Abstract

We construct a convolution-thresholding approximation scheme which is motivated by kinetic type of modeling and is consistent with the level-set equation for the generalised mean curvature evolution. Conditions for the monotonicity of the scheme are found and the convergence of the approximations to the corresponding viscosity solution is proved. We also discuss some aspects of the numerical implementation of such schemes and present several numerical tests.

Keywords: convolution-thresholding scheme, viscosity solution, level-set equation, curvature flow, kinetic modeling

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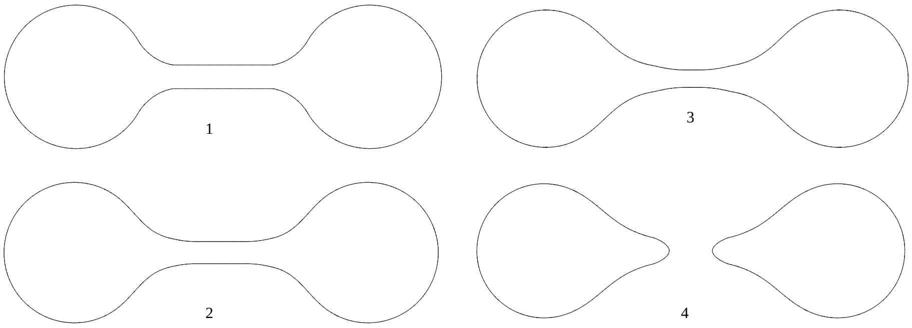

Figure 1: Example: a topological type changes during the evolution.

1 Background

During the last 20 years a considerable amount of mathematical work was devoted to the study of the following challenging problem. Consider a smooth hypersurface in $\mathbb{R}^n$. We let each point of the hypersurface move in the direction of the normal with the velocity proportional to the mean curvature at that point. If we denote the surface at time t by Γ_t, then the primary task is to study geometric properties of $\{\Gamma_t\}_{t>0}$ in terms of Γ_0.

Classical methods of differential geometry give a reasonable description of the evolution only in case $n = 2$. Under this condition a smooth curve Γ_0 remains smooth as time progresses. In the three dimensional case, however, one can easily find a smooth initial surface, which will develop singularities after some finite time. The classical example (see [32]) of such a surface is a boundary of a "dumb-bell" region, as illustrated in Figure 1. The cylindrical part in the middle will collapse at some critical time moment t_* , while the ends will still bound some finite volumes. After this moment, the classical motion by mean curvature is undefined. Even if we would extend the evolution for times $t > t_*$, we would expect the surface to be a boundary of two separate regions. This implies the change of the topological type of Γ_t. The parametrisation of Γ_0 develops a singularity in finite time.

As this example illustrates, in order to define the evolution for all $t > 0$, one has to come up with an approach general enough to deal with singularities and topological complications. There have been three successful attempts to do so: the varifold approach, the level-set approach and the phase field method.

A varifold generalisation of the problem was done by Brakke in [7]. The author recasts the problem in arbitrary codimension into settings of geometric measure theory. In this paper, an appropriate varifold subsolution for all $t > 0$ is defined and constructed. Brakke establishes many geometrical as well as some regularity properties of the solution. However, as shown by Angenent, Ilmanen and Chopp in [2], even if the initial data is smooth, the evolution may possess non-uniqueness after the formation of a singularity.

An alternative approach to the extension of the surface evolution past singularities is the so called level-set approach. It was suggested in the physical literature [27] and was extensively developed by Osher and Sethian [28]. The main idea of this method is to evolve some continuous function $g : \mathbb{R}^n \mapsto \mathbb{R}$ in such way that $\Gamma_t \subset \mathbb{R}^n$ would always be a level-set of $g(x,t)$ i.e.

$$\Gamma_t = \{x \in \mathbb{R}^n : g(x,t) = 0\} \ \forall t \geq 0.$$

Let us now formally derive the evolution equation for g. We assume that g is smooth and consider $x \in \Gamma_0$. The unit normal vector field of each level set of g is $\nu(x,t) = \frac{Dg}{|Dg|}$. Hence the ($(n-1)$ times) mean curvature vector field is

$$\frac{dx}{dt} = -\frac{1}{n-1} div(\nu)\,\nu.$$

Using the fact, that Γ_0 is a level-set i.e. $g(x,t) = const$, we write

$$0 = \frac{d}{dt} g((x(t), t) = -(Dg \cdot \nu)\, div(\nu) + g_t$$

Setting now $\nu(x,t) = \frac{Dg}{|Dg|}$, we obtain

$$g_t = |Dg|\, div\left(\frac{Dg}{|Dg|}\right). \qquad (1)$$

Let us make the following observation. Let $g(x,t)$ satisfy (1) and $u = \Psi(g)$, where Ψ is smooth and monotone. Then a direct calculation shows, that u also satisfies (1). This property reflects the fact, that the evolution of each level set depends only on its geometrical properties (namely mean curvature) and not on other level-sets. The equation is invariant with respect to monotone transformations of g.

Using reasoning similar to the one above, we can derive the evolution equation for a function u with each point of a level-set moving along the normal with velocity equal to some function G of the mean curvature. This, so called generalised level-set equation, or generalised mean curvature evolution PDE is

$$u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right). \tag{2}$$

This equation is of degenerate parabolic type. The existence and uniqueness of generalised viscosity solutions, see [14] to the Cauchy problem

$$\begin{cases} u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right) & \text{in } \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{3}$$

was investigated in [19], [12], [23].

Let us mention some applied problems, where such an evolution arises in a natural way.

We begin with a fast reaction-slow diffusion problem

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} V_u(u) \quad u(x,0,\epsilon) = g(x) \quad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

where the potential $V : \mathbb{R} \mapsto \mathbb{R}$ has several local minima $u_1, u_2, \ldots, u_k$. The asymptotic behaviour of u for small ϵ was studied by Rubinstein, Stenberg and Keller in [30]. They showed that $u(x,t,\epsilon)$ tends to u_j at those points x where $g(x)$ is in the basin of attraction of u_j for the pure reaction equation $u_t = -\epsilon^{-1} V_u(u)$. This implies, that the solution tends to be piecewise constant. The study of the boundary between two regions where the solution is equal to say u_i and u_j shows that it moves along the normal direction towards the region with greater u with the velocity that is proportional to $[V] = V(u_i) - V(u_i)$. In the situation when $[V] = 0$, the second order asymptotic analysis gives the velocity of the front equal to ϵk, where k is the mean curvature of the front.

Connections between the reaction-diffusion problem and curvature flows has been intensively investigated (see [3], [18], [24]). Moreover, such singular limits were used as a kind of definition for curvature flows past singularities in the phase-field approach [16], [9].

For example, the following result established by Katsoulakis and Souganidis in [24] shows, that for some types of chemical potentials, the evolution of the reaction front in some sense converges to one, defined by the level set equation (1). They study the reaction-diffusion problem in the form

$$u_t^\epsilon - \Delta u^\epsilon + \frac{1}{\epsilon^2} V_u(u^\epsilon), \; u^\epsilon(x,0) = g^\epsilon.$$

where

1. V is a double-well potential with wells of equal depth, and the two local minimisers of V are denoted by u_+ and u_-, and the local maximiser is u_0.

2. the functions $g^\epsilon \in L^\infty(\mathbb{R}^n)$ are periodic, and there exists an open set $\Omega_0 \in \mathbb{R}^n$ and a closed periodic $(n-1)$-dimensional hypersurface Γ_0 with the following properties:

   (a) $\mathbb{R}^n = \Omega_0 \cup \bar{\Omega}_0^c \cup \Gamma_0$

   (b) for each $\delta > 0$, there exists $\eta > 0$ such that the following inequalities hold,

$$\begin{aligned} \liminf_{\epsilon \mapsto 0} g^\epsilon &\geq u_0 + \eta \text{ in } \Omega_0 \cap \{r : \operatorname{dist}(r, \Gamma_0) \geq \delta\} \\ \limsup_{\epsilon \mapsto 0} g^\epsilon &\leq u_0 - \eta \text{ in } \bar{\Omega}_0^c \cap \{r : \operatorname{dist}(r, \Gamma_0) \geq \delta\}. \end{aligned}$$

They have established that for each sequence $\epsilon_k \mapsto 0$ there exist a subsequence $\epsilon_{k_l} \mapsto 0$ and subsets $\Gamma, P, N \subset \mathbb{R}^n \times [0, \infty)$ satisfying

$$\Gamma \cup P \cup N = \mathbb{R}^n \times [0, \infty), \quad \Gamma \cap P = \Gamma \cap N = P \cap N = \emptyset,$$

such that $\Gamma_t = \Gamma \cap \{t\}$ is a generalised mean curvature evolution starting at Γ_0 and, as $\epsilon_{k_l} \mapsto 0$,

$$u^{\epsilon_{k_l}} \mapsto \begin{cases} u_+ & \text{locally uniformly in } P \\ u_- & \text{locally uniformly in } N. \end{cases}$$

A more direct application of the mean curvature flow PDE can be found in [13]. The authors compute numerically minimal surfaces in $\mathbb{R}^3$. For a given closed curve (or a set of closed curves), they compute the surface(s), which has zero mean curvature at each point and this curve as a boundary. Such surfaces are obtained as a long time limit of the initial surface that moves by mean curvature and contains the given curve at each time. This process was organised by putting an appropriate interpolatory boundary conditions on the curve.

The most important area, where an evolution of type (2) arises is in image processing. One of the most common tasks here is to reduce noise in an image. This is often called "smoothing". It is also common to represent a black and white image as a real function of two spatial variables $u : \mathbb{R}^2 \mapsto \mathbb{R}$. The smoothing operation H should satisfy the following fundamental axioms of image processing (see [1]):

1. it has to commute with the contrast changes i.e.

$$\Psi(Hu) = H(\Psi u)$$

where $\Psi : \mathbb{R} \mapsto \mathbb{R}$ monotone.

2. H has to commute with the affine transforms of the image plane.

In image processing one wishes to consider not only one smoothing operator, but a family $\{H_t\}_{t\geq 0}$. This family is called "scale space", and t-scale parameter. Depending on the value of the scale parameter, we get "more" or "less" smooth image $H_t(u)(x) = u(x,t)$. The result by Alvarez et al. in [1] says, that if the scale space satisfies some stability and regularity properties, then $u(x,t)$is the viscosity solution of (2) for some nondecreasing continuous G and an initial image as u_0.

1.1 Kinetic generated schemes and convolution-thresholding dynamics

In the present work we construct and study a class of approximations to the general curvature flows (2), so called convolution-thresholding dynamics. We suggest here a flexible generalisation of this type of dynamics motivated by a kinetic approach that suites well for approximation of general curvature flows.

Let us consider a convolution generated motion of a hypersurface in $\mathbb{R}^n$. By this we mean the following. Assume, that initially the surface under consideration is a boundary of a compact set $C \in \mathbb{R}^n$. Take a compactly supported function $\tilde{\rho} : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ (in fact, one may also take $\tilde{\rho}$ with unbounded support, but fast decreasing for large x). We define $\rho : \mathbb{R}^n \mapsto \mathbb{R}_+$,

$$\rho(x) = \frac{1}{h^{n/2}} \tilde{\rho}\left(|x| / \sqrt{h}\right)$$

and introduce a convolution

$$M(C)(x,h) = \int_{\mathbb{R}^n} \chi_C(y)\, \rho(x-y)\, dy.$$

Now $M(C)(x,h)$ is a function of x, and we can define a new position of the surface as a boundary of the set

$$\mathcal{H}_h C = \{x \in \mathbb{R}^n : F(M(C)(x,h)) \leq 0\}, \tag{4}$$

where F is some (thresholding) function. The next step is to introduce an operator on the space of bounded functions $\mathbb{B}(\mathbb{R}^n)$: $H(h) : \mathbb{B}(\mathbb{R}^n) \mapsto \mathbb{B}(\mathbb{R}^n)$ by

$$[H(h)u](x) = \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \geq \lambda]\}. \tag{5}$$

The purpose of the present study is for a given function G in (3), to find a corresponding thresholding function F in (4), so that $H(h)^n g(x)$ would converge to the unique viscosity solution of (3).

This problem was solved by Evans in [17] and Ishii in [21] in the case when when G is linear. This solution discussed in Section 3.

