows. □

Lemma 5.3 *For all n such that $\lfloor n/20 \rfloor \geq u$*

$$\mathbf{P}\left(\exists t \in [0,\tau] : X(t) \in E_n \cap F_n\right) \leq C^5 e^2 n^{-3/2}$$

where C comes from Lemma 5.1.

Proof: Let $Z_n := \int_0^\tau \mathbf{1}_{\{X(t) \in E_n \cap F_n\}} dt$. Fubini's Theorem and Lemma 5.2 imply that $\mathbf{E}[Z_n] \leq C^5 n^{-5/2}$. Now by the memoryless property of τ, we have

$$\mathbf{E}[Z_n] = \mathbf{P}(Z_n > 0)\mathbf{E}[Z_n | \, Z_n > 0].$$

If we can show that

$$\mathbf{E}[Z_n] = \mathbf{P}(Z_n > 0)\mathbf{E}[Z_n | \, Z_n > 0] \geq \frac{1}{e^2 n} \tag{28}$$

we will obtain, as desired,

$$\mathbf{P}(Z_n > 0) \leq C^5 e^2 n^{-3/2}.$$

To show equation (28), let $\sigma := \inf\{t \geq 0 : X(t) \in E_n \cap F_n \text{ occurs }\}$. Note that on the event $Z_n > 0$, necessarily $\sigma \in [0,\tau]$, By the strong Markov property and the memoryless property of τ, the probability that for all $k = 1, 2, \ldots, n$, the variable $X_k(t)$ is not refreshed during $[\sigma, \sigma + 1/n]$ and $\tau > \sigma + 1/n$ is exactly $e^{-\frac{n+1}{n}}$ and on this event, $Z_n \geq 1/n$. Hence

$$\mathbf{P}[Z_n \geq 1/n | \, Z_n > 0] \geq e^{-2},$$

which immediately gives equation (28). □

Lemma 5.4

$$\mathbf{P}(\forall t : X(t) \in E_n \text{ i.o.}) = 1$$

Proof: Denote the five intervals $(1/20, 1/10), (2/10, 3/10), (4/10, 5/10), (6/10, 7/10)$ and $(8/10, 9/10)$ by $I_1, \ldots, I_5$. For each such i, choose $a_i, b_i \in I_i$ with $a_i \neq b_i$. Choose ten numbers $f(a_1), \ldots, f(a_5)$ and $f(b_1), \ldots, f(b_5)$ such that $f(a_i)f(b_i) < 0$ for each i and such that the polygonal function f with the above values at $a_1, \ldots, a_5$ and $b_1, \ldots, b_5$ satisfies $\int_0^1 |f'(x)|^2 dx \leq 1$. The lemma now follows from Corollary 2.3. □

Proof of Theorem 1.11: Lemma 5.3 and the Borel–Cantelli lemma imply that if $B_n := \{\exists t \in [0,\tau] : X(t) \in E_n \cap F_n\}$, then

$$\mathbf{P}(B_n \text{ i.o.}) = 0.$$

This together with Lemma 5.4 and the fact that $F_{n+1}(t) \subseteq F_n(t)$ yields

$$\mathbf{P}(\forall t \in [0,\tau] : X(t) \in F_n \text{ for only finitely many } n) = 1,$$

which implies (27), and the theorem follows. $\square$

We now turn to our generalization of Theorem 1.12. Let G be a discrete abelian group with identity element 0 and let ν be a probability measure on G which is symmetric in the sense that $\nu(g) = \nu(-g)$ for all $g \in G$. Define the process $\{X_n(t)\}_{t \geq 0}$ as at the end of the introduction and again let $S_n(t) := \sum_{k=1}^{n} X_k(t)$, noting that for fixed t, this is simply a usual random walk on G with step size distribution given by ν which we will denote by $\{S_n\}$. We will let P_x denote the probabilities when the (nondynamical) random walk $\{S_n\}$ begins at location $x \in G$ while $\mathbf{P}$ again denotes the probability measure on the underlying probability space on which the dynamical process is defined. Our general result (which clearly includes Theorem 1.12) is the following. Let $p_n := P(S_n = 0)$.

