Pricing Double Barrier Options with Error Control

Per Hörfelt*
Dept. of Mathematics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
e-mail:perh@math.chalmers.se

Abstract

The pricing formulas for double barrier options with rebate can be expressed as (infinite) series, see Kunitomo et al. (1992). This paper derives error estimates for the truncation error that appears when these series are approximated with a partial sum.

Key words: Option pricing, double barrier options, truncation error.

JEL classification: G13

AMS 2000 Mathematics subject classification: 91B28

1 Introduction

A double barrier option is an option which has a barrier above and below the initial price of the underlying asset, and the option is activated (knocked in) or extinguished (knocked out) as soon as one of the barrier is hit. A double barrier option is sometimes combined with a rebate options. This is an option which compensates for the loss that occurs if the knock-out option is knocked out or the knock-in option never is knocked in.

Several people have analysed the pricing of double barrier options with zero rebate. Kunitomo and Ikeda [7] calculated its value using the Levy formula. They express the price of a double barrier option as series of normal distributions. The formulas by Kunitomo et al. can also be found in Sidenius [12].

Hui [4] solved the pricing problem using separation of variables and Pelsser [11] derived the value with the aid of contour integration. Both obtained formulas which describe the price of a double barrier option as Fourier series.

*The author would like to thank Christer Borell, Chalmers University of Technology, and Anna Areskoug and Håkan Norekrans, Front Capital Systems AB, for their comments and suggestions.

Geman and Yor [3] derived expressions for the Laplace transform of the double barrier option price. They invert the Laplace transform numerically to obtain the option prices.

The rebate options corresponding to a double barrier option have been examined in Hui [4], Pelsser [11], and Siddenius [12]. In the two first mentioned papers the value of a rebate option is expressed as Fourier series, while in [12] the price is described as series of normal distributions.

The numerical characteristics for the two different *series* solutions have been compared in Hui et al. [6]. In that paper it is recommended that in a trading system, the solution involving standard normal distributions should be used, that is, the solution given by Kunitomo and Ikeda. The argument is that cancellation errors can appear in the Fourier series which may lead to substantial errors in the resulting theoretical values.

The main purpose of this paper is to continue to investigate Kunitomo-Ikeda's pricing formulas. We will derive error estimates for the truncation error that appears when the infinite series are approximated with a partial sum. Moreover, this paper shows that the rate of convergence for the truncation error decreases exponentially as the number of terms in the partial sum increases. The investigation will provide us with an algorithm to compute the price of a double barrier option which is easy to implement and gives control of the truncation error. We will as well consider the pricing of rebate options corresponding to a double barrier option. We will derive pricing formulas and establish error bounds for the truncation error. It should be emphasised that a similar research have been made independently by Lou, see [9].

The remainder of this paper is structured as follows. In the next section we will derive analytical expressions for certain distributions which involve stopping times associated with a Brownian motion. We will discuss numerical characteristics of the formulas obtained. Section 2 may be of independent interest since the distributions under consideration appear in other financial problems, for instance the pricing of one touch double barrier binary options, see Hui [4]. Section 3 discusses the pricing of double barrier options with zero rebate and Section 4 deals with the pricing of rebate options.

2 Hitting and Exit Times

This section computes various laws involving the first hitting time and the first exit time of a Brownian motion with drift. Take as given a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ carrying a Brownian motion $\{W_t\}_{t\geq 0}$. If $b \in \mathbb{R}$, the first hitting time of b, hereafter denoted $\lambda(b)$, is defined by

$$\lambda(b) = \inf\{t > 0\,;\, W_t = b\,\}.$$

Moreover, introduce a collection of probability measures $\{P^\theta\,;\,\theta\in\mathbb{R}\}$ given by

$$P^\theta(A)=E\big[e^{-\frac{1}{2}\theta^2+\theta W_1}1_A\big],\quad A\in\mathcal{F},$$

where E denotes expectation with respect to P. According to the Cameron-Martin theorem we have that the stochastic process W^θ, given by

$$W_t^\theta=W_t-\theta t,\quad 0\le t\le 1,$$

is a Brownian motion with respect to $(P^\theta,\{\mathcal{F}_t\}_{0\le t\le 1})$. This fact will frequently be used in the sequel.

The subject of the first part of this section is the following distribution,

$$F_+(a,b_1,b_2;\theta)=P^\theta(\,W_1\le a,\ \lambda(b_2)<\lambda(b_1),\ \lambda(b_2)\le 1\,),$$

defined for $b_1<0<b_2$, $a\le b_2$ and all $\theta\in\mathbb{R}$.

The key result in this section is the following lemma, the proof of which is based on an idea described in Andersson, see [1].

Lemma 2.1. *Suppose $b_1<0<b_2$, $a\le b_2$ and $\theta\in\mathbb{R}$. Set $\lambda^{(0)}=0$ and $\rho^{(0)}=0$ and define recursively the stopping times*

$$\lambda^{(n)}=\inf\{t>\rho^{(n-1)}\,;\,W_t=b_2\}$$

and

$$\rho^{(n)}=\inf\{t>\lambda^{(n-1)}\,;\,W_t=b_1\}$$

for $n\ge 1$. Then, for any $n\ge 1$,

$$\begin{aligned}F_+(a,b_1,b_2;\theta)=&\sum_{i=1}^{n}\big(P^\theta(\,W_1\le a,\ \lambda^{(2i-1)}\le 1\,)-P^\theta(\,W_1\le a,\ \lambda^{(2i)}\le 1\,)\big)\\&+P^\theta(\,W_1\le a,\ \lambda^{(2n+1)}\le 1,\ \lambda^{(1)}<\rho^{(1)}\,).\end{aligned}\tag{1}$$

