Preconditioning by incomplete factorization; an overview and an application to a linear elasticity problem.

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SUMMARY

Preconditioning by incomplete factorization is a well-known method for improving the performance of iterative methods. We give a short history of some main ideas in point incomplete factorizations and block incomplete factorizations. Some recent results and applications are also presented. Specially we consider the solution of the linear elasticity problem, where we also comment on the parallel solution of the preconditioning system.

KEY WORDS: preconditioning; incomplete factorization; conjugate gradient method; elasticity problem; parallel algorithms

1. INTRODUCTION

1.1. Background

We consider the iterative solution of large sparse systems of linear equations $Ax = b$, where A is a symmetric positive definite matrix. In some cases more general matrices will be considered.

It is well-known that the rate of convergence of the iterative methods very much depends on the spectrum of the coefficient matrix. In order to improve the rate of convergence, the linear system has to be transformed by a linear transformation in order to get a matrix with more favourable spectral properties as well as a linear system with the same solution as the original system. The transformation is performed by the matrix called the *preconditioner*.

With C, a symmetric positive definite preconditioner, the rate of convergence of the preconditioned conjugate gradient, PCG method is directed by $\kappa(C^{-1/2}AC^{-1/2})$ and also in some cases by the distribution of the eigenvalues, see for instance the papers by Axelsson and Lindskog [1] and van der Vorst and Sleijpen [2]. The preconditioner can be introduced in the conjugate gradient algorithm by the solution of a linear system $Ch = g$, see for instance the books by Axelsson and Barker [3] and by Golub and van Loan [4].

A good preconditioner C shall be an approximation to A and the system with matrix C shall be much easier to solve than the system with matrix A. Introducing a preconditioner

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in an iterative method will cause an extra cost for the construction and for applying the preconditioner in each iteration. For a preconditioner to be efficient, the rate of convergence must be improved enough so that the total amount of work decreases.

The present paper deals with preconditioners being incomplete LU-factorizations of the matrix A. The basic idea in incomplete factorizations is to allow fill elements only if certain conditions are satisfied, which are given by positions or by values. Before we give an overview of various incomplete factorization preconditioners in the following sections, we will here shortly give some historical remarks and also shortly mention some other frequently used preconditioning techniques. The remainder of the paper is organised as follows: In Section 2 we give some strategies in preconditioning by point incomplete factorization. In Section 3 we compare two recent ideas on point incomplete factorization preconditioning. Some examples of incomplete factorization based on block matrices are given in Section 4. In Section 5 we use incomplete factorization preconditioners for the solution of a linear elasticity problem.

1.2. Historical remarks

Some early ideas on the use of preconditioners are from the 1950's by Lanczos [5] and by Stiefel [6]. They suggested the use of polynomials as preconditioners. In 1956, Hestenes [7] formulated an algorithm equivalent with the use of preconditioners in the conjugate gradients. A simple preconditioner is the point Jacobi preconditioner, where $C = diag(A)$, the diagonal of A. No extra work is needed for the calculation of this preconditioner. Forsythe and Strauss [8] showed in 1955 that among all preconditioners on diagonal form, the point Jacobi is optimal or close to optimal in the sence of reducing the condition number for matrices with property A. In 1969 van der Sluis showed this for general sparse matrices, see [9]. In the 1960's, some early papers on incomplete factorization were presented, see Buleev [10] and Oliphant [11] . As we have mentioned above, the development of incomplete factorizations will be reported on in the following sections.

In the beginning of 1970's, Axelsson suggested to use the SSOR method as a preconditioner for the conjugate gradient method, see [12]. With the decomposition $A = D + L + L^T$, where $D = diag(A)$ and L and L^T are the strictly lower and upper triangular parts of A, the SSOR matrix is given by $C(\omega) = \frac{1}{2-\omega}(\frac{1}{\omega}D + L)(\frac{1}{\omega}D)^{-1}(\frac{1}{\omega}D + L)^T$. For an optimal value of ω the spectral condition number $\kappa(C^{-1/2}AC^{-1/2}) = \sqrt{\kappa(A)}$, for a certain class of model problems, and hence, the number of iterations will be reduced to a lower order. The construction of the matrix $C(\omega)$ needs no extra work. For details see for instance the book by Axelsson and Barker [3].

1.3. Some frequently used preconditioners

The examples of preconditioners, $diag(A)$, $SSOR$ and LU are approximations to the given matrix A. Another class of preconditioners is based on the idea of approximating the inverse of A. Members of this class are the polynomial preconditioners. If $A = I - B$ and $\rho(B) < 1$, $A^{-1} = \sum_{k=0}^{\infty} B^k$. Hence A^{-1} can be approximated by a finite number of terms in the sum, see a paper from 1979 by Dubois, Greenbaum and Rodrigue [13]. Another possibility given by van der Vorst in 1982, is to use Neumann expansions for approximating the inverse of the LU factors, see [14]. Methods of this type became popular in the 1980s in connection with the use of parallel computers. Another idea to construct a preconditioner being a direct approximation

to the inverse of A is to find a sparse matrix M which minimizes $\|AM - I\|$ for some norm. One reason for the investigation of the sparse approximate inverses is the fact that incomplete LU factorizations may result in an ill-conditioned factorization for indefinite or nonsymmetric matrices. Another reason is the use of parallel computers. An early proposal of this type of method was given in 1982 by Benson and Frederickson [15]. Many authors have considered methods of this kind. Here we just mention papers by Kolotilina and Yeremin from 1993 [16] and 1994 [17].

Other important preconditioners are the multilevel preconditioners and methods of domain decomposition type. These are however outside the scope of this paper.

2. SOME STRATEGIES IN PRECONDITIONING BY POINT INCOMPLETE FACTORIZATIONS

An incomplete factorization of A, $C = LU$ where L is a lower triangular matrix and U is an upper triangular matrix, is a modification of the Gaussian elimination of A such that fill-in is allowed only due to certain restrictions.

One way to define an incomplete factorization is to choose an index set S where fill-in is allowed. During the factorization, elements in all positions outside S are set to zero. With the set $S = \{(i,j);\ a_{i,j} \neq 0\}$ we have the Incomplete LU factorization ILU(0).

In 1977, Meijerink and van der Vorst proved the existence of the ILU factorization for M-matrices for any choice of S including the diagonal, see [18]. Further they showed that the ILU preconditioned conjugate gradient method could result in a fast solution method. The paper by Meijerink and van der Vorst became a breakthrough on preconditioners as well as on iterative methods.

The asymptotic rate of convergence when the Incomplete Cholesky, IC preconditioner is applied to elliptic, second order partial differential equations is the same as by use of the unpreconditioned matrix. To be more precise, the condition numbers $\kappa(C^{-1/2}AC^{-1/2})$ and $\kappa(A)$ are of the same order $O(h^{-2})$, $h \to 0$ with respect to the mesh size parameter h. This is called first order convergence rate, while $\kappa(C^{-1/2}AC^{-1/2}) = O(h^{-1})$ is called second order convergence rate.

