Pricing Discrete European Barrier Options Using Lattice Random Walks

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Abstract

This paper designs a numerical procedure to price discrete European barrier options in Black-Scholes model. The pricing problem is divided in a series of initial value problems, one for each monitoring time. Each initial value problem is solved by replacing the driving Brownian motion by a lattice random walk.

Some results from the theory of Besov spaces will be presented which show that the convergence rate of lattice methods for initial value problems depends on two factors, namely the smoothness of the initial value (or the value function) and the moments for the increments of the lattice random walk. This fact is used in order to obtain an efficient method to price discrete European barrier options. Numerical examples and comparisons with other methods are carried out to show that the proposed method yields fast and accurate results.

Key words: option pricing, discrete barrier options, lattice random walks, rate of convergence, Besov spaces.

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1 Introduction

A barrier option is activated (knocked in) or extinguished (knocked out) when the underlying asset price reaches a specified level, or barrier. It is common to assume that the underlying asset is continuously monitored against the barrier. However, for many traded barrier options the barrier(s)

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is (are) monitored only at specific dates. These options are usually referred to as *discrete* barrier options.

The problem of pricing discrete barrier options in Black-Scholes model has been discussed previously in the literature. A numerical procedure based on the trinomial method has been investigated by Broadie et al. [BGK]. Zwan et al. [ZVF] consider implicit finite difference methods, Ahn et al. [AFG] study an adaptive mesh model and Sullivan [Su] develops a method based on numerical quadrature. Approximation formulas have been derived in [BGK2] and [H]. The more general problem to compute Wiener integrals was probably first studied by Cameron [C] in 1951. More recently in 1999 Steinbauer [St] developed a quadrature formula for Wiener measure and discussed other numerical methods as well. For additional information about Monte Carlo methods, see [BBG] and [BGK].

The purpose of the present paper is twofold. One goal is to design an efficient numerical procedure to price discrete European barrier options. The procedure is based on the idea of replacing the driving Brownian motion in the underlying asset price with a lattice random walk. The procedure that is presented in this paper will be able to handle discrete double as well as discrete moving barrier options. In this way we extend a previous work by Broadie et al. [BGK] which treats a trinomial tree appropriate to price discrete barrier options provided that the option only has *one constant* barrier. Recall that a trinomial tree is a lattice random walk where the terms have at the most three possible outcomes. A numerical example indicates that the procedure developed in this paper is considerably more efficient for European styled options than the method proposed in [BGK]. However, it should be remarked that our method cannot, in contrast to the method designed by Broadie et al., be extended to price *American* barrier options.

The second aim with this paper is to continue the research by, among many others, Leisen et al. [LR] and Heston et al. [HZ] to characterize the rate of convergence of lattice methods for initial value problems. To this end we will present certain results from the theory of Besov spaces (see [BL]) which will be useful to construct an efficient method to price discrete barrier options. It should be emphasized that most previous research in this area in the financial literature is restricted to solely trinomial trees. This paper will, however, consider arbitrary lattice random walks with finite variance.

The paper is structured as follows. In Section 2 we give a brief introduction to the lattice method and make a first ansatz to price discrete barrier options. Section 3 discusses Besov spaces and characterizes the rate of convergence of lattice methods. In Section 4 we return to the problem of pricing discrete barrier options. Based on the results in Section 3 we improve the method discussed in Section 2. Section 5 shows some numerical examples. Finally, Section 6 concludes this paper with some suggestions for future research.

2 Preliminaries

Throughout it is assumed that the price of the underlying asset $\{S_t\}_{t\geq 0}$ evolves under the risk-neutral martingale measure according to

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})\,t+\sigma W_t}, \ t \geq 0,$$

where $\{W_t\}_{t\geq 0}$ is a normalized Brownian motion and where the risk free rate r, the volatility σ, and the initial price S_0 are assumed to be constants.

