PROPERTY $(\beta)_{\mathcal{E}}$ FOR TOEPLITZ OPERATORS WITH H^∞-SYMBOL

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ABSTRACT. Suppose that g is a tuple of bounded holomorphic functions on a strictly pseudoconvex domain D in $\mathbb{C}^m$ with smooth boundary. Viewed as a tuple of operators on the Hardy space $H^p(D)$, $1 \leq p < \infty$, g is shown to have property $(\beta)_{\mathcal{E}}$ and therefore g possess Bishop's property (β). In the case $m = 1$ it is proved that the same result also holds when $p = \infty$.

1. INTRODUCTION

Suppose that X is a Banach space and that $a = (a_1, \dots, a_n)$ is a commuting tuple of bounded linear operators on X. Let E be one of spaces X, $\mathcal{E}(\mathbb{C}^n, X)$ or $\mathcal{O}(U, X)$, where $U \subset \mathbb{C}^n$. Denote by $K_\bullet(z-a, E)$ the Koszul complex

$$0 \to \Lambda^n E \xrightarrow{\delta_{z-a}} \Lambda^{n-1} E \xrightarrow{\delta_{z-a}} \cdots \xrightarrow{\delta_{z-a}} \Lambda^0 E \to 0,$$

with boundary map

$$\delta_{z-a}(f s_I) = 2\pi i \sum_{k=1}^{p} (-1)^{k-1} (z_{i_k} - a_{i_k}) f s_{i_1} \wedge \cdots \wedge \widehat{s}_{i_k} \wedge \cdots \wedge s_{i_p},$$

where $I = (i_1, \dots, i_p)$ and p is an integer. Let $H_\bullet(z - a, E)$ be the corresponding homology groups.

The Taylor spectrum of a, $\sigma(a)$, is defined as the set of all $z \in \mathbb{C}^n$ such that $K_\bullet(z - a, X)$ is not exact. If for all Stein open sets U in $\mathbb{C}^n$ the natural quotient topology of $H_0(z - a, \mathcal{O}(U, X))$ is Hausdorff and $H_p(z - a, \mathcal{O}(U, X)) = 0$ for all $p > 0$, then a is said to have Bishop's property (β). It has property $(\beta)_{\mathcal{E}}$ if the natural quotient topology of $H_0(z - a, \mathcal{E}(\mathbb{C}^n, X))$ is Hausdorff and if $H_p(z - a, \mathcal{E}(\mathbb{C}^n, X)) = 0$ for all $p > 0$.

By Theorem 6.2.4 in [9], the tuple a has Bishop's property (β) if and only if there exists a decomposable resolution, that is, if and only if there are Banach spaces X_i and decomposable tuples (see [9] for the definition) of operators a_i on X_i such that

$$0 \to X \xrightarrow{d} X_0 \xrightarrow{d} \cdots \xrightarrow{d} X_r \to 0$$

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is exact, $da = a_0 d$ and $da_i = a_{i+1} d$. Property $(\beta)_{\mathcal{E}}$ is equivalent to the existence of a resolution of Fréchet spaces with Mittag-Leffler inverse limit of generalized scalar tuples (that is tuples which admit a continuous $C^\infty(\mathbb{C}^n)$-functional calculus), see Theorem 6.4.15 in [9]. Property $(\beta)_{\mathcal{E}}$ implies Bishop's property (β), see [9].

Suppose that D is a strictly pseudoconvex domain in $\mathbb{C}^m$ with smooth boundary. We consider the tuple $T_g = (T_{g_1}, \dots, T_{g_n})$, $g_k \in H^\infty(D)$, of operators on $H^p(D)$ defined by $T_{g_k} f = g_k f$, $f \in H^p(D)$. The main theorem of this paper is the following.

Theorem 1.1. *Suppose that D is a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^∞-boundary and that $g \in H^\infty(D)^n$. Then the tuple T_g of Toeplitz operators on $H^p(D)$, $1 \le p < \infty$, satisfies property $(\beta)_{\mathcal{E}}$, and thus Bishop's property (β).*

In case g has bounded derivative this theorem has previously been proved in [14, 16, 17]. In case D is the unit disc in $\mathbb{C}$, Theorem 1.1 also holds when $p = \infty$; this is proved in Section 4. As a corollary to Theorem 1.1 we have that T_g on the Bergman space $\mathcal{O}L^p(D)$ has property $(\beta)_{\mathcal{E}}$, see Corollary 3.4.

Let us recall how one can prove that T_g on the Bergman space $\mathcal{O}L^2(D)$ has property $(\beta)_{\mathcal{E}}$ under the extra assumption that g has bounded derivative. Define the Banach spaces B_k as the spaces of locally integrable $(0,k)$-forms u such that

$$\|u\|_{B_k} := \|u\|_{L^2(D)} + \left\|\bar\partial u\right\|_{L^2(D)} < \infty.$$

Since g has bounded derivate we have the inequality

$$\|(\varphi \circ g)u\|_{B_k} \lesssim \sup_{z \in g(D)} \left(|\varphi(z)| + \left|\bar\partial \varphi(z)\right|\right) \|u\|_{B_k}$$

for all $\varphi \in C^\infty(\mathbb{C}^n)$. Hence $\varphi \mapsto T_{\varphi \circ g}$ is a continuous $C^\infty(\mathbb{C}^n)$-functional calculus, where $T_{\varphi\circ g}$ denotes multiplication by $\varphi \circ g$ on B_k. Since we have the resolution

$$0 \to \mathcal{O}L^2(D) \to B_0 \xrightarrow{\bar\partial} B_1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} B_m \to 0$$

by Hörmander's L^2-estimate of the $\bar\partial$ equation, the tuple T_g on $\mathcal{O}L^2(D)$ has property $(\beta)_{\mathcal{E}}$ by the above mentioned Theorem 6.4.15 in [9].

To prove Theorem 1.1 we will construct a complex

$$0 \to H^p(D) \xrightarrow{i} B_0 \xrightarrow{\bar\partial} B_1 \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} B_m \to 0, \tag{1}$$

where B_k are Banach spaces of $(0,k)$-forms on D. The spaces B_k are defined in terms of tent norms. We prove that $\varphi \mapsto T_{\varphi\circ g}$ is a continuous $C^\infty(\mathbb{C}^n)$-functional calculus, where $T_{\varphi\circ g}$ denotes multiplication by $\varphi\circ g$ on B_k. If the complex (1) were exact the proof of Theorem 1.1 would be finished. As we can solve the $\bar\partial$-equation with appropiate estimates we will be able to prove that T_g on H^p has property $(\beta)_{\mathcal{E}}$ anyway. More precisely (1) is exact at B_k, $k \ge 3$. If $f \in B_2$ and $\bar\partial f = 0$ then there is a function u in another Banach space B_1' such that $\bar{\partial}u = f$. Mutiplication by g is a bounded operator on B_1'. If $f \in B_1$ and $f' \in B_1'$ such that $\bar{\partial}f + \bar{\partial}f' = 0$ then there is a solution $u \in L^p(\partial D)$ to the equation $\bar{\partial}_b u = f + f'$.