A modification of the ILU factorization, the MILU factorization, where entries which appear in positions outside S are not disregarded but added to the main diagonal of U was suggested by Gustafsson in 1978 [19]. This was a generalization of a so called generalized SSOR method for the standard five-point matrix given by Dupont, Kendall and Rachford [20] and by Axelsson [12]. In the thesis by Gustafsson from 1979 [21], the existence of the modified incomplete factorization is shown for diagonally dominant matrices and the theory is confirmed in a paper from 1983 [22].

The development of the basic ILU and MILU factorization methods include methods for allowing more fill-in in the factors in order to improve the performance. One possibility is to extend the set S of specific positions where fill-in is allowed. This is considered by Gustafsson [19] and by Meijerink and van det Vorst [18]. See also the book by Axelsson and Barker [3].

Another possibility is to use a drop by size condition, where the fill-in which is lower a given absolute value, the drop tolerance, is dropped. This idea was suggested in 1973 by Tuff and

Jennings [23] and also in 1983 by Østerby and Zlatev [24] and by Axelsson and Munksgaard [25]. For applications, see for instance papers from 1992 by D'Azevedo, Forsyth and Tang [26], [27].

The effect of orderings of the unknowns on the performance of the preconditioned conjugate gradient method is investigated by Duff and Meurant in 1989, see [28]. They considered for instance nested dissection, minimum degree and red-black orderings. The conclusion was that the ICCG method in general does not perform much better from reordering. However, in a recent paper by Benzi, Szyld and van Duin from 1999 [29], it is shown that certain reorderings for direct methods such that reverse Cuthill-McKee can be very beneficial for the solution of nonsymmetric linear systems by preconditioned Krylov subspace methods. The good performance is reported in the number of iterations and in the deviation of the preconditioned matrices from identity.

3. TWO RECENT IDEAS AND A COMPARISON

As is already mentioned, The IC factorization exists for M-matrices and the MIC factorization exists for diagonally dominant matrices. Many realistic applications will of course be excluded by these restrictions. In this section we present two recent ideas being remedies to these restrictions. At first we present the idea by Gustafsson from 1996 [30], where a modification on element level is performed. The assembled matrix corresponding to these modified element matrices then becomes a weakly diagonally dominant M-matrix and is hence appropriate for incomplete factorization. Next, we consider a recent preconditioning method presented by Kaporin in 1998 [31], which is shown to exist for general symmetric positive definite matrices. A comparison of the methods will also be done.

3.1. Modification of element matrices giving an M-matrix

The method by Gustafsson is considered for symmetric and positive definite matrices A, arising in the finite element discretization of second order partial differential equations. By a premodification on element level, the assembled matrix denoted $A^{(m)}$ becomes a weakly diagonally dominant M-matrix. The premodification is performed if there exists positive off-diagonal entries in the element matrices. The positive off-diagonal entries are replaced by zeros and the corresponding values are added to the diagonal in the same row. The matrix $A^{(m)}$ can be factorized by the standard IC or MIC methods. The condition number of the preconditioned matrix is only increased by a constant factor independent of the mesh size parameter h. Thus, the IC factorization of $A^{(m)}$ will give $\kappa(C^{-1/2}AC^{-1/2}) = O(h^{-2})$, $h \to 0$ i.e. first order convergence and the MIC factorization will give $\kappa(C^{-1/2}AC^{-1/2}) = O(h^{-1})$, $h \to 0$, i.e. second order convergence. Hence, the rate of convergence is of the same order as for problems where IC and MIC factorization can be applied to A itself. We denote these methods EIC and EMIC, respectively, where E stands for element modification. It is evident that the E(M)IC preconditioned conjugate gradient method can be applied to a much wider range of problems than standard (M)IC. For details, see [30].

Computer experiments verify the good behaviour of the method. In one example, quadratic finite elements are used for the solution of an isotropic model problem with right isosceles triangles. The assembled matrix A without using element modification is neither a diagonally

dominant matrix nor an M-matrix. Hence, element modification is used and the IC and MIC factorizations are performed on $A^{(m)}$ instead of A. Other examples considered are anisotropic model problems discretized by bilinear finite elements, where the element matrices have some positive off-diagonal entries. Also, anisotropic problems on right isosceles triangulations are solved with various angles between the triangles and the coordinate system, such that the standard IC or MIC factorizations of A fail. For all examples, computer tests verify first order convergence for the EICCG method and second order of convergence for the EMICCG method.

A variant of element matrix modification is the so called partly element modification where a certain fraction of positive off-diagonal elements are moved to the diagonal. This technique could be seen as stabilization although it does not give an M-matrix in general.

3.2. Preconditioning by the $U^TU + U^TR + R^TU - S$ factorization

Here we shortly present the incomplete factorization method recently developed by Kaporin [31]. This method exists for general symmetric positive definite matrices. The preconditioner is denoted the Robust Incomplete Cholesky 2nd order Stabilized, RIC2S factorization. An earlier method of this kind but only of first order of accuracy was given by Ajiz ans Jennings in 1984 [32], the Robust Incomplete Cholesky method, RIC1.

For the systems of linear equations $Ax = b$, where A is a sparse and symmetric positive definite matrix, the preconditioning by Kaporin is based on the decomposition

$$A = U^TU + U^TR + R^TU - S$$

where U is an upper triangular sparse approximation to the Cholesky factor, R is strictly upper triangular and S is nonnegative definite. The matrices R and S are error matrices with small elements.

The preconditioning matrix is $C = U^TU$ and the factorization method is based on a drop tolerance, i. e. the amount of fill-in is restricted by some condition on its relative size. It is also based on a control of pivot entries. The properties of the decomposition implies that pivot breakdowns will not occur and that U is relatively well-conditioned.

It is shown that the condition numbers for the preconditioned matrices satisfy $\kappa_1 \leq 2 + c_1\frac{\tau}{\lambda_{min}(A)}$ for the RIC1-method and $\kappa_2 \leq 4 + c_2\frac{\tau^2}{\lambda_{min}(A)}$ for the RIC2S-method. Here τ is the drop tolerance parameter.

According to these estimates, RIC1 is said to be of first order of approximation and RIC2S is said to be of second order of approximation. It follows from these estimates however, that both methods are first order of convergence rate with respect to the mesh-size parameter h. This is so since these estimates of the condition numbers assume that $diag(A) = I$ and then $\lambda_{min} = O(h^2)$, $h \to 0$.

A general property of the method is that the cost for factorization is sensitive to the parameter τ and for the method to be efficient τ has to be adapted to each problem. It is shown that the RIC2S preconditioning essentially reduces the number of iterations in the PCG method with a relatively few number of nonzero elements in U.

