PROPERTY \((\beta)\epsilon\) FOR TOEPLITZ OPERATORS WITH \(H^{\infty}\)-SYMBOL

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ABSTRACT. Suppose that \(g\) is a tuple of bounded holomorphic functions on a strictly pseudoconvex domain \(D\) in \(\mathbb{C}^m\) with smooth boundary. Viewed as a tuple of operators on the Hardy space \(H^p(D), 1 \leq p < \infty\), \(g\) is shown to have property \((\beta)\epsilon\) and therefore \(g\) possess Bishop’s property \((\beta)\). In the case \(m = 1\) it is proved that the same result also holds when \(p = \infty\).

1. INTRODUCTION

Suppose that \(X\) is a Banach space and that \(a = (a_1, \ldots, a_n)\) is a commuting tuple of bounded linear operators on \(X\). Let \(E\) be one of spaces \(X, \mathcal{E}(\mathbb{C}^n, X)\) or \(\mathcal{O}(U, X)\), where \(U \subset \mathbb{C}^n\). Denote by \(K_{\bullet}(z-a, E)\) the Koszul complex

\[ 0 \rightarrow \Lambda^r E \xrightarrow{\delta_{z-a}} \Lambda^{r-1} E \xrightarrow{\delta_{z-a}} \cdots \xrightarrow{\delta_{z-a}} \Lambda^0 E \rightarrow 0, \]

with boundary map

\[ \delta_{z-a}(f s_I) = 2\pi i \sum_{k=1}^{p} (-1)^{k-1}(z_{i_k} - a_{i_k}) f s_{i_1} \wedge \cdots \wedge s_{i_k} \wedge \cdots \wedge s_{i_p}, \]

where \(I = (i_1, \ldots, i_p)\) and \(p\) is an integer. Let \(H_{\bullet}(z-a, E)\) be the corresponding homology groups.

The Taylor spectrum of \(a, \sigma(a)\), is defined as the set of all \(z \in \mathbb{C}^n\) such that \(K_{\bullet}(z-a, X)\) is not exact. If for all Stein open sets \(U\) in \(\mathbb{C}^n\) the natural quotient topology of \(H_0(z-a, \mathcal{O}(U, X))\) is Hausdorff and \(H_p(z-a, \mathcal{O}(U, X)) = 0\) for all \(p > 0\), then \(a\) is said to have Bishop’s property \((\beta)\). It has property \((\beta)\epsilon\) if the natural quotient topology of \(H_0(z-a, \mathcal{E}(\mathbb{C}^n, X))\) is Hausdorff and if \(H_p(z-a, \mathcal{E}(\mathbb{C}^n, X)) = 0\) for all \(p > 0\).

By Theorem 6.2.4 in [9], the tuple \(a\) has Bishop’s property \((\beta)\) if and only if there exists a decomposable resolution, that is, if and only if there are Banach spaces \(X_i\) and decomposable tuples (see [9] for the definition) of operators \(a_i\) on \(X_i\) such that

\[ 0 \rightarrow X \xrightarrow{d} X_0 \xrightarrow{d} \cdots \xrightarrow{d} X_r \rightarrow 0 \]

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is exact, \( da = a_0 d \) and \( da_i = a_{i+1} d \). Property \((\beta)_{\xi}\) is equivalent to the existence of a resolution of Fréchet spaces with Mittag-Leffler inverse limit of generalized scalar tuples (that is tuples which admit a continuous \( C^\infty(C^n)\)-functional calculus), see Theorem 6.4.15 in [9]. Property \((\beta)_{\xi}\) implies Bishop’s property \((\beta)\), see [9].

Suppose that \( D \) is a strictly pseudoconvex domain in \( C^n \) with smooth boundary. We consider the tuple \( T_g = (T_{g_1}, \ldots, T_{g_n}), \ g_k \in H^\infty(D) \), of operators on \( H^p(D) \) defined by \( T_{g_k}f = g_k f, \ f \in H^p(D) \). The main theorem of this paper is the following.

**Theorem 1.1.** Suppose that \( D \) is a bounded strictly pseudoconvex domain in \( C^n \) with \( C^\infty \)-boundary and that \( g \in H^\infty(D)^n \). Then the tuple \( T_g \) of Toeplitz operators on \( H^p(D) \), \( 1 \leq p < \infty \), satisfies property \((\beta)_{\xi}\), and thus Bishop’s property \((\beta)\).

In case \( g \) has bounded derivative this theorem has previously been proved in [14, 16, 17]. In case \( D \) is the unit disc in \( \mathbb{C} \), Theorem 1.1 also holds when \( p = \infty \); this is proved in Section 4. As a corollary to Theorem 1.1 we have that \( T_g \) on the Bergman space \( OL^p(D) \) has property \((\beta)_{\xi}\), see Corollary 3.4.

Let us recall how one can prove that \( T_g \) on the Bergman space \( OL^2(D) \) has property \((\beta)_{\xi}\) under the extra assumption that \( g \) has bounded derivative. Define the Banach spaces \( B_k \) as the spaces of locally integrable \((0, k)\)-forms \( u \) such that

\[
\|u\|_{B_k} := \|u\|_{L^2(D)} + \|\bar{\partial}u\|_{L^2(D)} < \infty.
\]

Since \( g \) has bounded derivate we have the inequality

\[
\|(\varphi \circ g) u\|_{B_k} \lesssim \sup_{z \in g(D)} \left( |\varphi(z)| + |\bar{\partial}\varphi(z)| \right) \|u\|_{B_k}
\]

for all \( \varphi \in C^\infty(C^n) \). Hence \( \varphi \mapsto T_{\varphi g} \) is a continuous \( C^\infty(C^n)\)-functional calculus, where \( T_{\varphi g} \) denotes multiplication by \( \varphi \circ g \) on \( B_k \).

Since we have the resolution

\[
0 \rightarrow OL^2(D) \rightarrow B_0 \overset{\delta}{\rightarrow} B_1 \overset{\delta}{\rightarrow} \cdots \overset{\delta}{\rightarrow} B_m \rightarrow 0
\]

by Hörmander’s \( L^2 \)-estimate of the \( \bar{\partial} \) equation, the tuple \( T_g \) on \( OL^2(D) \) has property \((\beta)_{\xi}\) by the above mentioned Theorem 6.4.15 in [9].

