CONVERGENCE FOR THE SQUARE ROOT OF THE POISSON KERNEL IN RANK ONE SYMMETRIC SPACES AGAINST BOUNDARY FUNCTIONS WITH REGULARITY

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ABSTRACT. We consider the square root of the Poisson kernel in a Riemannian symmetric space of real rank one. It is known that for a boundary function in L^p one has convergence in a region strictly larger than the admissible regions and also that the region increases with p. We consider boundary data in Sobolev spaces L^p_α, and prove convergence within even larger regions which increases with p and α.

1. Introduction

Let $X = G/K$ be a Riemannian symmetric space of the noncompact type and of rank 1. (The notation is explained in Section 2.) On X we consider the λ-Poisson operator $P_\lambda f(g{\cdot}o) = \int_{K/M} f(kM)P^{\lambda+\rho}(kM, g)dkM$, where $P(kM, g)$ is the Poisson kernel of G/K, $f \in L^p(K/M)$ and $\lambda + \rho \in \mathfrak{a}$. We know that $P_\lambda f$ satisfies the equation

$$\Delta P_\lambda f = (|\lambda|^2 - |\rho|^2)P_\lambda f,$$

where Δ is the Laplace-Beltrami operator on G/K. If $\lambda \geq 0$ it is known that $P_\lambda f(g)$ does not necessarily converge to $f(kM)$ as g tends to kM. To obtain convergence we need to consider the normalization $\mathcal{P}_\lambda f = P_\lambda f/P_\lambda 1$. We know that $\mathcal{P}_\lambda f$ converges admissibly to f almost everywhere on the boundary for $f \in L^p$, $p \geq 1$.

In [Sjö88] Sjögren extends his previous L^1 result on the unit disk $\mathbb{U}$ [Sjö84] to general symmetric spaces X of rank 1. In X the weakly tangential regions are defined as

$$\{n_1 \exp(tH_0)n \cdot x : x \in \mathcal{C}, n \in B(Ct^q), t > c\}$$

where $\mathcal{C}$ is a compact subset of X, $q = \frac{1}{2\langle\rho,H_0\rangle}$ and $B(Ct^q)$ are balls in $\overline{N}$. It is the factor n which makes these regions larger than the ordinary admissible convergence regions. In a previous paper [Rön97] the first author proved that if X is a general symmetric space of rank 1 and $\lambda = 0$, we have convergence

Date: 17th December 2001.
[†]Supported by the Carl Trygger Foundation.
[‡]Supported by the European Commission TMR network "Harmonic Analysis".

in larger regions which we call L^p weakly tangential regions ($1 \leq p < \infty$). These regions increase with p, and coincide with the weakly tangential regions in [Sjö88] if $p = 1$. We would like to give an intuitive geometric view of the weakly tangential regions. If we only consider admissible convergence regions, that is, ignore the factor n in the above expression, we can consider the region as a tube along the curve $n_1 \exp tH_0$. This is because for each fixed value t_0, we dilate $n_1 \exp t_0 H_0$ with a fixed compact set $\mathcal{C}$. (Observe that the fact that we approach a boundary point when t tends to infinity, gives us the possibility to identify this boundary point with the asymptotic direction of $n_1 \exp tH_0$, which gives one unique boundary point to each tube.) If we now multiply $n_1 \exp tH_0$ with an arbitrary point $n \in B(Ct^q)$, we make a dilation of $n_1 \exp tH_0$ in certain directions determined by $\overline{N}$. The fact that the radius of B increases to infinity with t, ensures that this enlargement of the admissible region does not produce another (wider) tube. Instead we get a region which increases in width as t tends to infinity.

In the case of Euclidean spaces it has been observed by several authors that if one demand that the boundary function has some prescribed regularity (such as belonging to Sobolev or Besov spaces) the typical convergence regions will be larger than the ordinary nontangential convergence regions, it was proved in [NRS82] using potential theory, another proof was given in [NS84] and other related papers are [Dor86] and [Sue90]. This has also been studied in other settings by e.g. [CDS92]. The interesting fact is that the enlargement of the convergence regions both when one consider regular boundary functions and for the square root of the Poisson kernel is given by a simple multiplicative factor which increases with p.

We will investigate the situation for the square root of the Poisson kernel in a general symmetric space of rank 1 with boundary functions in the Sobolev spaces L^p_α, $1 < p < \infty$ and $\alpha > 0$. These spaces are defined in terms of the Bessel potentials $J_\alpha * f$ as in [Sak79]. In the Euclidean spaces one can easily adopt the proof in [NRS82], they consider the convolution with the Poisson kernel against potentials like $K * f$ where K is an unbounded L^1 function. The size of the convergence regions are determined by the L^q norm of $K * P_y$ (for f in L^p and q the dual exponent). However, it is easy to see that the Poisson kernel can be replaced by e.g. the square root of the Poisson kernel without any modifications of the proof, other than some trivial technical details, unfortunately their method does not extend to symmetric spaces.

