BRANCHING PROCESSES WITH DETERIORATING RANDOM ENVIRONMENTS

PETER JAGERS AND LU ZHUNWEI

ABSTRACT. In this paper, a BPRE (branching process in random environments) is presented, in which a population development in deteriorating environments is described. Some primary results on process growth and extinction probability are shown. At last, two simple numerical examples that attempt to simulate the probability model are given to help understand its mathematical framework.

1. INTRODUCTION

Consider a Galton-Watson branching process $\{Z_n\}$ in random environments $\bar{\zeta} = \{\zeta_n\}$. Here ζ_n is measurable with respect to $\mathcal{B}_n \supset \sigma(Z_0, \ldots, Z_n)$ for every $n = 0, 1, 2, \ldots$ and is called the environment of nth generation or season. Let $\phi_{\zeta_n}(s)$, $s \in [0,1]$ be the pgf (probability generating function) according to which every individual of the nth generation reproduces independently given $\mathcal{B}_n$. Clearly, ϕ_{ζ_n} is also determined by $\mathcal{B}_n$ and therefore dependent on $Z_0, \ldots, Z_n$. The dependence is assumed to be measurable. We denote by $\mu_{\zeta_j} = \phi'_{\zeta_j}(1)$ the average number of the offspring per individual in the jth generation. Again, μ_{ζ_j} is random and measurable with respect to $\mathcal{B}_j$. Noting that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, we can see that in general the random environment $\{\zeta_n\}$ is no longer iid (independent and identically distributed) and not even stationary. Now, we suppose $Z_0 = z_0 > 1$ and assume $\mu_{\zeta_j} \downarrow \mu$ *a.s.* $(j \to \infty)$, where μ is a positive random variable or, in particular cases, a positive constant. What can we say about $\{Z_n\}$?
In a simplified manner the above mathematical framework describes population development in deteriorating environments. The environment of a season is influenced by earlier population history but may also have independent exogenous components. But due to depletion of resourses or increasing pollution mean reproduction decreases.

2. FURTHER DESCRIPTION OF $\{Z_n\}$

Now let us put the assumption $\mu_{\zeta_j} \downarrow \mu$ *a.s.*$(j \to \infty)$ aside for the moment. Then in our mathematical model here, $\{Z_n\}$ is a branching process with a $\{\mathcal{B}_n\}$ adapted random environment $\bar{\zeta} = \{\zeta_n\}$. By the $\{Z_n\}$ we can understand the following evolving population of particles.
Assume that we start with Z_0 particles and denote the population size at time n by Z_n. The transition from Z_n to Z_{n+1} takes place as below. All the Z_n members

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of the nth generation live for a unit of time, after which each of them splits into a random number of $(n+1)$th generation particles independently according to the same offspring distribution with pgf $\phi_{\zeta_n}(s)$, where $\phi_{\zeta_n}(s)$ is chosen at random from a collection Φ of pgf's according to the information from $\mathcal{B}_n$, which is affected by the size of $Z_0, \dots, Z_n$ and maybe other factors due to the hypothesis $\mathcal{B}_n \supset \sigma(Z_0, \dots, Z_n)$.
Let $\{X_j^{(\zeta_n)}\}$, $j = 1, 2, \dots$ be an iid random variables sequence with the same pgf $\phi_{\zeta_n}(s)$. Then $\{Z_n\}$ can be expressed by the recursion formula as

$$Z_0 = z_0, \quad Z_{n+1} = \sum_{j=1}^{Z_n} X_j^{(\zeta_n)}, \quad (n \geq 0).$$