In a recent report by Axelsson et al, [33], the RIC2S preconditioned conjugate gradient method is used for the solution of some real-life problems, for instance a dam buildt on a rock massif and a bridge problem. We refer to [33] for a detailed description of the problems. In this short overview we will just refer some results for one of the most ill-conditioned problems, a part of the full 3D model of the bridge construction. The bridge is of homogenuous material

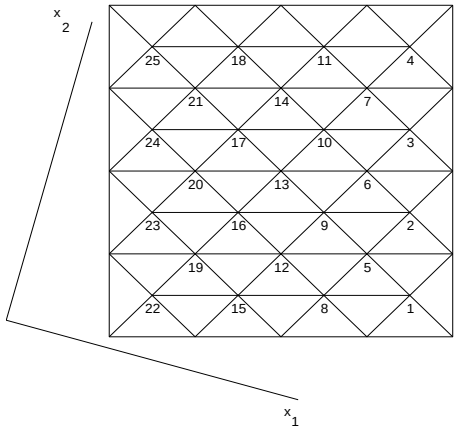

Figure 1. Discretization and nodenumbering

and there are no forces applied. The results show the characteristics of the method. The times for the construction of the preconditioner increases for lower values of τ i.e. for larger density of the matrix. However, the times for the iterations are about equal or somewhat decreasing for larger density leading to totally increasing times for larger density. With the optimal value of the threshold parameter, a significant decrease in time is shown compared to the Jacobi preconditioner.

3.3. A comparison of the EMIC and RIC2S

We compare the EMIC method with the RIC2S method. Most problems are also problems used as testproblems by Kaporin [31]. The results we present here are quite recent.

The testproblems are the following:

1. The 5-point approximation for a discontinuous problem $- \bigtriangledown (a \bigtriangledown u) = f$ on the unit square and with $a = 10^4$ inside the four subdomains [0.125,0.25]x[0.125,0.375], [0.125,0.25]x[0.500,0.875], [0.375,0.875]x[0.125,0.375] and [0.375,0.875]x[0.500,0.875] and $a = 1$ outside these subdomains.

2. The 13-point approximation for a biharmonic problem.

3. Quadratic finite element for Laplace equation on a uniform rightangled triangulation.

4. Linear finite element for the anisotropic problem $-u_{xx} - 0.001u_{yy} = f$, with a special triangulation and nodenumbering suitable for parallelization, see Figure 1. (For an explanation of the rotated coordinate system, see 5.3.)

We comment that Kaporin expects the RIC2S method to be preferable for testproblem 1, compared to incomplete factorization by position, because the elements in the subdiagonals of A differ quite a lot in size.

For the solution of testproblem 2 we use an idea presented in [34] for the finite element approximation of the biharmonic problem based on a mixed variable variational formulation. Here we just use $(U^TU)(U^TU)$ as a preconditioner for the 13-point matrix, where U^TU is a (M)IC factorization of the 5-point matrix. For testproblem 3, we present the number of

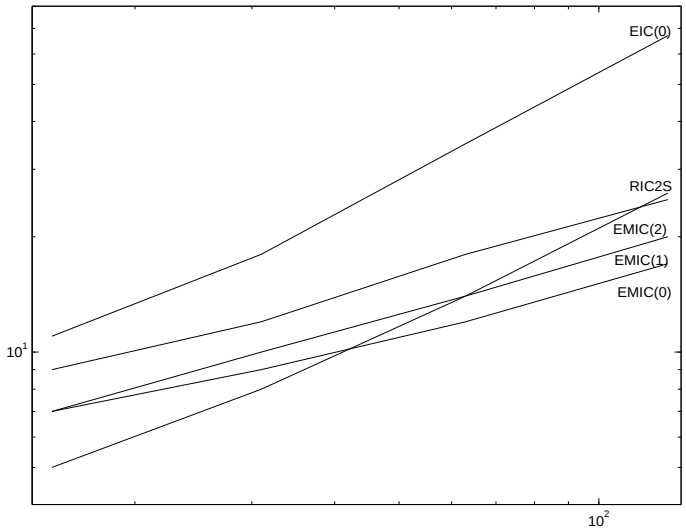

Figure 2. The number of iterations for testproblem 3.

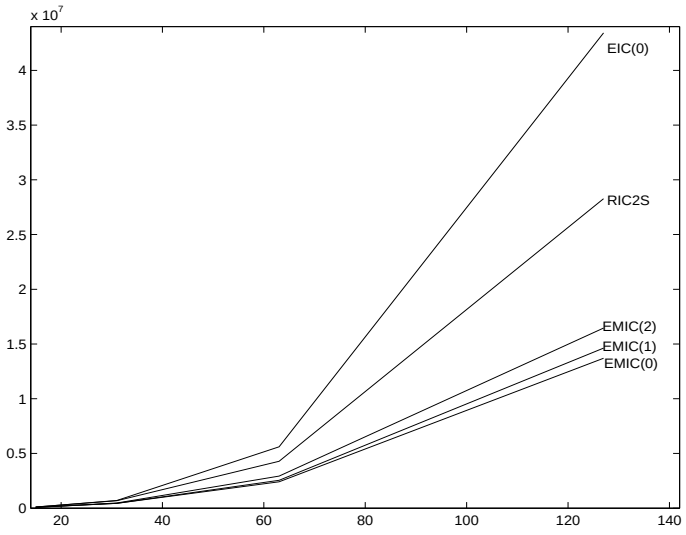

Figure 3. The total work for various methods and testproblem 3.

iterations , see Figure 2, and the total work for some methods, see Figure 3.

In comparison we present the work for different methods (with RIC2S as the norm) applied to the four testproblems, see Table I. Cholinc is a method in MATLAB, based on just dropping elements i.e. this method is not robust. For instance, the figure 0.97 for problem 2 is very sensitive for the choice of drop-parameter. In all tests the optimal drop-parameter (found by trial and error) has been used for RIC2S and Cholinc. For the E(M)IC methods, we had to use

Problem	RIC2S	Cholinc	EIC(0)	EMIC(0)	EMIC(1)	EMIC(2)
1	1	1.2	2.4	0.92	0.70	0.70
2	1	0.97	1.7	0.71	0.52	0.46
3	1	1.3	1.3	0.68	0.59	0.56
4	1	1.7	3.6(3.7)	1.80(2.0)	0.65(1.4)	0.69(1.6)

Table I. The work for different methods with RIC2S as the norm.

partly element modification for testproblem 4 in order to get the optimal performance. Within parenthesis we also give the figures for complete modification on element level. We have used a constant function as the right-hand side, the zero vector as starting approximation and a relative residual stopping criterion with $\epsilon = 10^{-4}$.