It should also be noted from [NRS82] that for every $f \in L^p(\mathbb{R}^n)$ one can find a kernel K such that $f = K * F$ for some $F \in L^p$, and thus every f in L^p does indeed have boundary limits in the translates of a tangential convergence region, but the region depends of course on the given function. (that any f has a tangential approach region which is not translation invariant was already noted in [Lit27]). This result does not follow in our setting since the class of kernels we consider is to small. If one could prove the boundedness of the maximal operator for the Banach algebra generated by the Bessel potentials then it would follow that any L^p function has a convergence region which is strictly larger than the L^p-weakly tangential convergence region.

For $f \in L^p_\alpha$, $1 < p < \infty$ and $\alpha > 0$, we will now extend the result in [NRS82] to obtain a.e. convergence of $\mathcal{P}_0 f$ to f in the L^p_α weakly tangential regions,

$$\{n_1 \exp(tH_0)n \cdot x : x \in \mathcal{C}, n \in B(Ct^{pq}e^{t\alpha p'q}),\ t > c,\ p' < p\}.$$

The reader should observe that the difference between these regions and the weakly L^p tangential are that the radius of the balls are now $Ct^{pq}e^{t\alpha p'q}$ instead of Ct^{pq}. We have the following result:

Theorem 1.1. *Let* $\operatorname{rank} X = 1$. *Let H_0 be a fixed element in $\mathfrak{a}_+$ and let $q = \frac{1}{2\langle \rho, H_0\rangle}$. Let $f \in L^p_\alpha$, $p > 1$ and $0 < \alpha p < D$, where D is the homogeneous dimension of the boundary. Then for almost all $n_1 \in \overline{N}$,*

$$\mathcal{P}_0 f(n_1 \exp(tH_0)n' \cdot x) \to f(k(n_1)M)$$

as $t \to \infty$, x stays in a compact subset of X and n' stays in the ball $B(Ct^{pq}e^{t\alpha p'q})$ for some fixed constant C and for all $p' < p$.

The condition $\alpha p < D$ is natural; if $\alpha p > D$ then the the function f will be continuous and we then have convergence within any region. We prove this result by establishing the usual maximal function estimate. The proof of this estimate is rather technical, and rests heavily on a lemma given by the first author in [Rön97], generalizing an earlier version by Sjögren in [Sjö88]. The main point is that we split the kernels into pieces with small support, which gives us control of the corresponding operators.

Lemma 1.2. *Let $L \subset \overline{N}$ be a compact set, and let $p > 1$ be given. Assume that the sublinear operators T_k, $k = 1, 2, \dots,$ are defined in $L^p(L)$, that they take values which are nonnegative measurable functions on L, and satisfy the following conditions for some $C > 0$:*

1. *Each T_k is of weak type (p, p) with constants at most C_1.*
2. *Each T_k is given by*

$$T_k f(n) = \sup_{i \in I_k} |f| * K_i(n),$$

where I_k is an index set and the kernels K_i satisfy $\operatorname{supp} K_i \subset B(\gamma_i)$

3. $\int K_i^*(n)dn \leq C_1$

Here, for $i \in I_k$, $K_i^*(n) = \sup_{n' \in B(\gamma_{k+N'})} K_i(nn')$, for some natural number N', and the positive numbers γ_k satisfy the following condition: γ_k is decreasing and $\gamma_k^{1/\beta} < \gamma_{k-C}$ for some $C(\beta) > 0$ and all fixed $\beta > 0$, where β will depend on the group $\overline{N}$.

Then the operator $Tf(x) = \sup_k T_k f$ *is of weak type (p,p) with constant only depending on* L, N' *and* C_1.

The structure of this paper is as follows: In section 2 we explain the notation and give the necessary structure theory of symmetric spaces, including a definition of λ-Poisson integrals. In section 3 we develop the expression for the kernel. In section 4 we establish the necessary estimates for these expressions of the kernel and in section 5 we use these estimates and Lemma 1.2 to prove Theorem 1.1.

2. Preliminaries

Let $X = G/K$ be a Riemannian symmetric space of real rank one. Then G is a semisimple Lie group with finite center, and K a maximal compact subgroup. The Lie algebras of G and K are $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let θ be the Cartan involution. The Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a linear subspace of $\mathfrak{g}$. This is the eigenspace decomposition of $\mathfrak{g}$ with respect to the Cartan involution θ, where $\mathfrak{k}$ are the elements with eigenvalue $+1$, and $\mathfrak{p}$ are the elements with eigenvalue -1. Let $\mathfrak{a}$ be a maximal abelian subgroup of $\mathfrak{p}$, and A the corresponding connected subgroup of G.

The real rank is the dimension of $\mathfrak{a}$; in our paper we assume it to be equal to one. By the adjoint representation ad we arrive at the root space decomposition of the Lie algebra, $\mathfrak{g} = \oplus \mathfrak{g}_\alpha$; if $X \in \mathfrak{g}_\alpha$, then $\operatorname{ad}(H)X = \alpha(H)X$, for $H \in \mathfrak{a}$. The nonzero α:s are called (restricted) roots. The dimension of the root spaces are m_α. Let $\mathfrak{a}_+$ be one of the components of the subset of $\mathfrak{a}$ where none of the roots vanishes; this is called the positive Weyl chamber. A root α is called positive if $\alpha(H)$ is positive for all H in $\mathfrak{a}_+$. Let α and possibly 2α be the positive roots. Let $\rho = (m_\alpha + 2m_{2\alpha})\alpha/2$ denote half the sum of the positive roots counted with multiplicity. The Killing form $\langle X, Y\rangle = \operatorname{Tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$ allows us to identify the dual of $\mathfrak{a}$ with $\mathfrak{a}$.