To prove Theorem 1.1 we will construct a complex

\[
(1) \quad 0 \rightarrow H^p(D) \overset{i}{\rightarrow} B_0 \overset{\delta}{\rightarrow} B_1 \overset{\delta}{\rightarrow} \cdots \overset{\delta}{\rightarrow} B_m \rightarrow 0,
\]

where \( B_k \) are Banach spaces of \((0, k)\)-forms on \( D \). The spaces \( B_k \) are defined in terms of tent norms. We prove that \( \varphi \mapsto T_{\varphi g} \) is a continuous \( C^\infty(C^n)\)-functional calculus, where \( T_{\varphi g} \) denotes multiplication by \( \varphi \circ g \) on \( B_k \). If the complex \((1)\) were exact the proof of Theorem 1.1 would be finished. As we can solve the \( \bar{\partial} \)-equation with appropriate estimates we will be able to prove that \( T_g \) on \( H^p \) has property \((\beta)_{\xi}\) anyway. More precisely \((1)\) is exact at \( B_k \), \( k \geq 3 \). If \( f \in B_2 \) and \( \bar{\partial}f = 0 \) then
there is a function $u$ in another Banach space $B'_1$ such that $\tilde{\partial} u = f$. Multiplication by $g$ is a bounded operator on $B'_1$. If $f \in B_1$ and $f' \in B'_1$ such that $\tilde{\partial} f + r \tilde{\partial} f' = 0$ then there is a solution $u \in L^p(\partial D)$ to the equation $\tilde{\partial} u = f + f'$.

The construction of the complex (1) in the case $p < \infty$ is inspired by the construction in [5] and in the case $p = \infty$ and $m = 1$ it is inspired by Tom Wolff’s proof of the corona theorem. Let us recall the proof of the $H^p$-corona theorem in the unit disc of $\mathbb{C}$. Suppose that $g = (g_1, \ldots, g_n) \in H^\infty(D)^n$, where $D$ is the unit disc in $\mathbb{C}$, and that $0 \not\in g(D)$. Consider the complex (1); the definitions of the $B_k$-spaces can be found in the beginning of Section 3 and Section 4. Suppose that $f \in H^p(D)$. Then the equation $\delta g u_1 = f$ has a solution in $K_1(g, B_0)$, namely $u_1 = \sum_k \tilde{g}_k s_k / |g|^2$. Hence $\delta g \partial u_1 = 0$ as $\delta g$ and $\partial$ anticommutate, and we can solve the equation $\delta g u_2 = \partial u_1$ by defining $u_2 \in K_2(g, B_1)$ as $u_1 \wedge \partial u_1$. Since $u_2$ satisfies the condition

$$\|(1 - |z|) u_2\|_{T_p^2} + \|(1 - |z|)^2 \partial u_2\|_{T_p^2} < \infty,$$

by a Wolf type estimate there is a solution $v$ in $K_2(g, L^p(\partial D))$ to the equation $\tilde{\partial} u_2 = v$ (here $T_p^2$ and $T_1^p$ denote certain tent spaces). Let $u_1' = u_1^* - \delta g v \in K_1(g, L^p(\partial D))$, where $u_1^*$ is the boundary values of $u_1$. Since $\tilde{\partial} u_1' = 0$ there is a holomorphic extension $U_1'$ of $u_1'$ to $D$ which satisfies the equation $\delta g U_1' = f$.

The above proof also yields that $\sigma(T_g) = \overline{g(D)}$; the exactness of higher order in the Koszul complex follows by similar reasoning. That $\sigma(T_g) = \overline{g(D)}$ is proved in [5] for the case $D$ strictly pseudoconvex and $p < \infty$. One main difference of the proof of that $T_g$ has property $(\beta)_\varepsilon$ and the proof of that $\sigma(T_g) = \overline{g(D)}$ is the following. As a substitution of the explicit choices of $u_1$ and $u_2$ one uses the fact that $T_g$ considered as an operator on $B_k$ has property $(\beta)_\varepsilon$, which in turn follows from the fact that $T_g$ on $B_k$ has a $C^\infty(\mathbb{C}^n)$-functional calculus.

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2. Preliminaries

Suppose that $D$ is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary given by a strictly plurisubharmonic defining function $\rho$. Let $r = -\rho$. All norms below are with respect to the metric

$$\Omega = r i \partial \bar{\partial} \log (1/r),$$

and we have

$$|f|^2 \sim r^2 \int_g |f|^2 + r |f \wedge \partial r|_g^2 + r |f \wedge \partial r|^2 + |f \wedge \partial r \wedge \partial r|^2,$$

where $\beta = i \partial \bar{\partial} r$, which is equivalent to the Euclidean metric.
The Hardy space $H^p$ is the Banach space of all holomorphic functions, $f$, on $D$ such that

$$\|f\|_{H^p} = \sup_{\varepsilon > 0} \int_{|z| = \varepsilon} |f(z)|^p \, d\sigma(z) < \infty,$$

where $\sigma$ is the surface measure. It is wellknown that a function $u$ in $L^p(\partial D)$ is the boundary value of a function $U$ in $H^p$ if and only if

$$\int_{\partial D} uh = 0$$

for all $h \in C^\infty_{\text{c}, n-1}(\bar{D})$ such that $\overline{\partial} h = 0$.

Let $d(\cdot, \cdot)$ be the Koranyi pseudometric on $\partial D$ and let $z'$ be the point on $\partial D$ closest to $z \in D_\varepsilon$, where $D_\varepsilon$ is a small enough neighbourhood of $\partial D$ in $D$. For a point $\zeta$ on the boundary let

$$A_\zeta = \{z \in D_\varepsilon : d(z', \zeta) < r(z)\} \cup (D \setminus D_\varepsilon).$$

For a ball $B$ defined by $B = \{z \in \partial D : d(z, \zeta) < t\}$ let, for small $t$,

$$\bar{B} = \{z \in D_\varepsilon : d(z', \zeta) < t - r(z)\},$$

and for large $t$

$$\bar{B} = \{z \in D_\varepsilon : d(z', \zeta) < t - r(z)\} \cup (D \setminus D_\varepsilon).$$

A function $f$ is in the tent space $T^p_q$, where $p < \infty$ and $q < \infty$, if

$$\|f\|_{T^p_q} := \left( \int_{\partial D} \left( \int_{z \in A_\zeta} |f(z)|^q \, r(z)^{-m-1} \right)^{p/q} \, d\sigma(\zeta) \right)^{1/p} < \infty.$$

The function $f$ is in $T^p_{\infty}$ if $f$ is continuous with limits along $A_\zeta$ at the boundary almost everywhere and such that

$$\|f\|_{T^p_{\infty}} := \left( \int_{\partial D} \sup_{z \in A_\zeta} |f(z)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty.$$

A function $f$ is in $T^\infty_q$ if

$$\|f\|_{T^\infty_q} := \left\| \sup_{z \in A_\zeta} \left( \frac{1}{|B|} \int_{z \in B} |f(z)|^q \, r(z)^{-1} \right)^{1/q} \right\|_{L^\infty(\partial D)} < \infty.$$

Note that $f \in T^p_q$ if and only if $r^{-1/p} f \in L^p(D)$ by Fubini’s theorem. From [8] we have the inequality

(2) \[ \int_D |fg| r^{-1} \lesssim \|f\|_{T^p_q} \|g\|_{T^{p'}_{q'}} \]

for $1 \leq p, q \leq \infty$, where $p'$ and $q'$ denote dual exponents. By [8] $T^p_{q'}$, where $1 \leq p < \infty$ and $1 < q < \infty$, is the dual of $T^p_q$ with respect to the
pairing
\[ \langle f, g \rangle \rightarrow \int_D f g r^{-1}. \]