3.4. Conclusions

We have compared two different techniques for pointwise incomplete incomplete factorization; RIC2S based on a drop tolerance and EMIC based on fix sparsity pattern. Both techniques are applicable to general symmetric positive definite systems.

We conclude that for our testproblems, a proper implementation of the EMIC method is faster and simpler than the RIC2S method. Before stating a final recommendation of method, however, more general, realistic problems have to be implemented and tested.

4. SOME EXAMPLES OF INCOMPLETE FACTORIZATIONS BASED ON BLOCK MATRICES

A block partitioning of a matrix A can be made in a natural way for instance when the domain of the problem is a grid of $n \times n$ points in 2D or $n \times n \times n$ in 3D. The block partitioning is then based on the grid lines in 2D or grid planes in 3D. The size of the blocks and the number of blocks increase with increasing line/plane size.

A block partitioning can also be based on the physical problem. For instance, the linear elasticity problem, which we will consider in Section 5, naturally gives a 2×2 block matrix in 2D and a 3×3 block matrix in 3D. In Section 5 we will compare various block preconditioners for the 2D problem.

4.1. Block tridiagonal matrices

The block partitioning, based on lines of grid points often results in a block tridiagonal structure if a regular grid is used.

Let us assume that the matrix A is a block tridiagonal, symmetric and positive definite matrix,

$$A = \begin{bmatrix} A_1 & B_1 & & \\ B_1 & \cdot & \cdot & \\ & \cdot & \cdot & B_{n-1} \\ & & B_{n-1} & A_n \end{bmatrix}.$$

In performing a complete factorization on block level, the pivot blocks have to be inverted. Consider the block factorization

$$A = (D + L)D^{-1}(D + L^T).$$

Here $D = diag(D_1, ..., D_n)$ is the block diagonal matrix of pivot blocks, where

$$D_1 = A_1, \ D_i = A_i - B_{i-1}D_{i-1}^{-1}B_{i-1}^T, \ i = 2, ..., n.$$

The matrix L is the strictly lower block triangular part of A. In practice the matrices $A_i, \ i = 1, ..., n$ are often sparse, typically bandmatrices. However, the inverse matrices D_i^{-1} become full matrices.

In an *incomplete* block factorization one wants to preserve a sparse structure in calculating approximations to the matrices $D_i, \ i = 1, ..., n$. Hence, the inverses $D_i^{-1}, \ i = 1, ..., n-1$ have to be approximated by sparse matrices. The use of the incomplete block factorization as a preconditioner will then cause an easily solvable preconditioning system.

A general proof of the existence of incomplete block methods, when the coefficient matrix is an M-matrix, is given by Axelsson [35] in 1986.

We will here shortly mention two forms of incomplete block factorizations.

In the *standard* form of the incomplete block factorization the preconditioning matrix C is as follows,

$$C = (X + L)X^{-1}(X + L^T) = (X + L)(I + X^{-1}L^T),$$

where $X = diag(X_1, ..., X_n)$ is a block diagonal matrix satisfying the relations

$$\begin{aligned} X_1 &= A_1 \\ X_i &= A_i - B_{i-1}\Omega_{i-1}(X_{i-1}^{-1})B_{i-1}^T, \ i = 2, ..., n. \end{aligned}$$

Here $\Omega_{i-1}(X_{i-1}^{-1})$ is a sparse approximation of the inverse of X_{i-1}. Examples of $\Omega_{i-1}(X_{i-1}^{-1})$ are given later. In the solution of the preconditioning system, linear systems with matrices $X_i, \ i = 1, ..., n$ have to be solved in each iteration for instance by (incomplete) Cholesky factorization of X_i. Eventually the matrices X_i have to be approximated by sparse, symmetric positive definite matrices $\Omega_i'(X_i)$ in order to control the sparsity pattern and hence the computational complexity.

This method is studied by many authors. Two basic references are by Concus, Golub and Meurant [36] and by Axelsson [37]. See also the papers by Axelsson and Polman [38] and Kolotilina and Polman [39].

In a paper from 1992 [40], Kolotilina and Yeremin show the existence of the incomplete block factorization and the positive definitness of X and hence of C under certain general assumptions on the sparse approximations of the pivot blocks and their inverses. For details and for choice of sparse approximations to the inverse of a symmetric positive definite matrix by factorized sparse approximate inverses we refer to [40].

In the *inverse free* form of the incomplete block factorization, the preconditioning matrix is given by

$$C = (Y^{-1} + L)(I + YL^T),$$

where $Y = diag(Y_1, ..., Y_n)$ is a block diagonal matrix given by

$$\begin{aligned} Y_i &= \Omega_i(S_i^{-1}), \quad i \geq 1 \\ S_1 &= A_1 \\ S_i &= A_i - B_{i-1}Y_{i-1}B_{i-1}^T, \ i \geq 2. \end{aligned}$$

Also here $\Omega_i(S_i^{-1})$ is a sparse approximation to the inverse of the matrix S_i. Solving the preconditioning system with matrix C here involves multiplications with the Y_i blocks and hence the method is on inverse free form.

This method was introduced by Axelsson in 1985 [37] and is also studied by for instance Axelsson and Polman [38] and Kolotilina and Polman [39] and by Kolotilina and Yeremin [41].

The existence of the incomplete factorization and positive definiteness of the preconditioning matrix is shown by Kolotilina and Yeremin [40] for general assumptions on the sparse approximations of the inverses. The approximate inverses may for instance be calculated as the main tridiagonal part of the inverse. This choice is done by Concus, Golub and Meurant [36]. Here, also a modified version is given, where the row sum is preserved by modifying the diagonal. Chan and Meurant [42] showed in a paper from 1990 an improved condition number for the latter method.

Factorized sparse approximate inverses were introduced by Kolotilina and Yeremin 1993, [16] and may be used as sparse approximations of the inverses. Kolotilina and Yeremin have shown that these approximations satisfy the conditions for existence and positive definiteness of the preconditioner, which was given in their paper [40]. In a paper that has just appeared Kharchenko et al have presented a new method called AINV-A for the construction of sparse approximate inverse preconditioners for positive definite matrices, see [43].

Related methods, for instance using row sum criterions are presented and analysed by many authors, see for instance [35], [38], [44] [45] and [46].

4.2. Unstructured matrices

In a recent paper by Kolotilina, Nikishin and Yeremin from 2000 [47] incomplete LU-factorizations based on general non-singular unstructured matrices are presented. The methods denoted IBBLU use a block partitioning of the given matrix, i.e. $A = \{A_{ij}\}_{i,j=1}^n$ and compute a lower block triangular matrix $L = \{L_{ij}\}_{i,j=1}^n$ and an upper block triangular matrix $U = \{U_{ij}\}_{i,j=1}^n$.