We can deduce the transition probabilities given $\mathcal{B}_n$ as

$$P\{Z_{n+1} = j | Z_n = i; \mathcal{B}_n\} = \begin{cases} \sum_{n_1+\dots+n_i=j} \prod_{k=1}^{i} \frac{1}{n_k!} \phi_{\zeta_n}^{(n_k)}(0) & i \geq 1, j \geq 0, \\ \delta_{0j} & i = 0, j \geq 0, \end{cases}$$

which is an i-fold convolution of the distribution law $\{p_k(\zeta_n) = \frac{1}{k!}\phi_{\zeta_n}^{(k)}(0)\}$, $k = 0, 1, \dots$ corresponding to the pgf $\phi_{\zeta_n}(s)$. By taking expectation, we get the transition probabilities of $\{Z_n\}$ as

$$P\{Z_{n+1} = j | Z_n = i\} = \begin{cases} \sum_{n_1+\dots+n_i=j} \mathbb{E}[\prod_{k=1}^{i} \frac{1}{n_k!} \phi_{\zeta_n}^{(n_k)}(0)] & i \geq 1, j \geq 0, \\ \delta_{0j} & i = 0, j \geq 0. \end{cases}$$

Furthermore, noting that $\mathcal{B}_n$ gives us Z_n and the environment ζ_n, we obtain that

$$\mathbb{E}[s^{Z_{n+1}}] = \mathbb{E}[\mathbb{E}[s^{Z_{n+1}} | \mathcal{B}_n]] = \mathbb{E}[\mathbb{E}[s^{X_1^{(\zeta_n)} + \dots + X_{Z_n}^{(\zeta_n)}} | \mathcal{B}_n]] = \mathbb{E}[\phi_{\zeta_n}(s)]^{Z_n}, \quad (n \geq 0)$$

where we use the same sign $\mathbb{E}$ all different of expectations according to context. Since the environment ζ_n of the nth generation includes the information about $Z_0, \dots, Z_n$ and $\zeta_0, \dots, \zeta_{n-1}$, it may well happen that

$$\mathbb{E}[s^{Z_{n+1}} | \mathcal{B}_n] = [\phi_{\zeta_n}(s)]^{Z_n} \neq [\phi_{\zeta_0}(\phi_{\zeta_1}(\dots(\phi_{\zeta_n}(s))\dots))]^{z_0}, \quad (n \geq 1)$$

i.e. the pgf of Z_n can no longer in general be expressed by some expectation for an iteration of random pgf's. Clearly, in our model, the $\{Z_n\}$ is different from that of the either Smith-Wilkinson BPRE model or the Athreya-Karlin BPRE model, in which the random environment $\bar{\zeta}$ is an iid random variables sequence or a stationary ergodic process (see [1],[2],[3]). It is more complicated here. In particular cases, if $\{Z_n\}$ is a Markov chain, then it degenerates to population-size-dependent branching process or population-size-dependent branching proces with random environments (see [4],[5]).

3. Process growth

Write $W_n = \frac{Z_n}{\mu^n}$, $W_n^* = \frac{Z_n}{z_0 \prod_{i=0}^{n-1} \mu_{\zeta_i}}$ and $Y_n = z_0 \prod_{i=0}^{n-1} \frac{\mu_{\zeta_i}}{\mu}$, in which the products $\prod_{i=0}^{n-1} \mu_{\zeta_i}$ in W_n^* and $\prod_{i=0}^{n-1} \frac{\mu_{\zeta_i}}{\mu}$ in Y_n are defined by 1 as $n = 0$. Then we have

Theorem 1. *$\{W_n^*\}$ is a non-negative martingale with respect to $\{\mathcal{B}_n\}$, and $0 \leq \lim_{n\to\infty} W_n^* = W^* < +\infty$ a.s. exists.*

Proof. $W_n^* \geq 0$ for all $n \geq 0$ is obvious by its definition. That $\{W_n^*\}$ is a martingale with respect to $\{\mathcal{B}_n\}$ follows by

$$\begin{aligned}\mathbb{E}[W_{n+1}^*|\mathcal{B}_n] &= \mathbb{E}[\frac{Z_{n+1}}{z_0\prod_{i=0}^n \mu_{\zeta_i}}|\mathcal{B}_n] = \mathbb{E}[\frac{\sum_{j=1}^{Z_n} X_j^{(\zeta_n)}}{z_0\prod_{i=0}^n \mu_{\zeta_i}}|\mathcal{B}_n] \\ &= \frac{Z_n\mu_{\zeta_n}}{z_0\prod_{i=0}^n \mu_{\zeta_i}} = \frac{Z_n}{z_0\prod_{i=0}^{n-1} \mu_{\zeta_i}} = W_n^* \quad (n \geq 0).\end{aligned}$$