Suppose that \( f \in T^p_{q_0}, \ g \in T^q_{q_1} \) and let \( q = (q_0^{-1} + q_1^{-1})^{-1} \). Then for all \( h \in T^q_{q'} \) we have
\[ \int_D |fg h| r^{-1} \lesssim \|fh\|_{T_{q_1}^q} \|g\|_{T_{q_1}^q} \leq \|f\|_{T_{q_1}^{q_0}} \|g\|_{T_{q_1}^{q_1}} \|h\|_{T_{q'}^{q'}} \]
by (2) and Hölder's inequality. Thus by the duality for \( T^q_{q'} \) we get the inequality
\[ \|fg\|_{T^p_{q_0}} \lesssim \|f\|_{T^p_{q_0}} \|g\|_{T^q_{q_1}} \]
for \( 1 < p \) and \( 1 < q < \infty \). Since the inequality (3) is equivalent to
\[ \|fg\|_{T^p_{q_0}} \lesssim \|f\|_{T^p_{q_0}} \|g\|_{T^q_{q_1}} \]
for \( 0 < t < \infty \), (3) holds if \( 0 < p, q_0, q_1 \).

We will use the inequality (see [12])
\[ \|f\|_{T^p_{q_0}} \lesssim \|f\|_{H^p}, \ p > 0 \]
and (see e.g. [7] for \( p < \infty \) and [3] for \( p = \infty \))
\[ \|r^{1/2} \partial f\|_{T^p_{q_0}} \lesssim \|f\|_{H^p}, \ p > 0. \]
Moreover, we use that \( \|\partial f\| \lesssim r^{-1/2} \) if \( f \in H^{\infty} \).

There is an integral operator \( K : C_0^{\infty}(\bar{D}) \rightarrow C_{0,q}(\bar{D}), q \geq 0, \) see [5], such that \( \partial K u + K \partial u = u, \ u \in C_0^{\infty}(\bar{D}), s \geq 1, \)
\[ \|ru\|_{T^p_{q_0}} \lesssim \|ru\|_{T^p_{q_0}} \quad \text{and} \quad \|K u\|_{L^p(\partial D)} \lesssim \|r^{1/2} u\|_{T^p_{q_0}} \]
if \( \tau > 0 \) and \( 1 \leq p < \infty \). Furthermore,
\[ \|K u\|_{L^p(\partial D)} \lesssim \|r^{1/2} u\|_{T^p_{q_0}} + \|ru\|_{T^p_{q_0}}. \]
To see that the inequality (6) follows from [5], note that by the definition of \( W^{1-1/p} \) in [1], \( \|ru\|_{T^p_{q_0}} = \|u\|_{W^{1-1/p}}. \) By [4] the adjoint \( P \) of \( K \)
satisfies
\[ \|P \psi\|_{L^\infty(D)} \lesssim \|\psi\|_{L^\infty(\partial D)} \quad \text{and} \quad \|r^{1/2} \mathcal{L} P \psi\|_{L^2(\partial D)} \lesssim \|\psi\|_{L^2(\partial D)} \]
(whence \( \mathcal{L} \) is an arbitrary smooth \((1,0)\)-vectorfield). The \( L^2 \)-result is proven by means of a \( T1 \)-theorem of Christ and Journé. By [10] it now follows that
\[ \|P \psi\|_{T^p_{q_0}} \lesssim \|\psi\|_{L^p(\partial D)}, \quad p > 1, \]
and
\[ \|ru\|_{T^p_{q_0}} \lesssim \|\psi\|_{L^p(\partial D)}, \quad p > 1. \]
The inequality (7) follows from (8) and (9).
In section 4 we use completed tensor products of locally convex Hausdorff spaces, see e.g. Appendix 1 in [9]. Suppose that $E$ and $F$ are locally convex Hausdorff spaces. We denote by $L(E, F)$ the space of all continuous and linear maps from $E$ to $F$. The topology $\pi$ on $E \otimes F$ is defined as the finest locally convex topology such that the canonical bilinear map $E \times F \to E \otimes F$ is continuous. We denote by $E \otimes_{\pi} F$, the space $E \otimes F$ with the topology $\pi$ and we denote the completion of $E \otimes_{\pi} F$ with $E \otimes_{\pi} F$. There is another topology on $E \otimes F$, the topology $c$; in case $E$ is nuclear this topology coincides with the topology $\pi$ and we therefore omit the index $\pi$ in this case. The Fréchet space $\mathcal{E}(\mathbb{C}^n)$ is nuclear and we have the isomorphism $\mathcal{E} (\mathbb{C}^n, E) \cong \mathcal{E} (\mathbb{C}^n) \hat{\otimes} E$.

3. PROPERTY (β)ξ FOR TOEPLITZ OPERATORS WITH $H^\infty$-SYMBOL ON $H^p$

First we need to define the sequence (1) and prove that there is a continuous $C^\infty (\mathbb{C}^n)$-functional calculus on each of the spaces $B_k$.

Define the norms $\| \cdot \|_{B_k}$, $k \geq 0$, by

$$\| u \|_{B_0} = \| u \|_{L^\infty} + \| r^{1/2} du \|_{L^p_{\bar{\partial}}} + \| r \bar{\partial} u \|_{L^p_{\partial}}$$

on $C^\infty (\bar{D})$,

$$\| u \|_{B_1} = \| r^{1/2} u \|_{L^p_{\bar{\partial}}} + \| r du \|_{L^p_{\partial}}$$

on $C^\infty_{0,1} (\bar{D})$ and

$$\| u \|_{B_k} = \| r^{k/2} u \|_{L^p_{\bar{\partial}}} + \| r^{k/2+1/2} \bar{\partial} u \|_{L^p_{\partial}}$$

on $C^\infty_{0,k} (\bar{D})$ for $k \geq 2$. Let $B_k$ be the completion of $C^\infty_{0,k} (\bar{D})$ with respect to the norm $\| \cdot \|_{B_k}$. We also define $B'_k$ as the completion of $C^\infty_{0,1} (\bar{D})$ with respect to the norm $\| \cdot \|_{B'_k}$, defined by

$$\| u \|_{B'_k} = \| r^{1/2} u \|_{L^p_{\bar{\partial}}} + \| r \bar{\partial} u \|_{L^p_{\partial}}$$

The injection $i : H^p \to B_0$ is well defined and continuous by (4) and (5). That $\bar{\partial} : B_k \to B_{k+1}$, $k \geq 0$ is continuous follows immediately from the definitions. Thus we have defined a complex

$$0 \to H^p (D) \overset{i}{\to} B_0 \overset{\bar{\partial}}{\to} B_1 \overset{\bar{\partial}}{\to} \cdots \overset{\bar{\partial}}{\to} B_m \to 0.$$