The off-diagonal blocks $L_{ij},\ i > j$ and $U_{ij},\ i < j$ are stored explicitly, whereas the diagonal blocks L_{ii} and U_{ii}, which are lower and upper triangular respectively, are not stored explicitly. Instead their sparse inverses $\tilde{L}_{ii}^{-1}$ and $\tilde{U}_{ii}^{-1}$ are computed and stored. Hence, L and U are represented in an *explicit-implicit* block form, which is an advantage when the incomplete LU-factorization $\tilde{A} = LU$ is used as a preconditioner. The solution of a preconditioning system reduces to multiplications of the sub-matrices $L_{ij},\ i \neq j$ and $U_{ij},\ i \neq j$ by vectors and multiplications of $\tilde{L}_{ii}^{-1}$ and $\tilde{U}_{ii}^{-1}$ by vectors, i.e. we have an inverse free solution process. This is obvious beneficial on parallel computers.

Without going into too much details of the calculation of the factors L and U we will mention some main points:

- The calculation is based on the principal submatrices of A,

$$A^{(1)} = A_{11},\ A^{(k)} = \begin{bmatrix} A^{(k-1)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{kk} \end{bmatrix},\ k = 2, ..., n$$

- The factors L and U are calculated recursively on principal submatrix level by calculating *factorized sparse approximate inverses* of the pivot blocks as follows, where $\tilde{\Omega}_k$, Ω_k and Ω_k' denote approximation operators:

· At first G_1 and F_1, an upper and a lower sparse triangular matrix are calculated,

$$G_1 F_1 = \tilde{\Omega}_1 \left[A_{11}^{-1}\right] \text{ and let } L^{(1)} = F_1^{-1},\ U^{(1)} = G_1^{-1}.$$

· Then for $k = 2, ..., n$ the following is computed

$$X^{(k)} = \Omega_k(A_{21}^{(k)}(U^{(k-1)})^{-1}),\ Y^{(k)} = \Omega_k'((L_1^{(k-1)})^{-1}A_{12}^{(k)})$$

and the factorization of the pivot block,

$$G_k F_k = \tilde{\Omega}_1 \left[(A_{kk} - X^{(k)}Y^{(k)})^{-1}\right]$$

after which

$$L^{(k)} = \begin{bmatrix} L^{(k-1)} & 0 \\ X^{(k)} & F_k^{-1} \end{bmatrix},\ U^{(k)} = \begin{bmatrix} U^{(k-1)} & Y^{(k)} \\ 0 & G_k^{-1} \end{bmatrix}$$

· The incomplete factorization is

$$\tilde{A} = LU \text{ where } L = L^{(n)} \text{ and } U = U^{(n)}.$$

We note that the IBBLU algorithm gives a pointwise incomplete LU-factorization. Theoretically it is shown that the preconditioning matrices are correctly defined and non-singular for M- and H-matrices under certain general assumptions on the approximation operators $\tilde{\Omega}_k$, Ω_k and Ω_k' $k = 1, 2, ..., n$.

Kolotilina, Nikishin and Yeremin have shown that for both symmetric and unsymmetric problems the IBLLU algorithm is competitive with the best available preconditioners. As is alraedy mentioned, the IBLLU algorithm is designed mainly for unstructured sparse matrices. For block tridiagonal matrices, the traditional incomplete block factorization is recommended since the IBBLU algorithm gives in general not a block diagonal error matrix in this case.

5. APPLICATION OF INCOMPLETE FACTORIZATION PRECONDITIONERS TO A LINEAR ELASTICITY PROBLEM

In this section we compare a number of preconditioners based on block incomplete factorization, which are combined with pointwise incomplete factorization, for the solution of the linear elasticity problem.

This is an overview of the main results presented in two recent papers, [48] and [49] and a paper under preparation [50].

5.1. The problem

We consider the plain strain case of the linear elasticity problem with isotropic material and Dirichlet boundary condition. For the formulation of the three-dimensional problem we refer to [48].

Let $u(x) = (u_1, u_2)$ denote the displacement vector , $\sigma = (\sigma_{ij})$ the stress tensor and $\epsilon = (\epsilon_{ij})$ the strain tensor, satisfying

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

in a domain Ω in R^2 with boundary Γ . The body is assumed to be fixed along the part Γ_D of the boundary and loaded by body forces $f(x) = (f_1, f_2)$ and boundary forces $g(x) = (g_1, g_2)$.

The equations and boundary conditions are as follows,

$$div\ \sigma + f = 0\ in\ \Omega$$

$$\sigma_{ij}(u) = \sum_{k,l=1}^{2} c_{ijkl}(x)\ \epsilon_{kl}(u)$$

$$u = u_0\ on\ \Gamma_D,\ \sigma \cdot n = g\ on\ \Gamma_N = \Gamma \setminus \Gamma_D.$$

Here n is the outward pointing normal on Γ_N . The material coefficients $c_{ijkl}(x)$ are bounded on Ω and satisfy the relations

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij},\ c_0 \sum_{i,j} \xi_{ij}^2 \leq \sum_{ijkl} c_{ijkl}(x)\xi_{ij}\xi_{kl}$$

for $c_0 > 0$ and all symmetric tensors ξ . Further, (in the isotropic case) $c_{iiii} = \lambda + 2\mu$, $c_{iijj} = \lambda$ and $c_{ijij} = c_{ijji} = \mu$, $i \neq j$, where λ and μ are the Lamé coefficients related to the contraction ratio ν and the modulus of elasticity E through $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$.

A variational formulation of the problem leads to the problem: find u such that $u - u_0 \in V$ and

$$a(u, v) = L(v)\ ,\ \forall v \in V \text{ where } V = \{v \in [H^1(\Omega)]^2\ ,\ v = 0\ on\ \Gamma_D\}$$

with $H^1(\Omega)$ being the usual Sobolev space with norm $\| \|_1$.

We consider the boundary conditions $\Gamma_D = \Gamma$ and $u_0 = 0$. Then the bilinear form can be formulated

$$a(u, v) = \int_\Omega \sum_i \frac{\partial u_i}{\partial x_i}\frac{\partial v_i}{\partial x_i} + \frac{1-\tilde{\nu}}{2} \sum_{j \neq i} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} + \frac{1+\tilde{\nu}}{2} \sum_{j \neq i} \frac{\partial u_j}{\partial x_j}\frac{\partial v_i}{\partial x_i}\, d\Omega,$$

where $\tilde{\nu} = \frac{\nu}{1-\nu}$. For details, see [48].

In practice $0 < \nu < \frac{1}{2}$ i. e. $0 < \tilde{\nu} < 1$ and $\nu \to \frac{1}{2}$, i. e. $\tilde{\nu} \to 1$ represents an incompressible material.