Noting that $\mathbb{E}[W_0^*] = 1$ for all $n \geq 0$, by the martingale convergence theorem, it must yield that $0 \leq \lim_{n\to\infty} W_n^* = W^* < +\infty$ *a.s.* exists. □

Theorem 2. *$\{W_n\}$ is a non-negative submartingale with respect to $\{\mathcal{B}_n\}$, and $0 \leq \lim_{n\to\infty} W_n = W \leq +\infty$ is well defined on the set $\{W^* \neq 0\} \cup \{Y \neq +\infty\}$.*

Proof. Since $W_n = \frac{Z_n}{\mu^n} \geq 0$ and $\frac{\mu_{\zeta_j}}{\mu} \geq 1$ *a.s.* $(j \geq 0)$ by the assumption of $\mu_{\zeta_j} \downarrow \mu$ *a.s.*, and noting that

$$\mathbb{E}[W_{n+1}|\mathcal{B}_n] = \mathbb{E}[\frac{Z_{n+1}}{\mu^{n+1}}|\mathcal{B}_n] = \frac{Z_n}{\mu^n}\frac{\mu_{\zeta_n}}{\mu} \geq \frac{Z_n}{\mu^n} = W_n \ a.s. \quad (n \geq 0),$$

we see that $\{W_n\}$ is a non-negative submartingale with respect to $\{\mathcal{B}_n\}$. From the assumption $z_0 > 1$ and $\frac{\mu_{\zeta_j}}{\mu} \geq 1$ *a.s.* $(j \geq 0)$, we know that $Y_{n+1} \geq Y_n \geq z_0$ *a.s.* for all $n \geq 0$. Then $z_0 \leq \lim_{n\to\infty} Y_n = Y \leq +\infty$ must exist. Therefore, since $W_n = W_n^* Y_n$, $0 \leq \lim_{n\to\infty} W_n = W = W^*Y \leq +\infty$ is well defined unless $W^* = 0$ and $Y = +\infty$ as required. □

Remark. 1. However, even on the set $\{W^* = 0\} \cap \{Y = +\infty\}$, i.e. when $\lim_{n\to\infty} W_n^* = 0$ and $\lim_{n\to\infty} Y_n = +\infty$, obviously, sometimes $0 \leq \lim_{n\to\infty} W_n = W \leq +\infty$ still is well defined. And then if $W_n^* = O(Y_n^{-1})$ *a.s.*, we still have

$$0 \leq \lim_{n\to\infty} W_n = \lim_{n\to\infty} W_n^* Y_n = W < +\infty \ a.s.$$

otherwise $W = +\infty$ *a.s.*.
2. Since $\{W_n\}$ is a non-negative submartingale with respect to $\{\mathcal{B}_n\}$, if

$$\sup_{n\geq 1} \mathbb{E}[W_n] = \lim_{n\to\infty} \mathbb{E}[W_n] < +\infty,$$

then by the submartingale convergence theorem, $0 \leq \lim_{n\to\infty} W_n = W < +\infty$ *a.s.* exists. Therefore, we know that $0 \leq W < +\infty$ *a.s.*, if (i) $\lim_{n\to\infty} \mathbb{E}[W_n] < +\infty$, or (ii) $Y < +\infty$, or (iii) $\lim_{n\to\infty} Y_n = +\infty$, but the limit $\lim_{n\to\infty} W_n$ exists and $W_n^* = O(Y_n^{-1})$ *a.s.*. Otherwise, either $W = +\infty$ *a.s.* or the limit $\lim_{n\to\infty} W_n$ does no exist.