**Lemma 3.1.** Suppose that $g \in H^\infty (D)^n$. Then one can define $T_{g_1} : B_k \to B_k$ by $T_{g_i} u = g_i u$, $1 \leq i \leq n$, for all $k \geq 0$. The tuple $T_g$ on $B_k$, $k \geq 0$, has a continuous $C^\infty (\mathbb{C}^n)$-functional calculus and property (β)ξ.
Proof. That \( T^u_\alpha \) can be defined on \( B_k \) follows from the calculation below (let \( \varphi(z) = z_i \) below). We begin with the case \( k = 0 \). Suppose that \( \varphi \in C^\infty(\mathbb{C}^n) \) and \( u \in C^\infty(\bar{D}) \). From (3) we have
\[
\left\| r^{1/2} u \partial g \right\|_{T^p_\infty} \lesssim \left\| u \right\|_{T^p_\infty} \left\| r^{1/2} \partial g \right\|_{T^p_\infty},
\]
\[
\left\| r \left| du \right| \partial g \right\|_{T^p_\infty} \lesssim \left\| r^{1/2} \left| du \right| \right\|_{T^p_\infty} \left\| r^{1/2} \partial g \right\|_{T^p_\infty}
\]
and
\[
\left\| ru \partial g \right\|_{T^p_\infty} \lesssim \left\| u \right\|_{T^p_\infty} \left\| r \partial g \right\|^2_{T^p_\infty}.
\]
Since \( \left\| r^{1/2} \partial g \right\|_{T^p_\infty} < \infty \) by the inequality (5) we thus get
\[
\left\| (\varphi \circ g) u \right\|_{B_0} \leq \sup_{z \in g(D)} |\varphi(z)| \left\| u \right\|_{B_0} + \left\| r^{1/2} d(\varphi \circ g) u \right\|_{T^p_\infty} + \left\| r \partial(\varphi \circ g) \right\|_{T^p_\infty} + \left\| r \partial \overline{\partial}(\varphi \circ g) u \right\|_{T^p_\infty} \lesssim \sup_{z \in g(D)} \left( |\varphi(z)| + |D\varphi(z)| + |D^2 \varphi(z)| \right) \left\| u \right\|_{B_0},
\]
where \( D\varphi \) and \( D^2 \varphi \) denotes all derivates of \( \varphi \) of order 1 and 2 respectively. Note that \( (\varphi \circ g) u \notin C^\infty(\bar{D}) \) in general. Let \( g_l \in C^\infty(\bar{D}) \) be such that \( g_l \to g \) in \( H^p(D)^n \) with \( g_l \) uniformly bounded as \( l \to \infty \) and suppose that \( u \) is fixed. We have the equalities
\[
d(\varphi \circ g_l - \varphi \circ g) = \sum_i \varphi_i \circ g_l \partial g_l^i - \varphi_i \circ g \partial g^i + \varphi_i \circ g_l \overline{\partial} g_l^i - \varphi_i \circ g \overline{\partial} g^i
\]
and
\[
\partial \overline{\partial}(\varphi \circ g_l - \varphi \circ g) = \sum_{i,j} \varphi_{ij} \circ g_l \partial g_l^i \partial g_l^j - \varphi_{ij} \circ g \partial g^i \partial g^j,
\]
where the index in \( \varphi_i \) denotes partial derivate and the upper index in \( g_l^i \) and \( g^i \) denotes \( i \)th component. Hence we get
\[
\left| d(\varphi \circ g_l - \varphi \circ g) \right| \leq |D\varphi \circ g_l| \left| \partial g_l - \partial g \right| + |D\varphi \circ g_l - D\varphi \circ g| \left| \partial g \right|,
\]
and
\[
\left| \partial \overline{\partial}(\varphi \circ g_l - \varphi \circ g) \right| \leq |D^2 \varphi \circ g_l| \left| \partial g_l - \partial g \right| \left( \left| \partial g_l \right| + \left| \partial g \right| \right) + |D^2 \varphi \circ g_l - D^2 \varphi \circ g| \left| \partial g \right|^2.
\]
By (4) we have
\[
\left\| (\varphi \circ g_l - \varphi \circ g) u \right\|_{T^p_\infty} + \left\| r^{1/2} (\varphi \circ g_l - \varphi \circ g) du \right\|_{T^p_\infty} + \left\| r (\varphi \circ g_l - \varphi \circ g) \partial \overline{\partial} u \right\|_{T^p_\infty} \lesssim \left\| \varphi \circ g_l - \varphi \circ g \right\|_{T^\infty_\infty} \lesssim \left\| g_l - g \right\|_{T^\infty_\infty} \lesssim \left\| g_l - g \right\|_{H^p}.
\]
We also have that
\[
\|r^{1/2}d(\varphi \circ g_l - \varphi \circ g)\|_{\mathcal{T}_2^p} + \|r |d(\varphi \circ g_l - \varphi \circ g)| |du|\|_{\mathcal{T}_2^p} \lesssim \\
\|r^{1/2}d(\varphi \circ g_l - \varphi \circ g)\|_{\mathcal{T}_2^p} \lesssim \|r^{1/2}|D\varphi \circ g_l| |\vartheta g_l| - \vartheta g|\|_{\mathcal{T}_2^p} + \|r^{1/2}|D\varphi \circ g_l - D\varphi \circ g| |\vartheta g|\| \lesssim \|g_l - g\|_{H^p}
\]
by (3),(4) and (5). Furthermore,
\[
\|r\vartheta (\varphi \circ g_l - \varphi \circ g)\|_{\mathcal{T}_2^p} \lesssim \|r|D^2\varphi \circ g_l| |\vartheta g_l - \vartheta g|/(|\vartheta g_l| + |\vartheta g|)|\|_{\mathcal{T}_2^p} + \|r|D^2\varphi \circ g_l - D^2\varphi \circ g| |\vartheta g|^2\|_{\mathcal{T}_2^p} \lesssim \|g_l - g\|_{H^p}
\]
by (3),(4) and (5). Thus
\[
\|(\varphi \circ g_l - \varphi \circ g)\|_{B_0} \to 0
\]
as \(l \to \infty\) and therefore we have that \((\varphi \circ g)u\) is in the completion of \(C^\infty(\bar{D})\) with respect to the norm \(\|\cdot\|_{B_0}\). We extend the map
\[
u \mapsto (\varphi \circ g)u : C^\infty(\bar{D}) \to B_0
\]
to a continuous map \(\varphi(T_g) : B_0 \to B_0\), bounded by a constant times
\[
\sup_{z \in \Sigma(D)} (|\varphi(z)| + |D\varphi(z)| + |D^2\varphi(z)|).
\]
Hence \(T_g\) on \(B_0\) has a continuous \(C^\infty(\mathbb{C}^n)\)-functional calculus.