Proof. Firstly, let $A=\{\lambda^{(1)}<\rho^{(1)}\}$ and $B_n=\{\lambda^{(n)}\le 1\}$. Note that for all $\omega\in A^c$ we have $\lambda^{(1)}(\omega)=\lambda^{(2)}(\omega)$, which implies $\rho^{(2)}(\omega)=\rho^{(3)}(\omega)$, which in turn implies $\lambda^{(3)}(\omega)=\lambda^{(4)}(\omega)$ and so forth. Hence, by induction on n it can be shown that for all $\omega\in A^c$ we have $\lambda^{(2n-1)}(\omega)=\lambda^{(2n)}(\omega)$ for any $n\ge 1$, and, accordingly from this

$$1_{B_{2n-1}}1_{A^c}=1_{B_{2n}}1_{A^c}\tag{2}$$

for every $n\ge 1$. By a similar argument one can prove that for all $\omega\in A$ and any $n\ge 1$ we have $\lambda^{(2n)}=\lambda^{(2n+1)}$, and, hence,

$$1_{B_{2n}}1_A=1_{B_{2n+1}}1_A\tag{3}$$

for every $n \geq 1$. Next observe that for any given sets C_1 and C_2 we have

$$1_{C_1} 1_{C_2} = 1_{C_1} - 1_{C_1} 1_{C_2^c}. \tag{4}$$

Successive applications of the equations (2), (3) and (4) yield

$$\begin{aligned} 1_{B_1} 1_A &= 1_{B_1} - 1_{B_1} 1_{A^c} \\ &= 1_{B_1} - 1_{B_2} 1_{A^c} \\ &= 1_{B_1} - 1_{B_2} + 1_{B_2} 1_A \\ &= 1_{B_1} - 1_{B_2} + 1_{B_3} 1_A \\ &\vdots \\ &= 1_{B_1} - 1_{B_2} + 1_{B_3} - \ldots - 1_{B_{2n}} + 1_{B_{2n+1}} 1_A \end{aligned}$$

for any $n \geq 1$. By integrating both sides over $\{W_1 \leq a\}$ with respect to P^θ we obtain equation (1). □

To give an explicit expression of the terms in the sum in equation (1) we will introduce some definitions. In what follows we let Φ denote the standard normal distribution function, i.e.

$$\Phi(x) = \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} \frac{d\xi}{\sqrt{2\pi}}.$$

Morever, let $\{U_t\}_{t \geq 0}$ be the Brownian semi-group, i.e.

$$(U_t\, f)(x) = E\big[f(x + W_t)\big],$$

where $f : \mathbb{R} \to \mathbb{R}$ is bounded and Borel measurable. Recall that if Λ is a stopping time with respect to $\{\mathcal{F}_t\}$ such that $\Lambda \leq T$ P-a.s., where T is a fixed positive number, then the strong Markov property for Brownian motion tells us

$$E[f(W_T) \mid \mathcal{F}_\Lambda](\omega) = (U_{T-\Lambda(\omega)} f)(W_{\Lambda(\omega)}(\omega)) \quad P\text{-a.s.}.$$

This fact will be used in the next lemma, which considers the special case $\theta = 0$.

Lemma 2.2. *Let $\lambda^{(i)}$ be defined as in Lemma 2.1 and suppose $a \leq b_2$. For any $i \geq 1$ it holds*

$$P(\, W_1 \leq a,\, \lambda^{(2i-1)} \leq 1\,) = \Phi(a - 2\alpha_1^{(i)}) \tag{5}$$

and

$$P(\, W_1 \leq a,\, \lambda^{(2i)} \leq 1\,) = \Phi(a - 2\alpha_2^{(i)}), \tag{6}$$

where $\alpha_1^{(i)} = i(b_2 - b_1) + b_1$ and $\alpha_2^{(i)} = i(b_2 - b_1)$.

Proof. We follow Karatzas et al. [8], p.95 and p.98. Firstly, fix a positive number $t \le 1$ and note that the symmetry of Brownian motion implies

$$\begin{aligned}\big(U_{1-t}1_{(-\infty,a]}\big)(b) &= P(\, b + W_{1-t} \le a\,) \\ &= P(\, b + W_{1-t} \ge 2b - a\,) \\ &= \big(U_{1-t}1_{[2b-a,\infty)}\big)(b)\end{aligned} \tag{7}$$

for any real numbers a and b.

The stopping time $\tau = \lambda^{(2i-1)} \wedge 1$ is obviously bounded for any $i \ge 1$. The strong Markov property in combination with equation (7) now implies for $\omega \in \{\lambda^{(2i-1)} < 1\}$,

$$\begin{aligned}E\big[1_{\{W_1 \le a\}} \mid \mathcal{F}_\tau\big](\omega) &= \big(U_{1-\tau(\omega)}1_{(-\infty,a]}\big)\big(W_{\tau(\omega)}(\omega)\big) \\ &= \big(U_{1-\tau(\omega)}1_{(-\infty,a]}\big)(b_2) \\ &= \big(U_{1-\tau(\omega)}1_{[2b_2-a,\infty)}\big)(b_2) \\ &= \big(U_{1-\tau(\omega)}1_{[2b_2-a,\infty)}\big)\big(W_{\tau(\omega)}(\omega)\big) \\ &= E\big[1_{\{W_1 \ge 2b_2-a\}} \mid \mathcal{F}_\tau\big](\omega).\end{aligned}$$

By integrating over $\{\tau < 1\} = \{\lambda^{(2i-1)} < 1\}$ we see that

$$P(\, W_1 \le a,\, \lambda^{(2i-1)} < 1\,) = P(\, W_1 \ge 2b_2 - a,\, \lambda^{(2i-1)} < 1\,). \tag{8}$$

Note that $\{\lambda^{(2i-1)} = 1\} \subset \{W_1 = b_2\}$ and thus $P(\,\lambda^{(2i-1)} = 1\,) = 0$. In combination with equation (8) this yields

$$P(\, W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = P(\, W_1 \ge 2b_2 - a,\, \lambda^{(2i-1)} \le 1\,).$$