Theorem 5.5 *Consider a symmetric random walk $\{S_n\}$ on an abelian group with identity 0 as above. Then*

$$\mathbf{P}(\exists t : \, S_n(t) = 0 \textit{ for infinitely many values of } n \,) = \begin{cases} 0 & \textit{if} \sum_{n=0}^{\infty} np_n < \infty \\ 1 & \textit{if} \sum_{n=0}^{\infty} np_n = \infty \,. \end{cases} \tag{29}$$

Again, the proof will use a series of lemmas. Since both sides of (29) trivially hold if $\{S_n\}$ is itself a recurrent random walk, we can assume that the random walk $\{S_n\}$ is transient. While not necessary, the arguments simplify slightly if, as usual, we run our process up until a random time τ which has an exponential distribution with mean 1. Therefore, we now let $Z_n := \int_0^\tau \mathbf{1}_{\{S_n(t)=0\}} dt$.

Lemma 5.6 *For any $k \geq 1$ and any $x \in G$, $P_x(S_k = 0) \leq \max\{p_{k-1}, p_k\}$, where p_{-1} is taken to be 0.*

Proof: For even k, the fact that $P_x(S_k = 0) \leq p_k$ is standard (see pg. 139 in [2]). For $k = 2m + 1$,

$$P_x(S_{2m+1} = 0) = \sum_{w \in G} P_x(S_{2m} = w) P_w(S_1 = 0) \leq p_{2m} \sum_{w \in G} P_0(S_1 = w) = p_{2m}.$$

$\square$

Lemma 5.7 *For $k, m \geq 0$,*

$$\int_0^\infty \mathbf{P}(S_k(0) = 0, S_{k+m}(t) = 0)e^{-t}dt \leq 2\frac{p_k}{k+1}\sum_{j=0}^{k+1} p_{j+m-1}. \tag{30}$$

Proof: By conditioning on the number of the variables $\{X_0, \ldots, X_{k-1}\}$ which update their value by time t (which has a binomial distribution with parameters k and $1-e^{-t}$), the left hand side of (30) equals

$$\int_0^\infty \sum_{j=0}^{k} \binom{k}{j} (1-e^{-t})^j (e^{-t})^{k-j} \sum_{x \in G} P_0(S_{k-j} = x) P_x(S_j = 0) P_x(S_{j+m} = 0) e^{-t} dt$$

which equals

$$\sum_{j=0}^{k} \sum_{x \in G} P_0(S_{k-j} = x) P_x(S_j = 0) P_x(S_{j+m} = 0) \int_0^\infty \binom{k}{j} (1-e^{-t})^j (e^{-t})^{k-j} e^{-t} dt.$$

The integral can be easily checked by induction to be $1/(k+1)$ for any $j \in \{0, \ldots, k\}$. Alternatively, this can be seen by noting that this integral is the probability that U_{k+1} is the $(j+1)$st smallest of $U_1, U_2, \ldots, U_{k+1}$ where $U_1, U_2, \ldots, U_{k+1}$ are k independent mean 1 exponential random variables. Next, by applying Lemma 5.6 to the term $P_x(S_{j+m} = 0)$, one bounds

$$\sum_{x \in G} P_0(S_{k-j} = x) P_x(S_j = 0) P_x(S_{j+m} = 0)$$

by $(p_{j+m-1} + p_{j+m})p_k$. Putting this together, the lemma follows. □

Lemma 5.8

(a) $\mathbf{P}(Z_n > 0) \leq e^2 n p_n$ *for all* $n \geq 1$.

(b) $\mathbf{P}(Z_n > 0) \geq \frac{(n+1)p_n}{2\sum_{j=0}^{n} p_j}$ *for all* $n \geq 1$.