The discretization is made by the finite element method on a triangulation Ω_h and with variation over the finite dimensional space

$$V_h = \{v_h \in V; v_h|_K \in [P_1(K)]^2\ \forall K \in \Omega_h\},$$

where $P_1(K)$ is the space of piecewise linear basis functions over the elements K. In a later subsection we will also consider hierarchical linear-quadratic finite elements.

With the set of basis functions for V_h, $\{\Phi_i^{(1)}, \Phi_i^{(2)}\}$, where $\Phi_i^{(1)} = (\Phi_i, 0)$, $\Phi_i^{(2)} = (0, \Phi_i)$ for the usual first degree polynomial basis functions $\{\Phi_i\}$, and the numbering in order $\{\Phi_i^{(1)}\}$, $\{\Phi_i^{(2)}\}$, the assembled matrix becomes

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}.$$

It is shown that A is not an M-matrix, since it has positive off-diagonal entries.

5.2. Block preconditioners

We first mention the block-diagonal preconditioners based on the separate displacement component of a, i. e.

$$\tilde{a}(u,v) = \int_\Omega \{\sum_i \frac{\partial u_i}{\partial x_i}\frac{\partial v_i}{\partial x_i} + \frac{1-\tilde{\nu}}{2}\sum_{j\neq i}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\}d\Omega.$$

The assembled matrix corresponding to $\tilde{a}$ is

$$C_D = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

When C_D is used as a preconditioner, the subsystems with matrices $A_{ii},\ i = 1,2$ are solved by the MIC(0) preconditioned conjugate gradient method, i. e. inner iterations are performed.

An alternative preconditioning, the approximate block-diagonal preconditioning method is obtained by replacing the matrices $A_{ii}, i = 1,2$ by their incomplete Cholesky factors, i. e. $\tilde{L}_1\tilde{L}_1^T,\ \tilde{L}_2\tilde{L}_2^T$ respectively. Then the preconditioner is

$$\tilde{C}_D = \begin{bmatrix} \tilde{L}_1 & 0 \\ 0 & \tilde{L}_2 \end{bmatrix}\begin{bmatrix} \tilde{L}_1^T & 0 \\ 0 & \tilde{L}_2^T \end{bmatrix}$$

In this case no inner iterations are performed.

Next we consider the full block incomplete factorization method

$$C_F = \begin{bmatrix} L_1 & 0 \\ A_{12}^T L_1^{-T} & L_2 \end{bmatrix}\begin{bmatrix} L_1^T & L_1^{-1}A_{12} \\ 0 & L_2^T \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & \bar{A}_{22} \end{bmatrix},$$

where $\bar{A}_{22} = A_{12}^T A_{11}^{-1} A_{12} + A_{22}$ and $L_1L_1^T = A_{11},\ L_2L_2^T = A_{22}$ are the Cholesky factors of A_{11} and A_{22}. In the solution of the preconditioning system $C_F r = g$ with matrix C_F, the following block system is solved,

$$\begin{aligned} A_{22}r_2 &= g_2 - A_{12}^T A_{11}^{-1} g_1 \\ A_{11}r_1 &= g_1 - A_{12}r_2 \end{aligned},$$

i. e. three inner systems have to be solved, two with matrix A_{11} and one with matrix A_{22}. Hence, inner iterations are performed.

If we replace A_{11} and A_{22} by their incomplete Cholesky factors $\tilde{L}_1\tilde{L}_1^T$ and $\tilde{L}_2\tilde{L}_2^T$, we get the approximate full block incomplete preconditioning method,

$$\tilde{C}_F = \begin{bmatrix} \tilde{L}_1 & 0 \\ A_{12}^T \tilde{L}_1^{-T} & \tilde{L}_2 \end{bmatrix}\begin{bmatrix} \tilde{L}_1^T & \tilde{L}_1^{-1}A_{12} \\ 0 & \tilde{L}_2^T \end{bmatrix},$$

where no inner iterations are performed. In [48] and [49] preconditioners based on a modified separate displacement component are also considered and we refer to these papers for details.

The condition numbers for the preconditioned matrices with preconditioners C_D and C_F are shown to be bounded as follows,

$$\kappa(C_D^{-1/2}AC_D^{-1/2}) \leq \frac{2}{1-\tilde{\nu}}, \quad \kappa(C_F^{-1/2}AC_F^{-1/2}) \leq \frac{(3-\tilde{\nu})^2}{8(1-\tilde{\nu})},$$

i. e. the condition numbers are independent of the discretization i. e. of the mesh size parameter. We observe that for almost incompressible materials ($\tilde{\nu} \approx 1$) the bound for the full block diagonal method is about a factor 4 smaller than the bound for the block diagonal method.

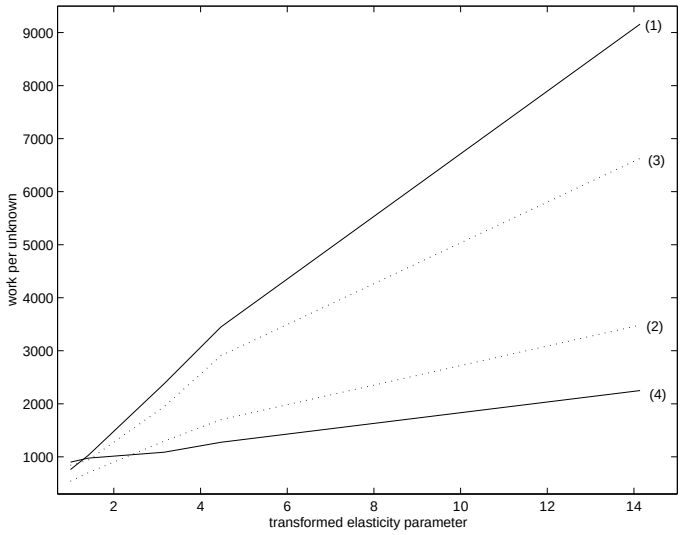

Figure 4. The work per unknown for the elasticity parameter $\tilde{\nu}$ near 1. In methods (1), (2), (3) and (4) we use C_D, $\tilde{C}_D$, C_F and $\tilde{C}_F$

5.3. Comparison between the preconditioners

The preconditioners are used for the solution of the problem with constant body force $f = 1$ on the unit square with various orientations θ, the angle between the x_1-axis and the hypotenuses in the triangulation of the domain. The reason for using different values of θ is given in Section 3.1. Further, a uniform right isosceles triangulation with nodenumbering giving a partially parallel solution method is used, see Figure 2.

We have compared the block-diagonal and the full block incomplete factorization preconditioners, specially for almost incompressible materials.

Numerical tests on the rate of convergence confirm the theory that the condition number for the full block incomplete factorization preconditioning method is about a factor 4 smaller than for the block diagonal method.

As is expected the approximate versions of the preconditioners, where no inner iterations are performed give a slower rate of convergence. However, the approximate block-diagonal and full block preconditioners give a smaller amount of work per iteration and in fact these methods give the smallest work per unknown with a certain relative stopping criterion for the iterations.