Suppose, that G is non-linear. As we show in Section 4, in this case one has to use two convolutions M_1 and M_2 and a thresholding function of two variables $F(M_1, M_2)$. This is necessary to ensure that the operator H is consistent with the PDE in (3). We also show, how to choose convolution kernels in order to get a monotone H. These two conditions-monotonicity and consistency-are crucial for the convergence.

The following kinetic approximation scheme can serve as a useful physical interpretation of the more general convolution-thresholding approximation methods for generalised curvature flows that we suggest in the present paper. This kinetic approximation is similar to the phase field approach but is more flexible.

Let $f(t,x,\xi)$ with $\xi \in \mathbb{R}^n$ be the distribution function having the sense of the amount of particles in the volume $dxdv$. The function $u(t,x) = \int f(t,x,\xi)\,d\xi$ has the sense of the mean density of the gas with distribution function f at position x at time t. General mean values like $M(t,x) = \int f(t,x,\xi)\,\lambda(\xi)\,d\xi$ with a weight function $\lambda(\xi) \geq 0$ are traditionally interpreted as macro parameters for a gas with the distribution function f.

We consider the following BGK type model kinetic equation for $f(t,x,\xi)$ with collision term consisting of two qualitatively different terms

$$\frac{\partial f}{\partial t} + \xi \nabla_x f = \frac{1}{Kn}\left(u\rho(\xi) - f\right) + \frac{1}{\gamma} f(\xi)\, G(u, M). \tag{6}$$

The first one is a usual relaxation term with normalised "Maxwellian" distribution $\rho(\xi)$, $\int \rho(\xi)\,d\xi = 1$. The second term describes a kind of chemical reaction that generates particles or eliminate them depending on the values of the density $u(t,x)$ and possibly of some other macroparameter $M(t,x)$. The relaxation term makes f tend to a local equilibrium $u(t,x)\,\rho(\xi)$.

The choice of the function $G(\rho, M,)$ determines qualitative properties of solutions.

A standard method for solving an equation of kinetic type is the splitting method. Instead of solving equation (6) one can sequentially solve on small time steps Δt the following simpler equations having a clear physical meaning:

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \nabla_x \xi f_1 = 0, & \qquad\qquad - \text{collisionless flow,} \\ f_1|_{t=0} &= F_0; \\ \frac{\partial f_2}{\partial t} = \frac{1}{\gamma} f_2(\xi) \cdot G(u, M), & \qquad\qquad - \text{chemical reaction} \\ f_2|_{t=\Delta t} &= f_1(\Delta t); \\ \frac{\partial f_3}{\partial t} = \frac{1}{Kn}(u\rho(\xi) - f_3), & \qquad\qquad - \text{relaxation} \\ f_3|_{t=2\Delta t} &= f_2(2\Delta t); \end{aligned} \tag{7}$$

where the macro parameters u, M ingoing in the equations are calculated for the actual approximation, correspondingly f_1, f_2 , f_3.

The first step means that the initial data $F_0\left(x, \xi\right)$ are transported from each point $x_0 \in R^n$ along the collisionless paths $x = x_0 + t\xi$ to the points $x = x_0 + \Delta t\xi$. The resulting distribution function $f_1\left(\Delta t, x, \xi\right)$ differs much from the locally equilibrium $u\left(\Delta t, x\right) \rho\left(\xi\right)$ function.

We choose parameters γ and Kn small in comparison with Δt. The solutions of the second and the third equations of the splitting scheme will be after the time Δt close to stationary solutions corresponding to zeroes of operators in the right hand sides of these equations. These zeroes determine geometrical properties and the character of the whole process.

Zeroes of the linear operator $\left(u\rho\left(\xi\right) - f\right)$ are evidently all functions of the form $u\left(x\right)\rho\left(\xi\right)$. We still do not concretize the function $\rho\left(\xi\right)$. The only requirement is the existence of moments of high enough order. Depending on what kind of surface dynamics we want to model we will impose more concrete conditions.

The integration of (7) with respect to ξ with weights 1 and $\lambda\left(\xi\right)$ gives the equations

$$\frac{\partial u}{\partial t} = \frac{1}{\gamma} uG\left(u, M\right)$$

$$\frac{\partial M}{\partial t} = \frac{1}{\gamma} M G(u, M)$$

We choose the chemical reaction term $G(u, M)$ such that this system of differential equations has two stable stationary points (u_1, M_1) and (u_2 , M_2) in the plane of macro parameters and one unstable stationary manifold determined by the equation $F(u, M) = 0$.

For $\gamma << 1$ the $u(t, x)$ and $M(t, x)$ reach after the finite time Δt one of the stationary states depending on the initial data at the point x . This process leads to the formation of two sets with constant density values $u_1$and u_2 an interface set Γ that we shell use for the approximation of generalised curvature flows. One can observe from here that only the thresholding function $F(u, M)$ is important for our approximations. Details of the chemical reaction term do not play any role for the our scheme.

At the third step of the splitting method the distribution function evolves independently at all space points x tending to a locally equilibrium stage of the form $\rho(x) M(\xi)$ depending on x only via the density $\rho(x)$. If the parameter $Kn << 1$ this convergence to equilibrium is fast and the equilibrium is achieved after the time Δt . Now we apply this procedure to particular initial data for modeling surface dynamics.

Consider the family $\Gamma(t)$ of surfaces with initial state $\Gamma(0)$ such that $\Gamma(0) = \partial C_0$ is a boundary for a compact set C_0 . We use here the notation $t = \Delta t$.

Take the initial distribution function $f_1(0, x, \xi) = F_0(x, \xi)$ in the form

$$F_0(x, \xi) = u_0 \cdot \rho(\xi),$$

where u_0 is the characteristic function of the set C_0 times the equilibrium Maxwellian distribution $\rho(\xi)$. Moments of different orders of the function $f_1(t, x, \xi)$ play a crucial role in the construction of our approximations to the differential characteristics of the surface $\Gamma(t)$.

We require here that the function $\rho(\xi)$ is non negative, dependent only on $|\xi|$ and has enough many moments.

We consider the function $f_1(\Delta t, x, \xi)$ and corresponding values of mean values ρ , M.

$$u_1(t, x) = \int_{R^n} u_0(x - t\xi) \rho(\xi) \, d\xi$$

The change of variables $y = x - t\xi$ gives

$$u_1(t,x) = \int_{R^n} u_0(y) \frac{1}{t^n} \rho((x-y)/t)\, dy = u_0 * m_t(x) \tag{8}$$

namely that the function $u_1(x)$ is a convolution of the initial density u_0 with the function $m_t(y) = \frac{1}{t^n}\rho(y/t)$.

The same proof shows similar convolution formulas for any weight $\lambda(\xi)$ with $m_\lambda(y) = \frac{1}{t^n}\lambda(y/t)\rho(y/t)$:

$$\begin{aligned} M_\lambda(t,x) &= \int_{R^n} \rho_0(x - t\xi)\lambda(\xi)\rho(\xi)\, d\xi = \\ &= \int_{R^n} \rho_0(y)\lambda((x-y)/t)\frac{1}{t^n}\rho((x-y)/t)\, dy = \\ &= (\rho_0 * m_{\lambda,t})(x) \end{aligned}$$

with $m_{\lambda,t} = \lambda(y/t) \cdot \dfrac{1}{t^n}\rho(y/t)$.

The result $f_3(3\Delta t, x, \xi)$ after the chemical reactions step and the relaxation step has the form $f_3(3\Delta t, x, \xi) = u_3(x)\rho(\xi)$ where u_3 is a characteristic function of a compact domain bounded by the surface where the thresholding function $F(u, M) \geq 0$.

We see that the kinetic splitting scheme generates a kind of convolution thresholding dynamics of surfaces with thresholding depending on two (or more) convolutions. The simpler example given at the beginning of the section is reproduced if we choose chemical reactions $G = G(u)$ depending only on the density u of the gas and take the time scaling $t = \sqrt{h}$.

This paper is organised as follows. After introducing the basic notions and stating some results for viscosity solutions in Section 2, we turn to the methods of approximation for such solutions (Section 3). The main of the present work result is described in Section 4, where we show how to construct F in order to get a convergence of the convolution-thresholding approximation to the viscosity solution of (3) with a non-linear function G. More precisely, the following local uniform convergence is proved

$$((H(t/m))^m g)(x) \mapsto u(x,t),\, m \mapsto \infty,$$

where H defined by (5) and $u(x,t)$ is the viscosity solution of

$$\begin{cases} u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right) & \text{in } \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

We use this construction for numerical calculation for some cases of the general curvature flows in two dimensions. The results and several approaches to the implementation are described in Section 5.

2 Viscosity solution framework

In the present work we will extensively use the theory of viscosity solutions to equations of the form:

$$F\left(x, u, Du, D^2u\right) = 0 \tag{9}$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n) \mapsto \mathbb{R}$ and $\mathbb{S}(n)$ is the set of symmetric $n \times n$ matrices.

We follow a paper [14] by Crandall, Ishii and Lions, where a comprehensive overview on the theory of viscosity solutions is given. They also present the basic tool in proving the existence of such solutions for the Dirichlet problem.

Equations with F satisfying additional fundamental monotonicity conditions are considered:

1. $F(x, r, p, X) \leq F(x, s, p, Y)$ whenever $r \leq s$
2. $F(x, r, p, X) \leq F(x, s, p, Y)$ whenever $Y \leq X$

   here $r, s \in \mathbb{R}$ $x, p \in \mathbb{R}^n$ $X, Y \in \mathbb{S}(n)$ and $\mathbb{S}(n)$ is equipped with its usual partial ordering.

To motivate the notion of a viscosity solution, we consider u as a classical subsolution of $F(x, u, Du, D^2u) = 0$, i. e.

$$F\left(x, u(x), Du(x), D^2u(x)\right) \leq 0 \ \ \forall x \in \mathbb{R}^n.$$

Suppose also that ϕ is C^2 and x_0 is a local maximum of $u-\phi$. Then $Du(x_0) = D\phi(x_0)$, $D^2u(x_0) \leq D^2\phi(x_0)$, and by (2)

$$F\left(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\right) \leq F\left(x_0, u(x_0), Du(x_0), D^2u(x_0)\right) \leq 0 \tag{10}$$

Since we want to have $u \notin C^2$, the latter inequality suggests defining an arbitrary function u to be some kind of generalised subsolution of $F = 0$, if

$$F\left(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)\right) \leq 0$$

whenever ϕ is in C^2 and x_0 is a local maximum of $u - \phi$. Another use of x_0 being a local maximum of $u - \phi$ is that for x near it, we have $u(x) \leq u(x_0) - \phi(x_0) + \phi(x)$, and a Taylor approximation implies

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), x - x_0 \rangle + o\left(|x - x_0|^2\right), \quad (11)$$

where $p = D\phi(x_0)$ and $X = D^2\phi x_0$. Now assume that (11) holds for some $p \in \mathbb{R}^n$ and $X \in \mathbb{S}(n)$. If also u is smooth, then $p = Du$ and $D^2u \leq X$. So if u is a classical subsolution of $F = 0$, then $F(x_0, u(x_0), p, X) \leq 0$ whenever (11) holds. This observation motivates the following auxiliary definition:

Definition 1. *Let $\mathcal{O} \in \mathbb{R}^n$, $u : \mathcal{O} \mapsto \mathbb{R}$ and $x_0 \in \mathcal{O}$. The following map from $\mathcal{O}$ to the subsets of $\mathbb{R}^n \times \mathbb{S}(n)$ is called the second order superjet of u at x_0*

$$\begin{aligned} J^{2,+}_{\mathcal{O}} u(x_0) &= \{(p, X) \in \mathbb{R}^n \times \mathbb{S}(n) : \\ u(x) &\leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2} \langle X(x - x_0), x - x_0 \rangle + o\left(|x - x_0|^2\right) \\ \textit{as } x &\mapsto x_0 \,\} \end{aligned}$$

We might as well repeat the motivation with reverse inequality sign to arrive at

Definition 2. *The second order subjet of u at x_0 is*

$$J^{2,-}_{\mathcal{O}} u(x_0) = -J^{2,+}_{\mathcal{O}}(-u)(x_0)$$

Having these notions we are ready to introduce the viscosity solutions.