The theoretical price v at time $t = 0$ of a discrete barrier knock-out call/put with maturity date T and strike price K is given by

$$v = e^{-rT} E\big[\Psi_K(S_T)\, 1_{\{\, H_l(t) < S_t < H_u(t),\, t\in M\,\}}\big] \tag{1}$$

with $\Psi_K(x) = (x-K)^+$ or $\Psi_K(x) = (K-x)^+$, respectively. The set $M = \{t_1, t_2, \dots, t_{m-1}, t_m\}$ denotes the monitoring dates. We assume that $0 < t_1 < t_2 \ldots < t_m = T$. The functions $H_l : M \to [0,\infty)$ and $H_u : M \to (0,\infty]$, where $H_l < H_u$, describe the barrier levels at the monitoring dates M.

This paper will focus on the pricing of knock-*out* options. The values of the corresponding knock-in options then follow by the fact that the sum of two otherwise identical in- and out- call (put) options is a plain call (put) option.

Next we will introduce some definitions. Let $\nu = r - \sigma^2/2$ and put for each $i = 1, 2, \dots, m$,

$$a_i = \frac{\ln\big(H_l(t_i)/S_0\big) - \nu t_i}{\sigma} \quad \text{and} \quad b_i = \frac{\ln\big(H_u(t_i)/S_0\big) - \nu t_i}{\sigma}.$$

Let for $i = 1, 2, \dots, m$ the function $\chi_i : \mathbb{R} \to \mathbb{R}$ be the indicator function of the interval $I_i = (a_i, b_i)$, that is

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \in I_i, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, suppose that

$$\tau_i = t_i - t_{i-1}, \quad i = 1, \dots, m,$$

where $t_0 = 0$.

Now, let the class of operators $\{U_t\}_{t\geq 0}$ denote the Brownian semi-group, i.e.

$$(U_t f)(x) = E\big[f(x + W_t)\big], \quad x \in \mathbb{R}, \tag{2}$$

where $f : \mathbb{R} \to \mathbb{R}$, and assume that

$$v_{m-1}(x) = e^{-rT}\, U_{\tau_m}\Big(\Psi_K\big(S_0 e^{\nu T + \sigma(\cdot)}\big)\chi_m(\cdot)\Big)(x), \quad x \in \mathbb{R}. \tag{3}$$

If we define recursively

$$v_{i-1}(x) = \big(U_{\tau_i}(v_i\,\chi_i)\big)(x), \quad x \in \mathbb{R},\ 1 \le i \le m-1, \tag{4}$$

then the Markov property for Brownian motion tells us that the theoretical value v of a discrete barrier option equals $v = v_0(0)$.

Thus, one way to compute the theoretical price of a discrete barrier option is to determine the functions v_i, $i = 0, 1, \dots, m-1$. We are thereby led to the problem to compute the function

$$x \mapsto (U_\tau f)(x) \tag{5}$$

for a given function f and fixed $\tau > 0$. This problem may alternatively be expressed in terms of partial differential equations. Namely, under some appropriate assumptions on f the function $u(x,t) = (U_t f)(x)$, $x \in \mathbb{R}$, $t \in [0,\tau]$, is the solution to the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} & \text{in } \mathbb{R} \times (0,\tau], \\ u|_{t=0} = f & \text{on } \mathbb{R}, \end{cases}$$

see for instance [KS]. Thus, the function $x \mapsto (U_\tau f)(x)$ equals the solution of the initial value problem for the heat equation at a fixed time τ.