Furthermore, we have

Theorem 3. *On the set $\{\sum_{j=0}^{\infty}(\mu_{\zeta_j} - \mu) < +\infty\} \cup \{Z_n \to 0\}$, $0 \leq \lim_{n\to\infty} W_n = W < +\infty$ a.s. exists.*

Proof. First, we know that $\sum_{j=0}^{\infty}(\mu_{\zeta_j} - \mu) < +\infty$ *a.s.* is equivalent to $\prod_{j=0}^{\infty} \frac{\mu_{\zeta_j}}{\mu} < +\infty$ *a.s.* or $Y = z_0\prod_{j=0}^{\infty} \frac{\mu_{\zeta_j}}{\mu} < +\infty$ *a.s.*. Recalling Theorem 2, on the set $\{\sum_{j=0}^{\infty}(\mu_{\zeta_j} - \mu) < +\infty\}$, then we have

$$0 \leq \lim_{n\to\infty} W_n = W = W^*Y < +\infty \ a.s. \quad \text{exists.}$$

Secondly, if $Z_n \to 0$, then there must a.s. exist some positive integer-valued random variable N, such that $Z_{n+N} = 0$ for all $n \geq 0$. On the other hand, we know that

$\mu^n > 0$ for all $n \geq 0$ since $\mu > 0$. Therefore, $W_{n+N} = \frac{Z_{n+N}}{\mu^{n+N}} = 0$ for all $n \geq 0$, which implies that, on the set $\{Z_n \to 0\}$, $\lim_{n\to\infty} W_n = 0$ and the proof is complete. □

Corollary 1. *If there exist a constant* $\epsilon > 0$ *and a non-negative random variable* X*, such that* $(\mu_{\zeta_j} - \mu) \leq \frac{X}{j^{1+\epsilon}}$ *a.s. for all sufficiently large* j*, then* $\lim_{n\to\infty} W_n = W < +\infty$ *a.s..*

Proof. Since $(\mu_{\zeta_j} - \mu) \leq \frac{X}{j^{1+\epsilon}}$ *a.s.* for all sufficiently large j, we have $\sum_{j=0}^{\infty}(\mu_{\zeta_j} - \mu) < +\infty$ *a.s.* and the assertion follows Theorem 3. □

4. About the extinction probability

Let $q = P\{Z_n \to 0\}$ be the extinction probability of $\{Z_n\}$. We know that

$$\begin{aligned} q &= P\{Z_n = 0, \text{ for some } n\} = P\{\cup_{n=1}^{\infty}\{Z_n = 0\}\} \\ &= \lim_{n\to\infty} P\{Z_n = 0\} = \lim_{n\to\infty} \mathbb{E}[s^{Z_n}]_{s=0}. \end{aligned}$$

First, let us look at the case of $\mu < 1$ *a.s.*. Clearly, if $\lim_{n\to\infty} W_n = W < +\infty$ is well defined, then

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} \mu^n \lim_{n\to\infty} W_n = 0 \cdot W = 0 \ a.s.,$$

i.e. $q = P\{Z_n \to 0\} = 1$. The following theorem does not use information about $\{W_n\}$.

Theorem 4. *If* $\mu < 1$ *a.s., then* $q = 1$.

Proof. Since $\mu_{\zeta_j} \downarrow \mu < 1$ *a.s.*, we can define a stopping time by

$$n_0 = \min\{j : \mu_{\zeta_j} < 1\}.$$

Then

$$\mu_{\zeta_{n_0+n}} < 1 \text{ for all } n \geq 0.$$

Conditionally upon $\mathcal{B}_{n_0}$, $\mathbb{E}[Z_{n_0}|\mathcal{B}_{n_0}] = Z_{n_0}$ and $\mathbb{E}[Z_{n_0+1}|\mathcal{B}_{n_0}] = Z_{n_0}\mu_{\zeta_{n_0}}$. If $\mathbb{E}[Z_{n_0+n}|\mathcal{B}_{n_0}] \leq Z_{n_0}\mu_{\zeta_{n_0}}^n$, then

$$\begin{aligned} & \mathbb{E}[Z_{n_0+n+1}|\mathcal{B}_{n_0}] = \mathbb{E}[\mathbb{E}[Z_{n_0+n+1}|\mathcal{B}_{n_0+n}]|\mathcal{B}_{n_0}] \\ = \ & \mathbb{E}[Z_{n_0+n}\mu_{\zeta_{n_0+n}}|\mathcal{B}_{n_0}] \leq \mu_{\zeta_{n_0}}\mathbb{E}[Z_{n_0+n}|\mathcal{B}_{n_0}] \leq Z_{n_0}\mu_{\zeta_{n_0}}^{n+1}. \end{aligned}$$