In particular, if $\{S_n\}$ is transient, then there exists a constant C such that $np_n/C \leq \mathbf{P}(Z_n > 0) \leq Cnp_n$ for all $n \geq 1$.

Proof: Clearly for $n \geq 1$,

$$\mathbf{P}(Z_n > 0) = \frac{\mathbf{E}[Z_n]}{\mathbf{E}[Z_n | \, Z_n > 0]}. \tag{31}$$

Next, trivially, $\mathbf{E}[Z_n] = p_n$ by Fubini's Theorem. We next show that for $n \geq 1$,

$$\mathbf{E}[Z_n | \, Z_n > 0] \geq \frac{1}{e^2 n}, \tag{32}$$

from which (a) will follow. To show this, let $\sigma := \inf\{t \geq 0 : S_n(t) = 0\}$. Note that conditioned on the event $\{Z_n > 0\}$, $\sigma \in [0, \tau)$. By the strong Markov property and the memoryless property of τ, the probability that for all $k = 1, \ldots, n$, $X_k(t)$ does not change its value for $t \in [\sigma, \sigma + 1/n]$ and $\tau > \sigma + 1/n$ is exactly $1/e(1/e)^{1/n}$ and if this occurs, then $Z_n \geq 1/n$. Hence

$$\mathbf{P}[Z_n \geq 1/n | \, Z_n > 0] \geq 1/e^2,$$

which immediately yields (32).

We go on to prove (b). By (31) and the fact that $\mathbf{E}[Z_n] = p_n$, it suffices to show that

$$\mathbf{E}[Z_n | \, Z_n > 0] \leq \frac{2}{n+1} \sum_{j=0}^{n} p_j.$$

By stopping the first time t at which the process is such that $S_n(t) = 0$, we get (again using the strong Markov property and the memoryless property of τ) that

$$\mathbf{E}[Z_n | \, Z_n > 0] = \int_0^\infty \mathbf{P}(S_n(t) = 0 | \, S_n(0) = 0) e^{-t} dt$$

and hence, by Lemma 5.7, is at most $2 \sum_{j=0}^{n} p_j/(n+1)$, as desired. $\square$

Proof of Theorem 5.5 in the case where the sum in (29) converges: Assume that $\sum_{n=0}^{\infty} np_n < \infty$. This assumption together with Lemma 5.8 (a) then implies that $\sum_{n=0}^{\infty} \mathbf{P}(Z_n > 0) < \infty$ and hence by Borel-Cantelli, there are no times $t \in [0, \tau]$ such that $S_n(t) = 0$ for infinitely many values of n. It easily follows that

$$\mathbf{P}(\exists t : \, S_n(t) = 0 \text{ for infinitely many values of } n\,) = 0,$$

as desired. $\square$

The other case $\sum_{n=0}^{\infty} np_n = \infty$ is more difficult. Let Z_n be as above and let $W_n := \sum_{k=0}^{n} kZ_k$. Note that $\sum_{n=0}^{\infty} np_n = \infty$ is equivalent to $\lim_{n\to\infty} \mathbf{E}[W_n] = \infty$. A key step is to establish the following lemma.

Lemma 5.9 *There exists a constant C such that $\mathbf{E}[W_n^2] \leq C\mathbf{E}[W_n]^2$ for all n.*

Proof: Since the process $\{S_n(t)\}$ is reversible,

$$\begin{aligned} \mathbf{E}[Z_k Z_{k+m}] &= \mathbf{E} \int_0^\tau \int_0^\tau \mathbf{1}_{\{S_k(s)=0\}} \mathbf{1}_{\{S_{k+m}(t)=0\}} \, dt \, ds \\ &= 2 \int_0^\infty \int_s^\infty \mathbf{P}[S_k(s) = 0, S_{k+m}(t) = 0] e^{-t} \, dt \, ds \,. \end{aligned} \tag{33}$$

Replacing t by $t + s$ and noting that $\mathbf{P}[S_k(s) = 0, S_{k+m}(s+t) = 0]$ in independent of s, it follows that the above is equal to $2 \int_0^\infty \mathbf{P}(S_k(0) = 0, S_{k+m}(t) = 0) e^{-t} dt$ which, by Lemma 5.7, is at most $4p_k \sum_{j=0}^{k+1} p_{j+m-1}/(k+1)$.