One well-known approach to estimate the function in equation (5) is given by the binomial method. That is, the Brownian motion in equation (2) is replaced with a binomial distributed random walk. In this paper we will consider an extension of this method which can be described as follows. Let ζ be a lattice random variable with $P(\zeta \in \mathbb{Z}) = 1$, $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$,

$$E[\,\zeta\,] = 0 \quad \text{and} \quad 0 < \lambda = \operatorname{Var}(\zeta) < \infty.$$

Furthermore, suppose $\zeta_1, \dots, \zeta_n$ are stochastic independent copies of ζ and set

$$(U_t^{(\zeta,h)} f)(x) = E\big[f(x + h\sum_{i=1}^{n} \zeta_i)\big], \quad \text{with } n = \frac{t}{h^2\lambda}, \tag{6}$$

for all $h > 0$, $t \in R_h = \lambda h^2 \mathbb{N}$, $\mathbb{N} = \{0, 1, \dots\}$, and each $x \in \mathbb{R}$. We use the convention that if $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ then $cA = \{x\,;\, x = ca,\ a \in A\}$. In addition, in the sequel we assume that $\sum_{i=k}^{k-1} a_i = 0$ for any $k \in \mathbb{Z}$. According to the central limit theorem the sequence of random variables

$$\{h_n \sum_{i=1}^{n} \zeta_i\}_{n=1}^{\infty}, \quad \text{with } h_n = \sqrt{\frac{\tau}{n\lambda}},$$

will converge in distribution to W_τ. This gives us reason to believe that $U_\tau^{(\zeta,h_n)} f$ might be a good approximation of $U_\tau f$ for sufficiently large values of n. However, as we will show in Section 4, this first ansatz can be improved considerably.

3 Rate of Convergence

As a consequence of the continuous mapping theorem (see [D], p.87), $U_\tau^{(\zeta,h_n)}f$ will converge point wise to $U_\tau f$ as $n \to \infty$ if f is bounded and if the set of all discontinuity points of f has Lebesgue measure 0. This section discusses the *rate of convergence* as the number of terms in the random walk tends to infinity. Thus, we want to state necessary and sufficient conditions on f and ζ such that, given $x \in \mathbb{R}$ and $\tau > 0$,

$$\left|(U_\tau^{(\zeta,h_n)}f)(x) - (U_\tau f)(x)\right| = O(n^{-\alpha}), \quad \text{as } n \to \infty, \tag{7}$$

for some $\alpha > 0$.

There are lots of contributions to this problem in the literature. For instance Berry and Esseen, [Be] and [Es], consider the special case when f is piecewise constant and von Bahr [Bahr] when the initial value is a polynomial. Diener et al. [DD] and Leimar et al. [LR] analyse the binomial tree for initial values corresponding to the payoff function of a call option. Kreiss et al. [KTW], Heston et al. [HZ] and Walsh et al. [WW] examine, among other things, the dependence between the smoothness of f and the convergence rate of the difference in equation (7). Butzer et al. [BHW] investigate the convergence rate when the initial values f are differentiable functions. Finally, Löfström [Lö] presents sharp estimates on the difference in equation (7) uniformly for all $x \in \mathbb{R}$ if f belongs to a so called Besov space.

Before we comment on these papers any further we will introduce some definitions. The next definition has its origin in the theory on finite difference methods (see [RM]).

Definition 1. *Let ζ be a lattice random variable with $0 < \lambda = Var(\zeta) < \infty$. We will say that ζ is* **consistent** *of order μ, where μ is an integer, if*

$$E\left[\, e^{i\xi\zeta} \,\right] = E\left[\, e^{i\xi W_\lambda} \,\right] + O(\xi^{\mu+2}), \quad \text{as } \xi \to 0,$$

where i is the imaginary unit and $\xi \in \mathbb{R}$.

It is well-known that there is a close connection between the consistency number and the moments of a random variable. On one hand, if ζ has an absolute moment of order $\mu + 1$, i.e. $E[\,|\zeta|^{\mu+1}\,] < \infty$, and ζ is consistent of order μ then $E[\,\zeta^k\,] = E[\,W_\lambda^k\,]$ for all positive integers $k \le \mu + 1$ (see [D], p. 101). On the other hand, if ζ has a moment of order $\mu + 2$ and $E[\,\zeta^k\,] = E[\,W_\lambda^k\,]$ for all positive integers $k \le \mu + 1$ then ζ is consistent of order μ (see [Lu], p. 23).