By induction over n, thus $\mathbb{E}[Z_{n_0+n}|\mathcal{B}_{n_0}] \leq Z_{n_0}\mu_{\zeta_{n_0}}^n$ for all $n \geq 0$. Letting $n \to \infty$, we get $\mathbb{E}[Z_{n_0+n}|\mathcal{B}_{n_0}] \to 0$, which together with Fatou's lemma implies $Z_{n_0+n} \to 0$ *a.s.*. Clearly $Z_n \to 0$ if and only if $Z_{n_0+n} \to 0$. Therefore, we obtain $Z_n \to 0$ *a.s.*, i.e. $q = P\{Z_n \to 0\} = 1$. The proof is complete. □

Secondly, let us look at the case of $\mu > 1$ *a.s.*. Clearly, if $0 < \lim_{n\to\infty} W_n = W \leq +\infty$ is well defined, noting that $W_n = \frac{Z_n}{\mu^n}$ and $\mu > 1$ *a.s.*, then we have

$$\lim_{n\to\infty} Z_n = \lim_{n\to\infty} \mu^n \lim_{n\to\infty} W_n = +\infty \ a.s..$$

Thus $q = 0 < 1$ holds. However, if we strengthen the condition $\mu_{\zeta_j} \geq \mu$ for $\forall j \geq 0$ to

$$\sum_{k>i} p_k(\zeta_j) \geq \sum_{k>i} p_k(\xi) \ \text{ for } \forall i, j \geq 0, \qquad (*)$$

where $\{p_k(\xi)\}$ is a random probability distribution law dependent on some random environment ξ with expectation $\mu_\xi \stackrel{D}{=} \mu$ and assume that $\mathbb{E}[-\log(1-p_0(\xi))] < +\infty$, then as Theorem 4, the following theorem also does not use information about W_n.

Theorem 5. *Under the assumption above, if $\mu \geq 1$ a.s. and $P\{\mu > 1\} > 0$, then $q < 1$.*

Proof. From the assumption together with the result of [1], we know that there is an iid random environment $\bar{\xi} = \{\xi_n\}$, such that $\xi_n \stackrel{D}{=} \xi$ for $\forall n \geq 0$ and thus

$$\sum_{k=1}^{\infty} kp_k(\xi_n) = \mu_{\xi_n} \stackrel{D}{=} \mu, \text{ for } \forall n \geq 0;$$

and

$$\sum_{k>i} p_k(\zeta_j) \stackrel{D}{\geq} \sum_{k>i} p_k(\xi_j), \text{ for } \forall i, j \geq 0.$$

Further choose $\theta > 1$ so that $P\{\mu > \theta\} > 0$. Then

$$\text{(i)} \quad \mathbb{E}[\log \mu_{\xi_j}] = \mathbb{E}[\log \mu_\xi] \geq \mathbb{E}[\log(\mu \wedge \theta)] \geq \theta P\{\mu > \theta\} > 0 \text{ for } \forall j \geq 0;$$

and

$$\text{(ii)} \quad \mathbb{E}[-\log(1 - p_0(\xi_j))] < +\infty, \text{ for } \forall j \geq 0,$$

so that the supercriticality holds. In other words, the reproduction in the environment ξ_j is dominated by that in environment ζ_j stochastically for any $j \geq 0$ so that if $\{Z_n^*\}$ is a BPRE with initial population size $Z_0^* = z_0 = Z_0$ living in the random environment $\bar{\xi} = \{\xi_n\}$, then it must be supercritical, and

$$Z_n \stackrel{D}{\geq} Z_n^*, \text{ for } \forall n \geq 0.$$

Hence

$$\begin{aligned} q &= P\{Z_n \to 0\} = \lim_{n\to\infty} P\{Z_n = 0\} = \lim_{n\to\infty} P\{Z_n \leq 0\} \\ &\leq \lim_{n\to\infty} P\{Z_n^* \leq 0\} = \lim_{n\to\infty} P\{Z_n^* = 0\} = q^* < 1, \end{aligned}$$

as required. □

At the last, let us look at the case of $\mu = 1$ *a.s.*. In order to avoid trivialities and additional absorbing states, we shall assume that $0 < p_0(\zeta_n) + p_1(\zeta_n) < 1$ for all $n \geq 0$. Then we have