Since $b_2 \le 2b_2 - a$ we find

$$\{\, W_1 \ge 2b_2 - a,\, \lambda^{(2i-1)} \le 1\,\} = \{\, W_1 \ge 2b_2 - a,\, \rho^{(2i-2)} \le 1\,\}$$

and therefore

$$P(\, W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = P(\, W_1 \ge 2b_2 - a,\, \rho^{(2i-2)} \le 1\,). \tag{9}$$

Using the fact that $2b_2 - a \ge b_1$ and the symmetry of Brownian motion, equation (9) gives

$$\begin{aligned}&P(\, W_1 \ge 2b_2 - a, \rho^{(2i-2)} \le 1\,) \\ &\qquad = P(\, W_1 \le 2(b_1 - b_2) + a,\, \lambda^{(2i-3)} \le 1\,)\end{aligned} \tag{10}$$

for $i \ge 2$. The equation (5) now follows by induction on i.

By replacing $\lambda^{(2i-1)}$ by $\lambda^{(2i)}$ and $\rho^{(2i-2)}$ by $\rho^{(2i-1)}$ in the equations (9) and (10) we also get equation (6). $\square$

Next we extend Lemma 2.2 to arbitrary $\theta \in \mathbb{R}$.

Lemma 2.3. *Let $\lambda^{(i)}$ be defined as in Lemma 2.1 and suppose $a \le b_2$. For any $i \ge 1$ and any $\theta \in \mathbb{R}$ it holds*

$$P^\theta(\, W_1 \le a,\, \lambda^{(2i-1)} \le 1\,) = e^{2\theta\alpha_1^{(i)}}\,\Phi(a - 2\alpha_1^{(i)} - \theta)$$

and

$$P^\theta(\, W_1 \le a,\, \lambda^{(2i)} \le 1\,) = e^{2\theta\alpha_2^{(i)}}\,\Phi(a - 2\alpha_2^{(i)} - \theta) \tag{11}$$

where $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ are defined as in Lemma 2.3.

Proof. Fix an integer $j \ge 1$. Observe that

$$\begin{aligned}
P^\theta(\, W_1 \le a,\, \lambda^{(j)} \le 1\,) =& E\big[\exp(-\frac{1}{2}\,\theta^2 + \theta W_1) 1_{\{W_1 \le a,\, \lambda^{(j)} \le 1\}}\big] \\
=& \int_{-\infty}^{a} \exp(-\frac{1}{2}\,\theta^2 + \theta x) P(W_1 \in dx,\, \lambda^{(j)} \le 1).
\end{aligned}$$

Introduce the constant α, defined by

$$\alpha = \begin{cases} \alpha_1^{(i)},\ i = (j+1)/2, & \text{if } j \text{ is an odd number,} \\ \alpha_2^{(i)},\ i = j/2, & \text{if } j \text{ is an even number,} \end{cases}$$

From the previous lemma we get

$$\frac{d}{dx} P(\, W_1 \le x,\, \lambda^{(j)} \le 1\,) = \varphi(x - 2\alpha),$$

where $\varphi(x) = \frac{d}{dx}\Phi(x)$. Thus

$$\begin{aligned}
P^\theta(\, W_1 \le a,\, \lambda^{(j)} \le 1\,) =& \int_{-\infty}^{a} \exp(-\frac{1}{2}\,\theta^2 + \theta x)\,\varphi(x - 2\alpha) dx \\
=& \int_{-\infty}^{a} \exp\big(-\frac{1}{2}\,\theta^2 + \theta x - \frac{1}{2}(x - 2\alpha)^2\big) \frac{dx}{\sqrt{2\pi}} \\
=& \int_{-\infty}^{a} \exp\big(2\theta\alpha - \frac{1}{2}(x - 2\alpha - \theta)^2\big) \frac{dx}{\sqrt{2\pi}} \\
=& e^{2\theta\alpha} \int_{-\infty}^{a} \varphi(x - 2\alpha - \theta) dx \\
=& e^{2\theta\alpha}\,\Phi(a - 2\alpha - \theta),
\end{aligned}$$

which proves Lemma 2.3. □

Next we will focus on the remainder term

$$P^\theta(\,W_1 \le a,\ \lambda^{(2n+1)} \le 1,\ \lambda^{(1)} < \rho^{(1)}\,)$$

in the expression of F_+, given in equation (1). Observe that

$$\begin{aligned} P^\theta(\,W_1 \le a,\ \lambda^{(2n+1)} \le 1,\ \lambda^{(1)} < \rho^{(1)}\,) &\le P^\theta(\,W_1 \le a,\ \lambda^{(2n)} \le 1\,) \\ &= e^{2\theta\alpha_2^{(n)}}\Phi(a - 2\alpha_2^{(n)} - \theta), \end{aligned}$$

according to equation (11). By using the following well known inequality

$$\Phi(x) \le -\frac{1}{x}e^{-x^2/2}, \quad x < 0,$$

we have that the remainder term is bounded by

$$\begin{aligned} &P^\theta(\,W_1 \le a, \lambda^{(2n+1)} \le 1,\ \lambda^{(1)} < \rho^{(1)}\,) \\ &\qquad \le \frac{-1}{a - 2\alpha_2^{(n)} - \theta}\,\exp\big(2\theta\alpha_2^{(n)} - \frac{(a - 2\alpha_2^{(n)} - \theta)^2}{2}\,\big) \end{aligned}$$

if $a < 2\alpha_2^{(n)} + \theta$, which implies that, given $\gamma < 2$,

$$P^\theta(W_1 \le a,\ \lambda_1^{(2n+1)} \le 1,\ \lambda^{(1)} < \rho^{(1)}) = o(e^{-n^\gamma})$$

as n tends to infinity. This result in combination with Lemmas 2.1 and 2.3 imply the next proposition.