The construction of the complex (1) in the case $p < \infty$ is inspired by the construction in [5] and in the case $p = \infty$ and $m = 1$ it is inspired by Tom Wolff's proof of the corona theorem. Let us recall the proof of the H^p-corona theorem in the unit disc of $\mathbb{C}$. Suppose that $g = (g_1, \dots, g_n) \in H^\infty(D)^n$, where D is the unit disc in $\mathbb{C}$, and that $0 \notin \overline{g(D)}$. Consider the complex (1); the definitions of the B_k-spaces can be found in the beginning of Section 3 and Section 4. Suppose that $f \in H^p(D)$. Then the equation $\delta_g u_1 = f$ has a solution in $K_1(g, B_0)$, namely $u_1 = \sum_k \bar{g}_k f s_k / |g|^2$. Hence $\delta_g \bar{\partial} u_1 = 0$ as δ_g and $\bar{\partial}$ anticommute, and we can solve the equation $\delta_g u_2 = \bar{\partial} u_1$ by defining $u_2 \in K_2(g, B_1)$ as $u_1 \wedge \bar{\partial} u_1$. Since u_2 satisfies the condition

$$\|(1 - |z|)\, u_2\|_{T_2^p} + \left\|(1 - |z|)^2\, \partial u_2\right\|_{T_1^p} < \infty,$$

by a Wolff type estimate there is a solution v in $K_2(g, L^p(\partial D))$ to the equation $\bar{\partial}_b v = u_2$ (here T_2^p and T_1^p denote certain tent spaces). Let $u_1' = u_1^* - \delta_g v \in K_1(g, L^p(\partial D))$, where u_1^* is the boundary values of u_1. Since $\bar{\partial}_b u_1' = 0$ there is a holomorphic extension U_1' of u_1' to D which satisfies the equation $\delta_g U_1' = f$.

The above proof also yields that $\sigma(T_g) = \overline{g(D)}$; the exactness of higher order in the Koszul complex follows by similar resoning. That $\sigma(T_g) = \overline{g(D)}$ is proved in [5] for the case D strictly pseudoconvex and $p < \infty$. One main difference of the proof of that T_g has property $(\beta)_\mathcal{E}$ and the proof of that $\sigma(T_g) = \overline{g(D)}$ is the following. As a substitution of the explicit choices of u_1 and u_2 one uses the fact that T_g considered as an operator on B_k has property $(\beta)_\mathcal{E}$, which in turn follows from the fact that T_g on B_k has a $C^\infty(\mathbb{C}^n)$-functional calculus.

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2. Preliminaries

Suppose that D is a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^∞-boundary given by a strictly plurisubharmonic defining function ρ. Let $r = -\rho$. All norms below are with respect to the metric

$$\Omega = ri\partial\bar{\partial} \log\left(1/r\right),$$

and we have

$$|f|^2 \sim r^2 \, |f|_\beta^2 + r\, |f \wedge \partial r|_\beta^2 + r\left|f \wedge \bar{\partial} r\right|_\beta^2 + \left|f \wedge \partial r \wedge \bar{\partial} r\right|_\beta^2,$$

where $\beta = i\partial\bar{\partial} r$, which is equivalent to the Euclidean metric.

The Hardy space H^p is the Banach space of all holomorphic functions, f, on D such that

$$\|f\|_{H^p} = \sup_{\varepsilon>0} \int_{r(z)=\varepsilon} |f(z)|^p \, d\sigma(z) < \infty,$$

where σ is the surface measure. It is wellknown that a function u in $L^p(\partial D)$ is the boundary value of a function U in H^p if and only

$$\int_{\partial D} uh = 0$$

for all $h \in C^\infty_{n,n-1}(\bar{D})$ such that $\bar{\partial} h = 0$.

Let $d(\cdot,\cdot)$ be the Koranyi pseudometric on ∂D and let z' be the point on ∂D closest to $z \in D_\varepsilon$, where D_ε is a small enough neighbourhood of ∂D in D. For a point ζ on the boundary let

$$A_\zeta = \{z \in D_\varepsilon : d(z', \zeta) < r(z)\} \cup (D \setminus D_\varepsilon)\,.$$

For a ball B defined by $B = \{z \in \partial D : d(z,\zeta) < t\}$ let, for small t,

$$\hat{B} = \{z \in D_\varepsilon : d(z', \zeta) < t - r(z)\},$$

and for large t

$$\hat{B} = \{z \in D_\varepsilon : d(z', \zeta) < t - r(z)\} \cup (D \setminus D_\varepsilon)\,.$$

A function f is in the tent space T^p_q, where $p < \infty$ and $q < \infty$, if

$$\|f\|_{T^p_q} := \left(\int_{\partial D} \left(\int_{z \in A_\zeta} |f(z)|^q \, r(z)^{-m-1} \right)^{p/q} d\sigma(\zeta) \right)^{1/p} < \infty.$$

The function f is in T^p_∞ if f is continuous with limits along A_ζ at the boundary almost everywhere and such that

$$\|f\|_{T^p_\infty} := \left(\int_{\partial D} \sup_{z \in A_\zeta} |f(z)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty.$$

A function f is in T^∞_q if

$$\|f\|_{T^\infty_q} := \left\| \sup_{\cdot \in B} \left(\frac{1}{|B|} \int_{z \in \hat{B}} |f(z)|^q \, r(z)^{-1} \right)^{1/q} \right\|_{L^\infty(\partial D)} < \infty.$$