Let $\mathfrak{n}$ be the sum $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ of the root spaces. The sum over the root spaces corresponding to the negative roots is $\overline{\mathfrak{n}}$, which is also the image of $\mathfrak{n}$ under the Cartan involution θ. The connected subgroups of G associated with $\mathfrak{n}$ and $\overline{\mathfrak{n}}$, are N and $\overline{N}$, respectively. Any $n \in N$ can be written as $n = \exp(X_1)\exp(X_2)$, $X_i \in \mathfrak{g}_{i\alpha}$. For $\overline{N}$ there is a similar expression where the product is taken over the negative roots.

The Iwasawa decomposition is $G = KAN$. This means that any $g \in G$ can be uniquely written as $g = k(g)\exp(H(g))n(g)$, with $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. From the Iwasawa decomposition we obtain the $\overline{N}A$ model of the symmetric space. In this decomposition the group $\overline{N}$ corresponds to the Furstenberg boundary K/M. The decomposition $\overline{N}A$ is the description of the symmetric space we will work with.

We fix an H_0 in the positive Weyl chamber $\mathfrak{a}_+$, such that $\alpha(H_0) = 1$. The conjugate of n, n^H, is $\exp(H)n\exp(-H)$. We choose a homogeneous gauge $|n|$ in $\overline{N}$ as

$$|n| = (c^2|X_1|^4 + 4c|X_2|^2)^{1/4}.$$

Here $|X|$ is the norm coming from the Killing form, that is $|X| = (-\langle X, \theta X\rangle)^{1/2}$ for any $X \in \mathfrak{g}$ and $4c = (m_\alpha + 4m_{2\alpha})^{-1}$. The reason for the choice of c and the gauge will be clear after we have defined the Poisson kernel. We have $|n'n| \leq C(|n'| + |n|)$ for some $C \geq 1$, and $|n^{tH_0}| = e^{-t}|n|$.

The ball with radius r in $\overline{N}$ centered at the origin is $B_r = \{n; |n| < r\}$, and the ball with radius r centered at n is nB_r. The measure of the ball B_r is proportional to r^D, where $D = m_\alpha + 2m_{2\alpha}$ is the homogeneous dimension of $\overline{N}$.

In a rank one symmetric space, we have an explicit form for the Poisson kernel, see e.g. Theorem 3.8 page 414 in [Hel78]. If $n = \exp(X_1)\exp(X_2)$, where $X_i \in \mathfrak{g}_{-i\alpha}$ then

$$P(n) = \frac{1}{(1 + 2c|X_1|^2 + c^2|X_1|^4 + 4c|X_2|^2)^{D/2}}.$$

The last two terms in the denominator sum up to $|n|^4$, which is the reason for the definition of the homogeneous gauge. The Poisson integral of a function f in L^1 on the boundary $\overline{N}$ is then

$$Pf(n_1\exp(tH_0)) = e^{2\langle\rho,tH_0\rangle}\int_{\overline{N}} P(n^{-tH_0})f(n_1n)dn.$$

¿From this expression we also get the normalized λ-Poisson integrals. We are studying the operator related with the normalized square root of the Poisson kernel $\mathcal{P}_0$:

$$\mathcal{P}_0f(n_1\exp(tH_0)) = \frac{e^{2\langle\rho,tH_0\rangle}}{t}\int_{\overline{N}} P^{1/2}(n^{-tH_0})f(n_1n)dn$$

where t is coming from the normalization $P_0 1$.

3. Expressions for the kernel

We consider $\mathcal{P}_0F$ where F is a function in a Sobolev space, here given as the convolution of an L^p function with the Bessel potential J_α. J_α is defined as

$$J_\alpha = \int_0^\infty s^{\frac{\alpha}{2}-1}e^{-s}h_s(n)ds,$$

where the heat kernel $h_t(n)$ comes from the sublaplacian on the nilpotent Lie group $\overline{N}$ see [VSCC92].

The operator $\mathcal{P}_0 F$ is given by

$$\mathcal{P}_0|J_\alpha * f|(n_1\exp(tH_0)n') = \frac{e^{2\langle\rho,tH_0\rangle}}{t}\int (J_\alpha * f)(n)\tilde{P}^{1/2}((n^{-1}n_1(n')^{tH_0})^{-tH_0})dn =$$

$$= \frac{e^{2\langle\rho,tH_0\rangle}}{t}(J_\alpha * f) * \tilde{P}^{1/2}_{tH_0}(n_1(n')^{tH_0}),$$

where $\tilde{P}(n) = P(n^{-1})$ and $\tilde{P}^{1/2}_{tH_0}(n) = \tilde{P}^{1/2}(n^{-tH_0})$.