Proposition 2.1. *Suppose $b_1 < 0 < b_2$, $a \le b_2$ and let $\alpha_1^{(i)} = i(b_2 - b_1) + b_1$ and $\alpha_2^{(i)} = i(b_2 - b_1)$. If $\gamma < 2$ then*

$$\begin{aligned} F_+(a, b_1, b_2; \theta) = &\sum_{i=1}^{n}\big(e^{2\alpha_1^{(i)}\theta}\Phi(a - 2\alpha_1^{(i)} - \theta) - e^{2\alpha_2^{(i)}\theta}\Phi(a - 2\alpha_2^{(i)} - \theta)\big) \\ &+ R_{n+1} \end{aligned}$$

where $R_{n+1} = o(e^{-n^\gamma})$, $n \to \infty$, or more precisely

$$|R_{n+1}| \le e^{2\theta\alpha_2^{(n)}}\,\Phi(a - 2\alpha_2^{(n)} - \theta).$$

Proposition 2.1 shows how the formula for the distribution F_+ should be implemented in order to be able to control the truncation error. Set

$$p_1^{(i)} = e^{2\alpha_1^{(i)}\theta}\,\Phi(a - 2\alpha_1^{(i)} - \theta) \ \text{ and } \ p_2^{(i)} = e^{2\alpha_2^{(i)}\theta}\,\Phi(a - 2\alpha_2^{(i)} - \theta).$$

Let

$$F_+^{(\epsilon)}(a, b_1, b_2; \theta) = \sum_{\{i\,;\,p_2^{(i)} \ge \epsilon\}} (p_1^{(i)} - p_2^{(i)}), \tag{12}$$

where $\epsilon > 0$. Proposition 2.1 now yields

$$\left| F_+(a, b_1, b_2; \theta) - F_+^{(\epsilon)}(a, b_1, b_2; \theta) \right| < \epsilon. \tag{13}$$

Thus, if the desired accuracy is set to ϵ, then one has to add the terms $(p_1^{(i)} - p_2^{(i)})$, $i = 1, 2, \ldots$ until $p_2^{(i)} < \epsilon$. The result will then have the desired accuracy.

Of course, in practice there is one additional error source besides the truncation error. The standard normal distribution must be evaluated using some numerical method. However, there are efficient methods with very high accuracy to compute the normal distribution function, see e.g. Cody [2]. The error appearing from the approximation of the normal distribution function may therefore be considered to be negligible compared to the truncation error. Thus, in what follows we will solely focus the truncation error.

The remaining part of this section is devoted to introduce and to determine certain transition distributions that will be useful in the sequel. First recall that for any $\theta \in \mathbb{R}$,

$$F_+(a, b_1, b_2; \theta) = P^\theta(\, W_1 \le a,\ \lambda(b_2) < \lambda(b_1), \lambda(b_2) \le 1\,)$$

if $b_1 < 0 < b_2$ and $a \le b_2$. Now let for any $\theta \in \mathbb{R}$,

$$F_-(a, b_1, b_2; \theta) = P^\theta(\, W_1 \ge a,\ \lambda(b_1) < \lambda(b_2),\ \lambda(b_1) \le 1\,) \tag{14}$$

if $b_1 < 0 < b_2$ and $a \ge b_1$. Furthermore, given $b_1 < 0 < b_2$, set

$$\begin{aligned} G_+(b_1, b_2; \theta) &= P^\theta(\, \lambda(b_2) < \lambda(b_1),\ \lambda(b_2) \le 1\,), \\ G_-(b_1, b_2; \theta) &= P^\theta(\, \lambda(b_1) < \lambda(b_2),\ \lambda(b_1) \le 1\,) \end{aligned} \tag{15}$$

and

$$G_2(a_1, a_2, b_1, b_2; \theta) = P^\theta(\, a_1 < W_1 \le a_2,\ \lambda(b_1) \wedge \lambda(b_2) > 1\,) \tag{16}$$

if $b_1 \le a_1 \le a_2 \le b_2$. These distributions can be expressed with the aid of the function F_+, as we will show in the next proposition.

Proposition 2.2. *Let the functions F_-, G_+, G_- and G_2 be defined as in the equations* (14)-(16). *Then*

$$\begin{aligned} F_-(a, b_1, b_2; \theta) &= F_+(-a, -b_1, -b_2; -\theta)\,, \\ G_+(b_1, b_2; \theta) &= \Phi(\theta - b_2) \\ &\quad + F_+(b_2, b_1, b_2; \theta) - F_-(b_2, b_1, b_2; \theta)\,, \\ G_-(b_1, b_2; \theta) &= G_+(-b_1, -b_2; -\theta) \end{aligned}$$

and

$$\begin{aligned} G_2(a_1, a_2, b_1, b_2; \theta) = \Phi(a_2 - \theta) &- \Phi(a_1 - \theta) \\ &- F_+(a_2, b_1, b_2; \theta) + F_+(a_1, b_1, b_2; \theta) \\ &- F_-(a_2, b_1, b_2; \theta) + F_-(a_1, b_1, b_2; \theta). \end{aligned}$$

Proof. The expression for F_- follows at once from the symmetry of Brownian motion.

Below, we let $\lambda_1 = \lambda(b_1)$ and $\lambda_2 = \lambda(b_2)$. To prove the second equation, note that

$$\begin{aligned} G_+(b_1, b_2; \theta) =& P^\theta(\, W_1 \le b_2,\, \lambda_2 < \lambda_1,\, \lambda_2 \le 1\,) \\ &+ P^\theta(\, W_1 > b_2,\, \lambda_2 < \lambda_1,\, \lambda_2 \le 1\,) \\ =& F_+(b_2, b_1, b_2; \theta) \\ &+ P^\theta(\, W_1 > b_2,\, \lambda_2 < \lambda_1\,), \end{aligned} \tag{17}$$

since $\{W_2 > b_2\} \subset \{\lambda_2 \le 1\}$. It is obvious that $P^\theta(\, W_1 > b_1,\, \lambda_1 = \lambda_2) = 0$ and, accordingly from this,

$$\begin{aligned} P^\theta(\, W_1 > b_2,\, \lambda_2 < \lambda_1\,) =& P^\theta(\, W_1 > b_2\,) \\ &- P^\theta(\, W_1 > b_2,\, \lambda_1 < \lambda_2\,) \\ =& P^\theta(\, W_1 > b_2\,) \\ &- P^\theta(\, W_1 > b_2,\, \lambda_1 < \lambda_2,\, \lambda_1 \le 1\,) \end{aligned} \tag{18}$$

because $\{\lambda_1 < \lambda_2\} \cap \{W_1 > b_2\} \subset \{\lambda_1 \le 1\}$. The equations (17) and (18) now yield the expression for G_+. The expression for G_- follows from symmetry.