In Figure 4 we report on the total work per unknown for the problem with $\theta = \frac{\pi}{8}$ and $\tilde{\nu}$ close to 1. The parameter $\tilde{\nu}$ is transformed by $\sqrt{\frac{1}{1-\tilde{\nu}}}$ in order to discern the behaviour for $\tilde{\nu}$ close to 1. The problem size is about 4000 and the relative residual stopping criterion is 10^{-4} for the outer iterations and 10^{-1} for the inner iterations in the case of the C_D and C_F preconditioners.

The same good behaviour for the $\tilde{C}_D$ and $\tilde{C}_F$ preconditioners as is seen in Figure 4 is also shown for other problems, e.g. a discontinuous elasticity problem.

5.4. Block preconditioners by use of linear-quadratic finite elements

In a paper during preparation [50] we have generalized the methods to higher order by using the hierarchical two-level method presented in [51]. We consider two-level linear-quadratic finite elements, where the linear basis functions are associated with the vertex nodes and the quadratic basis functions are associated with the other nodes. If the latter basis functions are numbered first, the stiffness matrices $A_{ii}, i = 1, 2$ in C_D takes the form

$$A_{ii} = \begin{bmatrix} A_{ii}^{(2)} & G_{ii}^T \\ G_{ii} & A_{ii}^{(1)} \end{bmatrix}, \quad i = 1,\ 2.$$

The size of $A_{ii}^{(1)}$ is about $1/4$ of the size of A_{ii}. It is well known that the condition number $\kappa(A_{ii}^{(2)})$ is of order $O(1)$, $h \to 0$ and $\kappa(A_{ii}^{(1)})$ is of order $O(h^{-2})$, $h \to 0$.

The simplest preconditioner is chosen as the block diagonals of A_{ii}, giving the preconditioner denoted the diagonal-diagonal preconditioner

$$C_{DD} = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{(2)} & 0 & 0 & 0 \\ 0 & A_{11}^{(1)} & 0 & 0 \\ 0 & 0 & A_{22}^{(2)} & 0 \\ 0 & 0 & 0 & A_{22}^{(1)} \end{bmatrix}.$$

The condition number of the corresponding preconditioned matrix is shown to be independent of the mesh size parameter in the finite element discretization. The systems with matrices $A_{ii}^{(1)}$ can be solved by inner iterations by the PCG method or one can avoid the inner iterations by replacing the matrices by their modified incomplete factors giving the approximate diagonal-diagonal preconditioning matrix. The matrices $A_{ii}^{(2)}$ can simply be replaced by their diagonals.

The diagonal-diagonal preconditioners are compared to full-block incomplete factorization preconditioners. Here we mention just one of them,

$$C_{DF} = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix},$$

where F_{ii} are full block incomplete factorizations of A_{ii} of the Schur complement kind,

$$F_{ii} = \begin{bmatrix} L_i^{(2)} & 0 \\ G_{ii} {L_i^{(2)}}^T & L_i^{(1)} \end{bmatrix} \begin{bmatrix} {L_i^{(2)}}^T & {L_i^{(2)}}^{-1} G_{ii}^T \\ 0 & {L_i^{(1)}}^T \end{bmatrix} = \begin{bmatrix} A_{ii}^{(2)} & G_{ii}^T \\ G_{ii} & A_{ii}^{(1)} + G_{ii} {A_{ii}^{(2)}}^{-1} G_{ii}^T \end{bmatrix},$$

where $L_i^{(k)} {L_i^{(k)}}^T$ are (formal) complete Cholesky factorisations of $A_{ii}^{(k)}$ as before.

Numerical tests show also here that the simplest method, the approximate diagonal-diagonal preconditioner is the fastest method for our model problem with almost incompressible material.

5.5. Conclusions

We have demonstrated how a linear isotropic elasticity problem can efficiently be solved by the PCG method based on block-incomplete factorizations combined with incomplete pointwise factorization. The most efficient, and in fact also the simplest methods, show a moderate

increase in the required work per unknown when the problem turnes to become incompressible. In the papers [49] and [50] we have introduced proper nodenumberings for high degree of parallelism if the MIC(0) method is used. Computer tests confirm a fairly high degree of speed-up in practice.

REFERENCES

1. Axelsson O, Lindskog G. On the eigenvalue distribution of a class of preconditioning methods. *Numer.Math.* 1986; **48**:479–498.
2. van der Vorst H, Sleijpen G. The effect of incomplete decomposition preconditioning on the convergence of conjugate gradients. In *Incomplete Decomposition (ILU) - Algorithms, Theory and Applications, Proceedings of the Eighth GAMM-Seminar, Kiel,January 21-26, 1992, Notes on Numerical Fluid Dunamics,* Hackbusch W, Wittum G (eds). Vieweg: Braunschweig, 1993; 179–187.
3. Axelsson O, Barker A. Finite element solution of boundary value problems. Theory and computation. Academic Press: Orlando Florida, 1984;
4. Golub G H, van Loan C F. Matrix Computations. 2nd edition. The John Hopkins University Press: Baltimore, 1989
5. Lanczos C. Chebyshev polynomials in the solution of large-scale linear systems. In *Proceedings of the ACM,* 1952; 124–133.
6. Stiefel E L. Kernel polynomials in linear algebra and their applications. *U. S. National Bureau of Standards. Applied Mathematical Series* 1958; **49**:1–24.
7. Hestenes M R. The Conjugate-Gradient method for solving linear systems. In *Sympos. Applied Math. Numerical Analysis,* McGraw-Hill: New York, 1956; **VI**:83–102.
8. Forsythe G, Strauss E. On best conditioned matrices. In *Proc. Amer. MAth. Soc.,* 1955; 340–345.
9. van der Sluis A. Condition numers and equilibration of matrices. *Numer.Math.* 1969; **14**:14–23.
10. Buleev N I. A numerical method for the solution of two-dimensional and three-dimensional equation of diffusion. *Math. Sb.* 1960; **51**:227–238.
11. Oliphant T A. An extrapolation process for solving linear systems. *Quart. Appl. Math.* 1962; **20**:257–267.
12. Axelsson O. A generalized SSOR method. *BIT* 1972; **13**:443–467.
13. Dubois D F, Greenbaum A, Rodrigue G H. Approximating the inverse of a matrix for use in iterative algorithms on vector processors. *Computing* 1979; **22**:257–268
14. van der Vorst H A. A vectorizable variant of some ICCG methods. *SIAM J. Sci. Stat. Comput.* 1982; **3**:350–356
15. Benson M V, Frederickson P O. Iterative solution of large sparse linear systems arising in certain multidimensional approximation problems. *Utilitas Math.* 1982; **22**:127–140
16. Kolotilina L Yu, Yeremin A Yu. Factorized sparse approximate inverse preconditionings. I. Theory. *SIAM J. Matrix Anal. Applic.* 1993; **14**:45–58
17. Kolotilina L Yu, Yeremin A Yu. Factorized sparse approximate inverse preconditionings. II: Solution of 3D FE Systems on Massively Parallel Computers. *Elegant Mathematics,Inc. (USA)* 1992; Research report **EM-RR 3** *International J. for High Speed Computing.* 1995; **7**:191–215.
18. Meijerink J A, van der Vorst H A. An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix. *Math. Comp.* 1977; **31**:148–162
19. Gustafsson I. A class of first order factorization methods. *BIT* 1978; **18**:142–156
20. Dupont T, Kendall R, Rachford H. An approximate factorization procedure for solving self-adjoint elliptic difference equations. *SIAM J. Numer. Anal.* 1968; **5**:559–573
21. Gustafsson I. Stability and rate of convergence of modified incomplete Cholesky factorization methods. Department of Computer Sciences, Chalmers University of Technology, Goteborg, Sweden. Research report **79.02R**
22. Gustafsson I. Modified incomplete Cholesky (MIC) factorizations. In *Preconditioning Methods; Theory and Applications,* Evans D J (eds). Gordon and Breach: New York, 1983; 265–293.
23. Tuff A. D, Jennings A. An iterative method for large systems of linear structural equations. *Int. J. Numer. Meth.* 1973; **7**:175–183
24. Østerby O, Zlatev Z. Direct Methods for sparse matrices. Springer Verlag: New York, 1983
25. Axelsson O, Munksgaard N. Analysis of incomplete factorizations with fixed storage allocation. In *Preconditioning Methods; Theory and Applications,* Evans D J (eds). Gordon and Breach: New York, 1983; 219–241.
26. D'Azevedo E F, Forsyth P A, Tang W P. Drop tolerance preconditioning for incompressible viscous flow. *Int'l J. Comp. Math.* 1992; **44**:301–312