It follows that

$$\mathbf{E}[W_n^2] \le 2\sum_{k=0}^{n}\sum_{m=0}^{n} k(k+m)\mathbf{E}[Z_k Z_{k+m}]$$

$$\le 8\sum_{k=0}^{n}\sum_{m=0}^{n} k(k+m)\frac{p_k}{k+1}\sum_{j=0}^{k+1} p_{j+m-1}. \tag{34}$$

On the other hand, we have that

$$\mathbf{E}[W_n]^2 = \sum_{k=0}^{n}\sum_{m=0}^{n} kmp_kp_m. \tag{35}$$

One then checks by inspection that there exists a constant C, independent of n, such that for $r, s \ge 1$, the coefficient of p_rp_s in (34) is at most C times the coefficient of p_rp_s in (35), the coefficients of p_0p_s for $s = 0, 1, 2$ in (34) is at most C, and in addition that for $s \ge 3$, the coefficient of p_0p_s in (34) is at most C times the coefficient of p_2p_{s-2} in (34). From these facts, the statement of the lemma then easily follows. □

Proof of Theorem 5.5 in the case where the sum in (29) diverges: A one-sided Chebyshev inequality (see [11, Lemma 5.4]) implies that for all n,

$$\mathbf{P}\left(W_n > \frac{\mathbf{E}[W_n]}{2}\right) \ge \frac{1}{1+4C}$$

where C comes from Lemma 5.9. By Fatou's lemma, it follows, using the fact that $\lim_{n\to\infty}\mathbf{E}[W_n] = \infty$, that

$$\mathbf{P}[\lim_{n\to\infty} W_n = \infty] \ge \frac{1}{1+4C}.$$

Hence

$$\mathbf{P}\left(\sum_{k=0}^{\infty}\mathbf{1}_{\{Z_k>0\}} = \infty\right) \ge \frac{1}{1+4C}.$$

If

$$A := \left\{\sum_{k=0}^{\infty}\mathbf{1}_{\{Z_k>0\}} = \infty\right\},$$

and

$$B := \left\{\sum_{k=0}^{\infty}\mathbf{1}_{\{\int_0^1 \mathbf{1}_{\{S_k(t)=0\}}dt>0\}} = \infty\right\},$$

then the argument right above (13) shows that $\mathbf{P}(A) \le \mathbf{P}(B)/(1-e^{-1})$. It follows that $\mathbf{P}(B) > 0$ and hence by the Hewitt–Savage 0-1 law (see [7, p.174]) applied to the sequence $\{X_n(t) : t \in [0,1]\}_{n\in\mathbf{N}}$, we have $\mathbf{P}(B) = 1$.

We now show how $\mathbf{P}(B) = 1$ implies the existence of times t for which $S_n(t) = 0$ for infinitely many values of n. Let

$$V_k = \bigcup_{n\ge k}\{t \in (0,1) : S_n(t) = 0\}^o$$

where A^o again denotes the interior of A. It follows from the above that each V_k is dense and open a.s. Hence, by the Baire Category Theorem, $\cap_{k\geq 1} V_k$ is a.s. dense, completing the proof. □

In the remainder of this section, we will compute the Hausdorff dimension of the set of return times for certain random walks on abelian groups (Theorem 5.10). This result immediately implies Theorem 1.13.

Theorem 5.10 *Let $\{S_n\}$ be a symmetric random walk on an abelian group with identity 0. Assume that $P(S_n = 0) \asymp 1/n^{\beta+1}$ for even n. If $\beta \in (0, 1]$, then*

$$\dim_H(\{t : S_n(t) = 0 \text{ i.o.}\}) = 1 - \beta.$$

Remark 5.11 It was proved in [12] that for any symmetric finitely supported random walk on a group, the return probabilities either decay faster than any power, or satisfy a power law as above.

