ON SUMMABILITY OF MEASURES WITH THIN SPECTRA

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Abstract. We study different conditions on the set of roots of the Fourier transform of a measure on the Euclidean space, which yield that the measure is absolutely continuous with respect to the Lebesgue measure. We construct a monotone sequence in the real line with this property. We construct a closed subset of $\mathbb{R}^d$ which contains a lot of lines of some fixed direction, with the property that every measure with spectrum contained in this set is absolutely continuous. We also give examples of sets with such property that every measure with spectrum contained in them is locally L^p summable for suitable $p > 1$. We discuss some related problems; among them we show that if a measure on the real line is such that its Fourier transform vanishes on the sequence $(n^{1/k})_{n=1}^{\infty}$, then both its singular and absolutely continuous parts share this property.

1. Introduction.

According to the general uncertainty principle a distribution (a measure in our case) and its Fourier transform can not be both too concentrate. In particular, if the Fourier transform of a measure is supported on the set of a special form then it has no singular part. We call a set with this property a *Riesz set*. Many different sufficient conditions for Riesz sets are known - we refer to [M], [Sh], [A], [HJ], where the conditions for $\mathbb{T}^d$ are given - roughly speaking the set should be concentrated on a halfspace and it can not contain a line. Another sufficient condition (both for $\mathbb{R}^d$ and $\mathbb{T}^d$) is given in [R], where the set is required to be strongly antisymmetric. In the present paper we study fenomena which occur only in the non-compact setting. We give a new class of examples of Riesz sets on $\mathbb{R}^d$ which are symmetric and include a lot of lines.

In Section 2 we prove the following criterion inspired by the de Leeuw transference method, on which the examples of Riesz sets are based.

Theorem 1. *Suppose that $\alpha_j K \cap \mathbb{Z}^d$ is a Riesz set on $\mathbb{Z}^d$ for every $j = 1, 2, \dots$ for some $K \subset \mathbb{R}^d$, and a sequence $\alpha_j \to \infty$. Then K is a Riesz set on $\mathbb{R}^d$.*

As a direct application of the above criterion we prove that so called f-poles are Riesz sets for every $f : \mathbb{R}_+ \to \mathbb{R}_+$ which decrease to 0. For

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2000 *Mathematics Subject Classification:* 42B10, 42A55.
Supported in part by KVA grant 11 29 3 551 and KBN grant 2 P03A 036 14.

any positive, decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ we call the set *an f-pole* iff it is an image of the set $\{(x_1, x') \in \mathbb{R}^d : |x'| \leq f(|x_1|)\}$ under a linear transformation.

Corollary 1. *For any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ decreasing to 0, any f-pole is a Riesz set.*

We also show an example of the Riesz set in $\mathbb{R}^d$ $(d \geq 2)$ whose interior contains all lines of some direction with except of a set of small Lebesgue measure.

The formulation of Theorem 1 is in the spirit of the criterion given by Meyer for compact group (cf. [M]), however, instead of using the argument of topological nature, we rather transfer the results from tori to the Euclidean spaces.

In Section 3 we study special cases of Riesz sets for which the L^1-summability can be improved. It is easy to see that if the Fourier transform of a measure $\mu \in M(\mathbb{R}^d)$ is supported on a set $K \subset \mathbb{R}^d$ of a finite Lebesgue measure, then μ is a bounded continuous function, and it belongs to $L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. Moreover, $\|\mu\|_p \leq |K|^{\frac{p-1}{p}} \|\mu\|_M$. We consider the class of sets $K \subset \mathbb{R}^d$ such that the function assigning to $t \in \mathbb{R}$ the $(d-1)$-dimensional Lebesgue measure of the intersection of K with the hyperplane $\{x_1 = t\}$ is L^p-summable. We prove the following result.

Theorem 2. *Let $1 < p \leq 2$, $K \in \mathbb{R}^d$ and there exists $y \in \mathbb{R}^d$ such that the function*

$$h(t) = m_{d-1}\big(K \cap \{\xi :< y, \xi >= t\}\big)$$

belongs to $L^p(\mathbb{R})$. Then any finite measure with Fourier transform supported on K is locally $L^{p'}$-summable where $\frac{1}{p} + \frac{1}{p'} = 1$.

We also give in this section several results about sharpness of this statement. Among them, we show that there exists a Riesz set which is not a Hardy set, i.e. there exists a function with spectrum contained in it, which does not belong to the class $H^1(\mathbb{R}^d)$.

In Section 4 we study conditions on a sequence of zeros of the Fourier transform of a measure which imply that the measure is absolutely continuous. We call a sequence $\Lambda \subset \mathbb{R}^d$ a *co-Riesz sequence* iff every finite measure with Fourier transform vanishing on Λ is absolutely continuous with respect to the Lebesgue measure. We prove that the co-Riesz sequences exist: we construct a co-Riesz sequence on $\mathbb{R}$ with a sequence of differences tending to 0 arbitrarily slowly. More precisely, we prove the following result.

Theorem 3. *No matter how slowly the sequence r_n tends to 0, there exists a co-Riesz sequence Λ such that* $\operatorname{dist}(\lambda_n, \Lambda \backslash \{\lambda_n\}) > r_n$.

We also show that vanishing of the Fourier transform of a function on any sequence without concentration points does not guarantee any additional summability of the function (compare with the Theorem 2). We also study some properties of co-Riesz sequences and formulate some problems.

In Section 5 we apply the method developed in the previous sections to the so called co-Lebesgue sequences. We call a sequence $\Lambda \subset \mathbb{R}^d$ *a* *co-Lebesgue* iff for every measure $\mu \in M(\mathbb{R}^d)$ with the Fourier transform vanishing on Λ, the Fourier transforms of its singular and absolutely continuous parts also vanish on Λ. We formulate a criterion for being co-Lebesgue which is applied to the sequences $(n^{1/k})_{n=1}^{\infty}$ $(k = 2, 3, \dots)$ and $(\log n)_{n=1}^{\infty}$.