Note that $f \in T^p_p$ if and only if $r^{-1/p} f \in L^p(D)$ by Fubini's theorem. From [8] we have the inequality

$$\int_D |fg| \, r^{-1} \lesssim \|f\|_{T^p_q} \, \|g\|_{T^{p'}_{q'}} \tag{2}$$

for $1 \le p, q \le \infty$, where p' and q' denote dual exponents. By [8] $T^{p'}_{q'}$, where $1 \le p < \infty$ and $1 < q < \infty$, is the dual of T^p_q with respect to the pairing

$$\langle f, g\rangle \to \int_D fgr^{-1}.$$

Suppose that $f \in T^p_{q_0}$, $g \in T^\infty_{q_1}$ and let $q = \left(q_0^{-1} + q_1^{-1}\right)^{-1}$. Then for all $h \in T^{p'}_{q'}$ we have

$$\int_D |fgh|\, r^{-1} \lesssim \|fh\|_{T^1_{q_1'}} \|g\|_{T^\infty_{q_1}} \leq \|f\|_{T^p_{q_0}} \|g\|_{T^\infty_{q_1}} \|h\|_{T^{p'}_{q'}}$$

by (2) and Hölder's inequality. Thus by the duality for $T^{p'}_{q'}$ we get the inequality

$$\|fg\|_{T^p_q} \lesssim \|f\|_{T^p_{q_0}} \|g\|_{T^\infty_{q_1}} \tag{3}$$

for $1 < p$ and $1 < q < \infty$. Since the inequality (3) is equivalent to

$$\|fg\|_{T^{tp}_{tq}} \lesssim \|f\|_{T^{tp}_{tq_0}} \|g\|_{T^\infty_{tq_1}}$$

for $0 < t < \infty$, (3) holds if $0 < p, q_0, q_1$.

We will use the inequality (see [12])

$$\|f\|_{T^p_\infty} \lesssim \|f\|_{H^p}, \quad p > 0 \tag{4}$$

and (see e.g. [7] for $p < \infty$ and [3] for $p = \infty$)

$$\left\|r^{1/2}\partial f\right\|_{T^p_2} \lesssim \|f\|_{H^p}, \quad p > 0. \tag{5}$$

Moreover, we use that $|\partial f| \lesssim r^{-1/2}$ if $f \in H^\infty$.

There is an integral operator $K : C^\infty_{0,q+1}(\bar{D}) \to C_{0,q}(\bar{D})$, $q \geq 0$, see [5], such that $\bar{\partial}Ku + K\bar{\partial}u = u$, $u \in C^\infty_{0,s}(\bar{D})$, $s \geq 1$,

$$\|r^\tau Ku\|_{T^p_1} \lesssim \left\|r^{\tau+1/2}u\right\|_{T^p_1} \text{ and } \|Ku\|_{L^p(\partial D)} \lesssim \left\|r^{1/2}u\right\|_{T^p_1} \tag{6}$$

if $\tau > 0$ and $1 \leq p < \infty$. Furthermore,

$$\|Ku\|_{L^p(\partial D)} \lesssim \left\|r^{1/2}u\right\|_{T^p_2} + \|r\partial u\|_{T^p_1}. \tag{7}$$

To see that the inequality (6) follows from [5], note that by the definition of $W^{1-1/p}$ in [1], $\|ru\|_{T^p_1} = \|u\|_{W^{1-1/p}}$. By [4] the adjoint P of K satisfies

$$\|P\psi\|_{L^\infty(D)} \lesssim \|\psi\|_{L^\infty(\partial D)} \text{ and } \left\|r^{1/2}\mathcal{L}P\psi\right\|_{L^2(D)} \lesssim \|\psi\|_{L^2(\partial D)}$$

(where $\mathcal{L}$ is an arbitrary smooth $(1,0)$-vectorfield). The L^2-result is proven by means of a $T1$-theorem of Christ and Journé. By [10] it now follows that

$$\|P\psi\|_{T^p_\infty} \lesssim \|\psi\|_{L^p(\partial D)}, \quad p > 1, \tag{8}$$

and

$$\|r\mathcal{L}P\psi\|_{T^p_2} \lesssim \|\psi\|_{L^p(\partial D)}, \quad p > 1. \tag{9}$$

The inequality (7) follows from (8) and (9).

In section 4 we use completed tensor products of locally convex Hausdorff spaces, see e.g. Appendix 1 in [9]. Suppose that E and F are locally convex Hausdorff spaces. We denote by $L(E,F)$ the space of all continuous and linear maps from E to F. The topology π on $E \otimes F$ is defined as the finest locally convex topology such that the canonical bilinear map $E \times F \to E \otimes F$ is continuous. We denote by $E \otimes_\pi F$, the space $E \otimes F$ with the topology π and we denote the completion of $E \otimes_\pi F$ with $E \hat{\otimes}_\pi F$. There is another topology on $E \otimes F$, the topology ϵ; in case E is nuclear this topology coincides with the topology π and we therefore omit the index π in this case. The Fréchet space $\mathcal{E}(\mathbb{C}^n)$ is nuclear and we have the isomorphism $\mathcal{E}\left(\mathbb{C}^n, E\right) \cong \mathcal{E}\left(\mathbb{C}^n\right) \hat{\otimes} E$.

3. Property $(\beta)_{\mathcal{E}}$ for Toeplitz operators with H^∞-symbol on H^p

First we need to define the sequence (1) and prove that there is a continuous $C^\infty(\mathbb{C}^n)$-functional calculus on each of the spaces B_k.

Define the norms $\|\cdot\|_{B_k}$, $k \geq 0$, by

$$\|u\|_{B_0} = \|u\|_{T^p_\infty} + \left\|r^{1/2} du\right\|_{T^p_2} + \left\|r\partial\bar{\partial}u\right\|_{T^p_1} \tag{10}$$

on $C^\infty(\bar{D})$,

$$\|u\|_{B_1} = \left\|r^{1/2} u\right\|_{T^p_2} + \|r du\|_{T^p_1} \tag{11}$$

on $C^\infty_{0,1}(\bar{D})$ and

$$\|u\|_{B_k} = \left\|r^{k/2} u\right\|_{T^p_1} + \left\|r^{k/2+1/2}\bar{\partial}u\right\|_{T^p_1} \tag{12}$$

on $C^\infty_{0,k}(\bar{D})$ for $k \geq 2$. Let B_k be the completion of $C^\infty_{0,k}\left(\bar{D}\right)$ with respect to the norm $\|\cdot\|_{B_k}$. We also define B_1' as the completion of $C^\infty_{0,1}(\bar{D})$ with respect to the norm $\|\cdot\|_{B_1'}$, defined by

$$\|u\|_{B_1'} = \left\|r^{1/2} u\right\|_{T^p_1} + \left\|r\bar{\partial}u\right\|_{T^p_1}.$$

The injection $i : H^p \to B_0$ is well defined and continuous by (4) and (5). That $\bar{\partial} : B_k \to B_{k+1}$, $k \geq 0$ is continuous follows immediately from the definitions. Thus we have defined a complex

$$0 \to H^p(D) \xrightarrow{i} B_0 \xrightarrow{\bar{\partial}} B_1 \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} B_m \to 0. \tag{13}$$