Next we consider the case \(k = 1\). Suppose that \(\varphi \in C^\infty(\mathbb{C}^n)\) and \(u \in C^\infty_{0,1}(\bar{D})\). From (3) and (5) we have the inequality
\[
\|r |\vartheta g| |u|\|_{\mathcal{T}_2^p} \lesssim \|r^{1/2}\vartheta g\|_{\mathcal{T}_2^{1,\infty}} \|r^{1/2}u\|_{\mathcal{T}_2^p} \lesssim \|r^{1/2}u\|_{\mathcal{T}_2^p}.
\]
Hence we get
\[
\|(\varphi \circ g)u\|_{B_1} \leq \sup_{z \in \Sigma(D)} |\varphi(z)| \|u\|_{B_1} + \|r\vartheta (\varphi \circ g) \wedge u\|_{\mathcal{T}_2^p} \lesssim \sup_{z \in \Sigma(D)} (|\varphi(z)| + |D\varphi(z)|) \|u\|_{B_1}.
\]
As in the case \(k = 0\) we prove that \((\varphi \circ g)u\) is in the completion of \(C^\infty_{0,1}(\bar{D})\). When we extend the map
\[
u \mapsto (\varphi \circ g)u : C^\infty(\bar{D}) \to B_1
\]
by continuity to a map \(\varphi(T_g) : B_1 \to B_1\) bounded by
\[
\sup_{z \in \Sigma(D)} (|\varphi(z)| + |D\varphi(z)|)
\]
and hence we have proved that \(T_g\) on \(B_1\) has a \(C^\infty(\mathbb{C}^n)\)-functional calculus.
In case $k \geq 2$ we suppose that $\varphi \in C^\infty(\mathbb{C}^n)$ and $u \in C^\infty_{0,k}(\bar{D})$. Since $|\partial g| \preceq r^{-1/2}$ we have
\[
\| (\varphi \circ g)u \|_{B_k} \leq \sup_{z \in g(D)} |\varphi(z)| \| u \|_{B_k} + \| r^{k/2+1/2} \tilde{\partial}(\varphi \circ g) \wedge u \|_{\mathcal{T}_q} \preceq \\
\sup_{z \in g(D)} (|\varphi(z)| + |D\varphi(z)|) \| u \|_{B_k}.
\]
As in the case $k = 0$ it follows that $T_g$ on $B_k$, $k \geq 2$, has a $C^\infty(\mathbb{C}^n)$-functional calculus.

That each of the tuples $T_g$ has property $(\beta)_\mathcal{C}$ now follows from Proposition 6.4.13 in [9].

We can extend the integral operator $K : C^\infty_{0,k+1}(\bar{D}) \rightarrow C_{0,k}(\bar{D})$, $k \geq 1$, to a continuous operator $K : B_{k+1} \rightarrow B_k$, $k \geq 2$, and a continuous operator $K : B_2 \rightarrow B'_1$. This because
\[
\| r^{k/2} Ku \|_{\mathcal{T}_q} \preceq \| r^{k/2+1/2} u \|_{\mathcal{T}_q} \leq \| u \|_{B_{k+1}}
\]
and
\[
\| r^{k/2+1/2} \tilde{\partial} Ku \|_{\mathcal{T}_q} = \| r^{k/2+1/2} (u - K\tilde{\partial} u) \|_{\mathcal{T}_q} \preceq \| u \|_{B_{k+1}}
\]
for all $u \in C^\infty_{0,k+1}(\bar{D})$ by (6), (12) and (14). Also observe that $Ku$ is in the completion of $C^\infty_{0,k}(\bar{D})$ under the norm $\| \cdot \|_{B_k}$ (or $\| \cdot \|_{\mathcal{T}_q}$) by dominated convergence and the fact that one can find $f_1 \in C^\infty(\bar{D})$ such that $f_1 \rightarrow Ku$, $\tilde{\partial} f_1 \rightarrow \tilde{\partial} Ku$ pointwise and $|f_1|, |\tilde{\partial} f_1| \leq 1$ (as $Ku, \tilde{\partial} Ku \in C(\bar{D})$). Approximation in $B_{k+1}$ yields that $\tilde{\partial} Ku + K\tilde{\partial} u = u$ for all $u \in B_{k+1}$, $k \geq 1$. Thus the complex (13) is exact in higher degrees.

Extend $K : C^\infty(\bar{D}) \rightarrow C(\partial D)$ to continuous maps $K : B_1 \rightarrow L^p(\partial D)$ and $K : B'_1 \rightarrow L^p(\partial D)$, which is possible by (6) and (7).

Define the $(1,0)$-vector field $\mathcal{L}$ by the equation
\[
\mathcal{L} = \chi \sum \frac{|\partial r|^{-2} \partial r \partial}{\partial z_k \partial \bar{z}_k},
\]
where $\chi$ is equal to 1 in a neighbourhood of $\partial D$ and 0 on the set where $\partial r = 0$. Suppose that $u \in C^\infty(\bar{D})$ and let $f = \tilde{\partial} u$. By integration by parts we have
\[
\int_{\partial D} uh = \int_D f \wedge h =: V(f, h)
\]
and
\[
\int_{\partial D} uh = \int_D f \wedge h = \int_D O(r) f \wedge h + \int_D r \mathcal{L}(f \wedge h) =: W(f, h)
\]
for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\tilde{\partial} h = 0$. We extend $V$ to elements $f$ in $B'_1$ and $W$ to elements in $B_1$. We say that the equation $\tilde{\partial} hu = f + f'$,
where \( u \in L^p(\partial D) \), \( f \in B_1 \) and \( f' \in B'_1 \), holds if and only if
\[
\int_{\partial D} uh = W(f, h) + V(f', h)
\]
for all \( h \in C^\infty_{m,m-1}(\bar{D}) \) such that \( \bar{\partial} h = 0 \).

**Lemma 3.2.** If \( f \in B_1 \), \( f' \in B'_1 \) and \( \bar{\partial} f + \bar{\partial} f' = 0 \) then \( u = K f + K f' \) solves the equation \( \bar{\partial} u = f + f' \). Moreover, if \( \varphi \in H^\infty(D) \) then \( \bar{\partial} (\varphi u) = T_\varphi f + T_\varphi f' \).

**Proof.** Suppose that \( f, f' \in C^\infty_0(\bar{D}) \). Since \( \bar{\partial} K(f + f') + K \bar{\partial}(f + f') = f + f' \), we have
\[
\int_{\partial D} (K f + K f') h = W(f, h) + V(f', h) - \int_D K(\bar{\partial} f + \bar{\partial} f') \wedge h
\]
for all \( h \in C^\infty_{m,m-1}(\bar{D}) \) such that \( \bar{\partial} h = 0 \). For fixed \( h \), we can estimate each term of the above equality by a constant times \( \|f\|_{B_1} + \|f'\|_{B'_1} \).