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|$. By $\mathbb{T}^d$ we denote d-dimensional torus identified naturally with the cube $(-\frac{1}{2}, \frac{1}{2}]^d \subset \mathbb{R}^d$. All measures are supposed to be finite Borel measures. The space of finite Borel measures of bounded total variation on $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$. By $\|\cdot\|$ we denote the usual norm on this space, i.e. the total variation of a measure. We denote by μ_s the part of μ singular with respect to the Lebesgue measure (cf. [HR, Chapt. III, Th. 14.22]). If a measure μ is absolutely continuous with respect to the Lebesgue measure m_d, there exists a density $f \in L^1(\mathbb{R}^d)$ such that $d\mu = f\, dm_d$. In this case we write for shortness $\mu \in L^1(\mathbb{R}^d)$, i.e we identify the measure with its density. The restriction of a measure μ to a Borel set Ω is denoted by $\mu|\Omega$. By $\widehat{\mu}(\xi) = \int e^{-i2\pi\langle x, \xi\rangle}\, d\mu(x)$ we denote the Fourier transform of the measure $\mu \in M(\mathbb{R}^d)$. For $A, B \subset \mathbb{R}^d$ by $A + B$ we denote the Minkowski sum; rA denotes the set $\{ra \in \mathbb{R}^d : a \in A\}$ $(r \in \mathbb{R})$. By $\operatorname{dist}(x, A)$ we denote the distance between $x \in \mathbb{R}^d$ and the nonempty set $A \subset \mathbb{R}^d$. The symbol C (possibly with indexes) denotes a non-negative constant which can change in value from one occurrence to another.

2. Symmetric Riesz sets.

We begin with the proof of Theorem 1.

Proof of Theorem 1. Suppose that K is not a Riesz set. Then there exists $\mu \in M(\mathbb{R}^d)$ such that $\operatorname{supp}\widehat{\mu} \subset K$ and $\mu_s \neq 0$. Let us choose an integer j such that $|\mu_s|(\alpha_j I^d) > \frac{2}{3}\|\mu_s\|$ (here $I^d = \{x \in \mathbb{R}^d : -\frac{1}{2} < x_k \leq \frac{1}{2}\}$). Let $\nu \in M(\mathbb{T}^d)$ be the measure defined by $\nu(E) = \mu(\alpha_j E + \alpha_j \mathbb{Z}^d)$ for $E \subset \mathbb{T}^d$. It is easy to see that $\widehat{\nu}(\xi) = \widehat{\mu}(\frac{\xi}{\alpha_j})$ for every $\xi \in \mathbb{Z}^d$. Since $\operatorname{supp}\widehat{\mu} \subset K$, the Fourier transform of ν vanishes outside some Riesz subset of $\mathbb{Z}^d$. Hence $\nu_s = 0$. But $\nu_s(E) = \sum_{\xi \in \mathbb{Z}^d} \mu_s(\alpha_j E + \alpha_j \xi)$ and therefore

$$\|\nu_s\| \geq \|(\mu|\alpha_j I^d)_s\| - \|(\mu|\mathbb{R}^d \setminus \alpha_j I^d)_s\| > \frac{1}{3}\|\mu_s\| > 0.$$

This contradiction completes the proof. □

Corollary 1 is a direct consequence of Theorem 1. One just have to shift the f-pole in such a way that it does not contain any line with rational points.

Example 1. Given $\varepsilon > 0$ there exists a closed symmetric (with respect to the origin) set $E \subset L = \{x_1 = 0\} \subset \mathbb{R}^d$ and a Riesz set $K \subset \mathbb{R}^d$ such that $m_L(L \setminus E) < \varepsilon$ and $\mathbb{R} \times E \subset \operatorname{Int} K$.

Let $A \subset L$ be an open symmetric set of measure $m_L(A) < \varepsilon$ and $A_1 \subset A_2 \subset \cdots \subset A$ be a sequence of open symmetric sets such that

$\frac{1}{n}\mathbb{Z}^d \cap L \subset A_n$ and $\overline{A}_n \subset A$ for $n = 1, 2, \ldots$. Then we put $K = \mathbb{R}^d \setminus \bigcup F_n$ where

$$F_n = \{(x_1, x') \in \mathbb{R}^d : |x_1| > \frac{n}{1 + |x'|} - 1 \quad \text{and} \quad (0, x') \in A_n\}.$$

We put $E = L \setminus A$. Clearly $nK \cap \mathbb{Z}^d \subset \{|x_1| \leq \frac{n^3}{n+|x'|} - n\}$ which is a finite subset of $\mathbb{Z}^d$. Hence, by Theorem 1, K is a Riesz set. The remaining property $\mathbb{R} \times E \subset \operatorname{int} K$ is obvious. □

It might happen (however we do not know it) that strengthening of Corollary 1 is valid and every L^1 function with the Fourier transform supported by an f-pole is better then L^1 integrable (i.e. belongs to some fixed Orlicz space), and this could be the reason for being a Riesz set. The next result shows however that this possible improvement can not be uniform for all functions f.

Let Φ be a Young function which define the Orlicz norm on $\mathbb{R}^d$; we denote the corresponding Orlicz space by $L^\Phi(\mathbb{R}^d)$ (cf. [RR]). We say that a function f belongs to $L^\Phi_{loc}(\mathbb{R}^d)$ iff for every $x \in \mathbb{R}^d$ there exists a neighbourhood U such that $f \cdot \chi_U \in L^\Phi(\mathbb{R}^d)$.

Proposition 2. *Let the Young function Φ be such that $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$. Then there exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $F \in L^1(\mathbb{R}^d)$ with the Fourier transform supported on the f-pole K_f, such that $F \notin L^\Phi_{loc}(\mathbb{R}^d)$.*

Proof. Let $\psi \in C^\infty(\mathbb{R}^d)$ be a positive function such that $\|\psi\|_1 = 1$ and its Fourier transform $\widehat{\psi}$ is positive and supported on the unit cube I^d. We can get such a function as the square of an L^1 function with smooth positive Fourier transform supported on $\frac{1}{2}I^d$. Clearly we have $\psi(x) > \sigma > 0$ for $x \in rI^d$ for some $r > 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function decreasing to 0 (to be fixed later). For $n = 1, 2, \ldots$ we define ψ_n by

$$\widehat{\psi}_n(x_1, x') = \widehat{\psi}(\frac{x_1}{2^n}, \frac{x'}{f(2^n)}).$$

Note that

1) $\operatorname{supp} \widehat{\psi}_n \subset K_f$
2) $\psi_n \geq 0$;
3) $\psi_n > 2^n f^{d-1}(2^n)\sigma$ on $E_n = 2^{-n} r I \times \left(\frac{r}{f(2^n)}\right)^{d-1} I^{d-1}$;
4) $\|\psi_n\|_1 = 1$.