Proof of Theorem 5.10: Let $R := \{t \in [0, 1] : S_n(t) = 0 \text{ i.o. }\}$. We first show that $\dim_H(R) \leq 1 - \beta$. To do this, let $\alpha > 1 - \beta$ and we will show that $\mathcal{H}^\alpha(R) = 0$. For $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$, let $I_i^n := [(i-1)/n, i/n]$. Next, letting

$$U_i^n := \{S_n(t) = 0 \text{ for some } t \in I_i^n\},$$

we have that $\mathbf{P}(U_i^n) \leq C p_n$ for some constant C. It follows that

$$\mathbf{E}\left[\sum_{n,i} I_{\{U_i^n\}} n^{-\alpha}\right] \leq C \sum_{n=0}^{\infty} n^{1-\alpha} p_n < \infty$$

since $\alpha > 1-\beta$. Hence a.s. $\sum_{n,i} I_{\{U_i^n\}} n^{-\alpha} < \infty$. This easily implies that that $\mathcal{H}^\alpha(R) = 0$, as desired.

We next show that $\dim_H(R) \geq 1 - \beta$. By the codimension argument mentioned earlier (see [19]), it suffices to show that if $A \subseteq [0, 1]$ is closed with $\dim_H(A) > 1 - \beta$, then

$$\mathbf{P}(R \cap A \neq \emptyset) > 0.$$

Given such a set A, by Frostman's theorem, there exists a probability measure ρ on A such that

$$\int_A \int_A |t - s|^{-\beta} d\rho(t) d\rho(s) < \infty.$$

Let

$$Z_k = \int_0^1 \sum_{n=2^k+1}^{2^{k+1}} I_{\{S_n(t)=0\}} d\rho(t).$$

It is immediate that $\mathbf{E}[Z_k] \asymp 1/2^{k\beta}$. We now estimate $\mathbf{E}[Z_k^2]$. Proposition 5.12 below with $L = 2^k$ easily implies that that this second moment is at most

$$\frac{C}{2^{2k\beta}} \int_A \int_A |t-s|^{-\beta} d\rho(t) d\rho(s).$$

It follows that

$$\frac{(\mathbf{E}[Z_k])^2}{\mathbf{E}[Z_k^2]} \geq C$$

from which we conclude by the Cauchy–Schwartz inequality that

$$\mathbf{P}(Z_k > 0) \geq C.$$

By Fatou's lemma,

$$\mathbf{P}\left(\sum_k I_{\{Z_k>0\}} = \infty\right) \geq C.$$

From here, one proceeds as in Theorem 5.5 to complete the proof. □

Proposition 5.12 *There exists a constant C such that for any L and $t > 0$,*

$$\sum_{n=L+1}^{2L} \sum_{m=0}^{L} \mathbf{P}[S_n(0) = 0, S_{n+m}(t) = 0] \leq C(L^2 t)^{-\beta}. \tag{36}$$

Before proving this we isolate the following easy lemma.

Lemma 5.13 *For any $\beta \in (0,1]$, there exists a constant C such that for any $L \geq 1$ and any $t \in (0,1]$, if X has binomial distribution with parameters L and $1 - e^{-t}$, then*

$$E\left[\frac{1}{X^\beta} \wedge 1\right] \leq C(Lt)^{-\beta}.$$

Proof: Break the expectation over the set $\{X \leq [L(1-e^{-t})]/2\}$ and its complement $\{X > [L(1-e^{-t})]/2\}$. The probability of the former set is, by Chebyshev's inequality, at most

$$\frac{4\mathbf{Var}(X)}{E[X]^2} \leq \frac{4}{E[X]} = \frac{4}{L(1-e^{-t})} \leq \frac{C}{Lt}.$$