Definition 3. *Let F satisfy (2) , (1) and $\mathcal{O} \in \mathbb{R}^n$. A viscosity subsolution of (9) on $\mathcal{O}$ is a function $u \in USC(\mathcal{O})$ such that*

$$F(x, u(x), p, X) \leq 0 \textit{ for all } x \in \mathcal{O} \textit{ and } (p, X) \in J^{2,+}_{\mathcal{O}} u(x)$$

A viscosity supersolution of (9) on $\mathcal{O}$ is a function $u \in LSC(\mathcal{O})$ such that

$$F(x, u(x), p, X) \geq 0 \textit{ for all } x \in \mathcal{O} \textit{ and } (p, X) \in J^{2,-}_{\mathcal{O}} u(x)$$

Definition 4. *A continuous function u is a viscosity solution of (9) on $\mathcal{O}$ if it is both a viscosity sub- and supersolution.*

It is easy to show, that

$$\begin{aligned} J^{2,+}_{\mathcal{O}} u(x_0) &= \left\{ \left(D\phi(x_0), D^2\phi(x_0) \right) : \phi \in C^2 \right. \\ &\qquad \left. \text{and } \; u - \phi \; \text{ has a local maximum at } \; x_0 \right\} \end{aligned}$$

and, similarly,

$$\begin{aligned} J^{2,-}_{\mathcal{O}} u(x_0) &= \left\{ \left(D\phi(x_0), D^2\phi(x_0) \right) : \phi \in C^2 \right. \\ &\qquad \left. \text{and } \; u - \phi \; \text{ has a local minimum at } \; x_0 \right\} \end{aligned}$$

Using these relationships, one can reformulate the definition of a viscosity solution in terms of test functions.

The usual procedure of proving the existence and uniqueness of such solutions to the Dirichlet problem consists of two basic steps: establishing the comparison property and invoking Perron's method. Consider a problem:

$$\begin{cases} F\left(x, u, Du, D^2u\right) = 0 & \text{in } \; \mathcal{O} \\ u = 0 & \text{on } \; \partial\mathcal{O}. \end{cases} \tag{12}$$

We say that the comparison principle for this problem holds, if for subsolution w and a supersolution v we have $w \leq v$. Assuming now that $\bar{u}$ is a supersolution and $\underline{u}$-supersolution, the solution u , according to Perron's method is

$$u(x) = \sup\left\{ v(x) : \underline{u} \leq v \leq \bar{u} \; \text{ and } \; v \; \text{ is subsolution of (12) } \right\}.$$

Considering a "parabolic" problem

$$u_t + F\left(t, x, u, Du, D^2u\right) = 0, \tag{13}$$

we observe that if the map $(x, r, p, X) \mapsto (t, x, r, p, X)$ satisfies (2) and/or (1) for each $t \in [0, T]$, then so is (13) when considered as an equation in $n + 1$ independent variables.

2.1 Viscosity solutions of the mean curvature evolution PDE

Getting back to our equation (2), we perform a differentiation to rewrite it in the following form:

$$u_t = |Du|\, G\left(\frac{1}{|Du|}\, tr\left(\left(I - \frac{Du \otimes Du}{|Du|^2} \right) D^2u \right) \right)$$

We see, that in our case

$$F(x,r,p,X) = F(p,X) = -|p|\, G\left(\frac{1}{|p|}\, tr\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right)\right)$$

satisfies (1) and (2), provided that G is nondecreasing and $p \neq 0$. However, it is not defined for $p = 0$, and, therefore, our settings need a slight adjustment. This was done by Evans and Spruck in [19] and by Chen, Giga and Goto in [12]. In our presentation we will use a somewhat more general definition of viscosity solutions introduced by Ishii and Souganidis in [23] to allow a wider class of functions G in (2). They consider a new class of test functions and adapt the definition of viscosity solution for possible singularities of F at $p = 0$.

Let us begin by introducing an auxiliary subclass of $C^2([0,\infty))$. We say that $f : [0,\infty) \mapsto \mathbb{R}$ lies in $\mathcal{F} \subset C^2$ if $f(0) = f'(0) = f''(0) = 0$, $f''(r) > 0$ for $r > 0$ and the following limits hold

$$\lim \frac{f'(|p|)}{|p|} F(p,I) = \lim \frac{f'(|p|)}{|p|} F(p,-I) = 0.$$

As was shown in [23], this set of functions is a non-empty cone, provided that $F(p,X) \in C((\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}(n))$. The class of test functions $\mathcal{A}(F)$ depends on F and is defined as follows.

Definition 5. *A function ϕ is admissible if it is in $C^2(\mathbb{R}^n \times (0,T))$ and for each $\acute{z} = (\acute{x}, \acute{t})$ where $D\phi(\acute{z}) = 0$, there is $\delta > 0$, $f \in \mathcal{F}$ and $\omega \in C([0,\infty))$ such that $\omega = o(r)$ and for all $() \in B(\acute{z}, \delta)$*

$$\left|\phi(x,t) - \phi(\acute{z}) - \phi_t(\acute{z})\left(t - \acute{t}\right)\right| \leq f(|x - \acute{x}|) + \omega\left(\left|t - \acute{t}\right|\right).$$

The definition of viscosity solution becomes

Definition 6. *$u : \mathcal{O} \subset \mathbb{R}^n \times (0,T) \mapsto \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of $u_t = F(Du, D^2u)$ in $\mathcal{O}$ if $\phi < \infty$ and for all $\phi \in \mathcal{A}(F)$ and all local maximum points z of $u^* - \phi$,*

$$\begin{cases} \phi_t(z) \leq F\left(D\phi(z), D^2\phi(z)\right) & \text{if } D\phi(z) \neq 0 \\ \phi_t(z) \leq 0 & \text{otherwise.} \end{cases}$$

Likewise, $u : \mathcal{O} \mapsto \mathbb{R} \cup \{\infty\}$ is a viscosity supersolution in $\mathcal{O}$ if $\phi > -\infty$, and for all $\phi \in \mathcal{A}(F)$ and all local minimum points z of $u_* - \phi$,

$$\begin{cases} \phi_t(z) \geq F\left(D\phi(z), D^2\phi(z)\right) & \text{if } D\phi(z) \neq 0 \\ \phi_t(z) \geq 0 & \text{otherwise.} \end{cases}$$

Consequently, a viscosity solution is a function that is sub- and supersolution simultaneously.

Let us now consider the initial value problem

$$\begin{cases} u_t + F\left(Du, D^2u\right) = 0 & \text{in } \mathcal{O} \times (0,T) \\ u = u_0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{14}$$

We denote the space of bounded and uniformly continuous functions on $\mathbb{R}^n$ by $BUC\left(\mathbb{R}^n\right)$ and restrict ourselves to a special class of F: it has to be geometric i.e. for any $\lambda > 0, \mu \in \mathbb{R}$ and $(p, X) \in \mathcal{O}$

$$F\left(\lambda p, \lambda X + \mu p \otimes p\right) = \lambda F\left(p, X\right). \tag{15}$$

The result of Ishii and Souganidis presented in [23]

Theorem 1. *Assume that $g \in BUC\left(\mathbb{R}^n\right)$, F is continuous and (2) together with (15) hold.*

Then (14) has an unique viscosity solution in $BUC\left(\mathbb{R}^n \times (0,T)\right)$.

To be more explicit, we give a corollary of this theorem for the case of problem (3).

Theorem 2. *Assume, that G is continuous and nondecreasing. Then the initial value problem*

$$\begin{cases} u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right) & \textit{in } \mathbb{R}^n \times (0,T) \\ u = g\left(x\right) \in BUC\left(\mathbb{R}^n\right) & \textit{on } \mathbb{R}^n \times \{0\} \end{cases}$$

has an unique viscosity solution $u \in BUC\left(\mathbb{R}^n \times (0,T)\right)$.

3 Approximate methods for curvature flows and convergence results

3.1 Overview

In order to track the evolution of a function $u_0\left(x\right)$ defined by the Cauchy problem (3), one needs to construct numerical approximations $u_h\left(x,t\right)$ and prove, that these approximations converge in some sense to the unique viscosity solution $u(x,t)$ of (3).

In [28], the authors proposed a finite difference approximation scheme for the above problem. Their technique is based on the use of a Hamilton- Jacobi formulation and the technology developed for the solution of hyperbolic conservation laws. The convergence of such schemes was proved by Crandall and Lions in [15]. Their proof of convergence is based on Theorem 5.

Another class of approximations, so called Matheron filters, comes from the image analysis. Suppose $\mathbb{F} \in \mathbb{P}(\mathbb{R}^n)$, where $\mathbb{P}(\mathbb{R}^n)$ is a collection of all subsets of $\mathbb{R}^n$. Define

$$(Tu)(x) = \inf_{A \in \mathbb{F}} \sup_{y \in A} u(x+y). \tag{16}$$

The classical theorem of Matheron in [25] asserts, that for each monotone operator T, that commutes with translations and contrast changes, there exists a family of structuring elements $\mathbb{F}$ such that (16) holds.

The connection between such operators and the mean curvature evolution PDE was established in [11]. The authors show how to choose $\mathbb{F}$ so, that suitably rescaled iterated Matheron filters would converge to the unique viscosity solution of (14). This result was then extended for a wider class of functions G in (3) by Guichard and Morel in [20] in the 2D case and by Cao in [10] in higher dimensions.

Threshold dynamics models, introduced by Ishii, Pires and Souganidis in [22], lead to approximations of the solution of the Cauchy problem (14), where the right hand side can be interpreted as a general elliptic operator on a level set of the solution. This is a generalisation of the problem (14) with (1), but it does not include (3) as a special case.

In what follows, we will consider convolution generated approximations defined by (5). We start by presenting results by Evans [17] and Ishii [21] on convergence of these approximations. Then we consider the convolution-thresholding approximations where the thresholding function F in (4) depends on two different mean values $F(M_1(C), M_2(C))$. Theorems 7 and 8 state the convergence of these approximations to the unique viscosity solution to the problem (3).

3.2 Convolution generated approximations

Consider the mean curvature evolution PDE (1). The proof of the fact, that convolution generated approximations converge to its viscosity solution is due to Evans [17] and Ishii [21].

Evans uses the evolution of compact sets in $\mathbb{R}^n$ defined by Bence, Merriman and Osher in [5], [26] to construct the approximations to the following problem:

$$\begin{cases} u_t = |Du|\, div\left(\frac{Du}{|Du|}\right) & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g(x) \in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{17}$$

This evolution of compact sets is called the BMO method and can be formulated as follows:

1. given a compact $C_0 \in \mathbb{R}^n$, solve the heat equation with the initial data being the indicator function of it i.e.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = \chi_{C_0} & \text{on } \mathbb{R}^n \times \{0\} \end{cases}, \tag{18}$$

2. for a $t > 0$, define

$$C_t = \left\{ x \in \mathbb{R}^n : u(x,t) \geq \frac{1}{2} \right\} \tag{19}$$

3. replace C_0 with C_t and repeat the procedure.