Thus approximation in \( B_1 \) and \( B'_1 \) yields that if \( f \in B_1 \) and \( f' \in B'_1 \) then
\[
\int_{\partial D} uh = W(f, h) + V(f', h) - \int_D K(\bar{\partial} f + \bar{\partial} f') \wedge h
\]
for all \( h \in C^\infty_{m,m-1}(\bar{D}) \) such that \( \bar{\partial} h = 0 \). Hence the equation \( \bar{\partial} h u = f + f' \) holds since we also have that \( \bar{\partial} f + \bar{\partial} f' = 0 \). Suppose that \( \varphi_k \in C^\infty(\bar{D}) \cap \mathcal{O}(D) \) are chosen such that \( \varphi_k \to \varphi \) in \( H^1(D) \). Replace \( h \) in (15) by \( \varphi_k h \) and approximate to get
\[
\int_{\partial D} \varphi (K f + K f') h = W(f, h\varphi) + V(f', h\varphi) - \int_D \varphi K(\bar{\partial} f + \bar{\partial} f') \wedge h
\]
for all \( h \in C^\infty_{m,m-1}(\bar{D}) \) such that \( \bar{\partial} h = 0 \), if \( f, f' \in C^\infty_0(\bar{D}) \). We estimate the terms to the right,
\[
|W(f, h\varphi)| \lesssim \int_D r^{3/2} |f| |\varphi| r^{-1} + \int_D r |\bar{\partial} f| |\varphi| r^{-1} + \int_D r |f| |\partial \varphi| r^{-1} \lesssim \|f\|_{B_1} \|\varphi\|_{H^p'}
\]
and
\[
|V(f', h\varphi)| \lesssim \int_D r^{1/2} |f'| |\varphi| r^{-1} \lesssim \|f'\|_{B'_1} \|\varphi\|_{H^p'}
\]
and
\[
\left| \int_D \varphi K(\bar{\partial} f + \bar{\partial} f') \wedge h \right| \lesssim \|r^{1/2} K(\bar{\partial} f + \bar{\partial} f')\|_{T_1} \|\varphi\|_{T_{2\infty}'} \lesssim \|\bar{\partial} f + \bar{\partial} f'\|_{B_2} \|\varphi\|_{H^p'} \lesssim \left( \|f\|_{B_1} + \|f'\|_{B'_1} \right) \|\varphi\|_{H^p'}
\]
for fixed $h$ by (2), (4) and (5). Hence approximation in $B_1$ and $B'_1$ yields that

$$\int_{\partial D} u \varphi h = W(T_\varphi f, h) + V(T_\varphi f', h)$$

for all $f \in B_1, f' \in B'_1$ such that $\bar{\partial} f + \bar{\partial} f' = 0$ and $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$. □

Next we prove that functions in $B_0$ have boundary values in $L^p(\partial D)$.

**Lemma 3.3.** There is a continuous and linear operator $u \mapsto u^*$ from $B_0$ to $L^p(\partial D)$ such that $u^*$ is the restriction of $u$ to $\partial D$ if $u \in C^\infty(\bar{D})$ and $(T_f u)^* = f^* u^*$ if $f \in H^\infty(D)$.

**Proof.** Suppose that $u \in C^\infty(\bar{D})$. Then $\|u\|_{L^p(\partial D)} \leq \|u\|_{B_0}$ and hence the restriction operator can be extended to a continuous operator from $B_0$ to $L^p(\partial D)$. Suppose that $u \in B_0$ and $f \in H^\infty(D)$. Let $u_l \in C^\infty(\bar{D})$ and $f_k \in C^\infty(\bar{D}) \cap \mathcal{O}(D)$ be such that $u_l \to u$ in $B_0$ and $f_k \to f$ in $H^\infty(D)$ with $f_k$ uniformly bounded. Then

$$\|f^* u^* - (T_f u)^*\|_{L^p(\partial D)} \leq \|f^* u_l^* - f^* u^*\|_{L^p(\partial D)} + \|f^* u_l^* - f_k^* u^*\|_{L^p(\partial D)} + \|(f_k u_l)^* - (f u_l)^*\|_{L^p(\partial D)} + \|(f u_l)^* - (T_f u)^*\|_{L^p(\partial D)} \to 0$$

if one first let $k \to \infty$ and then $l \to \infty$. □

Note that if $u \in B_0$ then

$$\int_{\partial D} u^* h = W(\bar{\partial} u, h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$ by approximation in $B_0$ and Lemma 3.3.

**Proof of Theorem 1.1**

We want to prove that the complex $K_\bullet(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$ has vanishing homology groups of positive order and that

$$\sum_i (z_i - T_{g_i}) \mathcal{E}(\mathbb{C}^n, H^p)$$

is closed in $\mathcal{E}(\mathbb{C}^n, H^p)$.

Suppose that $u^k \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$ and that $\delta_{z-g} u^k \to u_0$ in $\mathcal{E}(\mathbb{C}^n, H^p)$. By Lemma 3.1 there is a $u_1 \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ such that $i u_0 = \delta_{z-g} u_1$. Again by Lemma 3.1 we can recursively find $u_i \in K_i(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-1}))$ such that $\delta_{z-g} u_{i+1} = \bar{\partial} u_i$ for $i \geq 1$. Then we have that $\bar{\partial} u_{m+1} = 0$. Define $v_{m+1} \in K_{m+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{m-2}))$ by $v_{m+1} = K u_{m+1}$. Recursively define $v_i, i \geq 2$, by $v_i = K u_i - K \delta_{z-g} v_{i+1}$. Thus $v_i \in K_i(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-2}))$ if $i \geq 4$, $v_3 \in \Lambda^3 \mathcal{E}(\mathbb{C}^n, B'_1)$ and the equation $\bar{\partial} v_i = u_i - \delta_{z-g} v_{i+1}$ holds for $i \geq 3$. Furthermore
\( v_2 \in \Lambda^2 \mathcal{E}(\mathbb{C}^n, L^p(\partial D)) \) satisfies the equation \( \tilde{\partial}_b v_2 = u_2 - \delta_{z_{-g}} v_3 \) by Lemma 3.2.

Let \( u_1' = u_1' - \delta_{z_{-g}} v_2 \). By Lemma 3.2 we have that \( \tilde{\partial}_b \delta_{z_{-g}} v_2 = \delta_{z_{-T_g}} u_2 \) and thus

\[
\int_{\partial D} \delta_{z_{-g}} v_2 h = W(\delta_{z_{-T_g}} u_2, h)
\]

for all \( h \in C_{m,m-1}^\infty(\bar{D}) \) such that \( \tilde{\partial} h = 0 \). Since by equation (16)

\[
\int_{\partial D} u_1' h = W(\tilde{\partial} u_1, h)
\]

we have proved that

\[
\int_{\partial D} u_1' h = 0
\]

for all \( h \in C_{m,m-1}^\infty(\bar{D}) \) such that \( \tilde{\partial} h = 0 \). Thus \( U_1' \in K(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p)) \), where \( U_1' \) is the unique holomorphic extension of \( u_1' \). Since \( u_0 = \delta_{z_{-T_g}} U_1' \) by Lemma 3.3 we have proved that

\[
\sum_i (z_i - T_{g_i}) \mathcal{E}(\mathbb{C}^n, H^p)
\]

is closed in \( \mathcal{E}(\mathbb{C}^n, H^p) \).