If ζ is consistent of order μ then it is also consistent of any order less than μ. For this reason one sometimes say that ζ is *exactly* consistent of order μ if ζ is consistent of order μ but not consistent of order $\mu + 1$.

In connection with a trinomial tree, each fixed increment of the underlying random walk equals 0 or ± 1. Suppose the symmetrical random variable η has (at the most) three possible outcomes. Thus

$$P(\eta = 0) = p_0 \quad \text{and} \quad P(\eta = 1) = P(\eta = -1) = p_1, \tag{8}$$

with $p_0 + 2p_1 = 1$. The random variable η can be exactly consistent of order 2 or 4. To see this, note that from Taylor's formula we have as $\xi \to 0$,

$$\begin{aligned} E\left[e^{i\xi\eta} \right] &= p_0 + 2p_1 \cos(\xi) \\ &= 1 - 2p_1 \frac{\xi^2}{2} + 2p_1 \frac{\xi^4}{24} + O(\xi^6) \\ &= 1 - \lambda \frac{\xi^2}{2} + \lambda \frac{\xi^4}{24} + O(\xi^6) \\ &= \exp(-\lambda \frac{\xi^2}{2}) + \lambda \frac{\xi^4}{8} \left(\frac{1}{3} - \lambda \right) + O(\xi^6). \end{aligned}$$

Since $E[\exp(i\xi W_\lambda)] = \exp(-\lambda\xi^2/2)$ we see that η is exactly consistent of order 4 if

$$\lambda = 1/3$$

or, equivalently,

$$p_0 = 2/3, \quad p_1 = 1/6.$$

For any other choice of λ in $(0, 1]$ the random variable η is exactly consistent of order 2. In particular, we see that a random variable only taking the values ± 1 is exactly consistent of order 2.

Next we will introduce certain Banach spaces known as *Besov spaces* and below denoted by B^s_∞, $s > 0$. The Besov spaces are subspaces of the Banach space C_0, where C_0 denotes the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to \pm\infty} f(x) = 0$$

equipped with the norm $\|f\|_{C_0} = \max_{x \in \mathbb{R}} |f(x)|$.

The norm in the Besov space B^s_∞, henceforth denoted $\| \cdot \|_{B^s_\infty}$, is given as follows. Set $s = k + \gamma$, where k is a nonnegative integer and $0 < \gamma \le 1$. If $0 < \gamma < 1$ then the norm will be defined in terms of a so called Hölder condition with exponent γ,

$$\|f\|_{B^{k+\gamma}_\infty} = \|f\|_{C_0} + \sup_{t>0} \frac{1}{t^\gamma} \|f^{(k)}(\cdot + t) - f^{(k)}(\cdot)\|_{C_0} \,,$$

where $f^{(k)}$ denotes the k:th derivative of f. If $\gamma = 1$ then the norm is defined in terms of a so called Zygmund condition,

$$\|f\|_{B_\infty^{k+1}} = \|f\|_{C_0} + \sup_{t>0} \frac{1}{t} \|f^{(k)}(\cdot + t) - 2f^{(k)}(\cdot) + f^{(k)}(\cdot - t)\|_{C_0} .$$

In the literature there exist many other equivalent definitions of the norm in the Besov space B_∞^s. The definition here is taken from Brenner et al. [BTW].

If $s_1 < s_2$ then $B_\infty^{s_1} \supset B_\infty^{s_2}$ (see [BL], p.142). Thus the functions in $B_\infty^{s_2}$ are generally smoother than the functions in $B_\infty^{s_1}$. Much more can of course be said about Besov spaces and the interested reader is referred to [BL], Chapters 6 and 7, and the references therein.