We would like to see this as a convolution between f and a kernel Q_t. We therefore need to rewrite the above expression as

$$\tilde{P}^{1/2}_{tH_0} * (J_\alpha * f)(n_1) = (\tilde{P}^{1/2}_{tH_0} * J_\alpha) * f(n_1),$$

and we get the following expression

$$\mathcal{P}_0|J_\alpha * f|(n_1\exp(tH_0)n') = \frac{e^{2\langle\rho,tH_0\rangle}}{t} f * (\tilde{P}^{1/2}_{tH_0} * J_\alpha)(n_1(n')^{tH_0}).$$

To estimate the kernel $\tilde{P}^{1/2}_{tH_0} * J_\alpha$ we first need to simplify the convolution of $\tilde{P}^{1/2}$ with the heat kernel:

$$\tilde{P}^{1/2}_{tH_0} * h_s(\bar{n}) = \int h_s(n)\tilde{P}^{1/2}_{tH_0}(n^{-1}\bar{n})dn =$$

$$= \int h_s(n)\tilde{P}^{1/2}((n^{-1}\bar{n})^{-tH_0})dn = \int h_s(n)\tilde{P}^{1/2}((n^{-tH_0})^{-1}\bar{n}^{-tH_0})dn =$$

$$= \int e^{Dt}h_{se^{2t}}(n^{-tH_0})\tilde{P}^{1/2}((n^{-tH_0})^{-1}\bar{n}^{-tH_0})dn =$$

$$= e^{-Dt}\int e^{Dt}h_{se^{2t}}(n)\tilde{P}^{1/2}(n^{-1}\bar{n}^{-tH_0})dn =$$

$$= \int h_{se^{2t}}(n)\tilde{P}^{1/2}(n^{-1}\bar{n}^{-tH_0})dn = \tilde{P}^{1/2} * h_{se^{2t}}(\bar{n}^{-tH_0}).$$

In this sequence of equalities we have used known properties of the heat kernel, like $h_{se^{2t}}(n^{-tH_0}) = e^{-Dt}h_s(n)$, see e.g [Sak79], and the standard variable substitution $n^{-tH_0} \to n$.

4. Estimates of the kernels

Our next step is to obtain estimates of the kernel above.

Lemma 4.1. *We consider $Q_t(n_1) = h_t * \tilde{P}^{1/2}(n_1)$; We have that Q_t satisfies the following estimates; If $|n_1| < 1$.*

$$Q_t(n_1) \lesssim t^{-D/2}(1 + \log t).$$

If $|n_1| > 1$;

$$Q_t(n_1) \lesssim |n_1|^{-D}(\log|n_1| + 1) \quad t << |n_1|^2 \text{ or } t \sim |n_1|^2$$
$$Q_t(n_1) \lesssim t^{-D/2}(1 + \log(t)) \qquad t >> |n_1|^2$$

Proof of Lemma 4.1. ¿From [Sak79] we see that the heat kernel can be estimated with

$$h_t(n) \lesssim \begin{cases} t^{-D/2} & |n|^2 < t \\ |n|^{-D} & |n|^2 > t \end{cases}$$

We divide the proof in two parts, depending on the size of n_1.

The case $|n_1| < 1$. We now prove the above estimates. We take $|n_1| < 1$ and split the integral in two part, according to the above estimates of the heat kernel. First we consider $|n|^2 < t$:

$$\int_{|n|^2<t} h_t(n)\tilde{P}^{1/2}(n^{-1}n_1)dn \lesssim t^{-D/2}\int_{|n|^2<t} \frac{1}{(1+|n^{-1}n_1|)^D}dn$$

This can easily be seen to be bounded by $t^{-D/2}\log(1+\sqrt{t})$. The other part can be estimated by

$$\int_{|n|^2>t} h_t(n)\tilde{P}^{1/2}(n^{-1}n_1)dn \lesssim \int_{|n|^2>t} \frac{1}{|n|^D}\frac{1}{(1+|n^{-1}n_1|)^D}dn$$

which can be bounded by the same expression by simple estimates $t^{-D/2}\log(1+\sqrt{t})$.

The case $|n_1| > 1$. Here we need a further division, where we compare the size of n_1 to t.

When $t << |n_1|^2$ we first look at the "local" part,

$$t^{-\frac{D}{2}}\int_{|n|^2<t} \frac{1}{(1+|n^{-1}n_1|)^D}dn.$$

This integral is easy to take care of, we just estimate the denominator with $|n_1|^D$ and get the desired estimate.

