On the decay of local energy of solutions to nonlinear Klein-Gordon equations

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Abstract

The content of this note was presented at "Linear and nonlinear hyperbolic equations", a conference in honor of Sergio Spagnolo on the occasion of his 60th birthday, held in Grado, Italy in September 2001.

The Klein-Gordon equation is the equation for relativistic wave-propagation

$$\text{(KG)} \qquad \begin{aligned} \partial_t^2 u_o - \Delta u_o + m^2 u_o = 0 \quad & x \in \mathbf{R}^n, t \geq 0 \\ u_o|_0 = \varphi,\ \partial_t u_o|_0 = \psi \quad & x \in \mathbf{R}^n \end{aligned}$$

where $m > 0, \Delta = \sum_{j=1}^n \partial_{x_j}^2$, $(n \geq 3)$. The nonlinear counterpart, extensively studied since early 1960's, is

$$\text{(NLKG)} \qquad \begin{aligned} \partial_t^2 u - \Delta u + m^2 u + f(u) = 0 \quad & x \in \mathbf{R}^n, t \geq 0 \\ u|_0 = \varphi,\ \partial_t u|_0 = \psi \quad & x \in \mathbf{R}^n \end{aligned}$$

where $f(u)$ is a nonlinear function, $f(u) \cong |u|^{\rho-1}u$; modified at 0 if necessary, to be smooth enough, and with

$$1 + \tfrac{4}{n} < \rho < \tfrac{n+2}{n-2} = \rho^*$$

The conditions on f will be made precise below.

Energy:We will assume that $F(u) = \int_0^u f(v)dv \geq 0$. The energy

$$E(t) = \tfrac{1}{2} \int (|\partial_x u|^2 + |\partial_t u|^2 + m^2|u|^2)dx + \int F(u)dx$$

is a conserved quantity, $E(t) = E(0)$.Let $X_e = H_2^1 \times L_2$ with norm $\|.\|_e$ defined by

$$\|u(t)\|_e^2 = \|u(t)\|_{H^1}^2 + \|\partial_t u(t)\|_{L_2}^2$$

Our assumptions on ρ and f imply that

$$E(t) \leq C\|u(t)\|_e^2$$

Correspondingly, for $\Omega \subset \mathbf{R^n}$ we defind the local energy and energy norm by

$$E_\Omega(t) = \tfrac{1}{2}\int_\Omega (|\partial_x u|^2 + |\partial_t u|^2 + m^2|u|^2 + 2F(u))dx$$

and $X_e(\Omega)$ is defined by replacing the global spaces and norms in the definition of X_e by the corresponding local spaces and norms. The local energy as defined is no longer a conserved quantity (to get conservation of energy, one has to work with the whole surface of the light cone).

Global estimates in space-time (**Strichartz estimate**) for the KG (Strichartz [19], Segal [15], ...). If the data φ, ψ belong to X_e, then

$$\|u_o\|_{L_p(L_p^{\frac{1}{2}})} \leq C(\|\varphi\|_{H^1} + \|\psi\|_{L_2}) \leq C\|u_o(0)\|_e$$

where $p \geq 2$, $\delta_p = \frac{1}{2} - \frac{1}{p} = \frac{1}{n+1}$. More general, but also much more complex, estimates that bound u_0 in $L_q(\mathbf{R}, H_p^s(\mathbf{R}^n))$ are available (Strichartz [19], Marshall-Strauss-Wainger [11], Brenner [2]; a good exposition is given by Ginibre and Velo 1995 [6]) . One such example we will use is that if the data belong to X_e then

$$u_o \in L_2(\mathbf{R}, H_p^\gamma(\mathbf{R}^n)),$$
$$where\ \delta = \tfrac{1}{2} - \tfrac{1}{p} \in (\tfrac{1}{n}, \tfrac{1}{n-1})$$
$$and\ \gamma = \tfrac{1}{2} - \tfrac{1}{n-1}.$$

Space-time integrals of solutions of NLKG. Let u_o be a solution of KG with the same data at $t = 0$ as u, the solution of NLKG. Assume that the data has finite energy (i.e. $u(0), \partial_t u(0)$ belongs to X_e). Then one example of a Strichartz-type estimate for the NLKG due to the author [4] is that if

(1)
$$\delta = \tfrac{1}{2} - \tfrac{1}{p}\ , 0 \leq \sigma \leq s \leq 1\ , \theta \in (0,1]\ ,$$
$$\frac{s-\sigma}{1-\sigma} < \rho - 1\ , (n+1+\theta) \leq 1 + s - \sigma,$$
$$(n-1-\theta)\delta < 1 < (n-1+\theta)\delta$$

then

$$if\ u_o \in L_q(\mathbf{R}, H_p^\sigma(\mathbf{R}^n))\ then\ u \in L_q(\mathbf{R}, H_p^\sigma(\mathbf{R}^n))$$