Following Evans, we denote $C_t = \mathcal{H}(t)\, C_0$ and call $\{\mathcal{H}(t)\}_{t\geq 0}$ the heat diffusion flow, where $\mathcal{H}(t)$ is regarded as a mapping on a collection of compact sets in $\mathbb{R}^n$. The exact solution of (18) can be written as a convolution

$$u(x,t) = M_t(|\cdot|) \star \chi_{C_0} = \frac{1}{(4\pi t)^{n/2}} \int_{C_0} e^{-|x-y|^2/4t} dy,$$

where

$$M(x) = (4\pi)^{-n/2}\, e^{-x^2/4}$$

and

$$M_t(x) = \frac{1}{t^{n/2}} M\left(x/\sqrt{t}\right).$$

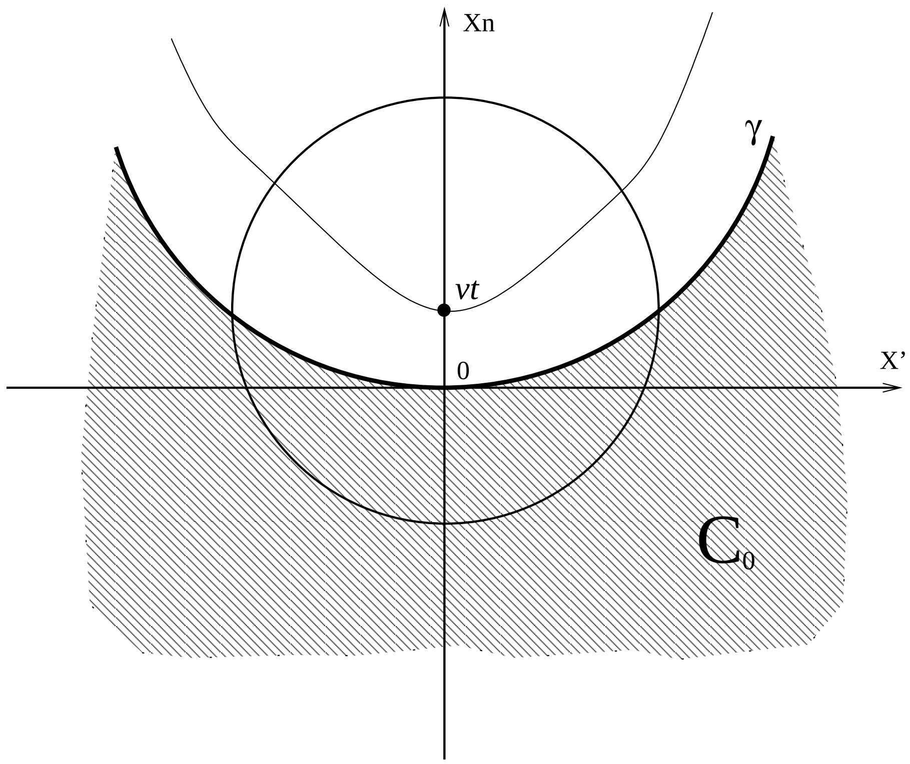

Figure 2: On the proof of the convergence of BMO

Note, that in terms of Section 1.1, this is a convolution generated motion of a surface with kernel $\rho = M(x)$ and thresholding $F(x) = x - 1/2$.

The proof that C_t approximates for small times t the mean curvature motion of the boundary Γ_0 of C_0 is relatively simple. Due to the importance of argumentation in the proof of this result for our further construction, we state it here.

Theorem 3. *(Evans, 93) Assume, that C_0 is compact subset of $\mathbb{R}^n$ with a smooth boundary ∂C_0. Furthermore, $x_0 \in \partial C_0$ and $\nu = \nu(x_0)$ is the outward unit normal vector at x_0. Select $v \in \mathbb{R}$, such that*

$$x_0 + vt\nu \in \partial \left[\mathcal{H}(t)\, C_0\right]. \tag{20}$$

Then

$$v = k + O(t) \quad \text{as} \quad t \mapsto 0,$$

where k is ($(n-1)$ times) the mean curvature of ∂C_0 at x_0.

Proof. Without loss of generality, we take $x_0 = 0$, $\nu = e_n = (0, 0, \ldots 1)$ and ∂C_0 near x_0 as a graph of a smooth function $\gamma : \mathbb{R}^{n-1} \mapsto \mathbb{R}$. We write

$\acute{x} = (x_1, x_2, \dots, x_{n-1})$. With the above notations the following equalities hold:

$$\gamma(0) = 0,\ D\gamma(0) = 0,\ k = \Delta\gamma(0).$$

Since (20),

$$\frac{1}{2} = u(0,0,\dots,vt,t) = (M_t(|\cdot|) \star \chi_{C_0})(0,0,\dots,vt,t)$$

In a sense, we intend to solve the above equation for v. Let us write it in a more explicit form:

$$\frac{1}{2} = \int_{C_0} M_t(|y - \vec{v}t|)\, dy.$$

The function $M(x)$ decreases rapidly for large values of $|x|$. This allows us (after rescaling of coordinates) to perform the integration over the unit cube Q, which contains a part of ∂C_0 that is represented by the graph of $\gamma(\acute{x})$.

$$\begin{aligned}
\frac{1}{2} &= \int_Q \chi_{C_0}(y)\, M_t(|y - \vec{v}t|)\, dy + O\left(e^{-\alpha/t}\right) = \\
&= \int_{\mathbb{R}^{n-1}} \int_{-1}^{\gamma(\acute{y})} M_t(|y - \vec{v}t|)\, dy_n d\acute{y} + O\left(e^{-\alpha/t}\right) = \\
&= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\gamma(\acute{y}) - vt} M_t(|z|)\, dz_n d\acute{z} + O\left(e^{-\alpha/t}\right) = \\
&= A_M + \int_{\mathbb{R}^{n-1}} \int_0^{t^{-1/2}\left[\gamma\left(t^{1/2}\acute{w}\right) - vt\right]} M(|w|)\, dw_n d\acute{w} + O\left(e^{-\alpha/t}\right).
\end{aligned}$$

Here we introduced the kernel dependent constant A_M:

$$A_M = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} M(|y|)\, dy_n d\acute{y}.$$

This constant happened to be exactly $\frac{1}{2}$ in our case:

$$\begin{aligned}
A_M &= \frac{\pi^{n/2}}{\Gamma(n/2)} \int_{-\infty}^{0} y^{n-1} M(y)\, dy \\
&= \frac{\pi^{n/2}}{\Gamma(n/2)} \int_{-\infty}^{0} \frac{y^{n-1}}{(4\pi)^{n/2}} e^{-y^2/4} dy = \frac{1}{2}.
\end{aligned}$$

In fact, this "coincidence" explains the choice of the thresholding constant in the BMO method in (19). Thus, the equation for v becomes:

$$\int_{\mathbb{R}^{n-1}} \left(\int_0^{t^{-1/2}\left[\gamma\left(t^{1/2}w\right)-vt\right]} M\left(|w|\right) dw_n \right) d\acute{w} = O\left(e^{-\alpha/t}\right). \qquad (21)$$

The next step is to estimate the remaining integral by expanding $\gamma\left(t^{1/2}\acute{w}\right)$ in a Taylor series,

$$\begin{aligned}
& t^{-1/2}\left[\gamma\left(t^{1/2}\acute{w}\right)-vt\right] \\
= \;& t^{-1/2}\left[\gamma\left(0\right)+t^{1/2}\gamma_{x_j}\left(0\right)w_j+\frac{t}{2}\gamma_{x_ix_j}\left(0\right)w_iw_j+\right. \\
+ \;& \left.\frac{t^{3/2}}{6}\gamma_{x_ix_jx_l}\left(0\right)w_iw_jw_l+O\left(t^2\left|\acute{w}\right|\right)-tv\right] \\
= \;& t^{1/2}\left[\frac{1}{2}\gamma_{x_ix_j}\left(0\right)w_iw_j+\frac{t^{1/2}}{6}\gamma_{x_ix_jx_l}\left(0\right)w_iw_jw_l-v+O\left(t\left|\acute{w}\right|\right)\right]
\end{aligned}$$

Now (21) implies

$$\int_{\mathbb{R}^{n-1}} M\left(\left|\acute{w}\right|\right)\left(\frac{1}{2}\gamma_{w_iw_j}\left(0\right)w_iw_j-v\right)d\acute{w}=O\left(t\right).$$

Using again the axial symmetry of the kernel and the fact, that

$$\int_{\mathbb{R}^{n-1}} z_i^2 e^{-\left|\acute{z}\right|^2/4}d\acute{z}=2\int_{\mathbb{R}^{n-1}} e^{-\left|\acute{z}\right|^2/4}d\acute{z},$$

we get

$$v=\Delta\gamma\left(0\right)+O\left(t\right).$$

□

Furthermore, Evans establishes a much stronger connection between the heat diffusion flow and the mean curvature evolution PDE. First, he defines the heat diffusion flow for continuous periodic functions:

$$\left[H\left(t\right)u\right]\left(x\right)=\sup\left\{\lambda\in R : x\in\mathcal{H}\left(t\right)\left[f\geq\lambda\right]\right\}. \qquad (22)$$

Then he employs the non-linear semigroup theory (Brezis and Pazy [8] theorem) to prove the following

Theorem 4.

$$\lim_{m\mapsto\infty}\left[H^m\left(\frac{t}{m}\right)u_0\right](x)=u(x,t),$$

where $u(x,t)$ is the unique viscosity solution of (17) and the convergence is locally uniform.

A generalisation of this result was proposed by Ishii in [21] . He observed, that one needs just a positive, axially symmetric convolution kernel to define the evolution of closed subsets of $\mathbb{R}^n$. Once this is done, one can define the evolution of functions by (22). Ishii proved, that this procedure gives approximations for the solution of the Cauchy problem

$$\begin{cases} u_t=a_0\,|Du|\,div\left(\frac{Du}{|Du|}\right) & \text{in } \mathbb{R}^n\times(0,\infty) \\ u=g(x)\in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n\times\{0\}, \end{cases} \tag{23}$$

where the constant $a_0>0$ depends on the kernel. Note, that the equation in (23) differs from (17) by time scaling a_0.

3.3 Convergence of approximation schemes by Barles and Souganidis

In what follows we will make use of a theorem by Barles and Souganidis proved in [4]. They consider a problem

$$F\left(x,u,Du,D^2u\right)=0 \text{ in } \bar{\Omega} \tag{24}$$

where $\Omega\in\mathbb{R}^n$ is open and $\bar{\Omega}$ its closure. The requirements on F are ellipticity (i.e. it has to satisfy (2)) and a strong uniqueness property i.e.

If $\quad u\in\mathbb{B}\left(\bar{\Omega}\right)$ is an upper semicontinious solution of (24) and
$v\in\mathbb{B}\left(\bar{\Omega}\right)$ is a lower semicontinious solution of (24),
then $\qquad u\le v$ on $\bar{\Omega}$.

Suppose now, that we have constructed an approximation scheme of the form

$$S\left(h,x,u^h(x),u^h\right)=0 \text{ in } \bar{\Omega} \tag{25}$$

with $S:\mathbb{R}^+\times\bar{\Omega}\times\mathbb{R}\times\mathbb{B}\left(\bar{\Omega}\right)\mapsto\mathbb{R}$ locally bounded. Assume also, that this scheme satisfies the following conditions:

1. stability

$$\begin{aligned} \text{For all } h &> 0, \text{ there exists a solution } u^h \in B\left(\bar{\Omega}\right) \text{ of (25)} \\ &\text{with bound independent of } h, \end{aligned}$$

2. monotonicity

$$\begin{aligned} S(h,x,t,u) &\le S(h,x,t,v) \\ \text{if } u &\ge v \text{ for all } h \ge 0,\ x \in \bar{\Omega},\ y \in \mathbb{R} \text{ and } u,v \in \mathbb{B}\left(\bar{\Omega}\right), \end{aligned}$$

3. consistency

$$\begin{aligned} \limsup_{\substack{h\mapsto 0\\ y\mapsto x\\ \xi\mapsto 0}} \frac{S(h,y,\phi(y),\phi+\xi)}{h} &\le \bar{F}\left(x,\phi(x),D\phi(x),D^2\phi(x)\right) \\ \liminf_{\substack{h\mapsto 0\\ y\mapsto x\\ \xi\mapsto 0}} \frac{S(h,y,\phi(y),\phi+\xi)}{h} &\ge \underline{F}\left(x,\phi(x),D\phi(x),D^2\phi(x)\right) \end{aligned}$$

for all $x \in \bar{\Omega}$ and $\phi \in C^\infty\left(\bar{\Omega}\right)$.

In this setting the result of Barles and Souganidis reads:

Theorem 5. *The solution u^h of a monotone, stable and consistent approximation scheme for (24) converges locally uniformly as $h \mapsto 0$ to the unique viscosity solution of the elliptic problem with strong uniqueness property.*

In order to base the proof of our main result on this theorem, we follow Pasquignon [29] and restate it in terms of the generalised mean curvature evolution PDE. Let $H(h)$ be the solution operator of our approximation $S(h,x,u,u(x)) = 0$, i.e.

$$\begin{aligned} u_h(x,(n+1)h) &= H(h)\,u_h(x,nh) = H(h)^{n+1}\,u_0(x), \\ u_h(x,0) &= u_0(x). \end{aligned}$$

Definition 7.