Then we put

$$F = \sum_{j=1}^{\infty} \frac{1}{j^2} \psi_{n_j},$$

where an increasing sequence of integers (n_j) will be fixed later. We are going to show that if f is chosen properly then $\int_{\varepsilon I^d} \Phi(\alpha|F|) = \infty$ for every $\varepsilon > 0$ and $\alpha > 0$. Put $\phi(t) = t^{-1}\Phi(t)$. Since $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$, we have $\phi(t) \to \infty$ as $t \to \infty$. Since Φ is superadditive, we have

$$\int_{\varepsilon I^d} \Phi(\alpha|F|) \geq \sum_j \int_{\varepsilon I^d} \Phi(\frac{\alpha \psi_{n_j}}{j^2}).$$

Thus, using properties 1) - 4), we get that for j such that $n_j > j^2(\alpha\sigma)^{-1}$, $\varepsilon f(2^{n_j}) < r$ and $2^{n_j} > \frac{r}{\varepsilon}$

$$\begin{aligned}\int_{\varepsilon I^d} \Phi(\frac{\alpha}{j^2}\psi_{n_j}) &\geq \int_{\varepsilon I^d \cap E_{n_j}} \Phi(\frac{\alpha}{j^2}2^{n_j} f^{d-1}(2^{n_j})\sigma) \\ &\geq \varepsilon^{d-1}\frac{r}{2^{n_j}}\Phi(\frac{\alpha}{j^2}2^{n_j} f^{d-1}(2^{n_j})\sigma) \\ &= \varepsilon^{d-1} r \frac{\alpha}{j^2}\sigma f^{d-1}(2^{n_j})\phi(\frac{\alpha}{j^2}2^{n_j} f^{d-1}(2^{n_j})\sigma).\end{aligned}$$

Put now

$$f(t) = \begin{cases} \max(\frac{1}{\log_2 t}, \phi^{-\frac{1}{d}}(\frac{t}{\log_2^d t})) & \text{for } t > 2^d, \\ \max(\frac{1}{d}, \phi^{-\frac{1}{d}}(\frac{2^d}{d^d})) & \text{for } t < 2^d. \end{cases}$$

Choose sequence (n_j) such that $\phi(2^{n_j} n_j^{-d}) > j^d$ and $n_j > j^3$. Then, using the above estimation and the definition of f, we get for large values of j

$$\begin{aligned}\int_{\varepsilon I^d} \Phi(\frac{\alpha}{j^2}\psi_{n_j}) &\geq \varepsilon^{d-1} r \frac{\alpha}{j^2}\sigma f^{d-1}(2^{n_j})\phi(2^{n_j} n_j^{-d}) \\ &\geq \varepsilon^{d-1} r\alpha\sigma j^{-2}\phi^{\frac{1}{d}}(2^{n_j} n_j^{-d}) \\ &\geq \varepsilon^{d-1} r\alpha\sigma \cdot j^{-1}.\end{aligned}$$

Hence the integral $\int_{\varepsilon I^d}\Phi(\alpha|F|)$ is estimated from below by a tail of a divergent series. □

Using now the well known fact that $H^1_+(\mathbb{R}^d) \subset (L\log L)_{loc}(\mathbb{R}^d)$ (cf. [St, Chapt. III.5.3]) and that the constructed function is positive, we get as a corollary that on $\mathbb{R}^d$ the class of Riesz sets is wider that the class of Hardy sets.

Corollary 2. *There exists an f-pole $K_f \subset \mathbb{R}^d$ which is not a Hardy set, i.e. there exists $F \in L^1(\mathbb{R}^d)$ with Fourier transform supported by K_f such that $F \notin H^1(\mathbb{R}^d)$.* □

3. Proof of Theorem 2.

We can assume that $y = (1, 0, \ldots, 0)$. Let $\mu \in M(\mathbb{R}^d)$ satisfies $\operatorname{supp}\widehat{\mu} \subset K$. For $t > 0$ we put $\mu_t = \mu * P_t$, where $\{P_t\}_{t>0}$ are Poisson kernels. Clearly $\mu_t \in L^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\|\mu_t\|_1 \leq \|\mu\|$. It is also clear that $\operatorname{supp}\widehat{\mu}_t \subset K$ and $\widehat{\mu}_t \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. We have

$$\begin{aligned}\mu_t(x) &= \int_{\mathbb{R}^d} \widehat{\mu}_t(\xi) e^{2\pi i\langle(x_1,x'),\xi\rangle}\, d\xi \\ &= \int_{-\infty}^{\infty} Pr_1(\widehat{\mu}_t(\xi)e^{2\pi i\langle x',\xi'\rangle})(\xi_1)e^{2\pi i x_1\xi_1}\, d\xi_1 \\ &= (Pr_1(\widehat{\mu}_t(\xi_1,\cdot)e^{2\pi i\langle x',\cdot\rangle}))^\wedge(-x_1),\end{aligned}$$

where

$$Pr_1(f)(s) = \int\limits_{\{x_1=s\}} f(x_1,x')\, dm_{d-1}(x').$$

Since $|\widehat{\mu}_t(\xi)e^{2\pi i\langle x',\xi'\rangle}| \leq \|\mu_t\|_1 \leq \|\mu\|$, and $m_{d-1}(\{x \in K : x_1 = s\}) = h(s)$, we get

$$\|Pr_1(\widehat{\mu}_t(\xi_1,\cdot)e^{2\pi i\langle x',\cdot\rangle})\|_{L^p(d\xi_1)} \leq \|\mu\| \cdot \|h\|_p.$$

Hence, if $\frac{1}{p} + \frac{1}{p'} = 1$, by the Hausdorff-Young inequality,

$$\|(Pr_1(\widehat{\mu}_t(\xi_1,\cdot)e^{2\pi i\langle x',\cdot\rangle}))^\wedge\|_{p'} \leq \|\mu\| \cdot \|h\|_p.$$

Thus $\|\mu_t(\cdot,x')\|_{p'} \leq \|\mu\| \cdot \|h\|_p$ for every $x' \in \mathbb{R}^{d-1}$.