Lemma 3.1. *Suppose that $g \in H^\infty(D)^n$. Then one can define $T_{g_i} : B_k \to B_k$ by $T_{g_i}u = g_i u$, $1 \leq i \leq n$, for all $k \geq 0$. The tuple T_g on B_k, $k \geq 0$, has a continuous $C^\infty(\mathbb{C}^n)$-functional calculus and property $(\beta)_{\mathcal{E}}$.*

Proof. That T_{g_i} can be defined on B_k follows from the calculation below (let $\varphi(z) = z_i$ below). We begin with the case $k = 0$. Suppose that $\varphi \in C^\infty(\mathbb{C}^n)$ and $u \in C^\infty(\bar{D})$. From (3) we have

$$\left\|r^{1/2} u \partial g\right\|_{T_2^p} \lesssim \|u\|_{T_\infty^p} \left\|r^{1/2} \partial g\right\|_{T_2^\infty},$$

$$\|r\,|du|\,|\partial g|\|_{T_1^p} \lesssim \left\|r^{1/2} du\right\|_{T_2^p} \left\|r^{1/2} \partial g\right\|_{T_2^\infty}$$

and

$$\left\|ru\,|\partial g|^2\right\|_{T_1^p} \lesssim \|u\|_{T_\infty^p} \left\|r\,|\partial g|^2\right\|_{T_1^\infty}.$$

Since $\left\|r^{1/2}\partial g\right\|_{T_2^\infty} < \infty$ by the inequality (5) we thus get

$$\|(\varphi \circ g) u\|_{B_0} \leq \sup_{z \in g(D)} |\varphi(z)|\,\|u\|_{B_0} + \left\|r^{1/2} d(\varphi \circ g) u\right\|_{T_2^p} +$$

$$\left\|r\bar{\partial}(\varphi \circ g) \wedge \partial u\right\|_{T_1^p} + \left\|r\partial(\varphi \circ g) \wedge \bar{\partial} u\right\|_{T_1^p} + \left\|r\partial\bar{\partial}(\varphi \circ g) u\right\|_{T_1^p} \lesssim$$

$$\sup_{z \in g(D)} \left(|\varphi(z)| + |D\varphi(z)| + \left|D^2\varphi(z)\right|\right) \|u\|_{B_0},$$

where $D\varphi$ and $D^2\varphi$ denotes all derivates of φ of order 1 and 2 respectively. Note that $(\varphi \circ g)u \notin C^\infty(\bar{D})$ in general. Let $g_l \in C^\infty(\bar{D})^n \cap \mathcal{O}(D)^n$ be such that $g_l \to g$ in $H^p(D)^n$ with g_l uniformly bounded as $l \to \infty$ and suppose that u is fixed. We have the equalities

$$d(\varphi \circ g_l - \varphi \circ g) = \sum_i \varphi_i \circ g_l \partial g_l^i - \varphi_i \circ g \partial g^i + \varphi_{\bar{i}} \circ g_l \overline{\partial g_l^i} - \varphi_{\bar{i}} \circ g \overline{\partial g^i}$$

and

$$\partial\bar{\partial}(\varphi \circ g_l - \varphi \circ g) = \sum_{i,j} \varphi_{\bar{i}j} \circ g_l \partial g_l^j \wedge \overline{\partial g_l^i} - \varphi_{\bar{i}j} \circ g \partial g^j \wedge \overline{\partial g^i},$$

where the index in φ_i denotes partial derivate and the upper index in g_l^i and g^i denotes i:th component. Hence we get

$$|d(\varphi \circ g_l - \varphi \circ g)| \leq |D\varphi \circ g_l|\,|\partial g_l - \partial g| + |D\varphi \circ g_l - D\varphi \circ g|\,|\partial g|,$$

and

$$\left|\partial\bar{\partial}(\varphi \circ g_l - \varphi \circ g)\right| \leq \left|D^2\varphi \circ g_l\right| |\partial g_l - \partial g| \left(|\partial g_l| + |\partial g|\right) +$$

$$\left|D^2\varphi \circ g_l - D^2\varphi \circ g\right| |\partial g|^2.$$

By (4) we have

$$\|(\varphi \circ g_l - \varphi \circ g) u\|_{T_\infty^p} + \left\|r^{1/2}(\varphi \circ g_l - \varphi \circ g) du\right\|_{T_2^p}$$

$$\left\|r(\varphi \circ g_l - \varphi \circ g)\partial\bar{\partial} u\right\|_{T_1^p} \lesssim \|\varphi \circ g_l - \varphi \circ g\|_{T_\infty^p} \lesssim \|g_l - g\|_{T_\infty^p} \lesssim$$

$$\|g_l - g\|_{H^p}.$$

We also have that

$$\left\|r^{1/2}d\left(\varphi\circ g_l-\varphi\circ g\right)u\right\|_{T_2^p}+\left\|r\left|d\left(\varphi\circ g_l-\varphi\circ g\right)\right|\left|du\right|\right\|_{T_1^p}\lesssim$$

$$\left\|r^{1/2}d\left(\varphi\circ g_l-\varphi\circ g\right)\right\|_{T_2^p}\lesssim\left\|r^{1/2}\left|D\varphi\circ g_l\right|\left|\partial g_l-\partial g\right|\right\|_{T_2^p}+$$

$$\left\|r^{1/2}\left|D\varphi\circ g_l-D\varphi\circ g\right|\left|\partial g\right|\right\|_{T_2^p}\lesssim\left\|g_l-g\right\|_{H^p}$$

by (3),(4) and (5). Furthermore,

$$\left\|r\partial\bar\partial\left(\varphi\circ g_l-\varphi\circ g\right)u\right\|_{T_1^p}\lesssim\left\|r\left|D^2\varphi\circ g_l\right|\left|\partial g_l-\partial g\right|\left(\left|\partial g_l\right|+\left|\partial g\right|\right)\right\|_{T_1^p}+$$

$$\left\|r\left|D^2\varphi\circ g_l-D^2\varphi\circ g\right|\left|\partial g\right|^2\right\|_{T_1^p}\lesssim\left\|g_l-g\right\|_{H^p}$$

by (3),(4) and (5). Thus

$$\left\|\left(\varphi\circ g_l-\varphi\circ g\right)u\right\|_{B_0}\to 0$$

as $l\to\infty$ and therefore we have that $(\varphi\circ g)u$ is in the completion of $C^\infty(\bar D)$ with respect to the norm $\|\cdot\|_{B_0}$. We extend the map

$$u\mapsto(\varphi\circ g)u:C^\infty(\bar D)\to B_0$$

to a continuous map $\varphi(T_g):B_0\to B_0$, bounded by a constant times

$$\sup_{z\in g(D)}\left(\left|\varphi(z)\right|+\left|D\varphi(z)\right|+\left|D^2\varphi(z)\right|\right).$$

Hence T_g on B_0 has a continuous $C^\infty\left(\mathbb{C}^n\right)$-functional calculus.