Since $\frac{1}{X^\beta} \wedge 1$ is at most 1, the expectation over the first set is at most C/Lt. The expectation over the second set is easily seen to be at most $C(Lt)^{-\beta}$. Since $\beta \leq 1$, this completes the proof. □

Proof of Proposition 5.12: Throughout this proof, C will denote an arbitrary constant whose value might change each time it appears. Note that $p_j \leq C/j^{\beta+1}$ for all j. Letting $Y_{n,p}$ denote a Binomial random variable with parameters n and p, the left-hand side of (36) becomes

$$\sum_{n=L+1}^{2L} \sum_{m=0}^{L} \sum_j \mathbf{P}(Y_{n,1-e^{-t}} = j) p_n p_{j+m}. \tag{37}$$

Since $Y_{n,1-e^{-t}}$ stochastically dominates $Y_{L,1-e^{-t}}$ for $n \geq L$, $p_j \leq C/j^{\beta+1}$ for all j, and $1/j^{\beta+1}$ is decreasing in j, we have that for any $m \in [0, L]$,

$$\sum_j \mathbf{P}(Y_{n,1-e^{-t}} = j)p_{j+m} \leq \sum_j \mathbf{P}(Y_{L,1-e^{-t}} = j)\frac{C}{(j+m)^{\beta+1}} \wedge 1.$$

Hence (37) is at most

$$\frac{CL}{L^{\beta+1}} \sum_j \mathbf{P}(Y_{L,1-e^{-t}} = j) \sum_{m=0}^{L} \frac{C}{(j+m)^{\beta+1}} \wedge 1.$$

Note next that $\sum_{m=0}^{L} \frac{C}{(j+m)^{\beta+1}} \wedge 1 \leq C(j^{-\beta}) \wedge 1$ from which it follows that the above is at most

$$\frac{C}{L^{\beta}} \sum_j \mathbf{P}(Y_{L,1-e^{-t}} = j)(j^{-\beta} \wedge 1).$$

Lemma 5.13 now completes the proof. □

6 Concluding remarks and open problems

1. The dynamical sensitivity and stability discussed here parallel, to some extent, the notions of noise sensitivity and stability studied in [3]. For instance, dynamical stability of the law of large numbers corresponds to noise stability of the majority function in [3], while dynamical sensitivity of run tests in Section 3, corresponds to the noise sensitivity of the Boolean function determining whether the length of the longest run in a finite binary sequence exceeds its median. It remains a challenge to establish more formal connections between dynamical sensitivity and noise sensitivity.

2. In this paper, we considered equilibrium dynamics with 1–dimensional time. It is possible to extend the dynamics to multi-dimensional time, for instance along the lines suggested in [5] and in [10].

3. We conjecture that recurrence of simple random walk in $\mathbf{Z}^2$ is sensitive. One motivation for this conjecture is the result of Adelman, Burdzy and Pemantle [1] who showed that projecting spatial Brownian motion to certain (random) planes can yield a transient process.

4. Is there a precise relationship between almost sure properties of sequences which are dynamically stable for simple random walk, and properties of paths which hold quasi-everywhere in Wiener space? (cf. [16] and Theorem 1.12 here).

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Itai Benjamini
Faculty of Math. and Computer Science
The Weizmann Institute of Science
Rehovot 76100
Israel
`itai@wisdom.weizmann.ac.il`
`www.wisdom.weizmann.ac.il/~itai`

Olle Häggström
Dept. of Mathematical Statistics
Göteborg University
412 96 Gothenburg
Sweden
`olleh@math.chalmers.se`
`www.math.chalmers.se/~olleh`

Yuval Peres
Dept. of Statistics, UC Berkeley
Berkeley, CA 94720 and
Institute of Mathematics
Hebrew University,
Jerusalem, Israel
`peres@stat.berkeley.edu`
`www.stat.berkeley.edu/~peres`

Jeffrey E. Steif
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-1060 and
Chalmers University of Technology
Gothenburg, Sweden
`steif@math.gatech.edu`
`www.math.chalmers.se/~steif`