Next we consider the "global" part

$$\int_{|n|^2>t} \frac{1}{|n|^D}\frac{1}{(1+|n^{-1}n_1|)^D}dn.$$

We can take care of the contribution of this part by dividing the integration area into three parts;

(1) $\sqrt{t} < |n| < |n_1|/2$. This part is easy, because here $|n^{-1}n_1| \sim |n_1|$. A polar coordinate substitution implies that this part of the integral can be estimated with
$$|n_1|^{-D}\log\frac{|n_1|}{2\sqrt{t}}.$$
(2) $|n| \sim |n_1|$. This again falls into two categories, either $|n^{-1}n_1| \sim |n_1|$ which can easily be seen to give a contribution $|n_1|^{-D}$, or $|n^{-1}n_1|$ is small, which require a further decomposition. We take the last case

to be $\{n; |n^{-1}n_1| < |n_1|/2\}$. We then make the decomposition, $A_k = \{n; |n^{-1}n_1| \sim 2^{-k}|n_1|\} \cap \{n; |n^{-1}n_1| < |n_1|/2\}$. This gives us

$$\int_{\cup A_k} \frac{1}{|n|^D} \frac{1}{(1+|n^{-1}n_1|)^D} dn \sim$$

$$\sim \sum_{-\infty}^{-1} \int_{A_k} \frac{|n_1|^{-D}}{(1+2^k|n_1|)^D} dn \sim \sum_{-\infty}^{-N} 2^{kD} + \sum_{-N}^{-1} \frac{2^{kD}}{2^{kD}|n_1|^D}$$

The best choice of N is to make both sums approximately equal, and this is obtained if we chose $2^{-N}|n_1| \sim 1$. This finally gives the estimate

$$\int_{\cup A_k} \frac{1}{|n|^D} \frac{1}{(1+|n^{-1}n_1|)^D} dn \sim |n_1|^{-D}(1+\log_+|n_1|)$$

(3) $|n| > 2|n_1|$, as in the above cases, this part of the integral can be estimated with $|n_1|^{-D}$.

When $t \sim |n_1|^2$, we again begin with the "local" part

$$t^{-\frac{D}{2}} \int_{|n|^2<t} \frac{1}{(1+|n^{-1}n_1|)^D} dn.$$

The integral can be split into two parts that can easily be handled, the first part is where the denominator $1+|n^{-1}n_1| \sim 1+|n_1|$. This is approximately half of the ball and here one gets that the integral can be estimated with $|n_1|^{-D} \sim t^{-D/2}$.

The other part can be estimated using dyadic annulus as above, and the integral gives the contribution $|n_1|^{-D}\log_+|n_1| \sim t^{-D/2}\log_+|n_1|$.

It remains to consider the "global" integral

$$\int_{|n|^2>t} \frac{1}{|n|^D} \frac{1}{(1+|n^{-1}n_1|)^D} dn.$$

For the part where $|n|$ is sufficiently large i.e. $|n| > 2|n_1|$ we can forget the n_1 part, and the integral is approximately $t^{-D/2} \sim |n_1|^{-D}$.

When $|n_1| \lesssim |n| < 2|n_1|$ we split the integral into two parts, depending on the value of $|n^{-1}n_1|$, (remember that here $|n| \sim |n_1|$). The part where $|n^{-1}n_1| \sim |n_1|$ can easily be evaluated and one gets the contribution $(1+|n_1|)^D$. The other part follows by using the dyadic annulus $A_k = \{n; |n^{-1}n_1| \sim 2^k|n_1|\} \cap \{n : |n_1| < |n| < 2|n_1|\}$.

$$\frac{1}{|n_1|^D} \int_{\cup A_k} \frac{1}{(1+|n^{-1}n_1|)^D} dn \sim \frac{1}{|n_1|^D} \sum_{-\infty}^{1} \int_{A_k} \frac{1}{(1+2^k|n_1|)^D} dn =$$

$$= \frac{1}{|n_1|^D} \sum_{-\infty}^{-N} 2^{kD}|n_1|^D + \frac{1}{|n_1|^D} \sum_{-N}^{1} \frac{(2^k|n_1|)^D}{2^{kD}|n_1|^D},$$

where N is chosen so $2^{-N}|n_1| \sim 1$. This sums up to $|n_1|^{-D}(1+\log_+|n_1|)$.

When $t >> |n_1|^2$, we start with the "local" part

$$t^{-\frac{D}{2}} \int_{|n|^2<t} \frac{1}{(1+|n^{-1}n_1|)^D} dn.$$

Since $|n_1|$ is comparatively small, we divide the ball $\{|n|^2 < t\}$ into dyadic annulus defined by $|n^{-1}n_1| \sim 2^k(|n_1|+\sqrt{t})$ where $k = -\infty \dots 0$. This gives the estimate

$$t^{-D/2} \int_{|n|<\sqrt{t}} \frac{1}{(1+|n^{-1}n_1|)^D} dn \sim t^{-D/2} \sum_{-\infty}^{0} \int_{A_k} \frac{dn}{(1+2^k(|n_1|+\sqrt{t}))^D} =$$

$$t^{-D/2} \sum_{-\infty}^{-N} 2^{kD} + t^{-D/2} \sum_{-N}^{0} \frac{2^{kD}(|n_1|+\sqrt{t})^D}{2^{kD}(|n_1|+\sqrt{t})^D} \leq t^{-D/2}(1+\log(|n_1|+\sqrt{t})),$$

where N is chosen so $2^{-N}(|n_1|+\sqrt{t}) \sim 1$.