Theorem 6. *If $\mu = 1$ a.s., $\sum_{j=0}^{\infty}(\mu_{\zeta_j} - 1) < +\infty$ a.s. and there is a constant $c > 0$ such that $p_0(\zeta_k) > c$ for all k from some $k_0 > 0$ onwards, then $q = 1$.*

Proof. First, noting that $0 < c < p_0(\zeta_k) < 1$, then for any constant $x > 0$, we always can take $\delta = c^x > 0$ such that

$$P\{Z_n \to 0 | \mathcal{B}_k\} \geq (p_0(\zeta_k))^{Z_k} \geq c^x = \delta, \text{ on } \{Z_k \leq x\}.$$

Therefore, by Theorem 2 in [6], we have

$$P\{Z_n \to 0 \text{ or } Z_n \to +\infty\} = 1. \qquad (**)$$

Moreover, since $\mu = 1$ *a.s.* $\Rightarrow W_n = Z_n$ *a.s.*, noting that $\sum_{j=0}^{\infty}(\mu_{\zeta_j} - 1) < +\infty$ *a.s.*, by Theorem 3, we know that $0 \leq \lim_{n\to\infty} Z_n = Z < +\infty$ *a.s.* exists, which together with $(**)$ tells us $q = P\{Z_n \to 0\} = 1$ as required. □

It is easily to obtain that

Corollary 2. *If $\mu = 1$ a.s. and there exist a non-negative random variable X and constants $\epsilon, c > 0$, such that $\mu_{\zeta_j} - 1 \leq \frac{X}{j^{1+\epsilon}}$ for all sufficiently large j and $p_0(\zeta_k) > c$ for all k from some $k_0 > 0$ onwards respectively, then $q = 1$.*

Corollary 3. *If there exists some $J > 0$ such that $\mu_{\zeta_j} = 1$ a.s. for all $j \geq J$, then $q = 1$.*

We conjecture that if $\mu \geq 1$ *a.s.* with $P\{\mu > 1\} > 0$, then $q < 1$ is probably always true.

5. Two numerical examples

In order to explain our probability model and help to understand its mathematical framework we provide two simple numerical examples here.

Example 1. In this example, we consider a population development in some environment, where the total amount of resource consumption will determine the reproduction. We can persume that the total amount of resource consumption is proportional to the accumulated number of the individuals who lived before or at time n and with the decrease of the amount of available resource the reproducing size will be reduced.
In our BPRE mathematical model we can suppose that the random reproducing pgf in nth generation has the form $\phi_{\zeta_n}(s) = \phi_{Z_0+\cdots+Z_n}(s)$, i.e. $\zeta_n = Z_0 + \cdots + Z_n$, and Φ, the collection of pgf's is $\Phi = \{\phi_m(s),\ m = 0, 1, 2, \dots\}$. Clearly, $\{\zeta_n\}$ here is non-decreasing, since Z_n is non-negative. By the definition of deteriorating environment, we should assume that $\mu_m \downarrow$ as $m \to \infty$.
One can easily define $\phi_m(s)$ in various forms for meeting different cases discussed above in Sections 3 and 4 of the paper and construct simple counterexamples to illustrate that the pgf of Z_n can not be expressed by iteration of pgf's of $\phi_{\zeta_0}, \phi_{\zeta_1}, \cdots, \phi_{\zeta_{n-1}}$.