It remains to determine G_2. Observe that

$$\begin{aligned} G_2(a_1, a_2, b_1, b_2; \theta) =& P^\theta(\, a_1 < W_1 \le a_2\,) \\ &- P^\theta(\, a_1 < W_1 \le a_2,\, \lambda_1 \wedge \lambda_2 \le 1\,) \\ =& P^\theta(\, W_1 \le a_2) - P^\theta(W_1 \le a_1\,) \\ &- P^\theta(\, W_1 \le a_2,\, \lambda_1 \wedge \lambda_2 \le 1\,) \\ &+ P^\theta(\, W_1 \le a_1,\, \lambda_1 \wedge \lambda_2 \le 1\,). \end{aligned}$$

The expression for G_2 is now a consequence of the fact that

$$\begin{aligned} P^\theta(\, W_1 \le a,\, \lambda_1 \wedge \lambda_2 \le 1\,) =& P^\theta(\, W_1 \le a,\, \lambda_2 < \lambda_1,\, \lambda_2 \le 1) \\ &+ P^\theta(\, W_1 \le a,\, \lambda_1 < \lambda_2,\, \lambda_1 \le 1\,) \end{aligned}$$

for any number a. ☐

Next we will introduce the "truncated" counterparts to F_-, G_+, G_- and G_2. Let $F_+^{(\epsilon)}$ be defined as in equation (12) and set

$$F_-^{(\epsilon)}(a, b_1, b_1; \theta) = F_+^{(\epsilon)}(-a, -b_1, -b_1; -\theta) \tag{19}$$

if $b_1 < 0 < b_2$ and $a \geq b_1$. Furthermore, given $b_1 < 0 < b_2$, set

$$\begin{aligned} G_+^{(\epsilon)}(b_1, b_2; \theta) &= \Phi(\theta - b_2) \\ &\quad + F_+^{(\epsilon)}(b_2, b_1, b_2; \theta) - F_-^{(\epsilon)}(b_2, b_1, b_2; \theta), \\ G_-^{(\epsilon)}(b_1, b_2; \theta) &= G_+^{(\epsilon)}(-b_1, -b_2; -\theta) \end{aligned} \tag{20}$$

and

$$\begin{aligned} G_2^{(\epsilon)}(a_1, a_2, b_1, b_2; \theta) =& \Phi(a_2 - \theta) - \Phi(a_1 - \theta) \\ & - F_+^{(\epsilon)}(a_2, b_1, b_2; \theta) + F_+^{(\epsilon)}(a_1, b_1, b_2; \theta) \\ & - F_-^{(\epsilon)}(a_2, b_1, b_2; \theta) + F_-^{(\epsilon)}(a_1, b_1, b_2; \theta) \end{aligned} \tag{21}$$

if $b_1 \leq a_1 \leq a_2 \leq b_2$.

From equation (13) and Proposition 2.2 we now get the following error estimates,

$$\begin{aligned} \left| G_+(b_1, b_2; \theta) - G_+^{(\epsilon)}(b_1, b_2; \theta) \right| &< 2\epsilon, \\ \left| G_-(b_1, b_2; \theta) - G_-^{(\epsilon)}(b_1, b_2; \theta) \right| &< 2\epsilon, \end{aligned} \tag{22}$$

and

$$\left| G_2(a_1, a_2, b_1, b_2; \theta) - G_2^{(\epsilon)}(a_1, a_2, b_1, b_2; \theta) \right| < 4\epsilon. \tag{23}$$

In the remaining sections it will be shown that the value of a double barrier options with rebate can be expressed as a linear combination of the functions G_+, G_- and G_2.

3 Pricing Double Barrier Options with Zero Rebate

The purpose of this section is to calculate the theoretical value of a double barrier option with zero rebate. To be more specific, our aim is to express the price in terms of the function G_2, defined in the previous section. The rebate options will be treated in the next section.

Because the market is assumed to be free of arbitrage, the following relation must hold

$$v = v_{ki} + v_{ko},$$

where v denotes the theoretical value of a (call/put) option and $[v_{ki}/v_{ko}]$ denotes the value of a [knock-in/knock-out] (call/put) option with zero rebate and with the same option parameters as the (call/put) option. Moreover, the barrier options are presumed to have the barriers at the same level. Since the theoretical values of calls and puts are known it is enough to solely price knock-in *or* knock-out options. We will henceforth focus on knock-out options.

We assume throughout that the price of the underlying asset $\{S_t\}_{t\geq 0}$ evolves under the risk-neutral martingale measure according to

$$S_t = S_0 e^{(r-q-\sigma^2/2)t+\sigma W_t}, \ t \geq 0,$$

where the risk free rate r, the dividend yield q and the volatility σ are assumed to be constants. Moreover, let the constants K, T, H_1 and H_2 denote strike price, time of expiration, lower barrier, and upper barrier, respectively.

In what follows, let

$$\tau(H) = \inf\{t > 0\,;\, S_t = H\}, \quad H > 0.$$

We are now in the position to establish the theoretical value of the double barrier knock-out options. The proof of the next theorem is based on a technique often referred to as "change of numeraire", see Musiela et al. (1998).