Next we consider the case $k=1$. Suppose that $\varphi\in C^\infty(\mathbb{C}^n)$ and $u\in C^\infty_{0,1}(\bar D)$. From (3) and (5) we have the inequality

$$\left\|r\left|\partial g\right|\left|u\right|\right\|_{T_1^p}\lesssim\left\|r^{1/2}\partial g\right\|_{T_2^\infty}\left\|r^{1/2}u\right\|_{T_2^p}\lesssim\left\|r^{1/2}u\right\|_{T_2^p}.$$

Hence we get

$$\|(\varphi\circ g)u\|_{B_1}\leq\sup_{z\in g(D)}\left|\varphi(z)\right|\|u\|_{B_1}+\left\|rd(\varphi\circ g)\wedge u\right\|_{T_1^p}\lesssim$$

$$\sup_{z\in g(D)}\left(\left|\varphi(z)\right|+\left|D\varphi(z)\right|\right)\|u\|_{B_1}.$$

As in the case $k=0$ we prove that $(\varphi\circ g)u$ is in the completion of $C^\infty_{0,1}(\bar D)$. When we extend the map

$$u\mapsto(\varphi\circ g)u:C^\infty(\bar D)\to B_1$$

by continuity to a map $\varphi(T_g):B_1\to B_1$ bounded by

$$\sup_{z\in g(D)}\left(\left|\varphi(z)\right|+\left|D\varphi(z)\right|\right)$$

and hence we have proved that T_g on B_1 has a $C^\infty\left(\mathbb{C}^n\right)$-functional calculus.

In case $k \geq 2$ we suppose that $\varphi \in C^\infty(\mathbb{C}^n)$ and $u \in C^\infty_{0,k}(\bar{D})$. Since $|\partial g| \lesssim r^{-1/2}$ we have

$$\|(\varphi \circ g)u\|_{B_k} \leq \sup_{z \in g(D)} |\varphi(z)| \, \|u\|_{B_k} + \left\| r^{k/2+1/2} \bar{\partial}(\varphi \circ g) \wedge u \right\|_{T_1^p} \lesssim$$

$$\sup_{z \in g(D)} \left(|\varphi(z)| + |D\varphi(z)| \right) \|u\|_{B_k} .$$

As in the case $k = 0$ it follows that T_g on B_k, $k \geq 2$, has a $C^\infty(\mathbb{C}^n)$-functional calculus.

That each of the tuples T_g has property $(\beta)_{\mathcal{E}}$ now follows from Proposition 6.4.13 in [9]. $\square$

We can extend the integral operator $K : C^\infty_{0,k+1}(\bar{D}) \to C_{0,k}(\bar{D})$, $k \geq 1$, to a continuous operator $K : B_{k+1} \to B_k$, $k \geq 2$, and a continuous operator $K : B_2 \to B_1'$. This because

$$\left\| r^{k/2} K u \right\|_{T_1^p} \lesssim \left\| r^{k/2+1/2} u \right\|_{T_1^p} \leq \|u\|_{B_{k+1}} \tag{14}$$

and

$$\left\| r^{k/2+1/2} \bar{\partial} K u \right\|_{T_1^p} = \left\| r^{k/2+1/2} (u - K\bar{\partial} u) \right\|_{T_1^p} \lesssim \|u\|_{B_{k+1}}$$

for all $u \in C^\infty_{0,k+1}(\bar{D})$ by (6), (12) and (14). Also observe that Ku is in the completion of $C^\infty_{0,k}(\bar{D})$ under the norm $\|\cdot\|_{B_k}$ (or $\|\cdot\|_{B_1'}$) by dominated convergence and the fact that one can find $f_l \in C^\infty_{0,k}(\bar{D})$ such that $f_l \to Ku, \bar{\partial} f_l \to \bar{\partial} K u$ pointwise and $|f_l|, \left|\bar{\partial} f_l\right| \lesssim 1$ (as $Ku, \bar{\partial} K u \in C(\bar{D})$). Approximation in B_{k+1} yields that $\bar{\partial} K u + K \bar{\partial} u = u$ for all $u \in B_{k+1}$, $k \geq 1$. Thus the complex (13) is exact in higher degrees.

Extend $K : C^\infty_{0,1}(\bar{D}) \to C(\partial D)$ to continuous maps $K : B_1 \to L^p(\partial D)$ and $K : B_1' \to L^p(\partial D)$, which is possible by (6) and (7). Define the $(1,0)$-vector field $\mathcal{L}$ by the equation

$$\mathcal{L} = \chi \sum |\partial r|^{-2} \frac{\partial r}{\partial \bar{z}_k} \frac{\partial}{\partial z_k},$$

where χ is equal to 1 in a neighbourhood of ∂D and 0 on the set where $\partial r = 0$. Suppose that $u \in C^\infty(\bar{D})$ and let $f = \bar{\partial} u$. By integration by parts we have

$$\int_{\partial D} uh = \int_D f \wedge h =: V(f,h)$$

and

$$\int_{\partial D} uh = \int_D f \wedge h = \int_D O(r) f \wedge h + \int_D r \mathcal{L}(f \wedge h) =: W(f,h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$. We extend V to elements f in B_1' and W to elements in B_1. We say that the equation $\bar{\partial}_b u = f + f'$, where $u \in L^p(\partial D)$, $f \in B_1$ and $f' \in B_1'$, holds if and only if

$$\int_{\partial D} uh = W(f,h) + V(f',h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar\partial h = 0$.