In the sequel we will sometimes use the term *local* Zygmund condition. A function f will be said to satisfy a local Zygmund condition if for all $c > 0$ and each $x \in \mathbb{R}$ we have

$$\sup_{0<t<c} \frac{1}{t} \big| f(x+t) - 2f(x) + f(x-t) \big| < \infty.$$

Note that if $f^{(k)} \in C_0$ and $f^{(k)}$ satisfies a local Zygmund condition then $f \in B_\infty^{k+1}$.

We shall introduce yet another Banach space, below denoted A_∞^s, $s > 0$. The space A_∞^s is a subspace to C_0 with norm

$$\|f\|_{A_\infty^s} = \|f\|_{C_0} + \sup_{0<h<1} \sup_{t \in R_h} \frac{1}{h^s} \|U_t^{(\zeta,h)} f - U_t f\|_{C_0},$$

where $R_h = \lambda h^2 \, \mathbb{N}$.

The following striking result is due to Löfström (see [Lö], p.408).

Theorem 1. *Suppose that ζ is exactly consistent of order μ. Then*

$$A_\infty^\varsigma = B_\infty^\varsigma , \quad 0 < \varsigma \le \mu,$$

with equivalent norms. Moreover, if $f \in C_0$ and

$$\sup_{t \in R_h} \| U_t^{(\zeta,h)} f - U_t f \|_{C_0} = o\big(h^\mu\big), \quad \text{as } h \to 0,$$

then $f \equiv 0$.

Thus, the convergence rate is closely related to the smoothness of the initial value f and to the moments of ζ. In particular, if $f \in B_\infty^s$ and if ζ is consistent of order μ, Theorem 1 yields that there is for each $\varsigma \le \min(\mu, s)$ a constant C, independent of f and n, such that

$$\| U_\tau^{(\zeta,h_n)} f - U_\tau f \|_{C_0} \le \frac{C}{n^{\varsigma/2}} \|f\|_{B_\infty^\varsigma}, \quad n > \frac{\tau}{\lambda}.$$

Here we recall that

$$h_n = \sqrt{\frac{\tau}{n\lambda}}.$$

Consequently, if $f \in B^s_\infty$ and if ζ is consistent of order μ then

$$|\,(U_\tau^{(\zeta,h_n)} f)(x) - (U_\tau f)(x)\,| = O(n^{-\alpha}), \quad \text{as } n \to \infty, \tag{9}$$

where

$$\alpha = \frac{1}{2}\min(\mu, s).$$

By applying this result to the trinomial tree it follows that if f has a derivative f' belonging to C_0 and satisfying a local Zygmund condition, then

$$|\,(U_\tau^{(\eta,h_n)} f)(x) - (U_\tau f)(x)\,| = O(1/n), \quad \text{as } n \to \infty,$$

for any $x \in \mathbb{R}$. The random variable η is defined as in equation (8). If, in addition, the derivative of order three $f^{(3)}$ is in C_0 and satisfies a local Zygmund condition and, furthermore, η is consistent of order 4 then

$$|\,(U_\tau^{(\eta,h_n)} f)(x) - (U_\tau f)(x)\,| = O(1/n^2), \quad \text{as } n \to \infty.$$

In the literature there are results similar to the one in the equation (9) derived from Taylor expansions of the initial value f or $U_t f$. See for instance the work by Butzer et al. [BHW] or, for the trinomial tree, Heston and Zhou [HZ]. However, the results by Butzer et al. and Heston et al. require more local regularity of the initial value than in equation (9). For other results along these lines, see [WW] or [KTW].

One can note as well that Theorem 1 states that if the uniform error on the whole lattice equals $O(h^\varsigma)$, $0 < \varsigma \le \mu$, then $f \in B^\varsigma_\infty$. The next proposition, which is taken from [TW], shows a similar result for a fixed time.

Proposition 1. *Suppose that ζ is consistent of order μ and let $1 < \varsigma \le \mu$. Assume that for a fixed $f \in C_0$ holds*

$$\|U_\tau^{(\zeta,h_n)} f - U_\tau f\|_{C_0} = O(h_n^\varsigma), \quad \text{as } n \to \infty.$$

Then $f \in B^{\varsigma-1}_\infty$.