Let $y = (y_1, y') \in \mathbb{R}^d$ and $U = \mathbb{R}\times\Omega$ be an open neighborhood of y such that $\Omega \in \mathbb{R}^{d-1}$ is an open neighborhood of y' with finite $(d-1)$-dimensional Lebesgue measure. Then

$$\begin{aligned}\int_U |\mu_t|^{p'} dm_d &= \int_\Omega \int_{\mathbb{R}} |\mu_t(x_1,x')|^{p'} dx_1 dx' \\ &\leq m_{d-1}(\Omega) \cdot \|\mu\|^{p'} \|h\|_p^{p'}.\end{aligned}$$

Hence there exists $C > 0$ such that for $t > 0$,

$$\|\mu_t\|_{L^{p'}(U)} \leq C.$$

By the assumption $(\mu_t)_{|U} \to \mu_{|U}$ in the $*$-weak topology. Since $\|(\mu_t)_{|U}\|_{p'}$ is bounded for $t > 0$, we get that $\mu_{|U} \in L^{p'}(U)$. □

The f-pole with $f(t) = t^{-\frac{q-1}{q(d-1)}}$ is called a q-pole.

Corollary 3. *Let $2 \leq p < \infty$. If the support of the Fourier transform of a measure μ is contained in a finite union of q-poles, where $q > p$, then $\mu \in L^p_{loc}(\mathbb{R}^d)$.*

Corollary 3 gives another proof that q-poles are Riesz sets for $q > 2$. However, by applying Theorem 2, one can construct Riesz sets which do not seem to be treated by Theorem 1.

Example. Let $K \in \mathbb{R}^d$ be any q-pole ($q > 2$) which does not contain a line orthogonal to the first coordinate. Let $K_n = K \cap \{n \leq x_1 \leq n+1\}$. If $(r_n)_{n=-\infty}^{\infty} \subset \mathbb{R}^d$ is any sequence with bounded first coordinate then the set $\bigcup_{n\in\mathbb{Z}}(K_n + r_n)$ satisfies the assumption of Theorem 2 for $p > q'$.

Corollary 3 shows that every q-pole is a "local" Λ_p for every $q > p \geq 2$. Next remark shows that (a) $loc\Lambda_q \not\subset \Lambda_p$ for $2 < q < \infty$ and $1 < p < \infty$, and (b) $loc\Lambda_p \neq loc\Lambda_q$ for $p, q \geq 2$ and $p \neq q$.

Proposition 3.

a) *Let $1 < q < \infty$. There exists a function $F \in L^1(\mathbb{R}^d)$ with the Fourier transform supported on q-pole, such that $F \notin L^p(\mathbb{R}^d)$ for any $1 < p < \infty$.*

b) *Let $1 < q < \infty$. There exists a function F with the Fourier transform supported on q-pole, such that $F \notin L^p_{loc}(\mathbb{R}^d)$ for every $p > q$.*

Proof. We use the function F constructed in the proof of Proposition 2. This time $f(t) = \min(1, t^{-\frac{q-1}{q(d-1)}})$ and $n_j = j$. Hence

$$\begin{aligned}
\|F\|_p &\geq j^{-2}\|\psi_{n_j}\|_p \\
&\geq j^{-2}\big(|E_{n_j}| \cdot (2^{n_j} f^{d-1}(2^{n_j})\sigma)^p\big)^{1/p} \\
&= j^{-2} r^{d/p} \sigma \big(2^{n_j} f^{d-1}(2^{n_j})\big)^{\frac{p-1}{p}} \\
&= j^{-2} r^{d/p} \sigma 2^{n_j(\frac{1}{q}\cdot\frac{p-1}{p})} \to \infty
\end{aligned}$$

as $j \to \infty$. This proves part (a). For part (b) we have

$$\begin{aligned}
\int_{\varepsilon I^d} |F|^p &\geq j^{-2} \int_{\varepsilon I^d} |\psi_{n_j}|^p \\
&\geq j^{-2} |\varepsilon I^d \cap E_{n_j}| \cdot \big(2^{n_j} f^{d-1}(2^{n_j})\sigma\big)^p \\
&= j^{-2} \varepsilon^{d-1} r \sigma^p 2^{n_j(\frac{p}{q}-1)} \to \infty
\end{aligned}$$

as $j \to \infty$ for any fixed $\varepsilon > 0$. □

4. co-Riesz sequences on $\mathbb{R}$.

We call $\Lambda = (\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}^d$ *a co-Riesz sequence* iff every measure $\mu \in M(\mathbb{R}^d)$ such that $\widehat{\mu}(\lambda) = 0$ for $\lambda \in \Lambda$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

We do not know if every sequence $\Lambda = (\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}$ such that $\lim_{|x|\to\infty} \operatorname{dist}(\Lambda, x) = 0$ (resp. $\lim_{x\to\infty} \operatorname{dist}(\Lambda, x) = 0$) is a co-Riesz sequence. (Note that if Λ is monotone then the second condition is equivalent to $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$).

On the other hand we can show that some very particular sequences from this class are indeed co-Riesz. This is in the case when Theorem 1 could be applied. Note that, despite the fact that Theorem 1 is formulated for Riesz sets which by definition are closed, it remains valid in this setting - in the proof of Theorem 1 we only use the values of the Fourier transform at the points from some special (countable) set.