Suppose that \( u_k \in K_k(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p)) \) is \( \delta_{z_{-T_g}} \) closed. Then there is a \( u_{k+1} \in K_{k+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0)) \) such that \( u_k = \delta_{z_{-T_g}} u_{k+1} \). Let \( u_{i+1} \in K_{i+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-k})) \) solve the equation \( \delta_{z_{-T_g}} u_{i+1} = \tilde{\partial} u_i \).

Then we have that \( \tilde{\partial} u_{m+k+1} = 0 \). Let \( v_{m+k+1} = K u_{m+k+1} \) and \( v_i = K u_i - K \delta_{z_{-T_g}} v_{i+1} \). Thus \( \tilde{\partial} v_i = u_i - \delta_{z_{-T_g}} v_{i+1} \) and \( \tilde{\partial} u_{k+2} = u_{k+2} - \delta_{z_{-T_g}} v_{k+3} \) since \( \tilde{\partial}(u_i - \delta_{z_{-T_g}} v_{i+1}) = 0 \). Define \( u_{k+1}' \) by the equation

\[
u_{k+1}' = u_{k+1}' - \delta_{z_{-T_g}} v_{k+2} \]

As in the case above we see that \( U_{k+1}' \) is a solution of the equation \( u_k = \delta_{z_{-T_g}} U_{k+1}' \), and hence the theorem is proved.

\[ \square \]

We now prove the analogue of Theorem 1.1 with the Hardy space replaced by the Bergman space. In the case of when \( g \) has bounded derivative this is proved in Theorem 8.1.5 in [9].

**Corollary 3.4.** Suppose that \( D \) is a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^\infty \) boundary and that \( g \in H^\infty(\bar{D})^n \). Then the tuple \( T_g \) of Toeplitz operators on the Bergman space \( \mathcal{O}L^p(D) \), \( 1 \leq p < \infty \), satisfies property \( (\beta) \) and Bishop’s property \( (\beta) \).

**Proof.** Let \( \rho \) be a strictly plurisubharmonic defining function for \( D \) and let \( \tilde{D} = \{(v,w) \in \mathbb{C}^{m+1} : \rho(v) + |w|^2 < 0 \} \). Define the operators \( P : H^p(\bar{D}) \rightarrow \mathcal{O}L^p(D) \) and \( I : \mathcal{O}L^p(D) \rightarrow H^p(\bar{D}) \) by \( P f(v) = f(v,0) \) and \( I f(v,w) = f(v) \) respectively. The operator \( P \) is continuous by the Carleson-Hörmander inequality since the measure with mass uniformly
distributed on \( \tilde{D} \cap \{ w = 0 \} \) is a Carleson measure. The operator \( I \) is continuous since
\[
\int_{\partial \tilde{D}} |f(v)|^p \sigma(v, w) \sim \lim_{\varepsilon \to 0} \int_D \left( -\rho(v) - |w|^2 \right)^{\varepsilon-1} |f(v)|^p \sim \\
\lim_{\varepsilon \to 0} \int_D \left( -\rho(v) \right)^{\varepsilon} |f(v)|^p = \int_D |f(v)|^p ,
\]
where \( \sigma \) is the surface measure. Let \( \tilde{g}(v, w) = g(v) \). Then \( T_{\tilde{g}} \) has property \( (\beta)_{\varepsilon} \) and since \( PI = id, T_{\tilde{g}} I = IT_{\tilde{g}} \) and \( PT_{\tilde{g}} = T_{\tilde{g}} P \) it is easy to see that \( T_{\tilde{g}} \) has property \( (\beta)_{\varepsilon} \).

4. **Property \( (\beta)_{\varepsilon} \) for Toeplitz Operators with \( H^\infty \)-Symbol on the Unit Disc**

In this section we will use the Euclidean norm. Let \( r(w) = 1 - |w|^2 \) and let \( D \) be the unit disc in \( \mathbb{C} \). Let \( B_0 \) be the Banach space of all functions \( u \in L^\infty(D) \) such that
\[
\| u \|_{B_0} = \| u \|_{L^\infty(D)} + \| ru \|_{L^\infty(D)} + \| r^2 \partial u \|_{T^\infty} + \| r^2 \partial \bar{u} \|_{T^\infty} < \infty.
\]
Since \( \| ru \|_{L^\infty(D)} < \infty \), \( B_0 \) consists of continuous functions on \( D \). We define \( B_1 \) as the Banach space of all locally integrable \((0, 1)\)-forms \( u \) such that
\[
\| u \|_{B_1} = \| ru \|_{L^\infty(D)} + \| r^2 \partial u \|_{T^\infty} + \| r^2 \partial \bar{u} \|_{T^\infty} < \infty.
\]

Suppose that \( u \in C^\infty(\overline{D}) \) and \( h \in C^\infty(\partial D) \). Then the Wolff trick (see the proof of Theorem 1.1) yields
\[
\int_{\partial D} uhdw = \int_D \tilde{\partial}(uP hdw) = \\
\int_D O(r) \tilde{\partial}(uP hdw) + \int_D r \mathcal{L} \tilde{\partial}(uP hdw) := S(u, h),
\]
where \( Ph \) is the Poisson integral of \( h \).

As in Section 3 we need to know that functions in \( B_0 \) has well defined boundary values.

**Lemma 4.1.** If \( u \in B_0 \) then there is a \( u^* \in L^\infty(\partial D) \) such that
\[
\int_{\partial D} u^* hdw = S(u, h)
\]
for all \( h \in L^2(\partial D) \) and \((fu)^* = f^*u^* \) if \( f \in H^\infty(D) \).

**Proof.** We have the estimate
\[
|S(u, h)| \lesssim \| u \|_{B_0} \| h \|_{L^2(\partial D)}.
\]
Hence there is a function \( u^* \in L^2(\partial D) \) such that
\[
\int_{\partial D} u^* \operatorname{hdw} = S(u, h)
\]
for all \( h \in L^2(\partial D) \). Suppose that \( h \in C^\infty(\partial D) \). Let \( u_t \) be the dilation 
\( u_t(w) = u(tw) \). Since
\[
|S(u_t - u, h)| \lesssim \int_D |u_t - u| + \int_D r |d(u_t - u)|^2 + \int_D r |\partial \bar{\partial}(u_t - u)|
\]
for fixed \( h \) we have that
\[
\int_{\partial D} u^*_t \operatorname{hdw} \rightarrow \int_{\partial D} u^* \operatorname{hdw}
\]
as \( t \nearrow 1 \). Therefore \( \|u^*\|_{L^\infty(\partial D)} \leq \|u\|_{B_1} \) since \( u^*_t \) is uniformly bounded by \( \|u\|_{L^\infty(D)} \). Let \( f_s(w) = f(sw) \) be the dilation of \( f \). Then we have that
\[
\int_{\partial D} f^*_s u^*_t \operatorname{hdw} = \int_{\partial D} (f^*_s - f^*) u^*_t \operatorname{hdw} + \int_{\partial D} f^* u^*_t \operatorname{hdw} \rightarrow \int_{\partial D} f^* u^* \operatorname{hdw}
\]
as \( s, t \nearrow 1 \), by dominated convergence. Since we also have
\[
\int_{\partial D} (f u)^*_t \operatorname{hdw} \rightarrow \int_{\partial D} (f u)^* \operatorname{hdw}
\]
as \( t \nearrow 1 \) we see that \( (f u)^* = f^* u^* \).