Next we will focus on the important special case when the initial value $f_\kappa(x) = \max(e^x - \kappa, 0)$, $\kappa > 0$, and the lattice variable $\zeta = \epsilon$, where ϵ denotes a symmetrical random variable with only two outcomes ± 1. It is known, see for instance [DD], [LR] and [WW], that

$$|(U_\tau^{(\epsilon,h_n)} f_\kappa)(x) - (U_\tau f_\kappa)(x)| = O(1/n), \quad \text{as } n \to \infty, \tag{10}$$

for any fixed $x \in \mathbb{R}$. Thus, for the payoff function of a call option the binomial method converges point-wise as $O(1/n)$. In [WW] it is shown that this result is the best possible in the sense that there exists a constant C, independent of n, such that

$$(U_\tau^{(\epsilon,h_n)} f_\kappa)(x) - (U_\tau f_\kappa)(x) \sim C/n,$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as n tends to infinity.

Finally, we consider the convergence rate when the initial value is discontinuous. For this problem the next famous theorem by Berry and Esseen is of great value (see [Be] or [Es]).

Theorem 2. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with mean 0, variance 1 and finite absolute third moment. There is a constant C only depending on the third absolute moment such that the distribution function $F_n(x) = P(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \le x)$ satisfies*

$$|F_n(x) - \Phi(x)| \le C/\sqrt{n}$$

for any x, where Φ is the standard normal distribution function.

Let χ_A denote the indicator function for the set $A \subseteq \mathbb{R}$ and let $f_a = \chi_{(-\infty,a]}$. Theorem 2 gives us

$$|\,(U_\tau^{(\zeta,h_n)} f_a)(x) - (U_\tau f_a)(x)\,| = O(1/\sqrt{n}), \quad \text{as } n \to \infty, \tag{11}$$

for any x and any ζ such that $E[|\zeta|^3] < \infty$. It is possible to show that the convergence rate in equation (11) cannot be better than $1/\sqrt{n}$ for the special case $\zeta = \epsilon$ and $x = a$. More precisely, we have

$$|\,(U_\tau^{(\epsilon,h_{2n})} f_a)(a) - (U_\tau f_a)(a)\,| \sim \frac{1}{2\sqrt{\pi n}},$$

see e.g. [Es] or [D] p.126.

We are now ready to deal with the main problem of this paper.

4 Pricing Discrete European Barrier Options Using Lattice Random Walks

In this section we will return to the problem of estimating the price v of a discrete barrier option. Recall from Section 2 that $v = v_0(0)$, where the function v_0 is given by the following recursion scheme

$$\begin{cases} v_{m-1} = e^{-rT}\, U_{\tau_m}\Big(\Psi_K\big(S_0 e^{\nu T + \sigma(\cdot)}\big)\chi_m(\cdot)\Big), & \\ v_{i-1} = U_{\tau_i}\,(v_i\,\chi_i), & \text{for } 1 \le i \le m-1. \end{cases} \tag{12}$$

Firstly, the function $v_{m-1}(x)$ can easily be expressed in terms of the standard normal distribution function. So in what follows we consider the function v_{m-1} as known.