Proof of Theorem 3. Without loss of generality we can assume that (r_j) is a non-increasing sequence consisting of powers of 2. Moreover we can assume that $\sum r_j = \infty$. Then we put $\lambda_n = \sum_{j=0}^{n} r_j$ for $n = 1, 2, \ldots$. It is easy to check that for every $n = 1, 2, \ldots$ the intersection $2^n(\mathbb{R} \setminus \Lambda) \cap \mathbb{Z}$ is a set contained in a halfline. Hence it follows from the theorem of F. and M. Riesz (cf. [HJ, 1.1.3, p.13]) and Theorem 1 that Λ is a co-Riesz sequence. □

Remark. An obvious modification of the proof of Theorem 3 gives its analogue for several variables. Namely one can prove that for every sequence (r_n) decreasing to 0 there exists a co-Riesz sequence $\Lambda = (\lambda_n) \subset \mathbb{R}^d$ such that $\mathrm{dist}(\lambda_n, \Lambda \setminus \{\lambda_n\}) > r_n$ for $n = 1, 2, \dots$.

It appears that there is no estimation for the growth of the distribution of the values of a function with the Fourier transform vanishing on a sequence Λ.

Proposition 4. *For every sequence $\Lambda \subset \mathbb{R}^d$ with no condensation points, and every Young function Φ such that $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$, there exists $f \in L^1(\mathbb{R}^d) \setminus L^\Phi_{loc}(\mathbb{R}^d)$ such that $\widehat{f}(\lambda) = 0$ for $\lambda \in \Lambda$.*

Proof. Put $\phi(t) = t^{-1}\Phi(t)$. Since $L^1(\mathbb{R}^d) \neq L^\Phi(\mathbb{R}^d)$, $\phi(t) \to \infty$ as $t \to \infty$. Let ψ be the function from the proof of Proposition 2 which is, additionally, uniformly decreasing to 0 at infinity. Let $x_0 = (1, 0, \dots, 0) \in \mathbb{R}^d$ and put for $m, k \in \mathbb{Z}$

$$f_{m,k}(x) = 2^{md}\psi(2^m x) - 2^{md}\psi(2^m x + 2^m k x_0).$$

Obviously $\widehat{f}_{m,k}(\xi) = (1 - e^{i2\pi k\langle \xi, x_0\rangle})\widehat{\psi}(\xi 2^{-m})$. Thus the Fourier transform of $f_{m,k}$ is supported on the cube $2^m I^d$ and its absolute value is less then $|1 - e^{i2\pi k\langle \xi, x_0\rangle}|$ for $\xi \in \mathbb{R}^d$. Since $\psi > 0$ and ψ is uniformly decreasing at infinity, for every $m \in \mathbb{Z}$ there exists $k = k(m) \in \mathbb{Z}$ such that $f_{m,k}$ is positive on the cube I^d and $f_{m,k}(x) > 2^{md}\sigma$ for $x \in r2^{-m}I^d$. Since the set $\Lambda \cap 2^m I^d$ is finite, for every $\varepsilon > 0$ we can always find an (arbitrarily large) integer N such that $\mathrm{dist}(N\langle \lambda, x_0\rangle, \mathbb{Z}) < \varepsilon$ for $\lambda \in \Lambda \cap 2^m I^d$. Hence there exists integer k_m such that we have $\widehat{f}_{m,k_m}(\lambda) < \frac{1}{M}$ for $\lambda \in \Lambda \cap 2^m I^d$, where $M = \#(\Lambda \cap 2^m I^d)$. Clearly $\widehat{f}_{m,k_m}(\lambda) = 0$ for $\lambda \in \Lambda \setminus 2^m I^d$. Put

$$h_1 = \sum \frac{1}{n^2} f_{m_n, k_{m_n}},$$

where the numbers m_n $(n = 1, 2, \dots)$ are going to be fixed later. We prove that for every $\alpha > 0$ the function $\Phi(\alpha h_1)$ is not integrable on any fixed neighbourhood of the origin, say aI^d (note that we can assume that h_1 is positive in aI^d). Since Φ is a superadditive function,

$$\int_{aI^d} \Phi(\alpha h_1) \geq \sum \int_{aI^d} \Phi(\frac{\alpha}{n^2} f_{m_n, k_{m_n}}).$$

If $r2^{-m_n} < a$ we have

$$\begin{aligned} \int_{aI^d} \Phi(\frac{\alpha}{n^2} f_{m_n, k_{m_n}}) &\geq \int_{r2^{-m_n} I^d} \Phi(\frac{\alpha\sigma}{n^2} 2^{m_n d}) \\ &= 2r\frac{\alpha\sigma}{n^2}\phi(\frac{\alpha\sigma}{n^2} 2^{m_n d}). \end{aligned}$$

If m_n is chosen to satisfy $\phi(n^{-3}2^{m_n d}) > n$, then for n sufficiently large

$$\int_{aI^d} \Phi(\frac{\alpha}{n^2} f_{m_n, k_{m_n}}) \geq 2r\alpha\sigma n^{-1}.$$

Thus $h_1 \notin L^{\Phi}_{loc}(\mathbb{R}^d)$. Let us consider now the functions g_{m_n} defined by

$$\widehat{g}_{m_n}(\xi) = \sum_{\lambda \in \Lambda \cap 2^{m_n} I^d} \widehat{f}_{m_n,k_{m_n}}(\lambda) \widehat{\psi}(\sqrt{d}\ \frac{\xi - \lambda}{\tau_{m_n}}),$$

where $\tau_m = \min\{1, |\lambda - \lambda'| : \quad \lambda, \lambda' \in \Lambda \cap (2^m + 1)I^d, \quad \lambda \neq \lambda'\}$. Then, obviously, $\widehat{g}_{m_n}(\lambda) = \widehat{f}_{m_n,k_{m_n}}(\lambda)$ for all $\lambda \in \Lambda$. Also

$$\|g_{m_n}\|_1 \leq \sum_{\lambda \in \Lambda \cap 2^{m_n} I^d} |\widehat{f}_{m_n,k_{m_n}}(\lambda)| < 1,$$

and

$$\begin{aligned} \|g_{m_n}\|_\infty &\leq \|\widehat{g}_{m_n}\|_1 \\ &\leq \sum_{\lambda \in \Lambda \cap 2^{m_n} I^d} |\widehat{f}_{m_n,k_{m_n}}(\lambda)| \tau^d_{m_n} \sqrt{d^{-d}} \|\widehat{\psi}\|_1 \\ &\leq \sqrt{d^{-d}} \|\widehat{\psi}\|_1. \end{aligned}$$