In view of our previous example, if the data belong to X_e, then the conditions above are satisfied for $0 \leq \sigma \leq \gamma$ and $\frac{1}{n} < \delta < \frac{1}{n-1}$, and hence

$$u \in L_2(\mathbf{R}, \mathbf{H}^\gamma_\mathbf{p}(\mathbf{R}^\mathbf{n})).$$

The following is a result on (local) energy decay: Let $\Omega_t = \{\epsilon(t)t \leq |x| \leq (1-\epsilon(t))t\}$, where $0 < \epsilon(t) < 1$, $\epsilon(t) \to 0$, *as* $t \to \infty$.Let $Y_t = H^1_2(\mathbf{R}^n \setminus \Omega_t)$. Then

Energy decay (Strichartz 1981 [20]). *Let u_o be a finite energy solution of the Klein Gordon equation.Then*

$$\|u_o(t)\|_{Y_t} \to 0 \ , as \ t \to \infty$$

Earlier Morawetz (1968) [12] proved a result about Energy decay on compact subsets Ω of $\mathbf{R}^\mathbf{n}$:

Local Energy Decay. *Let n =3 and assume that u is a solution of the NLKG, which is locally a classical solution. Then*

$$E_\Omega(t) \in L_1 \ and \ E_\Omega(t) \to 0 \ as \ t \to \infty$$

In particular,

$$\|u(t)\|_{L_2(\Omega)} \in L_2 \ and \ tends \ to \ 0 \ as \ t \to \infty$$

We will study the behavior of u(t), a solution of NLKG with finite energy data. Let Ω be a compact subset of $\mathbf{R}^\mathbf{n}$. Then

$$\|u(t)\|_{L_2(\Omega)} \leq |\Omega|^\delta \|u(t)\|_{L_p(\Omega)}$$

where as before, $\delta = \frac{1}{2} - \frac{1}{p}$. If $\frac{1}{n} < \delta < \frac{1}{n-1}$ then as mentioned above $\|u(t)\|_{L_p} \in L_2$, which proves that

$$\|u(t)\|_{L_2(\Omega)} \in L_2$$

In addition, we may use the existence of solutions of the Klein-Gordon equation approximating u in the following sense

Scattering (Brenner 1983-86 [2], [3], [4]). *There exists an everywhere defined scattering operator on X_e for the NLKG.*

In particular there is a solution u_+ of the Klein-Gordon equation with finite energy such that

$$\|u(t) - u_+(t)\|_e \to 0 \ , as \ t \to \infty$$

We conclude, using that $u_+(t)$ is uniformly continuous in H^1_2, that u is uniformly continuous in $L_2(\Omega)$. The integrability then implies that

$$\|u(t)\|_{L_2(\Omega)} \to 0 \ as \ t \to \infty$$

Notice that this result, as well as the extension of the Energy Decay Theorem to the solution of the NLKG, follows from that theorem and the Scattering Theorem above. We have

$$\|u(t)\|_{Y_t} \to 0 \ , as \ t \to \infty$$

where again $Y_t = H^1_2(\mathbf{R}^n \setminus \Omega_t)$, with $\Omega_t = \{\epsilon(t)t \leq |x| \leq (1-\epsilon(t))t\}$, where $0 < \epsilon(t) < 1$, $\epsilon(t) \to 0$, $as \ t \to \infty$.

There has been a number of results on pointwise decay in $L_p(\mathbf{R^n})$ of solutions of the NLKG over the years by e.g. Strauss 1968 [17], Morawetz and Strauss 1972 [13],Pecher 1974 [14] 1976, Brenner 1981-1985 (see e.g. [3]) ...

Pointwise L_p Decay (Brenner [4]). . *Let u be a solution of NLKG with sufficiently nice data (data which have sufficiently many derivatives in L_1 , say).Then for $\delta = \frac{1}{2} - \frac{1}{p} < \min(\frac{1}{n-1}, \frac{\rho-1}{4})$, $p \geq 2$, then*

$$\|u(t)\|_{H^1_p} \leq C(1+t)^{-n\delta}$$

The work of Grillakis ([8], [9]) on classical solutions for the nonlinear wave equation for critical exponents $\rho = \rho^*$ proves that we may use $\delta = \frac{1}{2}$ for n=3 (and $p = \infty$).The choice $\delta = \frac{1}{n-1}$ probably also holds for $n \leq 6$, even in the critical case $\rho = \rho^*$. Using the Pontwise L_p decay and our previous estimate of u in $L_p(\Omega)$, we get

Local Pointwise L_p Decay. *Let $n \geq 3$, and let $\delta < \frac{1}{n-1}$, with equality for n = 3.Then*

$$\|u(t)\|_{H^1_2(\Omega)} \leq C(\frac{|\Omega|^{\frac{1}{n}}}{1+t})^{n\delta}$$

for sufficiently "nice" data.

Example: Let $\Omega = \Omega(t)$ with $|\Omega(t)| \leq t^{\alpha n}$ with$0 \leq \alpha < 1$. Then

$$\|u(t)\|_{H^1_2(\Omega(t))} \leq Ct^{-(1-\alpha)n\delta}$$

Notice that only the size, not the actual position of Ω is involved.