27. D'Azevedo E F, Forsyth P A, Tang W P. Ordering methods for preconditioned conjugate gradient methods applied to unstructured grid problems. *SIAM J. Matrix Anal. Applic.* 1992; **13**(3):944–961
28. Duff I S, Meurant G A. The effect of orderings on preconditioned conjugate gradients. *BIT* 1989; **29**:635–657
29. Benzi M, Szyld D B, van Duin A. Orderings for incomplete factorization preconditioning of nonsymmetric problems. *SIAM J. on Scientific Computing* 1999; **20**:1652–1670
30. Gustafsson I. An incomplete factorization preconditioning method based on modification of element matrices. *BIT* 1996; **36**(1):86–100.
31. Kaporin I. E. High quality preconditioning of a general symmetric positive definite matrix based on its $U^TU + U^TR + R^TU$-decomposition. *Numerical Linear Algebra with Applications* 1998; **5**:483–509
32. Ajiz M A, Jennings A. A robust incomplete Cholesky-conjugate gradient algorithm. *Int. J. Numer. Methods Eng.* 1984; **20**:949–966.
33. Axelsson O, Kaporin I, Kucherov A, Neytcheva M. Comparison of algebraic solution methods on a set of benchmark problems in linear elasticity. Private communication.
34. Gustafsson I. A preconditioned iterative method for the solution of the Biharmonic problem. *IMA Journal of Numerical Analysis* 1984; **4**:55–67.
35. Axelsson O. A general incomplete block-matrix factorization method. *Linear Algebra Appl.* 1986; **74**:179–190.
36. Concus P, Golub G, Meurant G. Block preconditioning for the conjugate gradient method. *SIAM J. Sci. Statist. Comput.* 1985; **6**:220–252.
37. Axelsson O. Incomplete block matrix factorization preconditioning methods. The ultimate answer? *J. Comput. Appl.Math.* 1985; **12,13**:3–18.
38. Axelsson O, Polman B. On approximate factorization methods for block-matrices suitable for vector and parallel processors. *Linear Algebra Appl.* 1986; **77**:3–26.
39. Kolotilina L. Yu, Polman B. On incomplete block factorization methods of generalized SSOR type for H-matrices. *Linear Algebra and Appl.* 1992; **177**:111–136.
40. Kolotilina L. Yu, Yeremin A. Yu. Incomplete Block Factorizations as Preconditioners for Sparse SPD Matrices. In *Recent advances in Iterative Methods. The IMA Volumes in Mathematics and its Applications,* Golub G, Greenbaum A, Luskin M (eds). Springer-Verlag, 1994; **60**:119–133.
41. Kolotilina L. Yu, Yeremin A. Yu. Incomplete Block Factorization methods for complex-structure matrices. *J. Soviet Math.* 1989; **47**:2821–2834.
42. Chan T F, Meurant G A. Fourier analysis of block preconditioners. CAM Report 90-04. University of California, Los Angeles. Feb. 1990.
43. Kharchenko S. A, Kolotilina L. Yu, Nikishin A. A, Yeremin A. Yu. A robust AINV-type method for constructing sparse approximate inverse preconditioners in factored form. *Numerical Linear Algebra with Applications* 2001; **8**:165–179.
44. Beauwens R, Bouzid M B. Existence and conditioning properties of sparse approximate block factorizations. *SIAM Numer. Anal.* 1988. **25**:941–956.
45. Magolu M M. Modified block-approximate factorization strategies. *Numer. Math.* 1992; **61**:91–110.
46. Magolu M M. Empirically modified block incomplete factorizations. In *Incomplete Decomposition (ILU) - Algorithms, Theory and Applications, Notes on Numerical Fluid Mechanics,* Hackbusch W, Wittum G (eds). Vieweg: Braunschweig, 1993; **41**:78–87.
47. Kolotilina L Yu, Nikishin A A, Yeremin A Yu. Factorized sparse approximate inverse preconditionings. IV: simple approaches to rising efficiency. *Numerical Linear Algebra* to appear.
48. Gustafsson I, Lindskog G. On parallel solution of linear elasticity problems, Part I: Theory. *Numerical Linear Algebra with Applications* 1998; **5**:123–139.
49. Gustafsson I, Lindskog G. On parallel solution of linear elasticity problems, Part II: Methods and some computer experiments. Report 2000:32, Department of Mathematics, Chalmers University of Technology, Göteborg, Sweden. *Numerical Linear Algebra with Applications*, to appear.
50. Gustafsson I, Lindskog G. On parallel solution of linear elasticity problems, Part III: Higher order finite elements. Submitted to *Numerical Linear Algebra with Applications.*
51. Axelsson O, Gustafsson I. Preconditioning and two-level multigrid methods of arbitrary degree of approximation. *Mathematics of Computation* 1983; **40**:219–242.