Thus the function $h_2 = \sum \frac{1}{n^2} g_{m_n}$ is bounded, summable, and $\widehat{h}_1(\lambda) = \widehat{h}_2(\lambda)$ for all $\lambda \in \Lambda$. Hence the function $f = h_1 - h_2 \in L^1(\mathbb{R}^d)$, its Fourier transform vanishes on Λ and f is still not in $L^{\Phi}_{loc}(\mathbb{R}^d)$. □

Remark. If $\Phi(x) = x \log(x+1)$ then one can modify the above construction to get the function which does not belong to $H^1(\mathbb{R})$.

Proof. After some minor modification we can assume that h_2 is continuously differentiable. Indeed, during the construction, when we define numbers k_m, we replace the condition $\widehat{f}_{m,k_m}(\lambda) < \frac{1}{M}$ by the condition $\widehat{f}_{m,k_m}(\lambda) < \frac{1}{(2^m+1)M}$. Then we get the estimation on the gradient of g_{m_n} in the same way as the estimate for the sup norm of g_{m_n} in the proof above. Let χ be a smooth function supported on $2I^d$ such that $\chi \equiv 1$ on I^d. Since the function

$$h_3(x) = h_2(x)\chi(x) - h_2(x - 3x_0)\chi(x - 3x_0)$$

belongs to the space H^1, the function f belongs to H^1 iff $f + h_3$ does. The function $f + h_3$ is positive on the cube I^d, because it coincides there with h_1. Hence, by [St, Chapt. III.5.3], the restriction $(f + h_3)|_{I^d} = h_1|_{I^d}$ should agree with an $L \log L$-summable function on every compact subset of I^d, which is not the case. □

In the proof of Theorem 3 the arithmetic relations between elements of Λ were crucial. On the other hand, the set of non co-Riesz sequences is open in the following sense.

Proposition 5. *For any finite measure $\mu \in M(\mathbb{R})$ and a sequence $\Lambda = (\lambda_n) \subset \mathbb{R}$ which has no condensation points, there exists a sequence of positive numbers (r_n) such that for every sequence $\Lambda' = (\lambda') \subset \mathbb{R}$ such that $|\lambda_n - \lambda'_n| < r_n$ $(n = 1, 2, \ldots)$, there exists a measure $\mu' \in M(\mathbb{R})$ such that $\widehat{\mu}(\lambda_n) = \widehat{\mu}'(\lambda'_n)$ for $n = 1, 2, \ldots$ and $\mu_s = \mu'_s$.*

Proof. Without loss of generality we suppose that $\|\mu\| = 1$. Let $f_\mu(r) = |\mu|(\mathbb{R} \setminus [-r, r])$. We need the following lemma.

Lemma 1. *Given $c \in (0,1)$, $r > 0$, $x \in \mathbb{R}$ and a measure $\mu \in M(\mathbb{R})$ with $\|\mu\| = 1$, there exists a measure $\nu = \nu(c,r,x) \in M(\mathbb{R})$ absolutely continuous with respect to Lebesgue measure, such that* $\operatorname{supp}\widehat{\nu} \subset [x-2r, x+2r]$, $\widehat{\mu}(x) = (\mu-\nu)^\wedge(y)$ *for every* $y \in (x-r, x+r)$, *and* $\|\nu\| < C\big(c+f_\mu(\frac{c}{r})\big)$, *where the constant C does not depend from c, r, x and μ.*

We show first how Proposition 5 follows from the lemma. Let (r_n) and (c_n) be sequences of positive numbers such that the intervals $[\lambda_n - 2r_n, \lambda_n + 2r_n]$ are pairwise disjoint, $\sum c_n < \infty$ and $\sum f_\mu(\frac{c_n}{r_n}) < \infty$. Then $\mu' = \mu - \sum \nu(c_n, r_n, \lambda_n)$ is a finite measure which satisfies all the requirements.

Proof of Lemma 1. Let $\psi \in L^1(\mathbb{R})$ be such that $\|\psi\|_1 < C$, $\|\psi'\|_\infty < C$, $|\psi'(x)| < \frac{C}{|x|^2}$, $\operatorname{supp}\widehat{\psi} \subset [-2,2]$ and $\widehat{\psi}(x) = 1$ for $x \in [-1,1]$. Put $\rho_R(x) = \sup_{y\in[x-R,x+R]} |\psi'(y)|$. It is easy to see that $\|\rho_R\|_1 < C_1 \max(1,R)$. We denote $\psi_t(x) = t\psi(tx)$. Without loss of generality we suppose that $x = 0$. We consider first the case $\widehat{\mu}(0) = 0$. Then we have $\|\mu * \psi_r\| \leq C_2(c + f(\frac{c}{r})$. Indeed, since $\widehat{\mu}(0) = 0$, we can represent $\mu = \mu^R + \mu_R$ where μ_R is supported on the interval $[-R,R]$, $\int_{\mathbb{R}} d\mu_R = 0$ and $\|\mu^R\| < 2f(R)$. Then we estimate the convolution separately for μ_R and μ^R:

$$\|\mu^R * \psi_r\| \leq \|\mu^R\| \cdot \|\psi_r\|_1 \leq C \cdot f(R),$$

$$|(\mu_R * \psi_r)(x)| = |\int \psi_r(x-\cdot) - \psi_r(x)\, d\mu_R| \leq R \sup_{y\in[x-R,x+R]} |\psi_r'(y)| \cdot \|\mu_R\|.$$

Since $\sup_{y\in[x-R,x+R]} |\psi_r'(y)| = r^2 \rho_{rR}(rx)$ we get

$$\|\mu_R * \psi_r(x)\|_1 \leq CrR\max(1,rR)\|\mu\|.$$

Putting $R = \frac{c}{r}$ we get the desired estimation. Hence the measure $\nu = \mu * \psi_r$ satisfies the conditions of the lemma. If $\widehat{\mu}(0) \neq 0$ we put $\nu = (\mu - (\int d\mu)\delta_0) * \psi_r$. □

Remarks. 1) It follows from the proof that if Λ satisfies assumption of Proposition 5 and $\widehat{\mu}(\lambda) = 0$ for $\lambda \in \Lambda$ then there exists $\mu' \in M(\mathbb{R})$ such that $\mu_s = \mu_s'$ and $\widehat{\mu}'$ vanishes on some open set containing Λ.