Theorem 3.1. *Set*

$$\phi(x) = \frac{\ln(x/S_0)}{\sigma\sqrt{T}}, \quad x > 0.$$

and let $c = \phi(K)$, $d_1 = \phi(H_1)$, *and* $d_2 = \phi(H_2)$. *Let moreover*

$$\theta_0 = \frac{(r-q-\sigma^2/2)\sqrt{T}}{\sigma} \ \textit{and}\ \theta_1 = \theta_0 + \sigma\sqrt{T}.$$

If $K < H_2$ *then the theoretical value* v_{koc} *at time* $t = 0$ *of a double-barrier knock-out call option is given by*

$$\begin{aligned} v_{koc} =& S_0 e^{-qT} G_2(\max(c, d_1), d_2, d_1, d_2; \theta_1) \\ & - K e^{-rT} G_2(\max(c, d_1), d_2, d_1, d_2; \theta_0)\,, \end{aligned}$$

where G_2 *is defined as in equation* (16). *If* $K \geq H_2$, $v_{koc} = 0$.

If $K > H_1$ *the theoretical value* v_{kop} *of a double barrier knock-out put option at time* $t = 0$ *is given by*

$$\begin{aligned} v_{kop} =& K e^{-rT} G_2(d_1, \min(c, d_2), d_1, d_2; \theta_0) \\ & - S_0 e^{-qT} G_2(d_1, \min(c, d_2), d_1, d_2; \theta_1)\,. \end{aligned}$$

If $K \leq H_1$, $v_{kop} = 0$.

Proof. The theoretical value of a double barrier knock-out call can be written as

$$\begin{aligned} v_{koc} =& e^{-rT} E\big[\max(S_T - K, 0) 1_{\{\tau(H_1)>T,\, \tau(H_2)>T\}}\big] \\ =& S_0 e^{-qT} E\big[e^{-(\sigma^2/2)\,T+\sigma W_T} 1_{\{S_T \geq K, \tau(H_1)>T,\, \tau(H_2)>T\}}\big] \\ &- K e^{-rT} E\big[1_{\{S_T \geq K, \tau(H_1)>T,\, \tau(H_2)>T\}}\big]. \end{aligned}$$

Let $\tilde{P}$ be a measure on $\mathcal{F}$, defined by the Radon-Nikodym derivate $\tilde{P} = e^{-(\sigma^2/2)\,T+\sigma W_T} dP$. We now find

$$\begin{aligned} v_{koc} =& S_0 e^{-qT} \tilde{P}(\, S_T \geq K,\, \tau(H_1) > T,\, \tau(H_2) > T\,) \\ &- K e^{-rT} P(\, S_T \geq K,\, \tau(H_1) > T,\, \tau(H_2) > T\,). \end{aligned} \tag{24}$$

Set $k = \ln(K/S_0)$, $h_1 = \ln(H_1/S_0)$, $h_2 = \ln(H_2/S_0)$ and let $\eta \in \mathbb{R}$. The scaling property for Brownian motion and the Cameron-Martin theorem give

$$\begin{aligned} &P\big(\, (\eta T + \sigma W_T) \leq k,\, \min_{0\leq t\leq T} (\eta t + \sigma W_t) > h_1,\, \max_{0\leq t\leq T} (\eta t + \sigma W_t) < h_2\,\big) \\ &\quad = P\big(\, (\theta + W_1) \leq c,\, \min_{0\leq t\leq 1} (\theta t + W_t) > d_1\, \max_{0\leq t\leq 1} (\theta t + W_t) < d_2\,\big) \\ &\quad = P^{\theta}(\, W_1 \leq c,\, \lambda(d_1) > 1,\, \lambda(d_2) > 1\,), \end{aligned} \tag{25}$$

where c, d_1 and d_2 are defined as above and where

$$\theta = \frac{\eta\sqrt{T}}{\sigma}.$$

Let θ_0 and θ_1 be defined as above. If we replace η by $r - q - \sigma^2/2$ in equation (25) we obtain

$$\begin{aligned} &P(\, S_T \geq K, \tau(H_1) > T,\, \tau(H_2) > T\,) \\ &\qquad = P^{\theta_0}(W_T \geq c,\, \lambda(d_1) > 1,\, \lambda(d_2) > 1). \end{aligned}$$

The Cameron-Martin theorem gives that $\{\, W_t - \sigma t\,\}_{0\leq t\leq T}$ is a Brownian motion with respect to $(\tilde{P}, \{\mathcal{F}_t\}_{0\leq t\leq T})$. Thus, by setting $\eta = r - q + \sigma^2/2$ in equation (25) we get

$$\begin{aligned} &\tilde{P}(\, S_T \geq K, \tau(H_1) > T,\, \tau(H_2) > T\,) \\ &\qquad = P^{\theta_1}(W_T \geq c,\, \lambda(d_1) > 1,\, \lambda(d_2) > 1). \end{aligned}$$

The expression of v_{koc} now follows from equation (24) and the definition of G_2.

In a similar way one can prove the pricing formula for a double barrier knock-out put. □

We must emphasize that the expressions in Theorem 3.1 for the value of a double barrier option agree with the results derived in Kunitomo et al. (1992).

Introduce the "truncated" price $v_{koc}^{(\epsilon)}$ of a double barrier knock-out call by letting

$$\begin{aligned} v_{koc}^{(\epsilon)} =& S_0 e^{-qT} G_2^{(\epsilon)}(\max(c, d_1), d_2, d_1, d_2; \theta_1) \\ &- Ke^{-rT} G_2^{(\epsilon)}(\max(c, d_1), d_2, d_1, d_2; \theta_0), \end{aligned}$$

where c, d_1, d_2, θ_0, and θ_1 are defined as in Theorem 3.1 and where the function $G_2^{(\epsilon)}$ is defined as in equation (21). Then, according to equation (23),

$$|\, v_{koc} - v_{koc}^{(\epsilon)} \,| < 4(S_0 e^{-qT} + Ke^{-rT})\, \epsilon.$$

In other words, if one wants to determine an estimation $\tilde{v}_{koc}$ for the price of a double barrier knock-out call such that the truncation error is smaller than ϵ_0, then this value may be obtained by setting $\tilde{v}_{koc} = v_{koc}^{(\epsilon)}$, where

$$\epsilon = \frac{\epsilon_0}{4(S_0 e^{-qT} + Ke^{-rT})}.$$

With a similar argument one can derive error bounds for the approximation $v_{kop}^{(\epsilon)}$, defined in analogy to $v_{koc}^{(\epsilon)}$, for the price of a double barrier knock-out *put*.