1. *Consistency*
   An approximation operator $H(h)$, $h > 0$ is consistent with

$$\frac{\partial u}{\partial t} = |Du|\,G\left(div\,\frac{Du}{|Du|}\right),$$

if for any $\phi \in C^\infty\left(\bar{\Omega}\right)$ *and for any* $x \in \bar{\Omega}$*, the following holds,*

$$\frac{(H(h)\,\phi)(x) - \phi(x)}{h} = |D\phi|\, G\left(div\frac{Du}{|Du|}\right) + o_x(1) \;\; for \;\; D\phi \neq 0. \quad (26)$$

If the convergence of $o_x(1)$ *is locally uniform on sets, where* $Du \neq 0$*, then* $H(h)$ *is said to be uniformly consistent with the PDE.*

2. *Monotonicity*
   An operator $H(h)\,,\; h > 0$ *is locally monotone if there exists* $r > 0$ *such that for any functions* $u, v \in \mathbb{B}\left(\bar{\Omega}\right)$ *with* $u \geq v$ *on* $B(x,r) \setminus \{x\}$*, it holds*

   $$H(h)\,u(x) \geq H(h)\,v(x) + o(h)\,.$$

3. *Stability*
   An approximation scheme $H(h)$ *is stable if* $H(h)^n\, u \in \mathbb{B}\left(\bar{\Omega}\right)$ *for every* $u \in B\left(\bar{\Omega}\right), n \in \mathbb{N}, h > 0.$

The convergence result now reads:

Theorem 6. *Consider a monotone, stable approximation operator* $H(h)$ *that commutes with additions of constants (i.e* $H(h)(u + C) = H(h)\,u + C,\ \forall C \in \mathbb{R}$ *) and is consistent with (2). Suppose also, that*

$$\lim_{h \mapsto 0} \frac{H(h)\,(f(|x - x_0|))\,(x_0)}{h} = 0$$

for any $f \in \mathcal{F}(G)$*. Then* $u_h(x, nh)$ *converges locally uniformly to the unique viscosity solution* $u(x,t)$ *of (2) as* $nh \mapsto t$*.*

4 A convolution-thresholding method for a generalised curvature flow

4.1 Motivation

Let us turn back to our main problem posed in Section 1.1. We consider a convolution generated motion of a hypersurface in $\mathbb{R}^n$ defined by (4) and the corresponding evolution of an initially bounded function $g : \mathbb{R}^n \mapsto \mathbb{R}$ defined by (5). Consider also the initial value problem (3) with given G and g. We are looking for such a thresholding function F in (4), that $H^m_{t/m}g(x)$ would converge (in some sense) to the unique viscosity solution of (3).

For example, set $F(x) = x - 1/2$ and $\tilde{\rho}(x) = \frac{1}{(4\pi)^{n/2}}e^{-x^2/4}$ to get $\mathcal{H}_h$ and $H(h)$ as corresponding operators in section 3. Hence, Theorem 4 applies, and $H(h)^n u_0$ converges to the unique viscosity solution of (17).

We will see that, in fact, one has to use two convolutions M_1 and M_2 with different kernels and construct a thresholding function depending on two variables respectively to resolve this problem when G is not linear.

To see this, let us check whether $H(h)$ satisfies the conditions of the Theorem 6. More precisely, we look for requirements on F sufficient to fulfil the conditions of Theorem 6.

1. Stability
   Suppose $u(x) \in \mathbb{B}(\mathbb{R}^n)$. We want to show, that $H(h)u \in \mathbb{B}(\mathbb{R}^n)$. Intuitively, we require
   $$\mathcal{H}_h\mathbb{R}^n = \mathbb{R}^n, \tag{27}$$
   $$\mathcal{H}_h\emptyset = \emptyset, \tag{28}$$
   and denote $A = \max|u|$. With these settings $[u \leq A] = \mathbb{R}^n$ and
   $$-A \leq H(h)u(x) = \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \leq \lambda]\} \leq A.$$
   It remains to find out for which F the conditions (27) and (28) are satisfied. To do this, we substitute the corresponding sets into the definition of $\mathcal{H}$
   $$\begin{aligned} \mathcal{H}_h\mathbb{R}^n &= \{x \in \mathbb{R}^n : F(M\mathbb{R}^n(x,h)) \geq 0\} = \\ &= \left\{x \in \mathbb{R}^n : F\left(\int_{\mathbb{R}^n} \rho dx\right) \geq 0\right\} = \mathbb{R}^n \\ \mathcal{H}_h\emptyset &= \{x \in \mathbb{R}^n : F(M\emptyset(x,h)) \geq 0\} = \\ &= \{x \in \mathbb{R}^n : F(0) \geq 0\} = \emptyset. \end{aligned}$$
   Thus, the requirements on F become
   $$\begin{aligned} F\left(\int_{\mathbb{R}^n} \rho dx\right) &\geq 0 \\ , F(0) &< 0. \end{aligned}$$

2. Monotonicity

   Let us now show, that if $\mathcal{H}_h$ satisfies the so called inclusion principle, then H_h is monotonous.

   Lemma 1. *Assume, that $\mathcal{H}_h$ satisfies the inclusion principle i.e.*

   $$\forall C_1, C_2 \subseteq \mathbb{R}^n : C_1 \subseteq C_2 \text{ we have } \mathcal{H}_h C_1 \subseteq \mathcal{H}_h C_2,$$

   then H_h is monotone, that is

   $$\forall u, v \in \mathbb{C}(\mathbb{R}^n) : v \leq u \text{ we have } H_h(v) \leq H_h(u).$$

   Proof. Suppose, there exists x_0 s.t. $H(h)u(x_0) < H(h)v(x_0)$. We denote $\lambda_1 = H(h)u(x_0)$, $\lambda_2 = H(h)v(x_0)$ and $\epsilon = \frac{\lambda_2 - \lambda_1}{2} > 0$. Since $\lambda_1 + \epsilon < \inf\{\lambda \in \mathbb{R} : x_0 \in \mathcal{H}_h[v \leq \lambda]\}$, $x_0 \notin \mathcal{H}_h[v \leq \lambda_1 + \epsilon]$, but $\mathcal{H}_h[v \leq \lambda_1 + \epsilon] \supseteq \mathcal{H}_h[u \leq \lambda_1 + \epsilon]$. Therefore $x_0 \notin \mathcal{H}_h[u \leq \lambda_1 + \epsilon]$, which is in contradiction with the definition of λ_1. □

   This means, that we have to choose F s.t. the inclusion principle is satisfied. Using again the positivity of the convolution kernel ρ, we have $MC_1 \leq MC_2$. Hence, if F is non decreasing, the inclusion principle holds.

3. Consistency

   We sum up some calculations in the following

   Lemma 2. *Let $\phi \in C^\infty(\mathbb{R}^n)$ $\phi(0) = 0$ and $D\phi(0) = (0, 0, \ldots, \beta)$. Then the consistency of an operator $H(h)$ with (3) is equivalent to*

   $$\gamma(0) = hG(-\Delta\gamma(0)) + o(h), \tag{29}$$

   where $x_n = \gamma(\acute{x})$ is a parametrisation of the surface

   $$\{x \in \mathbb{R}^n : u(x) = H(h)u(0)\}$$

   near $\acute{x} = 0$.

   Proof. Without loss of generality, one can consider the consistency condition (26) only for ϕ as in the statement. We rewrite (26) in a more convenient form

   $$(H(h)\phi)(0) = h|D\phi(0)|G\left(\operatorname{div}\frac{Du}{|Du|}(0)\right) + o(h). \tag{30}$$

Looking closer at $\mathrm{div}\frac{Du}{|Du|}$, we write

$$div\left(\frac{D\phi}{|D\phi|}\right)=\frac{1}{|D\phi|}\sum_{i,j=1}^{n}\left(\delta_{i,j}-\frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_ix_j}.$$

Since $\phi(0)=0$ and $\phi_{x_i}(0)=\delta_{ni}\beta$,

$$\begin{aligned}\mathrm{div}\frac{D\phi}{|D\phi|}\bigg|_{x=0} &= \frac{1}{\beta}\left[\sum_{i=1}^{n}\phi_{x_ix_i}(0)-\frac{\phi_{x_n}(0)\,\phi_{x_n}(0)}{\beta^2}\phi_{x_nx_n}(0)\right]= \\ &= \frac{1}{\beta}\Delta^{'}\phi(0). \qquad (31)\end{aligned}$$

Here $\Delta^{'}\phi=\sum_{i=1}^{n-1}\phi_{x_ix_i}$. Our next step is to take small $\acute{x}$, namely $|\acute{x}|<Rh$. For such $\acute{x}$ we apply the inverse function theorem to ϕ,

$$H(h)\,\phi(0)=\phi(\acute{x},\gamma(\acute{x}))=\phi(0)+\beta\gamma(0)+O\left(h^2\right). \qquad (32)$$

Putting (32) and (31) into (30) we get

$$\gamma(0)=hG\left(\frac{1}{\beta}\Delta^{'}\phi(0)\right)+o(h). \qquad (33)$$

Furthermore, differentiating both sides of $H(h)\,\phi(0)=\phi(\acute{x},\gamma(\acute{x}))$ gives

$$\begin{aligned}\phi_{x_i}+\phi_{x_n}\gamma_{x_i} &= 0\\ \phi_{x_ix_j}+\phi_{x_ix_n}\gamma_{x_j}+\phi_{x_nx_j}\gamma_{x_i}+\phi_{x_nx_n}\gamma_{x_j}\gamma_{x_i}+\phi_{x_n}\gamma_{x_ix_j} &= 0\end{aligned}$$

for $j,i=1,\ldots,n-1$. We deduce $\gamma_{x_i}(0)=0$ from the first equality and rewrite the second one for $i=j$

$$\phi_{x_jx_j}(0)+\phi_{x_n}(0)\,\gamma x_jx_j(0)=0.$$

After a summation over j this becomes

$$\frac{1}{\beta}\Delta^{'}u(0)=-\Delta\gamma(0).$$

It remains to put this relation into (33) to get the desired equality (29). $\square$

Let us denote $\mu = H(h)\,\phi(0)$. We will see later that $\gamma(0)$ and $\Delta\gamma(0)$ are related to the quantity $M[\phi \leq \mu](0,h)$ through the thresholding

$$F\left(M[\phi \leq \mu](0,h)\right) = 0.$$

Let us show now, that a scheme with a thresholding depending only on one variable can be consistent with the PDE (2) only in the case of a linear G.