It only remains to deal with the integral over large n, where we can assume that $|n^{-1}n_1| \sim |n|$. For this case we get the estimate

$$\int_{|n|^2>t} \frac{1}{|n|^D} \frac{1}{(1+|n^{-1}n_1|)^D} dn \sim \int_{\sqrt{t}}^{\infty} r^{-2D} r^{D-1} dr \sim -\left[r^{-D}\right]_{\sqrt{t}}^{\infty} = t^{-D/2}.$$

□

5. Proof of the Theorem

We want to estimate the operator $M_{p',\alpha}f(n_1)$ which is defined as the supremum of the following expression, where we take the sup over n' and t satisfying $|n'| < t^{pq}e^{t\alpha p'q}$, $p' < p$:

$$\frac{e^{\langle 2\rho, tH_0\rangle}}{t} \int_0^{\infty} s^{\frac{\alpha}{2}-1} e^{-s} \int_{\overline{N}} f(n_1(n')^{-tH_0}n^{-1}) Q_{se^{2t}}(n^{-tH_0}) dnds.$$

We split the integral over $\overline{N}$ in two parts N_1 and N_2, where

$$N_1 = \{n : |n^{-tH_0}| \geq Ct^{p'q}e^{t\alpha p'q}\} \quad \text{and} \quad N_2 = \{n : |n^{-tH_0}| \leq Ct^{p'q}e^{t\alpha p'q}\}.$$

In the integral over N_1 we can get rid of n' and this leaves us with the usual Hardy-Littlewood maximal operator, which is known to be bounded on L^p.

This leaves us with the integral over N_2. We first decompose N_2 into dyadic annulus

$$A_k = \{n : 2^{k-1}t^q e^{t\alpha p'q} e^{-t} < |n| \leq 2^k t^q e^{t\alpha p'q} e^{-t}\}, \quad k = n(p',q,t), \dots, N(p',q,t)$$

where $N(p',q,t) = [q(p'-1)\log_2 t] = O(\log t)$ and the lower bound is such that we get $2^{n(p',q,t)}t^q e^{t\alpha p'q} \sim 1$, i.e. $n(p',q,t) \sim -t\alpha p'q$. Actually the first annulus should be replaced by a ball, but the value is constant (not depending on t) so not doing this will not have any significant effect on the estimates. We make a dyadic decomposition in t, $B_j = \{t : 2^{j-1} < t \leq 2^j\}$ for $j \geq N_0$ for some N_0 large enough. We also make the transformation $(n')^{H_0} \to n'$, and get

$$M_{p',\alpha}f(n_1) \lesssim \sup_{j\geq N_0} \sup_{\substack{t\in B_j \\ |n'|<2^{jp'q}e^{-t}e^{t\alpha p'q}}} \frac{e^{Dt}}{2^j}\times$$

$$\sum_{k=n(p',q,t)}^{N(p',q,t)} \int_0^\infty s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k} f(n_1n'n^{-1})Q_{se^{2t}}(n^{-tH_0})dnds \lesssim$$

$$\lesssim \sup_{j,n',t} \frac{e^{Dt}}{2^j}\sum_{n(p',q,t)}^{N(p',q,t)} \int_0^\infty s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k} f(n_1n'n^{-1})\frac{\log(|n^{-tH_0}|^2+se^{2t})}{(|n^{-tH_0}|^2+se^{2t})^{D/2}}dnds$$

Here we need to divide the s-integral into two pieces, in order to separate the cases when $|n^{-tH_0}|^2$ and se^{2t} is the dominant term in the kernel.

5.1. **Small s.** When $0 < s < a_ke^{-2t}$ and $a_k = 2^{2k}t^{2q}e^{2t\alpha p'q}$ the dominant term is $|n^{-tH_0}|^2$. We continue the above computation and call this part of the maximal operator $M^1_{p',\alpha}$.

$$M^1_{p',\alpha}f(n_1) \lesssim$$

$$\lesssim \sup_{j,n',t} \frac{e^{Dt}}{2^j}\sum_{n(p',q,t)}^{N(p',q,t)} \int_0^{a_ke^{-2t}} s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k} f(n_1n'n^{-1})\frac{\log(|n^{-tH_0}|^2)}{(|n^{-tH_0}|^2)^{D/2}}dnds \lesssim$$

$$\lesssim \sup_{j,n',t} \frac{e^{Dt}}{2^j}\sum_{n(p',q,t)}^{N(p',q,t)} \int_0^{a_ke^{-2t}} s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k} f(n_1n'n^{-1})\frac{\log(2^kt^qe^{t\alpha p'q})}{2^{kD}te^{t\alpha p'}}dnds \lesssim$$

$$\lesssim \sup_j \sup_{n',t} \frac{e^{Dt}}{2^j}\sum_{n(p',q,t)}^{N(p',q,t)} \int_0^{a_ke^{-2t}} s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k(n')^{-1}} f(n_1n^{-1})\frac{\log(2^kt^qe^{t\alpha p'q})}{2^{kD}te^{t\alpha p'}}dnds \equiv$$