4 Pricing Rebate Options

In this final section of this paper we will calculate the theoretical value of the rebate options. Let us first consider the rebate option belonging to a double-barrier knock-in option. A holder of this contract will receive a prespecified positive amount R (the rebate) at the maturity date T provided that the underlying asset price *never* crosses the barriers before or at the maturity date. Thus, the payoff at the maturity date T of the rebate option belonging to a double barrier knock-in option can be written as

$$R\, 1_{\{\, \tau(H_1) > T,\, \tau(H_2) > T\, \}},$$

where, as previous, $\tau(H) = \inf\{t > 0\, ;\, S_t = H\}$, $H > 0$. The theoretical value v_{kir} at time $t = 0$ of this option is therefore given by

$$\begin{aligned} v_{kir} &= e^{-rT} E\big[R 1_{\{\, \tau(H_1) > T,\, \tau(H_2) > T\, \}}\big] \\ &= Re^{-rT} P(\, \tau(H_1) > T,\, \tau(H_2) > T\,). \end{aligned}$$

Let θ_0, d_1, and d_2 be defined as in Theorem 3.1. The scaling property for Brownian motion and the Cameron-Martin theorem imply for any $t \geq 0$

$$P(\,\tau(H_1) > t,\, \tau(H_2) > t\,) = P^{\theta_0}(\lambda(d_1) > t/T,\, \lambda(d_2) > t/T) \qquad (26)$$

(cf. the proof of Theorem 3.1). Thus

$$\begin{aligned} v_{kir} &= Re^{-rT} P^{\theta_0}(\,\lambda(d_1) > 1,\, \lambda(d_2) > 1) \\ &= Re^{-rT} G_2(d_1, d_2, d_1, d_2; \theta_0). \end{aligned}$$

We turn now our attention to the rebate option corresponding to a double barrier knock-out call. Let for the sake of conciseness $\tau_1 = \tau(H_1)$ and $\tau_2 = \tau(H_2)$. The payoff at maturity T of this contract equals

$$R 1_{\{\,\tau_1 \wedge \tau_2 \leq T\}}.$$

The pay out will occur at the same time the barrier is reached, which is at time $\tau_1 \wedge \tau_2$. Thus, a holder of this contract will receive a prespecified positive amount R at the time when the underlying asset price breaches the barrier, provided that the asset price crosses the barrier before or at the maturity date T.

The theoretical value v_{kor} at time $t = 0$ of a rebate option corresponding to a double barrier knock-out option equals therefore

$$\begin{aligned} v_{kor} =& E\big[R \exp\big(-r(\tau_1 \wedge \tau_2)\big)\, 1_{\{\,\tau_1 \wedge \tau_2 \leq T\}}\big] \\ =& R\, E\big[\exp(-r\tau_1)\, 1_{\{\tau_1 < \tau_2,\, \tau_1 \leq T\}}\big] \\ &+ R\, E\big[\exp(-r\tau_2)\, 1_{\{\tau_2 < \tau_1,\, \tau_2 \leq T\}}\big]. \end{aligned}$$

To begin with, consider the expected value $E\big[\exp(-r\tau_1)\, 1_{\{\tau_1 < \tau_2,\, \tau_1 \leq T\}}\big]$. Set $\lambda_1 = \lambda(d_1)$ and $\lambda_2 = \lambda(d_2)$. Equation (26) yields

$$E\big[\exp(-r\tau_1)\, 1_{\{\tau_1 < \tau_2,\, \tau_1 \leq T\}}\big] = E^{\theta_0}\big[\exp(-r\,T\lambda_1)\, 1_{\{\lambda_1 < \lambda_2,\, \lambda_1 \leq 1\}}\big]\,,$$

where E^{θ_0} denotes expectation with respect to P^{θ_0}. Set

$$A = \{\lambda_1 < \lambda_2,\, \lambda_1 \leq 1\}.$$

From the definition of P^{θ_0} we have

$$E^{\theta_0}\big[\exp(-r\,T\lambda_1)\, 1_A\big] = E\big[\exp(-r\,T\lambda_1 - \frac{1}{2}\theta_0^2 + \theta_0 W_1\,)\, 1_A\,\big].$$

Recall that if $\{Z_t\}_{t \geq 0}$ is a $(P, \{\mathcal{F}\}_{t \geq 0})$-martingale and X is a bounded $\mathcal{F}_\Lambda$ measurable random variable, where Λ is a stopping time such that $\Lambda \leq 1$ P-a.s., then the optional sampling theorem implies

$$E\big[XZ_1\big] = E\big[XZ_\Lambda\big]. \qquad (27)$$

Since the random variable $\exp(-r\,T\lambda_1)\,1_A$ is $\mathcal{F}_{\lambda_1}$ measurable and bounded, equation (27) yields

$$E\big[\exp(-r\,T\lambda_1 - \frac{1}{2}\theta_0^2 + \theta_0 W_1\,)\,1_A\,\big]$$
$$= E\big[\exp(-rT\lambda_1 - \frac{1}{2}\theta_0^2\lambda_1 + \theta_0 W_{\lambda_1})\,1_A\,\big]$$
$$= E\big[\exp\big((\theta_0-\theta_2)W_{\lambda_1} - \frac{1}{2}\theta_2^2\lambda_1 + \theta_2 W_{\lambda_1}\big)\,1_A\,\big],$$

where $\theta_2 = \sqrt{\theta_0^2 + 2rT}$. Note that $W_{\lambda_1(\omega)}(\omega) = d_1$ if $\omega \in A$ and, accordingly from this,

$$E\big[\exp\big((\theta_0-\theta_2)W_{\lambda_1} - \frac{1}{2}\theta_2^2\lambda_1 + \theta_2 W_{\lambda_1}\big)\,1_A\,\big]$$
$$= \exp\big((\theta_0-\theta_2)d_1\,\big)\,E\big[\exp(-\frac{1}{2}\theta_2^2\lambda_1 + \theta_2 W_{\lambda_1}\big)\,1_A\,\big].$$