We begin by a calculation of $M[\phi \leq \mu](0,h)$

$$\begin{aligned}
M[\phi \leq \mu](0,h) &= \left(\chi_{[\phi\leq\mu]} \star \frac{1}{h^{\frac{n}{2}}}\tilde{\rho}\left(\frac{|\cdot|}{\sqrt{h}}\right)\right)(0) = \\
&= \int_{\mathbb{R}^n} \chi_{[\phi\leq\mu]}(y)\,\frac{1}{h^{\frac{n}{2}}}\tilde{\rho}\left(\frac{|y|}{\sqrt{h}}\right) dy \\
&= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\gamma(\acute{y})} \frac{1}{h^{\frac{n}{2}}}\tilde{\rho}\left(\frac{|y|}{\sqrt{h}}\right) dy_n d\acute{y} = \\
&= A + \int_{\mathbb{R}^{n-1}} \int_{0}^{\frac{1}{\sqrt{h}}\gamma\left(\sqrt{h}\acute{y}\right)} \tilde{\rho}(|y|)\, dy_n d\acute{y},
\end{aligned}$$

where

$$A = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} \tilde{\rho}(|y|)\, dy_n d\acute{y}. \tag{34}$$

Expanding $\gamma\left(\sqrt{h}\acute{y}\right)$ into Taylor series we get:

$$\begin{aligned}
\frac{1}{\sqrt{h}}\gamma\left(\sqrt{h}\acute{y}\right) = \sqrt{h}\frac{\gamma(0)}{h} \quad &+ \quad \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1} \gamma_{y_i y_j}(0)\, y_i y_j + \\
+\frac{h}{6}\sum_{i,j,m=1}^{n-1} \gamma_{y_i y_j y_m}(0)\, y_i y_j y_m \quad &+ \quad O\left(h^{3/2}\acute{y}^4\right).
\end{aligned}$$

Observing that $\gamma(0) = O\left(\sqrt{h}\right)$, we denote $\frac{\gamma(0)}{h} = v$. The expression for M becomes:

$$\begin{aligned}
&M[\phi \leq \mu](0,h) = A + \\
&+ \int_{\mathbb{R}^{n-1}} \tilde{\rho}(|\acute{y}|)\left[\sqrt{h}v + \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1} \gamma_{y_i y_j}(0)\, y_i y_j + O\left(h^{3/2}\acute{y}^4\right)\right] dy_n d\acute{y} \\
&= A + \sqrt{h}vC + \sqrt{h}\Delta\gamma(0)\, B + O\left(h^{3/2}\right).
\end{aligned} \tag{35}$$

Here

$$C=\int_{\mathbb{R}^{n-1}}\rho\left(\acute{y},0\right)d\acute{y}, \tag{36}$$

$$B=\frac{n-1}{2}\int_{\mathbb{R}^{n-1}}y_k^2\rho\left(\acute{y},0\right)d\acute{y}. \tag{37}$$

The thresholding condition becomes

$$F\left(A+\sqrt{h}vC+\sqrt{h}\Delta\gamma\left(0\right)B+O\left(h^{3/2}\right)\right)=0.$$

As was required by the inclusion principle, the function F is non decreasing. This implies, that F' exists almost everywhere, and we can write:

$$F\left(A\right)+F'\left(A\right)\left(\sqrt{h}Cv+\sqrt{h}\Delta\gamma\left(0\right)B\right)+o\left(h\right)=0.$$

Thus

$$v=\frac{\gamma\left(0\right)}{h}=-\frac{B}{C}\Delta\gamma\left(0\right)-\frac{F\left(A\right)}{\sqrt{h}CF'\left(A\right)}+o\left(\sqrt{h}\right).$$

Comparing this relationship with the one in Lemma 2, we see that the only G's we can resolve by thresholding dependent on one variable are the linear ones: $G\left(k\right)=\mathrm{const}\cdot k+\mathrm{const}$.

4.2 The convergence result for a wider class of G

To construct a thresholding function F such that the corresponding operator H (5) would be consistent with the generalised mean curvature PDE (2), we introduce F depending on two variables $F\left(M_1C\left(x,h\right),M_2C\left(x,h\right)\right)$, where

$$M_iC\left(x,h\right)=\int_C\rho_i\left(x-y\right)dy.$$

We will show that one can choose kernels $\tilde{\rho}_1$ and $\tilde{\rho}_2$ so, that the operator H will be consistent with the PDE in (3) even if G is non-linear.

For each ρ_i we write (35), i.e.

$$M_i\left[\phi\leq H\left(h\right)\phi\left(0\right)\right]\left(0,h\right)=A_i+\sqrt{h}vC_i+\sqrt{h}\Delta\gamma\left(0\right)B_i+O\left(h^{3/2}\right), \tag{38}$$

where $i = 1, 2$. This is a system of linear algebraic equation for $\Delta\gamma(0)$ and v. We choose the kernels so, that the determinant of this system is positive

$$D = C_1 B_2 - C_2 B_1 > 0,$$

denote $N_i = M_i [\phi \leq H(h)\phi(0)](0, h) - A_i$ and write the solution

$$\begin{aligned} v = \frac{\gamma(0)}{h} &= \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{C_1 B_2 - C_2 B_1} + O(h), \\ \Delta\gamma(0) &= \frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{C_1 B_2 - C_2 B_1} + O(h). \end{aligned}$$

Looking back at the Lemma 2, we see, that the operator H will be consistent with the PDE in (3) if we take

$$\begin{aligned} F(N_1, N_2) &= v - G(\Delta\gamma(0)) = \\ &= \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{D} - G\left(\frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{D} \right). \end{aligned} \tag{39}$$

Let us now take a look at the inclusion principle, which implies the monotonicity of the scheme. In the case of thresholding function of one variable, the inclusion principle holds when F is nondecreasing. Analogously, in the case of two variables we require

$$\frac{\partial F}{\partial N_1} = \frac{B_2}{D} - \frac{C_2}{D} G' \geq 0, \tag{40}$$

$$\frac{\partial F}{\partial N_2} = -\frac{B_1}{D} + \frac{C_1}{D} G' \geq 0. \tag{41}$$

This implies

$$\frac{B_1}{C_1} \leq G' \leq \frac{B_2}{C_2}. \tag{42}$$

therefore, at least for the time being, we consider G with bounded and positive derivative.

Next, we state some auxiliary results.

Lemma 3. *Suppose (40) and (41) hold and $\mathcal{H}$ is defined by (4), then $\forall h \in \mathbb{R}_+$,*

1. $\mathcal{H}(h)(\mathbb{R}^n) = \mathbb{R}^n$,
   $\mathcal{H}(h)(\emptyset) = \emptyset$,

2. $\forall a, b \in \mathbb{X} : a \subseteq b \Rightarrow \mathcal{H}(h)\, a \subseteq \mathcal{H}(h)\, b.$

Proof.

1. It is enough to show that $F(M_1(\mathbb{R}^n)(x,h), M_2(\mathbb{R}^n)(x,h)) \geq 0$ and $F(M_1(\emptyset)(x,h), M_2(\emptyset)(x,h)) < 0$. First we observe, that $F(A_1, A_2) = 0$, $M_i(\mathbb{R}^n)(x,h) \geq A_i$ and $M_i(\emptyset)(x,h) = 0 < A_i$. This, together with $\frac{\partial F}{\partial N_i} > 0$ gives the desired inequalities.
2. Since $M_i(b) \geq M_i(a)$, $F(M_1(b), M_2(b)) \geq F(M_1(a), M_2(a))$, therefore $[F(M_1(a), M_2(a)) \geq 0] \subseteq [F(M_1(b), M_2(b)) \geq 0]$, which is equivalent to $\mathcal{H}(h)\, a \subseteq \mathcal{H}(h)\, b$.

$\square$

Proposition 1. *Define H by (5) and $\mathcal{H}$ by (4), then for each $h > 0$ and $u \in \mathbb{B}(\mathbb{R}^n)$ one has $H(h)\, u \in \mathbb{B}(\mathbb{R}^n)$.*

Proof. Without loss of generality we assume, that $S_1 \leq u(x) \leq S_2$ for some $S_1, S_2 \in \mathbb{R}$. From $\forall h \in \mathbb{R}_+ \quad \mathcal{H}(h)(\mathbb{R}^n) = \mathbb{R}^n$ and $\mathcal{H}(h)(\emptyset) = \emptyset$ follows $x \in \mathcal{H}(h)[u \leq S_2]$ and $x \notin \mathcal{H}(h)[u \leq S_1]$. Therefore, we see that:

$$S_1 \leq H(h)\, u(x) = inf\, \{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u \leq \lambda]\} \leq S_2.$$

$\square$

With the results above, we are ready to state the convergence of the approximations $H(t/m)^m g$ to the unique viscosity solution of (3).

Theorem 7. *Let $H(h)$ be defined by*

$$[H(h)\, u](x) = \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \geq \lambda]\}$$

with

$$\mathcal{H}_h C = \{x \in \mathbb{R}^n : F(M_1(C)(x,h), M_2(C)(x,h)) \leq 0\},$$

where

$$F(N_1, N_2) = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{D} - G\left(\frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{D}\right),$$

where $\tilde{\rho}_1, \tilde{\rho}_2$ have compact support, and G is continuous nondecreasing satisfying (42).

Then

$$H^m_{\frac{t}{m}} g(x) \to u(x,t)$$

locally uniformly when $m \to \infty$. Here $u(x,t)$ is the unique viscosity solution of

$$\begin{cases} u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right) & \text{in } \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

with G satisfying (42).

Proof. Our aim is to show here that the operator $H(h)$ satisfies the conditions of the Theorem 6.

1. The monotonicity of H_h is ensured by Lemma 1 and Lemma 3.
2. The stability of H is exactly the result of the Proposition 1: $H(h)\, u \in \mathbb{B}\left(\bar{\Omega}\right)$
3. Another property of $H(h)$ that is required is that it must commute with addition of constant, i.e:

   $$\forall a \in \mathbb{R} \quad H(h)(u(x)+a) = H(h)\, u(x) + a$$

   This equality can be easily obtained from the very definition of $H(h)$:

   $$H(h)(u(x)+a) = inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u(x)+a \le \lambda]\} =$$

   $$= inf\{\beta + a \in \mathbb{R} : x \in \mathcal{H}(h)[u(x) \le \beta]\} = H(h)\, u(x) + a$$

4. The limit we are interested in is:

   $$\lim_{h \to 0} \frac{H(h)\, u(x_0)}{h} = 0$$

   for u of the form $u(x) = f(|x - x_0|)$, where $f \in C^2([0,\infty))$ with $f(0) = f'(0) = f''(0) = 0$ and $f''(r) > 0$ for $r > 0$.

It is enough to show, that this is true for $x_0 = 0$. First, we observe, that $\mathcal{H}_h^{-1}[\{0\}] = \{u \le \lambda_1\}$, where $\lambda_1 = H(h)\,u(0)$. Since both ρ_1 and ρ_2 have compact support, we can be sure, that there exists R s. t. $\left\{|x| \le R\sqrt{h}\right\} \supseteq \mathcal{H}_h^{-1}[\{0\}]$. Now we observe, that $\left\{|x| \le R\sqrt{h}\right\} = \{u \le \lambda_2\}$ for some $\lambda_2 > \lambda_1$. From the latter equality we deduce $\lambda_2 = O\left(h^{\frac{3}{2}}\right)$ and conclude by

$$\lim_{h\to 0} \frac{H(h)\,u(x_0)}{h} \le \lim_{h\to 0} \frac{O\left(h^{\frac{3}{2}}\right)}{h} = 0.$$

5. To see that our approximation operator is consistent with the PDE, we use Lemma 2 and essentially repeat the calculations from the motivation part. Due to Lemma 2, it is enough to prove the following

$$\gamma(0) = hG(-\Delta\gamma(0)) + o(h),$$

where $x_n = \gamma(\acute{x})$ is a parametrisation of the surface

$$\{x \in \mathbb{R}^n : u(x) = H(h)\,u(0)\}$$

near $\acute{x} = 0$. To show this, we use the fact that:

$$F(M_1[u \le \mu], M_2[u \le \mu])|_{x=0} = 0$$

We begin by writing the expressions for M_i in detail:

$$M_i = \left(\chi_{[u\le\mu]} \star \frac{1}{h^{\frac{n}{2}}}\rho_i\left(\frac{|\cdot|}{\sqrt{h}}\right)\right)(0) = \int_{\mathbb{R}^n} \chi_{[u\le\mu]}(y)\,\frac{1}{h^{\frac{n}{2}}}\rho_i\left(\frac{|y|}{\sqrt{h}}\right) dy =$$

$$= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\gamma(\acute{y})} \frac{1}{h^{\frac{n}{2}}}\rho_i\left(\frac{|y|}{\sqrt{h}}\right) dy_n d\acute{y} = A_i + \int_{\mathbb{R}^{n-1}} \int_0^{\frac{1}{\sqrt{h}}\gamma\left(\sqrt{h}\acute{y}\right)} \rho_i(|y|)\,dy_n d\acute{y}.$$

Here

$$A_i = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 \rho_i(|y|)\,dy_n d\acute{y}.$$

Expanding $\gamma(h\acute{y})$ in a Taylor series we get:

$$\frac{1}{\sqrt{h}}\gamma\left(\sqrt{h}\acute{y}\right) = \sqrt{h}\frac{\gamma(0)}{h} + \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1} \gamma_{y_iy_j}(0)\,y_iy_j +$$

$$+\frac{h}{6}\sum_{i,j,l=1}^{n-1} \gamma_{y_iy_jy_l}(0)\,y_iy_jy_l + O\left(h^{3/2}\acute{y}^4\right).$$

Observing that $\gamma(0) = O\left(\sqrt{h}\right)$, we denote $\frac{\gamma(0)}{h} = v$. The expression for M_i becomes:

$$\begin{aligned} M_i &= A_i + \\ &+ \int_{\mathbb{R}^{n-1}} \rho_i(\acute{y}, 0) \left[\sqrt{h} v + \frac{\sqrt{h}}{2} \sum_{i,j=1}^{n-1} \gamma_{y_i y_j}(0)\, y_i y_j + O\left(h^{3/2} \acute{y}^4\right) \right] dy_n d\acute{y} \\ &= A_i + \sqrt{h} v C_i + \sqrt{h} \Delta\gamma(0)\, B_i + O\left(h^{3/2}\right). \end{aligned} \tag{43}$$

Let us denote $k = \Delta\gamma(0)$.