Lemma 3.2. *If $f \in B_1$, $f' \in B_1'$ and $\bar\partial f + \bar\partial f' = 0$ then $u = Kf + Kf'$ solves the equation $\bar\partial_b u = f + f'$. Moreover, if $\varphi \in H^\infty(D)$ then $\bar\partial_b(\varphi u) = T_\varphi f + T_\varphi f'$.*

Proof. Suppose that $f, f' \in C^\infty_{0,1}(\bar{D})$. Since $\bar\partial K(f+f') + K\bar\partial(f+f') = f + f'$ we have

$$\int_{\partial D} (Kf + Kf')h = W(f,h) + V(f',h) - \int_D K(\bar\partial f + \bar\partial f') \wedge h \tag{15}$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar\partial h = 0$. For fixed h, we can estimate each term of the above equality by a constant times $\|f\|_{B_1} + \|f'\|_{B_1'}$. Thus approximation in B_1 and B_1' yields that if $f \in B_1$ and $f' \in B_1'$ then

$$\int_{\partial D} uh = W(f,h) + V(f',h) - \int_D K(\bar\partial f + \bar\partial f') \wedge h$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar\partial h = 0$. Hence the equation $\bar\partial_b u = f + f'$ holds since we also have that $\bar\partial f + \bar\partial f' = 0$. Suppose that $\varphi_k \in C^\infty(\bar{D}) \cap \mathcal{O}(D)$ are chosen such that $\varphi_k \to \varphi$ in $H^1(D)$. Replace h in (15) by $\varphi_k h$ and approximate to get

$$\int_{\partial D} \varphi\,(Kf + Kf')\,h = W(f, h\varphi) + V(f', h\varphi) - \int_D \varphi K(\bar\partial f + \bar\partial f') \wedge h$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar\partial h = 0$, if $f, f' \in C^\infty_{0,1}(\bar{D})$. We estimate the terms to the right,

$$|W(f,h\varphi)| \lesssim \int_D r^{3/2}\,|f|\,|\varphi|\,r^{-1} + \int_D r\,|\partial f|\,|\varphi|\,r^{-1} + \int_D r\,|f|\,|\partial\varphi|\,r^{-1} \lesssim$$

$$\|f\|_{B_1}\,\|\varphi\|_{H^{p'}}\,,$$

$$|V(f',h\varphi)| \lesssim \int_D r^{1/2}\,|f'|\,|\varphi|\,r^{-1} \lesssim \|f'\|_{B_1'}\,\|\varphi\|_{H^{p'}}$$

and

$$\left|\int_D \varphi K(\bar\partial f + \bar\partial f') \wedge h\right| \lesssim \left\|r^{1/2} K\left(\bar\partial f + \bar\partial f'\right)\right\|_{T_1^p} \|\varphi\|_{T_\infty^{p'}} \lesssim$$

$$\left\|\bar\partial f + \bar\partial f'\right\|_{B_2} \|\varphi\|_{H^{p'}} \lesssim \left(\|f\|_{B_1} + \|f'\|_{B_1'}\right) \|\varphi\|_{H^{p'}}$$

for fixed h by (2), (4) and (5). Hence approximation in B_1 and B_1' yields that

$$\int_{\partial D} u\varphi h = W(T_\varphi f, h) + V(T_\varphi f', h)$$

for all $f \in B_1$, $f' \in B_1'$ such that $\bar\partial f + \bar\partial f' = 0$ and $h \in C^\infty_{m,m-1}(\bar D)$ such that $\bar\partial h = 0$. $\square$

Next we prove that functions in B_0 has boundary values in $L^p(\partial D)$.

Lemma 3.3. *There is a continuous and linear operator $u \mapsto u^*$ from B_0 to $L^p(\partial D)$ such that u^* is the restriction of u to ∂D if $u \in C^\infty(\bar D)$ and $(T_f u)^* = f^* u^*$ if $f \in H^\infty(D)$.*

Proof. Suppose that $u \in C^\infty(\bar D)$. Then $\|u\|_{L^p(\partial D)} \leq \|u\|_{B_0}$ and hence the restriction operator can be extended to a continuous operator from B_0 to $L^p(\partial D)$. Suppose that $u \in B_0$ and $f \in H^\infty(D)$. Let $u_l \in C^\infty(\bar D)$ and $f_k \in C^\infty(\bar D) \cap \mathcal{O}(D)$ be such that $u_l \to u$ in B_0 and $f_k \to f$ in $H^p(D)$ with f_k uniformily bounded. Then

$$\|f^*u^* - (T_f u)^*\|_{L^p(\partial D)} \lesssim \|f^*u^* - f^*u_l^*\|_{L^p(\partial D)} + \|f^*u_l^* - f_k^*u_l^*\|_{L^p(\partial D)} +$$

$$\|(f_k u_l)^* - (f u_l)^*\|_{L^p(\partial D)} + \|(f u_l)^* - (T_f u)^*\|_{L^p(\partial D)} \to 0$$

if one first let $k \to \infty$ and then $l \to \infty$. $\square$

Note that if $u \in B_0$ then

$$\int_{\partial D} u^* h = W(\bar\partial u, h) \tag{16}$$

for all $h \in C^\infty_{m,m-1}(\bar D)$ such that $\bar\partial h = 0$ by approximation in B_0 and Lemma 3.3.

Proof of Theorem 1.1

We want to prove that the complex $K_\bullet\,(z - T_g, \mathcal{E}\,(\mathbb{C}^n, H^p))$ has vanishing homology groups of positive order and that

$$\sum_i (z_i - T_{g_i})\,\mathcal{E}\,(\mathbb{C}^n, H^p)$$

is closed in $\mathcal{E}\,(\mathbb{C}^n, H^p)$.

Suppose that $u^k \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$ and that $\delta_{z-g} u^k \to u_0$ in $\mathcal{E}\,(\mathbb{C}^n, H^p)$. By Lemma 3.1 there is a $u_1 \in K_1(z - T_g, \mathcal{E}\,(\mathbb{C}^n, B_0))$ such that $iu_0 = \delta_{z-T_g} u_1$. Again by Lemma 3.1 we can recursively find $u_i \in K_i(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-1}))$ such that $\delta_{z-T_g} u_{i+1} = \bar\partial u_i$ for $i \geq 1$. Then we have that $\bar\partial u_{m+1} = 0$. Define $v_{m+1} \in K_{m+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{m-2}))$ by $v_{m+1} = K u_{m+1}$. Recursively define v_i, $i \geq 2$, by $v_i = K u_i - K\delta_{z-T_g} v_{i+1}$. Thus $v_i \in K_i(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-2}))$ if $i \geq 4$, $v_3 \in \Lambda^3\mathcal{E}(\mathbb{C}^n, B_1')$ and the equation $\bar\partial v_i = u_i - \delta_{z-T_g} v_{i+1}$ holds for $i \geq 3$. Furthermore

$v_2 \in \Lambda^2\mathcal{E}(\mathbb{C}^n, L^p(\partial D))$ satisfies the equation $\bar{\partial}_b v_2 = u_2 - \delta_{z-T_g} v_3$ by Lemma 3.2.