$$\equiv C\sup_j T^1_j f(n_1)$$

where we made the linear change of variables $n'n^{-1} \to n^{-1}$ in the last inequality. Now we estimate the operators T^1_j with a sum of operators T^1_{jk} where we have moved the sup over n' and t inside the sum. We have

$$T^1_j \leq \frac{e^{-\alpha t}}{2^j}\sum_{k=n(p',q,t)}^{N(p',q,t)} T_{jk}$$

where

$$T^1_{jk}f(n_1) \equiv \sup_{\substack{|n'|\leq 2^{jqp'}e^{-t}e^{t\alpha p'q}\\ t\in B_j}} e^{\alpha t}\int_0^{a_k e^{-2t}} s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k(n')^{-1}} f(n_1n^{-1})\frac{2\log(2^k t^q e^{t\alpha p' q})}{2^{kD}te^{t\alpha p'}e^{-Dt}}dnds$$

The reason for introducing the factor $e^{-\alpha t}$ outside the sum is to get rid of an exponential factor below. We replace the integration region $A_k(n')^{-1}$ with the largest region occurring in the sum: $A_{N(p',q,t)}(n')^{-1}$ which is contained in a ball B with radius comparable to the radius of $A_{N(p',q,t)}$. Using that $|A_{N(p',q,t)}| \sim 2^{j(p'-1)}e^{t\alpha p'}e^{-Dt}t$, the same is also true for the ball B, and we get the following estimate;

$$\begin{aligned} T^1_{jk}f(n_1) \lesssim \sup_t \int_0^{a_k e^{-2t}} s^{\frac{\alpha}{2}-1}e^{-s}\frac{2^{j(p'-1)}\log(2^k t^q e^{t\alpha p' q})}{2^{kD}} \times \\ \frac{e^{\alpha t}}{2^{j(p'-1)}te^{t\alpha p'}e^{-Dt}}\int_B f(n_1n^{-1})dnds \lesssim \\ \lesssim e^{\alpha t}\int_0^{a_k e^{-2t}} s^{\frac{\alpha}{2}-1}e^{-s}ds\frac{2^{j(p'-1)}\log(2^k t^q e^{t\alpha p' q})}{2^{kD}}Mf(n_1) \end{aligned}$$

where $t \sim 2^j$. Here $Mf(n_1)$ is the usual Hardy-Littlewood maximal operator, which is well known to be bounded on L^p. To deal with the remaining integral we see that we can ignore the exponential factor without any loss, since s is small. The estimate of the integral together with the other constants will be a bound on the weak type (1,1) norm of the operator T^1_{jk}, and by routine calculations we get that the expression in front of the maximal function can be estimated by

$$\frac{1}{\alpha}a_k^{\alpha/2}e^{-\alpha 2^j}\frac{2^{j(p'-1)}\log(2^k 2^{jq}e^{2^j\alpha p' q})}{2^{kD}}e^{\alpha 2^j} \sim a_k^{\alpha/2}\frac{2^{j(p'-1)}\log(2^k 2^{jq}e^{2^j\alpha p' q})}{2^{kD}}$$

The logarithm can be estimated with $\alpha p' q j 2^j$ since we have the bound $k < c\log 2^j$. If we use this, together with the trivial L^∞ bound, Marcinkiewicz interpolation theorem gives the following bound on the operator in L^r;

$$\|T^1_{jk}\|_r \leq C(p',\alpha,r)\Big(a_k^{\alpha/2}\frac{j2^{j(p'-1)}2^j}{2^{kD}}\Big)^{1/r}$$

Now we will see that the operator T^1_j has L^r norm uniformly bounded with respect to j: first sum in k. The sum of the factors that do depend on k is

(remember that $\alpha < D$)

$$\sum_{k=n(p',q,t)}^{N(p',q,2^j)} \frac{2^{k\alpha/r}}{2^{kD/r}} = \sum_{k=n(p',q,2^j)}^{N(p',q,2^j)} 2^{k(\alpha-D)/r} \sim 2^{n(\alpha-D)/r} \sim (2^{jq}e^{2^j\alpha p'q})^{(D-\alpha)/r}.$$

The L^r norm of the operator T_j^1 can thus be estimated by

$$\frac{e^{-\alpha 2^j}}{2^j}\Big(2^{j(1-\alpha q)}e^{2^j\alpha p'-2^j\alpha^2 p'q}2^{j\alpha q}e^{2^j\alpha^2 p'q}2^{j(p'-1)}2^j\Big)^{1/r} \sim \frac{e^{-\alpha 2^j}}{2^j}\Big(2^{j(p'+1)}e^{2^j\alpha p'}\Big)^{1/r}.$$

If we consider the exponential factors we see that we in total have $e^{-\alpha 2^j+\alpha p' 2^j/r}$. The exponent will be negative if $r > p'$ and the exponential factors will kill the polynomial factors.

So we can bound the L^r-norm of T_j^1 independently of j. To check the rest of the conditions in the lemma is routine. A detailed description of this can be found in [Rön97]. We have thus completed this part of the proof and we continue to the case where s is large.