2) Proposition 5 can be easily extended to a multidimensional case.

The next result shows that in the previous proposition the sequence (r_n) could not be chosen uniformly for all measures.

Proposition 6. *For every positive sequence (r_j) there exist sequences $\Lambda = (\lambda_n)$ and $\Lambda' = (\lambda_n')$ and $\mu \in M(\mathbb{R})$ such that $|\lambda_j - \lambda_j'| < r_j$ and there is no $\mu' \in M(\mathbb{R})$ such that $\widehat{\mu}(\lambda_n) = \widehat{\mu}'(\lambda_n')$ for $n = 1, 2, \ldots$.*

Proof. We set $\Lambda' = \mathbb{Z}$. We index the sequences Λ and Λ' by integer numbers rather then natural ones. Let (a_n) be a decreasing sequence of positive numbers such that $\sum a_n < \infty$ and (k_n) be the sequence of positive integers such that $\sum k_n a_n^2 = \infty$. Put $b_n = a_j$ where j is the unique index such that $k_1 + \cdots + k_{j-1} < n \leq k_1 + \cdots + k_{j-1} + k_j$ (here we put $k_0 = 0$). Set

$$\lambda_m = \begin{cases} m & \text{if } m \neq 2^j,\ (j = 1, 2, \ldots) \\ m + \frac{1}{2}\omega_j^{-1} & \text{if } m = 2^n, \quad k_1 + \cdots + k_{j-1} < n \leq k_1 + \cdots + k_j, \end{cases}$$

where (ω_j) is a sequence of positive integers satisfying for $j = 1, 2, \dots$

1) $(2\omega_j)|\omega_{j+1}$;
2) $\omega_j^{-1} < \min\{r_{2^n} : n < k_1 + \dots + k_j\}$.

Let $\mu_n = (2i)^{-1}(\delta_{\omega_n} - \delta_{-\omega_n})$ and $\mu = \sum a_n \mu_n$ We have $\widehat{\mu}_n(t) = \sin \pi \omega_n t$. Hence $\widehat{\mu}_j(\lambda_{2^n})$ is positive for $n \le k_1 + \dots + k_{j-1}$, equals 1 for $k_1 + \dots + k_{j-1} < n \le k_1 + \dots + k_j$ and vanishes for $k_1 + \dots + k_j < n$. Thus $\widehat{\mu}(\lambda_{2^n}) > b_n$ for $n = 1, 2 \dots$. Clearly $\widehat{\mu}(\lambda_n) = 0$ for $n \ne 2^j$, $(j = 1, 2, \dots)$.

Suppose to the contrary that there exists a finite measure μ' such that $\widehat{\mu}'(j) = \widehat{\mu}(\lambda_j)$. By the de Leeuw transference theorem (cf. [deL], [StW, Chapt. VII, Th. 3.8]), there exists a bounded measure $\nu \in M(\mathbb{T})$ such that $\|\nu\|_{M(\mathbb{T})} \le \|\mu'\|_{M(\mathbb{R})}$ and $\widehat{\nu}(n) = \widehat{\mu}'(n)$ for $n = 1, 2, \dots$. But $\sum |\widehat{\nu}(n)|^2 = \sum b_j^2 = \infty$ which contradicts the fact that $A = \{2^k : k = 1, 2, \dots\}$ is a Λ_2 set, i.e. $\nu \in L^2(\mathbb{T})$ and $\|\nu\|_2 \le C\|\nu\|_M$ for every measure $\nu \in M(\mathbb{T})$ with the Fourier transform vanishing outside A. □

The above construction has one more application. We can use it to construct a sequence which does not allow co-balayage.

Proposition 7. *There exists a sequence $\Lambda = (\lambda_n)$ such that*

$$\inf_{\lambda \in \Lambda} \operatorname{dist}(\lambda, \Lambda \setminus \{\lambda\}) > 0,$$

and measure $\mu \in M(\mathbb{R})$ such that there is no measure $\mu' \in M(\mathbb{R})$ supported on a compact set such that $\widehat{\mu}(\lambda_n) = \widehat{\mu}'(\lambda_n)$ for $n = 1, 2, \dots$.

Proof. Let Λ and μ be the same as in the proof of Proposition 6 with one modification: the condition 2) on the sequence (ω_j) is replaced by another condition

2') $\omega_j^{-1} < j^{-1} a_j$.

Suppose that there exists $\mu' \in M(\mathbb{R})$ such that $\widehat{\mu}'(\lambda_n) = \widehat{\mu}(\lambda_n)$ for $n = 1, 2, \dots$ and $\operatorname{supp} \mu' \subset [-T, T]$ for some $T > 0$. Then the derivative of $\widehat{\mu}$ is bounded by $T \cdot \|\mu'\|_M$. Hence, for sufficiently large n, we have

$$\begin{aligned} |\widehat{\mu}'(2^n)| &\ge |\widehat{\mu}'(\lambda_{2^n})| - T \cdot \|\mu'\|_M \cdot |\lambda_{2^n} - 2^n| \\ &= |\widehat{\mu}'(\lambda_{2^n})| - T \cdot \|\mu'\|_M \cdot (2\omega_j)^{-1} \\ &> \frac{1}{2} |\widehat{\mu}'(\lambda_{2^n})|. \end{aligned}$$

Thus $\sum |\widehat{\mu}'(n)|^2 = \infty$, and we finish proceeding as in the proof of Proposition 6. □

Remarks. 1) Note that for $\Lambda = \mathbb{Z}$ measure μ' with properties postulated by Proposition 7 exists and it is supported be the interval of length 1. This is exactly what the de Leeuw theorem says.