Let $u_1' = u_1^* - \delta_{z-g^*} v_2$. By Lemma 3.2 we have that $\bar{\partial}_b \delta_{z-g^*} v_2 = \delta_{z-T_g} u_2$ and thus

$$\int_{\partial D} \delta_{z-g^*} v_2 h = W(\delta_{z-T_g} u_2, h)$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$. Since by equation (16)

$$\int_{\partial D} u_1^* h = W(\bar{\partial} u_1, h)$$

we have proved that

$$\int_{\partial D} u_1' h = 0$$

for all $h \in C^\infty_{m,m-1}(\bar{D})$ such that $\bar{\partial} h = 0$. Thus $U_1' \in K(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$, where U_1' is the unique holomorphic extension of u_1'. Since $u_0 = \delta_{z-T_g} U_1'$ by Lemma 3.3 we have proved that

$$\sum_i (z_i - T_{g_i})\, \mathcal{E}(\mathbb{C}^n, H^p)$$

is closed in $\mathcal{E}(\mathbb{C}^n, H^p)$.

Suppose that $u_k \in K_k(z - T_g, \mathcal{E}(\mathbb{C}^n, H^p))$ is δ_{z-T_g}-closed. Then there is a $u_{k+1} \in K_{k+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ such that $u_k = \delta_{z-T_g} u_{k+1}$. Let $u_{i+1} \in K_{i+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_{i-k}))$ solve the equation $\delta_{z-T_g} u_{i+1} = \bar{\partial} u_i$. Then we have that $\bar{\partial} u_{m+k+1} = 0$. Let $v_{m+k+1} = K u_{m+k+1}$ and $v_i = K u_i - K \delta_{z-T_g} v_{i+1}$. Thus $\bar{\partial} v_i = u_i - \delta_{z-T_g} v_{i+1}$ and $\bar{\partial}_b v_{k+2} = u_{k+2} - \delta_{z-T_g} v_{k+3}$ since $\bar{\partial}(u_i - \delta_{z-Tg} v_{i+1}) = 0$. Define u_{k+1}' by the equation $u_{k+1}' = u_{k+1}^* - \delta_{z-T_g} v_{k+2}$. As in the case above we see that U_{k+1}' is a solution of the equation $u_k = \delta_{z-T_g} U_{k+1}'$, and hence the theorem is proved.

□

We now prove the analogue of Theorem 1.1 with the Hardy space replaced by the Bergman space. In the case of when g has bounded derivate this is proved in Theorem 8.1.5 in [9].

Corollary 3.4. *Suppose that D is a bounded strictly pseudoconvex domain in $\mathbb{C}^m$ with C^∞-boundary and that $g \in H^\infty(D)^n$. Then the tuple T_g of Toeplitz operators on the Bergman space $\mathcal{O}L^p(D)$, $1 \le p < \infty$, satisfies property $(\beta)_{\mathcal{E}}$ and Bishop's property (β).*

Proof. Let ρ be a strictly plurisubharmonic defining function for D and let $\tilde{D} = \left\{(v, w) \in \mathbb{C}^{m+1} : \rho(v) + |w|^2 < 0\right\}$. Define the operators $P : H^p(\tilde{D}) \to \mathcal{O}L^p(D)$ and $I : \mathcal{O}L^p(D) \to H^p(\tilde{D})$ by $Pf(v) = f(v, 0)$ and $If(v, w) = f(v)$ respectively. The operator P is continuous by the Carleson-Hörmander inequality since the measure with mass uniformly distributed on $\tilde{D} \cap \{w = 0\}$ is a Carleson measure. The operator I is continuous since

$$\int_{\partial \tilde{D}} |f(v)|^p \, \sigma(v, w) \sim \lim_{\varepsilon \to 0} \varepsilon \int_{\tilde{D}} \left(-\rho(v) - |w|^2\right)^{\varepsilon - 1} |f(v)|^p \sim$$

$$\lim_{\varepsilon \to 0} \int_D (-\rho(v))^{\varepsilon} \, |f(v)|^p = \int_D |f(v)|^p \,,$$

where σ is the surface measure. Let $\tilde{g}(v, w) = g(v)$. Then $T_{\tilde{g}}$ has property $(\beta)_{\mathcal{E}}$ and since $PI = id, T_{\tilde{g}} I = IT_g$ and $PT_{\tilde{g}} = T_g P$ it is easy to see that T_g has property $(\beta)_{\mathcal{E}}$. □

4. Property $(\beta)_{\mathcal{E}}$ for Toeplitz operators with H^∞-symbol on the unit disc

In this section we will use the Euclidean norm. Let $r(w) = 1 - |w|^2$ and let D be the unit disc in $\mathbb{C}$. Let B_0 be the Banach space of all functions $u \in L^\infty(D)$ such that

$$\|u\|_{B_0} = \|u\|_{L^\infty(D)} + \|rdu\|_{L^\infty(D)} + \|rdu\|_{T_2^\infty} + \left\|r^2 \partial \bar{\partial} u\right\|_{T_1^\infty} < \infty.$$

Since $\|rdu\|_{L^\infty(D)} < \infty$, B_0 consists of continuous functions on D. We define B_1 as the Banach space of all locally integrable $(0, 1)$-forms u such that

$$\|u\|_{B_1} = \|ru\|_{L^\infty(D)} + \|ru\|_{T_2^\infty} + \left\|r^2 \partial u\right\|_{T_1^\infty} < \infty.$$

Suppose that $u \in C^\infty(\bar{D})$ and $h \in C^\infty(\partial D)$. Then the Wolff trick (see the proof of Theorem 1.1) yields

$$\int_{\partial D} uhdw = \int_D \bar{\partial}(uPhdw) =$$

$$\int_D O(r) \bar{\partial}(uPhdw) + \int_D r\mathcal{L}\bar{\partial}(uPhdw) := S(u, h),$$

where Ph is the Poisson integral of h.

As in Section 3 we need to know that functions in B_0 has well defined boundary values.

Lemma 4.1. *If $u \in B_0$ then there is a $u^* \in L^\infty(\partial D)$ such that*

$$\int_{\partial D} u^* hdw = S(u, h)$$

for all $h \in L^2(\partial D)$ and $(fu)^* = f^* u^*$ if $f \in H^\infty(D)$.