The largest δ-value for which we get decay results for solutions of the NLKG is determined by the singularities of the map

$$L_p \ni v \to E(t)v \in L_p, \ 1/p + 1/p^* = 1$$

where the solution of the Klein-Gordon equation is given by u(t) = F(t)ϕ +E(t)ψ, with $\phi = u(0)$, $\psi = \partial_t u(t)$. Much indicates that $\delta \leq \frac{1}{n-1}$, including results on classical solutions for critical case exponents ,$\rho = \rho^*$. Notice also that the maximal rate of decay for the solutions of the Klein-Gordon equation seems to be $\mathcal{O}(t^{-n/2})$ in $L_p(\Omega), \Omega \subset\subset \mathbf{R}^n$, $p \geq 2$. More should not be expected for sets $\Omega \subset \{|x| \leq t\}$, in contrast to the rapid decay of solutions of the wave equation in sets away from the light cone.

The following gives an example of a case when the maximal rate of the long range mean value is attained. Let $X = L_p(\mathbf{R}^n)$ where (1) holds and where

$$\mathcal{M}^t_{q,X} v(t) = (\tfrac{1}{t} \int_t^{2t} \|v(\tau)\|^q d\tau))^{\frac{1}{q}}$$

Theorem (Brenner, to appear). *Assume that $u \in X_e$ is a solution of the NLKG, and let u_o be corresponding solution of the Klein-Gordon equation. Let (1) be satisfied, and assume that $\mathcal{M}^t_{q,X} u_o(t)$ has maximal rate of decay. Then $\mathcal{M}^t_{q,X} u(t)$ also attains the maximal rate of decay, that is decays as $\mathcal{O}(t^{-n\delta})$.*

Similar results hold in the other cases when Strichartz' estimates are known to hold for the NLKG.

A local version of the long range estimates is (work in progress):Let $X = L_p(\Omega)$ $X' = L_p(\Omega')$ and let $\mathcal{M}^t_{q,X}$ be defined as above.Then

Local $L_q(L_p)$ Decay. *Let $\Omega \subset\subset \Omega' \subset\subset \mathbf{R}^n$, $n \geq 3$ and assume that $distance(\partial\Omega', \Omega) \geq t^\epsilon$ for some $\epsilon > 0$. Assume that (1) holds.Assume also that u_o has either maximal decay in $L_q(X')$ or else has uniform decay in this space. Then*

$$\mathcal{M}^t_{q,X} u(t) \leq$$
$$C\mathcal{M}^t_{q,X'} u_o(t) + \mathcal{O}(t^{-n\delta})(\int_0^t \|u_o(\tau)\|^q_{L_p(\Omega')} d\tau)^{\frac{1}{q}}$$

where u_o is a solution of the Klein-Gordon equation with the same data as u.

A1 Strichartz' estimates

Let u be a solution of the Klein-Gordon Equation with data in $X_e^{1/2} = H_2^{1/2} \times H_2^{-1/2}$.

Let $0 \leq \sigma \leq 1/2$, $2 < r \leq q$ and let $\delta_q = \frac{1}{2} - \frac{1}{q}$ and $\delta_r = \frac{1}{2} - \frac{1}{r}$.

Then

$$u_o \in L_r^{loc}(\mathbf{R}; H_q^{\frac{1}{2}-\sigma}(\mathbf{R}^n))$$

provided

$$
\begin{aligned}
&(\alpha) \quad (n+2)\delta_q &\geq 1+2\sigma \\
&(\beta) \quad n\delta_q + \delta_r &\leq 1+\sigma
\end{aligned}
$$

We may replace L_r^{loc} by L_r (global in time estimate) if in addition

$$(\gamma) \quad \sigma \quad \geq \delta_q - \delta_r \ (\geq 0)$$

If $r = 2$ (that is $\delta_r = 0$) equality in (β) and (γ) is not allowed. This could be handled by using weaker spaces, e.g. Besov spaces of suitable order.

For the wave equation , only (α) and (β) with equality matters (and local = global). In this case we use homogeneous Sobolev norms.

We may also replace L_r by H_r^s if we replace σ *by* $\sigma + s$.

A2 Conditions on the nonlinearity f

Let $f(u) \in C^1$ with $f(\mathbf{R}) \subseteq \mathbf{R}$ and assume that

(i) $F(u) = \int_0^u f(v)dv \geq 0$.

(ii)

$$
\begin{aligned}
|f'(u)| &< |u|^{\rho_0 - 1}, \qquad |u| \leq 1 \\
|f'(u)| &< |u|^{\rho_1 - 1}, \qquad |u| \geq 1
\end{aligned}
$$

where

$$1 + \tfrac{4}{n} < \rho_0,\, \rho_1 < \tfrac{n+2}{n-2} = \rho^*$$

(iii)

$$
\begin{aligned}
uf(u) - 2F(u) \geq \alpha F(u), \ \text{ some } \alpha > 0 \\
\text{and } F \text{ is not flat at 0 or } \infty
\end{aligned}
$$

The last condition ensures that we avoid local concentration of energy

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