2) We say that a sequence $\Lambda \subset \mathbb{R}$ has *de Leeuw property* iff for every measure $\mu \in M(\mathbb{R})$ there exists a measure $\mu' \in M(\mathbb{R})$ with compactly supported Fourier transform, such that $\widehat{\mu}'(\lambda) = \widehat{\mu}(\lambda)$ for every $\lambda \in \Lambda$. By the de Leeuw transference theorem, for every finite set $F \subset \mathbb{R}$ and $r \in \mathbb{R}$, any subset of the set $F + r\mathbb{Z}$ has the de Leeuw property. We do not know whether the converse is true.

3) It is much easier to construct a sequence Λ without the de Leeuw property if we skip the condition $\inf_{\lambda\in\Lambda} \operatorname{dist}(\lambda, \Lambda \setminus \{\lambda\}) > 0$. Moreover, every sequence Λ which contains an increasing subsequence (x_n) such that $\lim x_n = \infty$ and $\lim(x_{2n} - x_{2n+1}) = 0$, has no de Leeuw property. Indeed, let $\nu \in L^1(\mathbb{R})$ be a measure with Fourier transform supported on the interval $[-1, 1]$ such that $\nu(0) = 1$ and let $\nu_r \in M(\mathbb{R})$ be defined by $\widehat{\nu}_r(t) = \widehat{\nu}(\frac{t}{r})$. Passing, if necessary, to a subsequence we can assume that $\sum r_n^{1/2} < \infty$ where $r_n = x_{2n+1} - x_{2n} < x_{2n} - x_{2n-1}$ for $n = 1, 2, \dots$. Put $\mu = \sum r_n^{1/2} \nu_{r_n} e^{2\pi i x_{2n} t}$. Then we have $\|\mu\| < \|\nu\| \cdot \sum r_n^{1/2} < \infty$, $\widehat{\mu}(x_{2n}) = r_n^{1/2}$ and $\widehat{\mu}(x_{2n+1}) = 0$ for $n = 1, 2, \dots$. Hence the supremum of the derivative of $\widehat{\mu}$ on the interval (x_{2n}, x_{2n+1}) is greater then $r_n^{-1/2}$. Therefore the derivative of $\widehat{\mu}$ is unbounded, which means that μ is not compactly supported.

5. co-Lebesgue sequences.

We call the sequence $\Lambda \in \mathbb{R}^d$ *a co-Lebesgue* sequence iff for every measure $\mu \in M(\mathbb{R}^d)$ such that $\widehat{\mu}(\xi) = 0$ for $\xi \in \Lambda$ its singular part μ_s shares the same property, i.e. $\widehat{\mu}_s(\xi) = 0$ for $\xi \in \Lambda$. Clearly every co-Riesz sequence is co-Lebesgue. A slight modification of Theorem 1 allows to formulate the following criterion.

Proposition 8. *Assume that $\Lambda \in \mathbb{R}^d$ has the following property. For every $\xi \in \Lambda$ there exists $\alpha \in \mathbb{R}$ such that $\mathbb{Z}^d \setminus \alpha\Lambda$ is a Riesz set, and $\alpha\xi \in \mathbb{Z}^d$. Then Λ is a co-Lebesgue sequence.*

Proof. Let $\xi \in \Lambda$ and $\alpha \in \mathbb{R}$ be such that $\alpha\xi \in \mathbb{Z}^d$ and $\mathbb{Z}^d \setminus \alpha\Lambda$ is a Riesz set. Let $\nu \in M(\mathbb{T}^d)$ be the measure defined by $\nu(E) = \mu(\alpha E + \alpha\mathbb{Z}^d)$ for $E \subset \mathbb{T}^d$. Clearly $\nu_s(E) = \mu_s(\alpha E + \alpha\mathbb{Z}^d)$. It is easy to see that for every $k \in \mathbb{Z}^d$,

$$\widehat{\nu}(k) = \widehat{\mu}(\tfrac{1}{\alpha}k),$$

as well as

$$\widehat{\nu}_s(k) = \widehat{\mu}_s(\tfrac{1}{\alpha}k).$$

Since $\widehat{\mu}(\xi) = 0$ for $\xi \in \Lambda$, the Fourier transform of ν vanishes outside some Riesz subset of $\mathbb{Z}^d$. Hence, by the assumption, $\nu_s = 0$. Since $\alpha\xi \in \mathbb{Z}^d$, the above formula yields that $\mu_s(\xi) = \nu_s(\alpha\xi) = 0$. $\square$

Examples. 1) Let $k = 2, 3, \dots$. Then the sequence $\Lambda_k = (n^{1/k})_{n=1}^{\infty} \subset \mathbb{R}$ is co-Lebesgue one. Indeed, let $a \in \Lambda_k$. Then $a^k \in \mathbb{Z}$. Therefore $j^k a^k \in \mathbb{Z}$ for $j = 1, 2, \dots$. Hence $ja \in \Lambda_k$ for $j = 1, 2 \dots$. Therefore $\frac{1}{a}\Lambda_k \cap \mathbb{Z} = \mathbb{Z}_+$, and, by F. and M. Riesz theorem, $\mathbb{Z} \setminus (\frac{1}{a}\Lambda_k)$ is a Riesz set.

2) Let $\Lambda_0 = (\log n)_{n=1}^{\infty}$. If $a = \log m \in \Lambda_0$ then $na = \log m^n \in \Lambda$ for $n = 1, 2, \dots$ and hence, similarly as in Example 1, $\mathbb{Z} \setminus \frac{1}{a}\Lambda_0 = \mathbb{Z}_-$ is a Riesz set.

Remarks. 1) In fact, Proposition 8 together with the above example give something more, namely if $\widehat{\mu}(\xi) = 0$ for $\xi \in \Lambda_k$ then $\widehat{\mu}_s(\xi) = 0$ for $\xi \in \Lambda_k \cup -\Lambda_k$.

2) We do not know whether Λ_k are co-Riesz sequences.

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