Now we can express v and k in terms of M_i and constants A_i, B_i and C_i :

$$v = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{C_1 B_2 - C_2 B_1} + O(h),$$

$$k = \frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{C_1 B_2 - C_2 B_1} + O(h).$$

Since $F(M_1, M_2) = v - G(-k) = 0$, we have:

$$\frac{\gamma(0)}{h} = v = G(-\Delta\gamma(0)) + o(1),$$

or

$$\gamma(0) = hG(-\Delta\gamma(0)) + o(h).$$

$\square$

Remark 1. *As was already mentioned above, convolution kernels $\tilde{\rho}_i$ can also be taken with unbounded support. For example, the exponential decay for large arguments is sufficient in order for Theorem 7 to hold.*

The requirement (42) is quite restrictive. For instance, all functions of the form $G(x) = x^\alpha$ with $\alpha \neq 1$ do not satisfy it. Our next result shows, that it is enough to take G_ϵ satisfying (42) and uniformly close to G in order to approximate the solutions of (3).

Theorem 8. *Suppose $G_\epsilon \to G$ uniformly on $\mathbb{R}$. Denote by H_ϵ the operator with*

$$H^m_{\epsilon,\frac{t}{m}} u_0(x) \to u_\epsilon(x,t),$$

where $u_\epsilon(x,t)$ is the unique viscosity solution of

$$\frac{\partial u_\epsilon}{\partial t} = |Du_\epsilon| \, G_\epsilon \left(div \frac{Du_\epsilon}{|Du_\epsilon|} \right).$$

Then

$$H^m_{\epsilon,\frac{t}{m}} u_0(x) \to u(x,t)$$

locally uniformly as $m \to \infty$ and $\epsilon < 1/m$.

Proof. Since we know, that the approximation $H_{\epsilon,h}$ is consistent with the corresponding equation, for any test function the following holds:

$$\begin{aligned} \gamma(0) &= hG_\epsilon(-\Delta\gamma(0)) + o(h) \\ &= h(G(-\Delta\gamma(0)) + O(\epsilon)) + o(h) \\ &= hG(-\Delta\gamma(0)) + o(h), \end{aligned}$$

when $\epsilon = O(h)$. $\square$

5 Numerical implementation

As it was mentioned in Section 1, the need of tracking the motion of a surface evolving by its mean curvature arises in many applications.

This section is devoted to a description of numerical implementations of the convolution-thresholding scheme developed in Section 4.

We will always consider the evolution of a surface in $\mathbb{R}^n$, which is a level-set of a function $u : \mathbb{R}^n \mapsto \mathbb{R}$. In another words, we will calculate the evolution of u according to the Cauchy problem (3), but focus just on one particular isosurface of it. We also restrict our numerical computations to the two dimensional case $n = 2$.

Given a compact set $C \subset \mathbb{R}^n$, we fix convolution kernels ρ_1, ρ_2 and the time step h and approximate C_t at a time moment $t = mh$ by $(\mathcal{H}(h))^m C$. The algorithm of computations consists of the following steps:

1. Compute convolutions and the thresholding function

$$M_iC(x,h) = \int_{\mathbb{R}^n} \chi_C(y)\,\rho_i(x-y)\,dy \;\; i=1,2 \tag{44}$$
$$F(x,h) = F(M_1C(x,h), M_2C(x,h)). \tag{45}$$

2. Find the evolved set $\mathcal{H}(h)\,C = \{x \in \mathbb{R}^n : F(x,h) \geq 0\}$.

3. Repeat the procedure with the evolved set to get $\mathcal{H}^2(h)C$ and so on.

Due to the calculation of a convolution in (44) the main computational effort falls into the first step of the algorithm. We have implemented several different algorithms of calculation for this step. In what follows we describe the most efficient methods and make comments on the other ones.

5.1 Spectral method

The main idea of the spectral method is to use Fourier series to calculate the convolutions (44). The numerical aspects of this has been presented by Ruuth in [31]. He also proposes an efficient spatial discretization procedure to accurately approximate an isosurface with reasonable amount of points. We use this procedure to perform our calculations.

The general calculation procedure is as follows. We expand both kernels ρ_i and the characteristic function of a set C into the Fourier series to get the Fourier coefficients of convolutions $M_i(x,h)$. We are now in a position, when we can decide whether or not does a given point x belong to $\mathcal{H}(h)\,C$. We just have to evaluate the thresholding function at this point and compare it with zero. In another words, the surface position after time period h is an isosurface $\{x \in \mathbb{R}^n : F(M_1(x,h), M_2(x,h)) = 0\}$. However, two essential points need to be discussed here, namely:

1. Spatial discretization
First we observe, that the surface under consideration is always an isosurface of some function. From the very beginning, it is some isosurface of the initial data of (2), but after each application of $\mathcal{H}(h)$ it is the zero isosurface of the thresholding function F. Without loss of generality we assume that initially our surface is closed and is contained in a unit cube. Since the Fourier coefficients of the characteristic function

of C are needed, we have to integrate elementary functions (namely sines and cosines) over the region of a complicated shape C.

One approach to this problem is to choose a homogeneously distributed $N_p = N^n$ points in the cube. Using the usual discrete fast Fourier transform, the numerical cost of time each step of the algorithm would be $O(N^n \log N)$ operations. The problem however is that the position of the surface is determined with the accuracy $1/N$ and for precise calculations the number of points increases rapidly.

An alternative algorithm was developed by Ruuth in [31]. He proposes to use an adaptive spatial discretization. For the sake of simplicity, we describe this procedure in the case $n = 2$ (see Fig. 1):

(a) Choose a rough initial grid, i.e divide the unit square into $N_0 \times N_0$ equals squares and assign 1 to the points that are inside the curve and 0 to ones outside.

(b) Divide each square that has different values assigned to its corners into 4 equal squares and assign the corresponding values to the new grid points that appear.

(c) Refine also all the squares that have more than one grid point between their corners.

(d) Repeat steps 1b and 1c while the side of the smallest square is longer than a desired spatial accuracy.

In doing so, we significantly reduce the number of grid points. Besides that, the accurate piecewise linear approximation of the curve can be arranged. We divide each square that intersect the curve into two triangles and make a linear interpolation of F inside each triangle picking the zero isoline of it.

However, this adaptive refinement procedure introduces one technical complication, namely: standard discrete fast Fourier transform no longer applies, because the grid is not structured and not equally spaced. The way to proceed with the spectral method in this situation is in using unequally spaced fast Fourier transform algorithm developed by Beylkin in [6]. The author constructs an efficient procedure to compute Fourier coefficients of a function which is given on some set of points in $\mathbb{R}$. The method involves projecting the functions on a subspace of multiresolution analysis, and applies scaling functions from the spline family. The way to generalise the procedure to the functions on $\mathbb{R}^n$ is also given.

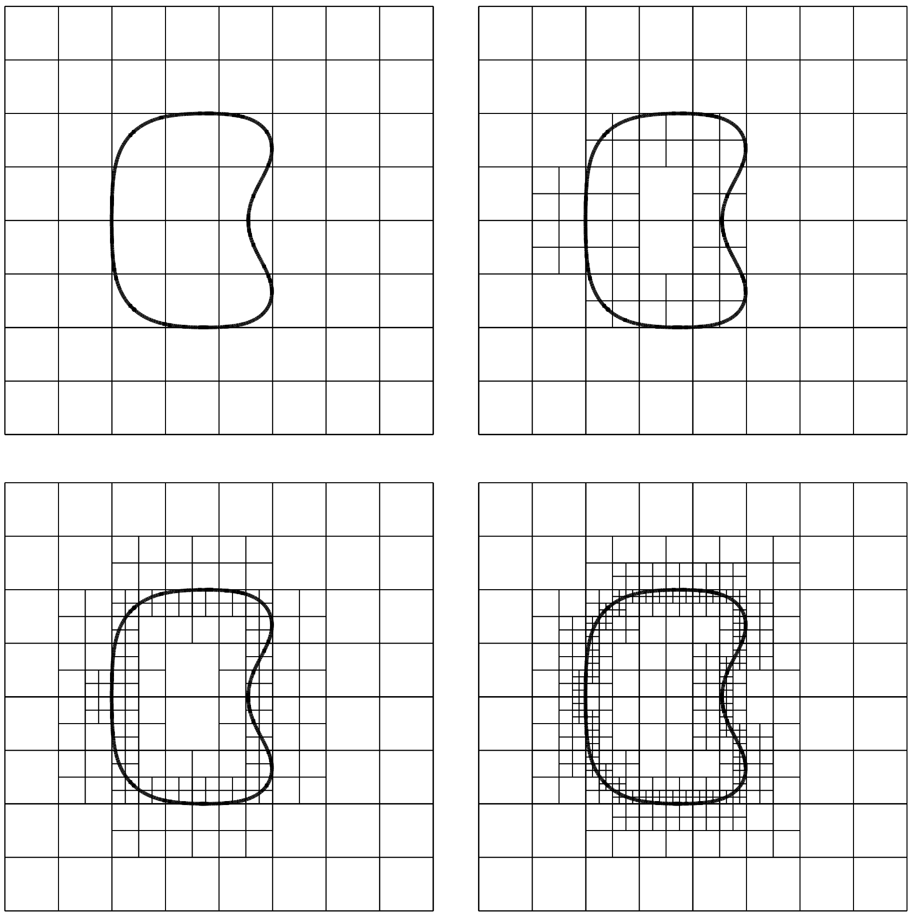

Figure 3: On the spatial discretization.

The Fourier coefficients of the characteristic function $\chi_C(x)$ of C are integrals over C. Using the procedure described above, for each mode we split the domain of integration into the squares. If the cell is entirely contained in C, the integrals over it can be computed analytically. A contribution to the corresponding integrals from the smallest squares that intersect the curve can also be added.

As it was calculated in [31], the numerical cost of the transform algorithm based on this method is

$$O\left(m^n N_p + N_f^n \log\left(N_f\right)\right), \tag{46}$$

where m is a constant depending on a desired accuracy in the calculation of the Fourier coefficients (in case $m = 23$ the accuracy is comparable with the machine truncation error) and N_f is a number of the Fourier modes along each axis.

In the case of homogeneous grid the number of operations is

$$O\left(N_{ph} \log N_{ph}^{1/n}\right), \tag{47}$$

where N_{ph} is the number of points in the homogeneous grid. Let us choose a spatial step $\Delta x = 2^{-M}$. In order to have such discretisation, the number of grid points in the case of homogeneous grid is $N_{ph} = 2^{nM}$ while in the case of adaptive grid-$N_p = C'L2^{(n-1)M}$, where L is $n-1$ dimensional Hausdorff measure of the boundary ∂C and C' is a constant of order one. Comparing (47) and (46), we see that in the case of adaptive discretisation one needs to perform $O\left(\frac{m^4 C'' L}{M 2^M} + \zeta\right)$ times less operations. Here $\zeta = \frac{N_f^n \log N_f}{2^{nM} M \log 2}$ is small. In our numerical experiments with different curves on the plane this acceleration factor turned out to be around $1 : 50$ on the first stages of the evolution. Since L decreases to zero as time progresses, the advantage of the adaptive grid becomes even bigger at large time values.

2. Time discretization
Since we expand a (discontinuous) indicator function $\chi_C(x)$ into a Fourier series, the corresponding coefficients decay slowly. To avoid large amounts of Fourier coefficients (large N_f) we choose smooth kernels ρ_i. In this case, even if the Fourier coefficients of χ_C decay slowly, the coefficients of the convolutions $M_i C$ will decay fast enough. For example, take $\rho(x) = \frac{1}{(4\pi)^{n/2}} e^{-x^2/4}$, then the estimate for the necessary

amount of coefficients reads (see [31]):

$$N_f \geq \left(\frac{\left| \ln \left(\frac{\epsilon}{16\pi L} \right) \right|}{\pi^2 h} \right)^{1/2}$$

where ϵ is the desired error in a position of the level-set $A_\rho = 1/2$ of the convolution M_ρ and L is the length of the curve.