Proof. We have the estimate

$$|S(u, h)| \lesssim \|u\|_{B_0} \|h\|_{L^2(\partial D)} \,.$$

Hence there is a function $u^* \in L^2(\partial D)$ such that

$$\int_{\partial D} u^* h dw = S(u, h)$$

for all $h \in L^2(\partial D)$. Suppose that $h \in C^\infty(\partial D)$. Let u_t be the dilation $u_t(w) = u(tw)$. Since

$$|S(u_t - u, h)| \lesssim \int_D |u_t - u| + \int_D r\, |d(u_t - u)|^2 + \int_D r\, \left|\partial\bar\partial(u_t - u)\right|$$

for fixed h we have that

$$\int_{\partial D} u_t^* h dw \to \int_{\partial D} u^* h dw$$

as $t \nearrow 1$. Therefore $\|u^*\|_{L^\infty(\partial D)} \leq \|u\|_{B_0}$ since u_t^* is uniformly bounded by $\|u\|_{L^\infty(D)}$. Let $f_s(w) = f(sw)$ be the dilation of f. Then we have that

$$\int_{\partial D} f_s^* u_t^* h dw = \int_{\partial D} (f_s^* - f^*) u_t^* h dw + \int_{\partial D} f^* u_t^* h dw \to \int_{\partial D} f^* u^* h dw$$

as $s, t \nearrow 1$, by dominated convergence. Since we also have

$$\int_{\partial D} (fu)_t^* h dw \to \int_{\partial D} (fu)^* h dw$$

as $t \nearrow 1$ we see that $(fu)^* = f^* u^*$. $\square$

Let

$$W(u, h) = \int_D O(r) u \wedge h dw + \int_D r \mathcal{L}(u \wedge h dw)$$

for $u \in B_1$ and $h \in H^1$, where $O(r)$ is the same $O(r)$ as in the definition of $S(u, h)$.

Lemma 4.2. *If $f \in \mathcal{E}(\mathbb{C}^n, B_1)$ then there is a $u \in \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$ such that $\bar\partial_b u = f$, that is*

$$\int_{\partial D} u(z) h dw = W(f(z), h)$$

for all $h \in H^1(D)$ and $z \in \mathbb{C}^n$.

Proof. Consider the bilinear map $W : B_1 \times H^1 \to \mathbb{C}$. This map is continuous since we have the estimate

$$|W(f, h)| \lesssim \|u\|_{B_1} \|h\|_{H^1},$$

which is used in Wolff's proof of the corona theorem. By the universal property for π-tensor products (see 41.3.(1) in [13]) there is a corresponding linear and continuous map W_1 from $B_1 \hat\otimes_\pi H^1$ to $\mathbb{C}$. Since

$$\mathcal{E}(\mathbb{C}^n, B_1) \cong \mathcal{E}(\mathbb{C}^n) \hat\otimes B_1 \cong L(\mathcal{E}'(\mathbb{C}^n), B_1)$$

by Appendix 1 in [9], $f \otimes id$ is a continuous map $\mathcal{E}'(\mathbb{C}^n) \hat\otimes H^1 \to B_1 \hat\otimes_\pi H^1$. Compose with the map W_1 to get a continuous functional on

$\mathcal{E}'(\mathbb{C}^n)\hat{\otimes}H^1$. The injection $\mathcal{E}'(\mathbb{C}^n)\hat{\otimes}H^1 \to \mathcal{E}'(\mathbb{C}^n)\hat{\otimes}L^1(\partial D)$ is a topological monomorphism, and hence we can extend with Hahn-Banach Theorem to a continuous functional on $\mathcal{E}'(\mathbb{C}^n)\hat{\otimes}L^1(\partial D)$. Since the dual of $\mathcal{E}'(\mathbb{C}^n)\hat{\otimes}L^1(\partial D)$ is isomorphic to $\mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$ by Theorem A1.12 in [9] we have a $u \in \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$. If $h \in H^1$ then

$$\int u(z)hdw = W(f(z), h)$$

and thus u is a solution to the equation $\bar{\partial}_b u = f$ in the sense of this lemma. $\square$

Theorem 4.3. *Let D be the unit disc in $\mathbb{C}$ and suppose that $g \in H^\infty(D)^n$. Then the tuple T_g of Toeplitz operators on $H^\infty(D)$ satisfies property $(\beta)_\mathcal{E}$, and thus Bishop's property (β).*

Proof. The tuple T_g considered as operators on B_0 or B_1 has a $C^\infty(\mathbb{C}^n)$-functional calculus (the proof of this is similar to Lemma 3.1). Hence they satisfies property $(\beta)_\mathcal{E}$ by Proposition 6.4.13 in [9]. Consider the well-defined complex

$$0 \to H^\infty \to B_0 \xrightarrow{\bar{\partial}} B_1 \to 0. \tag{17}$$

Suppose that $u^k \in \sum_i (z_i - T_{g_i})\mathcal{E}(\mathbb{C}^n, H^\infty)$ and $u^k \to u_0$ in $\mathcal{E}(\mathbb{C}^n, H^\infty)$. As T_g on B_0 has property $(\beta)_\mathcal{E}$ there is a $u_1 \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ such that $u_0 = \delta_{z-T_g}u_1$. Since T_g on B_1 has property $(\beta)_\mathcal{E}$, there is a $u_2 \in K_2(z - T_g, \mathcal{E}(\mathbb{C}^n, B_1))$ such that $\delta_{z-T_g}u_2 = \bar{\partial}u_1$. By Lemma 4.2 there is a $v \in \Lambda^2\mathcal{E}(\mathbb{C}^n, L^\infty(\partial D))$ such that

$$\int_{\partial D} vhdw = W(u_2, h)$$

for all $h \in H^1(D)$. Therefore we have that

$$\int_{\partial D} \delta_{z-g^*}vhdw = W(\delta_{z-T_g}u_2, h)$$

for all $h \in H^1(D)$. Define $u_1' \in K_1(z - g^*, \mathcal{E}(\mathbb{C}^n, L^\infty(\partial D)))$ by the equation $u_1' = u_1^* - \delta_{z-g^*}v$. Then

$$\int_{\partial D} u_1' hdw = 0$$

for all $h \in H^1$ since

$$\int_{\partial D} u_1^* hdw = S(u_1, h) = W(\bar{\partial}u_1, h)$$

by Lemma 4.1. Thus $U_1' \in K_1(z - T_g, \mathcal{E}(\mathbb{C}^n, H^\infty))$, where U_1' is the holomorphic extension. Since $u_0 = \delta_{z-T_g}U_1'$ by Lemma 4.1 we have proved that $\delta_{z-T_g}K_1(z - g, \mathcal{E}(\mathbb{C}^n, H^\infty))$ is closed.

Suppose that $u_k \in K_k(z - T_g, \mathcal{E}(\mathbb{C}^n, H^\infty))$ is δ_{z-T_g}-closed. Then there is a solution $u_{k+1} \in K_{k+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, B_0))$ to the equation $\delta_{z-T_g}u_{k+1} = u_k$ since T_g on B_0 has property $(\beta)_\mathcal{E}$. Continuing in exactly the same way as above we see that we can replace u_{k+1} with $U'_{k+1} \in K_{k+1}(z - T_g, \mathcal{E}(\mathbb{C}^n, H^\infty))$ such that $\delta_{z-T_g} U'_{k+1} = u_k$. Thus the theorem is proved. $\square$

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