Let
\[
W(u, h) = \int_D O(r) u \wedge \operatorname{hdw} + \int_D r \mathcal{L}(u \wedge \operatorname{hdw})
\]
for \( u \in B_1 \) and \( h \in H^1 \), where \( O(r) \) is the same \( O(r) \) as in the definition of \( S(u, h) \).

**Lemma 4.2.** If \( f \in \mathcal{E}(\mathbb{C}^n, B_1) \) then there is a \( u \in \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D)) \) such that \( \partial \bar{\partial} u = f \), that is
\[
\int_{\partial D} u(z) \operatorname{hdw} = W(f(z), h)
\]
for all \( h \in H^1(D) \) and \( z \in \mathbb{C}^n \).

**Proof.** Consider the bilinear map \( W : B_1 \times H^1 \rightarrow \mathbb{C} \). This map is continuous since we have the estimate
\[
|W(f, h)| \lesssim \|u\|_{B_1} \|h\|_{H^1},
\]
which is used in Wolff’s proof of the corona theorem. By the universal property for \( \pi \)-tensor products (see 41.3.1(1) in [13]) there is a corresponding linear and continuous map \( W_1 \) from \( B_1 \otimes_{\pi} H^1 \) to \( \mathbb{C} \). Since
\[
\mathcal{E}(\mathbb{C}^n, B_1) \equiv \mathcal{E}(\mathbb{C}^n) \otimes B_1 \equiv L(\mathcal{E}(\mathbb{C}^n), B_1)
\]
by Appendix 1 in [9], \( f \otimes \operatorname{id} \) is a continuous map \( \mathcal{E}(\mathbb{C}^n) \otimes H^1 \rightarrow B_1 \otimes_{\pi} H^1 \). Compose with the map \( W_1 \) to get a continuous functional on
$E'(\mathbb{C}^n) \otimes H^1$. The injection $E'(\mathbb{C}^n) \otimes H^1 \to E'(\mathbb{C}^n) \otimes L^1(\partial D)$ is a topological monomorphism, and hence we can extend with Hahn-Banach Theorem to a continuous functional on $E'(\mathbb{C}^n) \otimes L^1(\partial D)$. Since the dual of $E'(\mathbb{C}^n) \otimes L^1(\partial D)$ is isomorphic to $E(\mathbb{C}^n, L^\infty(\partial D))$ by Theorem A1.12 in [9] we have a $u \in E(\mathbb{C}^n, L^\infty(\partial D))$. If $h \in H^1$ then
\[
\int u(z)h\,dw = W(f(z), h)
\]
and thus $u$ is a solution to the equation $\overline{\partial} u = f$ in the sense of this lemma.

**Theorem 4.3.** Let $D$ be the unit disc in $\mathbb{C}$ and suppose that $g \in H^\infty(D)^n$. Then the tuple $T_g$ of Toeplitz operators on $H^\infty(D)$ satisfies property $(\beta)_\varepsilon$, and thus Bishop’s property $(\beta)$.

**Proof.** The tuple $T_g$ considered as operators on $B_0$ or $B_1$ has a $C^\infty(\mathbb{C}^n)$-functional calculus (the proof of this is similar to Lemma 3.1). Hence they satisfies property $(\beta)_\varepsilon$ by Proposition 6.4.13 in [9]. Consider the well-defined complex
\[
0 \to H^\infty \to B_0 \xrightarrow{\delta} B_1 \to 0.
\]
Suppose that $u^k \in \sum_i(z_i - T_{g_i})E(\mathbb{C}^n, H^\infty)$ and $u^k \to u_0$ in $E(\mathbb{C}^n, H^\infty)$. As $T_g$ on $B_0$ has property $(\beta)_\varepsilon$ there is a $u_1 \in K_1(z - T_g, E(\mathbb{C}^n, B_0))$ such that $u_0 = \delta_{z - T_g}u_1$. Since $T_g$ on $B_1$ has property $(\beta)_\varepsilon$, there is a $u_2 \in K_2(z - T_g, E(\mathbb{C}^n, B_1))$ such that $\delta_{z - T_g}u_2 = \overline{\partial} u_1$. By Lemma 4.2 there is a $v \in L^2 E(\mathbb{C}^n, L^\infty(\partial D))$ such that
\[
\int_{\partial D} v\,hdw = W(u_2, h)
\]
for all $h \in H^1(D)$. Therefore we have that
\[
\int_{\partial D} \delta_{z - g*}v\,hdw = W(\delta_{z - T_g}u_2, h)
\]
for all $h \in H^1(D)$. Define $u'_1 \in K_1(z - g^*, E(\mathbb{C}^n, L^\infty(\partial D)))$ by the equation $u'_1 = u_1^* - \delta_{z - g^*}v$. Then
\[
\int_{\partial D} u'_1\,hdw = 0
\]
for all $h \in H^1$ since
\[
\int_{\partial D} u'_1\,hdw = S(u_1, h) = W(\overline{\partial} u_1, h)
\]
by Lemma 4.1. Thus $U'_1 \in K_1(z - T_g, E(\mathbb{C}^n, H^\infty))$, where $U'_1$ is the holomorphic extension. Since $u_0 = \delta_{z - T_g}u'_1$ by Lemma 4.1 we have proved that $\delta_{z - T_g}K_1(z - g^*, E(\mathbb{C}^n, H^\infty))$ is closed.

Suppose that $u_k \in K_k(z - T_g, E(\mathbb{C}^n, H^\infty))$ is $\delta_{z - T_g}$-closed. Then there is a solution $u_{k+1} \in K_{k+1}(z - T_g, E(\mathbb{C}^n, B_0))$ to the equation $\delta_{z - T_g}u_{k+1} = u_k$ since $T_g$ on $B_0$ has property $(\beta)_\varepsilon$. Continuing in exactly
the same way as above we see that we can replace $u_{k+1}$ with $U'_{k+1} \in K_{k+1}(z - T_g, E(\mathbb{C}^n, H^\infty))$ such that $\delta_{z-T_g} U'_{k+1} = u_k$. Thus the theorem is proved.

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