5.2 Direct method

If ρ_1 and ρ_2 are simple enough, their convolutions with χ_C can be calculated explicitly. Let us choose

$$\begin{aligned} \tilde{\rho}_1 (x) &= \begin{cases} \frac{1}{|\mathcal{B}_1|} & \text{if } x < 1 \\ 0 & \text{otherwise} \end{cases} \\ \tilde{\rho}_2 (x) &= \frac{1}{\alpha^n} \tilde{\rho}_1 \left(\frac{x}{\alpha} \right) \end{aligned}$$

where $|\mathcal{B}_1|$ is a Lebesgue measure of an unit ball in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}_+,\ \alpha < 1$. In this case, the convolutions in (44) are proportional the measure of an intersection of C with a ball of radius $\sqrt{h}$ placed in the point x.

To be more explicit, we present expressions for the thresholding function $F(M_1, M_2))$ in the case $n = 2$:

$$F(M_1, M_2) = v - G(k),$$

where

$$\begin{aligned} v &= \frac{\pi\alpha (2\alpha M_1 - 2M_2 - \alpha + 1)}{4\sqrt{h} (\alpha^2 - 1)} \\ k &= \frac{-3\pi (2M_1 - 2\alpha M_2 + \alpha - 1)}{2\sqrt{h} (\alpha^2 - 1).} \end{aligned}$$

In this case convolutions M_1 and M_2 can be calculated as follows. We represent C as a disjoint union of squares and triangles using the adaptive grid procedure described in subsection 5.1 and calculate the area of intersection of the ball (suppρ) with each square and triangle. The numerical cost of each step of the evolution can be estimated by $O(N_p * N_i + N_p)$, where N_i is the number of points inside the ball of radius dt with the center at some grid point. When dt is large, the accuracy of the method is low, therefore one can take less grid points. Thus, N_i is entirely determined by the desired accuracy.

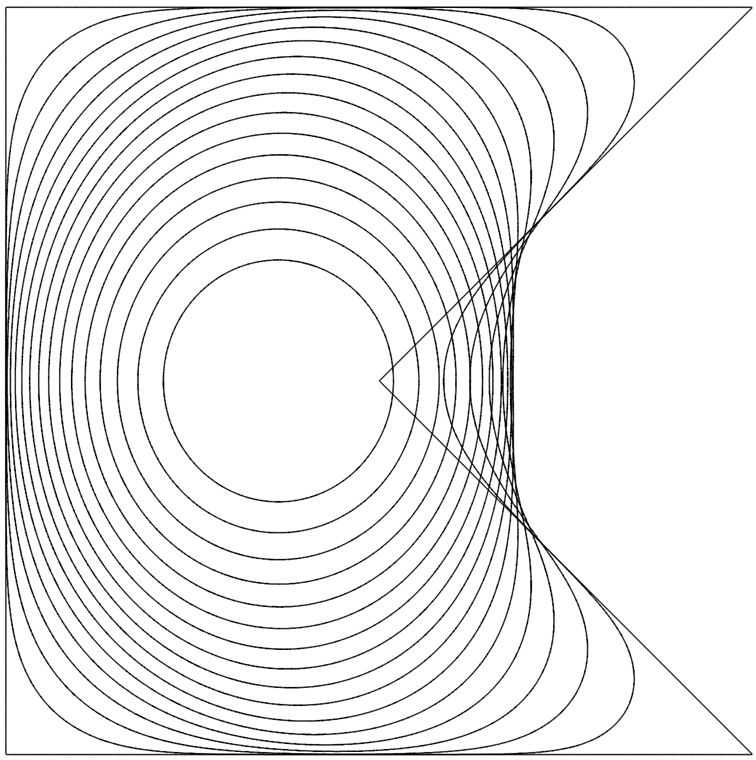

Figure 4: The mean curvature evolution of a non-smooth, non-convex curve

5.3 Computed examples

We present here some non-trivial examples of applications of the above algorithms in case of curve evolution.

Let us first look at the mean curvature evolution. In this case, according to the Von Neumann-Mullins parabolic law, the area enclosed by a simple curve which moves by mean curvature decays linearly in time i.e.

$$\frac{dS}{dt} = -2\pi.$$

Since the exact solution for the mean curvature evolution is known only for the case of a circle, the above relationship is our main criteria of the accuracy of a calculation.

Consider a non-convex, non-smooth initial curve depicted on Fig 4. The mean curvature evolution of this curve was calculated using the direct method with timestep values $dt = 1/600$ and $1/6000$. The shape of the curve is plotted

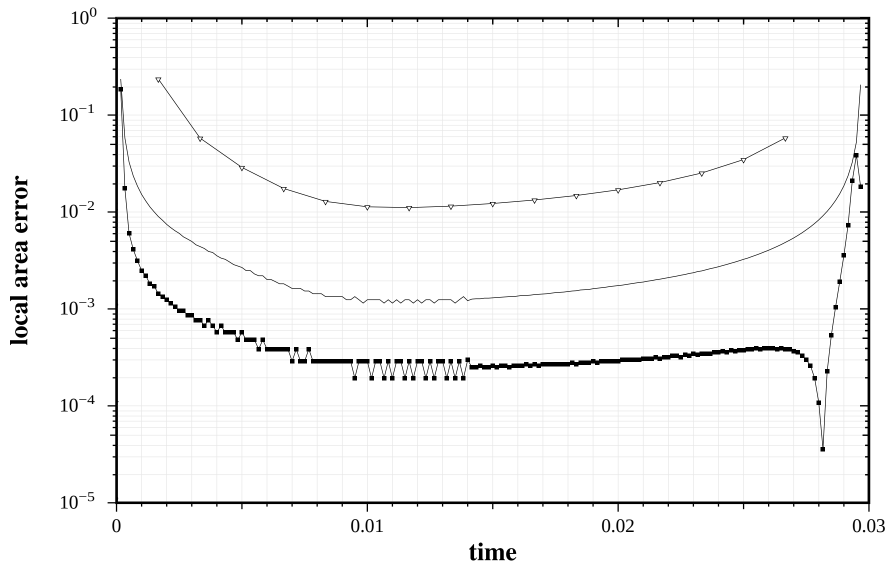

Figure 5: Local area error dependence on time. The first order method with timestep 1/600 – the line with triangle markers; the first order method with timestep 1/6000 – the thin line; the second order method with timestep 1/6000 – the line with square markers.

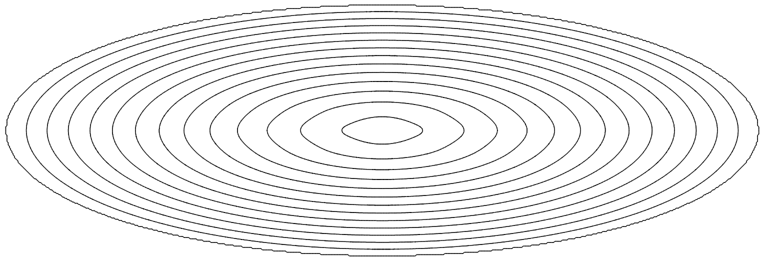

Figure 6: The evolution $v = k^{1/3}$ of an ellipse.

on the Fig. 4 for times $t = 1/600, 2/600, \cdots$, when calculated with the fine timestep. The comparison between local relative errors

$$e_i = \frac{|S_i - S_{i+1} - 2\pi dt|}{2\pi dt} \tag{48}$$

for calculations with different timesteps is seen on the Fig. 5. One can observe, that the error indeed depends linearly on the timestep: taking ten times smaller timestep we achieve ten times better accuracy.

The local area error dependence on time (48) depicted on the fig. 5 is in a sense typical. Since the curve is not smooth initially, we obtain quite high e_i values for the first several time steps. Calculating further, the error stays small until the curve turns into a small circle and begins to collapse rapidly. At these final stages of the evolution the size of the suppρ is comparable with the shape itself which makes the consistency result to fail.

Unfortunately, in the case $v = G(k)$ where G is non-linear, we have no criteria to estimate the error of the calculations. Let us illustrate the evolution with the velocity $v = k^{1/3}$. In this case, we can use the invariance mentioned in Section 1. For example, the excentricity e of the evolving ellipse must remain constant. The evolving ellipse is depicted on Figure 6 and Figure 7 shows its excentricity error

$$E_i = \frac{|e_i - e_0|}{e_0} \tag{49}$$

as a function of timestep i.

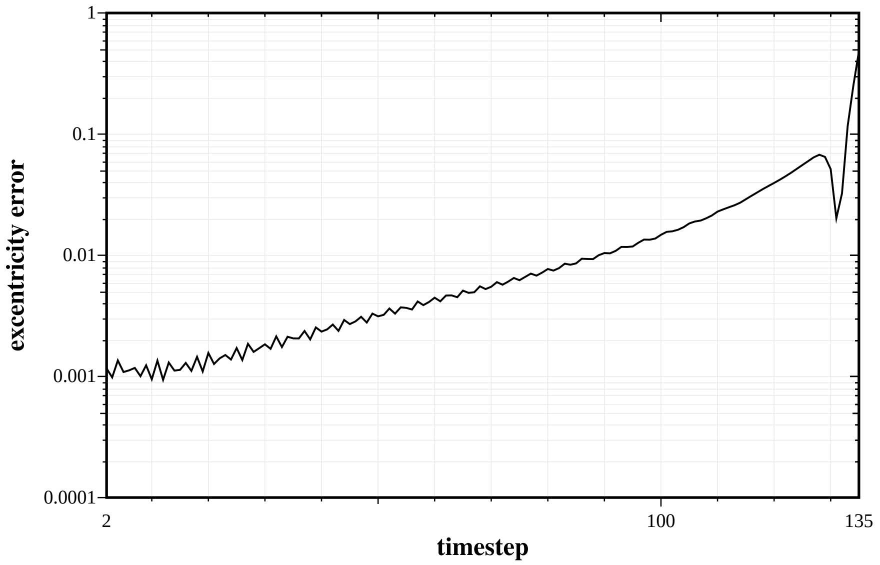

Figure 7: Excentricity error dependence on time

5.4 On the higher order schemes for the mean curvature motion

Let us now look at approximations to the mean curvature evolution. If the surface is smooth, the BMO method gives the first order approximation in time (see Theorem 3). A higher order scheme in time was proposed by Ruuth in [31]. The author uses an extrapolation argument to obtain the results. We show here, how to construct higher order approximations to the mean curvature evolution using the properties of the convolution kernels.

For the sake of simplicity, we present the construction in $\mathbb{R}^2$.We begin by writing the system of linear algebraic equations (38), where each equation is a relationship between the convolution value, mean curvature and the velocity on the evolved front. However, now we will keep an additional term in each equation with the fourth order derivatives to have the error $O\left(t^{5/2}\right)$:

$$M_1 \quad = \quad A_1 + \sqrt{h} v C_1 + \sqrt{h}\gamma''(0)\, B_1 + h\sqrt{h}\gamma''''(0)\, E_1 + O\left(h^{5/2}\right)$$

$$M_2 \quad = \quad A_2 + \sqrt{h} v C_2 + \sqrt{h} \gamma'' (0) \, B_2 + h \sqrt{h} \gamma'''' (0) \, E_2 + O \left(h^{5/2} \right)$$

We multiply the first equation by E_2, the second by E_1 and subtract one from the other to obtain

$$E_2 N_1 - E_1 N_2 = \sqrt{h} [(E_2 C_1 - E_1 C_2) v + (E_2 B_1 - E_1 B_2) \gamma'' (0)] + O \left(h^{5/2} \right).$$

This relationship motivates to take the thresholding function $F(N_1, N_2) = E_2 N_1 - E_1 N_2$ to have the mean curvature evolution with the second order accuracy for smooth curves. However this thresholding function does not simultaneously satisfy (40) and (41) and, therefore, the stability of the numerical scheme is not guaranteed.

The calculations with the above thresholding function were performed. Surprisingly, no sign of instability was observed in the numerical experiments and, as one can see on the Fig. 5, the accuracy was dramatically increased.

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