Moreover, according to the optional sampling theorem,

$$E\big[\exp(-\frac{1}{2}\theta_2^2\lambda_1 + \theta_2 W_{\lambda_1}\big)\,1_A\,\big] = P^{\theta_2}(A)$$
$$= G_-(d_1, d_2; \theta_2).$$

We can summarize this as follows

$$E\big[\exp(-r\tau_1)\,1_{\{\tau_1<\tau_2,\,\tau_1\le T\}}\big] = \exp\big((\theta_0-\theta_2)d_1\big)\,G_-(d_1, d_2; \theta_2).$$

In a similar way it can be shown that

$$E\big[\exp(-r\tau_2)\,1_{\{\tau_2<\tau_1,\,\tau_2\le T\}}\big] = \exp\big((\theta_0-\theta_2)d_2\big)\,G_+(d_1, d_2; \theta_2).$$

To sum up, we have

Theorem 4.1. *Let d_1, d_2, and θ_0 be defined as in Theorem 3.1 and suppose that G_2 is defined as in equation* (16). *The theoretical value v_{kir} at time $t = 0$ of a rebate option corresponding to a double barrier knock-in option is given by*

$$v_{kir} = Re^{-rT}\,G_2(d_1, d_2, d_1, d_2; \theta_0).$$

Let G_+ and G_- be defined as in equation (15). *The theoretical value v_{kor} at time $t = 0$ of a rebate option corresponding to a double-barrier knock-out option equals*

$$v_{kor} = R\,\exp\big(\,(\theta_0-\theta_2)\,d_1\big)\,G_-(d_1, d_2; \theta_2)$$
$$+\,R\,\exp\big(\,(\theta_0-\theta_2)\,d_2\big)\,G_+(d_1, d_2; \theta_2),$$

where $\theta_2 = \sqrt{\theta_0^2 + 2rT}$.

Now let

$$v_{kir}^{(\epsilon)} = Re^{-rT}\, G_2^{(\epsilon)}(d_1, d_2, d_1, d_2; \theta_0)$$

and

$$\begin{aligned} v_{kor}^{(\epsilon)} &= R\, \exp(\, (\theta_0 - \theta_2)\, d_1)\, G_-^{(\epsilon)}(d_1, d_2; \theta_2) \\ &\quad + R\, \exp(\, (\theta_0 - \theta_2)\, d_2)\, G_+^{(\epsilon)}(d_1, d_2; \theta_2), \end{aligned}$$

where d_1, d_2, θ_0, and θ_2 are defined as in Theorem 4.1 and where $G_+^{(\epsilon)}$, $G_-^{(\epsilon)}$, and $G_2^{(\epsilon)}$ are given by the equations (20)-(21). We now have, according to the equations (22) and (23),

$$|v_{kir} - v_{kir}^{(\epsilon)}| < 4Re^{-rT}\epsilon$$

and

$$|v_{kor} - v_{kor}^{(\epsilon)}| < 2R\big(\exp(\, (\theta_0 - \theta_2)\, d_1) + \exp(\, (\theta_0 - \theta_2)\, d_2)\big)\, \epsilon,$$

which give the desired error estimates.

5 Conclusions

We have addressed the problem of estimating the value of double barrier options with rebate. We have derived error bounds for the truncation error and described how the pricing formulas can be implemented so that one is assured that the resulting prices have a certain accuracy.

References

[1] Andersson, T. W. (1960) *A Modification of the Sequential Probability Ratio Test to Reduce the Sample Size* Ann. Math. Statist. **31**, 165-197

[2] Cody, W.J. (1969) *Rational Chebyshev Approximations for the Error Function* Math. Comp. **23**, 631-637

[3] Geman, H. and Yor, M. (1996) *Pricing and Hedging Double Barrier Options: A Probabilistic Approach* Math. Finance **6**, 365-378

[4] Hui, C.H. (1996) *One-Touch Double Barrier Binary Option Values* Appl. Financial Econ. **6**, 343-346

[5] Hui, C.H. (1997) *Time Dependent Barrier Option Values* J. Futures Market **17**, 667-688

[6] Hui, C.H., Lo, C.F. and Yuen, P.H. (2000): *Comment on 'Pricing Double Barrier Options Using Laplace Transforms' by Antoon Pelssner* Finance Stochast. **4**, 105-107

[7] Kunitomo, N. and Ikeda, M. (1992) *Pricing Options with Curved Boundaries* Math. Finance **2**, 275-298

[8] Karatzas, I. and Shreve, S. E. (1989) *Brownian Motion and Stochastic Analysis* (2nd Ed.) Springer-Verlag

[9] Lou, L.S.J. (2001) *Various Types of Double-Barrier Options* J. of Comp. Finance **4**, 125-138

[10] Musiela, M. and Rutkowski, M (1998) *Martingale Methods in Financial Modelling* (2nd Ed.) Springer-Verlag

[11] Pellser, A. (2000) *Pricing Double Barrier Options Using Laplace Transforms* Finance Stochast. **4**, 95-104

[12] Sidenius, J. (1998) *Double Barrier Options: Valuation by Path Counting* J. of Comp. Finance **1**, 63-79