Example 2. In this example, we intend to try to establish another BPRE model, where the environment deteriorates more slowly. Here we suppose that the resource can be expanded within certain limits in the initial period but can not resuscitate, which can describe that a group of individuals emigrate a closed place such as an isolated island. We hope that in our model if the population size keeps in some range then its corresponding reprodution shall be unchanged. Therefore, we can use the the number of population increases minus the number of decreases to set up a threshold restraining the development of the population. A concrete way of doing this as below:
Let $K \in Z_+$, $c \in R_+$, $\Phi = \{\phi_k(s)\}$, $k = 1, 2, \dots$, where $\phi_k(s)$ is the pgf of Poisson distribution with parameter $c + \frac{1}{k}$, i.e. $\phi_k(s) = e^{-(c+\frac{1}{k})(1-s)}$. Given $Z_0 = z_0 > 1$, take $\phi_{\zeta_0} = \phi_1$. Further, if $\phi_{\zeta_n} = \phi_j$, then if

$$|\{i : Z_i - Z_{i-1} > 0, i = 1, 2, \dots, n\}| - |\{i : Z_i - Z_{i-1} \leq 0, i = 1, 2, \dots, n\}| > K,$$

where $|\{\cdot\}|$ means the number of elements in the corresponding set $\{\cdot\}$, then we take $\phi_{\zeta_{n+1}} = \phi_{j+1}$, otherwise $\phi_{\zeta_{n+1}} = \phi_j$.

Here $\mathcal{B}_n = \sigma(Z_0, \dots, Z_n)$, $\bar{\zeta} = \{\zeta_n\}$ is adapted to $\{\mathcal{B}_n\}$, and is not iid and not stationary. Let $\mu = c$. Then we can see that

$$\lim_{j\to\infty} \mu_{\zeta j} = \lim_{k\to\infty} \phi'_k(1) = \lim_{k\to\infty} (c + \frac{1}{k}) = c = \mu,$$

and $\mu_{\zeta j}$ is monotonically decreasing to μ as we required.
Moreover, we can establish a more complicated model, in which ζ_n not only depends upon $Z_0, Z_1, \dots, Z_n$ but also upon a given random variable ξ_n. To the end, on the base of above example, we add a pgf set $\Phi^* = \{\phi_k^*(s)\}$, $k = 1, 2, \dots$, in which $\phi_k^*(s)$ is the pgf of Poisson distribution with parameter $c^* + \frac{1}{k}$, and an iid random variable sequence $\{\xi_n\}$ with the distribution law as $P\{\xi_n = c\} = p$, $P\{\xi_n = c^*\} = 1 - p$ $(0 < p < 1)$ for $n = 0, 1, 2, \dots$. We start according to value of ξ_0 to choose $\phi_{\zeta_0} = \phi_1$ or ϕ_1^*. In case $\phi_{\zeta_n} = \phi_j$ or ϕ_j^*, if

$$|\{i : Z_i - Z_{i-1} > 0, i = 1, 2, \dots, n\}| - |\{i : Z_i - Z_{i-1} \le 0, i = 1, 2, \dots, n\}| > K,$$

we choose $\phi_{\zeta_{n+1}} = \phi_{j+1}$ or ϕ_{j+1}^* at random according to the value of ξ_{n+1}, otherwise we let $\phi_{\zeta_{n+1}} = \phi_j$ or ϕ_j^* as the same as ϕ_{ζ_n}.
Here $\mathcal{B}_n = \sigma(Z_0, \dots, Z_n; \xi_n) \supset \sigma(Z_0, \dots, Z_n)$ and the pgf collection is $\Phi \cup \Phi^*$. We still have $\mu_{\zeta_j} \downarrow \mu$ *a.s.* $(j \to \infty)$, and where μ is a random variable with the distribution law as $P\{\mu = c\} = p$, $P\{\mu = c^*\} = 1 - p$.
Clearly, we can use the results in Sections 3 and 4 to deal with these mathematical models.

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Peter Jagers, Department of Mathematical Statistics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden
E-mail address: jagers@math.chalmers.se

Lu Zhunwei, Department of Mathematics, Taiyuan University of Technology, 030024, Taiyuan,Shanxi province,The People's Republic of China
E-mail address: luz1547@public.ty.sx.cn