5.2. **Large** s. This is the part where $s > a_k e^{-2t}$, with $a_k = 2^{2k}t^{2q}e^{2t\alpha p'q}$. Denote the corresponding maximal operator by $M^2_{p',\alpha}f(n_1)$.

$$\begin{aligned} M^2_{p',\alpha}f(n_1) &\lesssim \\ &\lesssim \sup_{j,n',t}\frac{e^{Dt}}{2^j}\sum_{k=n(p',q,t)}^{N(p',q,t)}\int_{a_k e^{-2t}}^{\infty} s^{\frac{\alpha}{2}-1}e^{-s}\int_{A_k} f(n_1 n' n^{-1})\frac{\log(se^{2t})}{s^{\frac{D}{2}}e^{Dt}}dnds \lesssim \\ &\lesssim \sup_{j,n',t}\frac{1}{2^j}\sum_{k=n(p',q,t)}^{N(p',q,t)}\int_{a_k e^{-2t}}^{\infty} s^{\frac{\alpha-D}{2}-1}e^{-s}\int_{A_k(n')^{-1}} f(n_1 n' n^{-1})\log(se^{2t})dnds \equiv \\ &\equiv C\sup_j T_j^2 f(n_1) \end{aligned}$$

In this series of inequalities we made a linear change of variables; $n'n^{-1} \to n^{-1}$. We again move the sup inside the sum and see that the operators T_j^2 can be estimated with a sum of operators T_{jk}^2:

$$T_j^2 \le \frac{e^{-\alpha t}}{2^j}\sum_{k=n(p',q,t)}^{N(p',q,t)} T_{jk}^2,$$

with

$$T_{jk}^2 f(n_1) \equiv \sup_{\substack{t,n' \\ t\in B_j,\ |n'|\le 2^{jq(p'-1)}e^{-t}t^q}} e^{\alpha t}\int_{a_k e^{-2t}}^{\infty} s^{\frac{\alpha-D}{2}-1}e^{-s}\int_{A_k(n')^{-1}} f(n_1 n^{-1})\log(se^{2t})dnds$$

We again replace the integration regions $A_k(n')^{-1}$ with a ball B, which contains $A_k(n')^{-1}$ for all k in the sum and has radius comparable to the radius of $A_{N(p',q,t)}$. For the volume we know $|A_{N(p',q,t)}| \sim 2^{j(p'-1)}e^{t\alpha p'}e^{-Dt}t$, and the same is also true for the ball B, and we get the following estimate;

$$T^2_{jk}f(n_1) \lesssim \sup_{t\in B_j} e^{\alpha t}\int_{a_k e^{-2t}}^{\infty} s^{\frac{\alpha-D}{2}-1}e^{-s}\log(se^{2t})2^{j(p'-1)}e^{t\alpha p'}e^{-Dt}tds\times$$
$$\frac{1}{2^{j(p'-1)}e^{t\alpha p'}e^{-Dt}t}\int_B f(n_1n^{-1})dn \lesssim$$
$$\lesssim e^{\alpha 2^j}\int_{a_k e^{-2\cdot 2^j}}^{\infty} s^{\frac{\alpha-D}{2}-1}e^{-s}\log(se^{2\cdot 2^j})2^{j(p'-1)}e^{2^j\alpha p'}e^{-D2^j}2^j dsMf(n_1)$$

Again $Mf(n_1)$ is bounded on L^p. To deal with the remaining integral we see that we can ignore the logarithmic factor without any loss, by choosing a slightly larger α. We can then get an estimate of the integral by using partial integration in s. The integral together with the other constants will be a bound on the weak type (1,1) norm of the operator T^2_{jk}, and we get that the norm is bounded by

$$a_k^{(\alpha-D)/2}e^{\alpha p'2^j}2^{jp'}.$$

If we use this together with Marcinkiewicz interpolation theorem, and the trivial L^∞ estimate, we get the following bound on the operator in L^r:

$$\|T^2_{jk}\|_r \lesssim \left(a_k^{(\alpha-D)/2}e^{\alpha p'2^j}2^{jp'}\right)^{1/r} \sim \left(\left(2^k2^{jq}e^{\alpha p'q2^j}\right)^{(\alpha-D)}e^{\alpha p'2^j}2^{jp'}\right)^{1/r}.$$

Now we can see that the operator T^2_j has L^r norm uniformly bounded with respect to j, because if we sum in k, the sum of the factors that do depend on k is

$$\sum_{k=n(p',q,t)}^{N(p',q,t)} 2^{k(\alpha-D)/r} \sim \left(2^{jq}e^{2^j\alpha p'q}\right)^{\frac{D-\alpha}{r}} = \left(2^{j(1-\alpha q)}e^{2^j\alpha p'-2^j\alpha^2p'q}\right)^{\frac{1}{r}}$$

This gives that the L^r norm of the operator T^2_j can thus be estimated by

$$\frac{e^{-\alpha 2^j}}{2^j}\left(2^{jp'}e^{2^j\alpha p'}\right)^{1/r}$$

which is bounded independently of j if $r \geq p'$. To check the rest of the conditions in the lemma is routine, just as in the first case.

This gives that $M^2_{p',\alpha}$ is bounded on L^r if $r > p'$ and the proof of the theorem is completed.

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