As already described in Section 2, for each $i = 1, 2, \ldots, m-1$ it is natural to estimate the function v_{i-1} by $U_{\tau_i}^{(\zeta,h)}(v_i\,\chi_i)$. There is, however, one disadvantage with this approach. Depending on the barriers, the function $v_i\,\chi_i$ is, for any $i = 1, \ldots, m-1$, discontinuous at the boundary points of the interval I_i. According to the discussion in the preceding section, a discontinuous initial value f may cause a slow convergence of the sequence $\{U_\tau^{(\zeta,h)} f\}$ as h tends to 0. But suppose for a moment that f can be written as

$$f = \phi - g, \tag{13}$$

where g is a function such that $U_\tau\, g$ can easily be evaluated analytically and ϕ is in some sense a smooth function. Then the discussion in Section 3 gives us strong reasons to believe that one will obtain a better estimate of $U_\tau f$ by using

$$U_\tau^{(\zeta,h)}\phi - U_\tau g$$

instead of $U_\tau^{(\zeta,h)} f$. Our next aim is to show how the functions $v_i\chi_i$, $i = 1, \ldots, m-1$, can be decomposed as in equation (13).

For the sake of simplicity, assume that I_i, $i = 1, 2, \ldots, m-1$, are bounded intervals, that is $a_i > -\infty$ and $b_i < \infty$. We will return to the special case when some of the intervals may be unbounded later on in this section.

Fix an i such that $i = 1, 2, \ldots, m-1$ and consider the functions

$$\psi_{a_i}(x) = e^{\gamma_{a_i}(x-a_i)} \sum_{k=0}^{d_i} \frac{\alpha_k^{(i)}}{k!}(x-a_i)^k \tag{14}$$

and

$$\psi_{b_i}(x) = e^{\gamma_{b_i}(x-b_i)} \sum_{k=0}^{d_i} \frac{\beta_k^{(i)}}{k!}(x-b_i)^k, \tag{15}$$

where d_i is a nonnegative integer. The constants γ_{a_i} and γ_{b_i} can be thought of as a positive and a negative number, respectively. However, we will for the moment put no restrictions on γ_{a_i} or γ_{b_i}. The coefficients $\alpha_k^{(i)}$ and $\beta_k^{(i)}$ above are chosen such that ψ_{a_i} and ψ_{b_i} equal the (analytic) function v_i and its first d_i derivatives at the points a_i and b_i, respectively. Thus

$$\frac{d^k v_i}{dx^k}(a_i) = \frac{d^k \psi_{a_i}}{dx^k}(a_i) \quad \text{and} \quad \frac{d^k v_i}{dx^k}(b_i) = \frac{d^k \psi_{b_i}}{dx^k}(b_i)$$

for each $k = 0, 1, \ldots, d_i$.

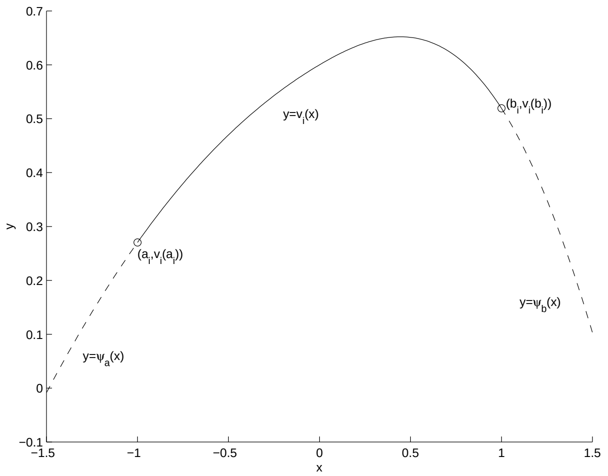

Figure 1: The functions v_i, ψ_{a_i} and ψ_{b_i} (cf. equations (12),(14), and (15), respectively). The solid line is the graph of $x \mapsto v_i(x)$ for $x \in I_i$. The dashed lines are the graphs of $x \mapsto \psi_{a_i}(x)$ and $x \mapsto \psi_{b_i}(x)$ for $x \leq a_i$ and $x \geq b_i$, respectively.

In practice, however, we will not (or rather cannot) differentiate the function v_i to estimate the coefficients $\alpha_k^{(i)}$ or $\beta_k^{(i)}$. Instead we will use numerical differentiation. This step is described in greater details in Appendix A.

Now set

$$g_i = \psi_{a_i}\, \chi_{(-\infty,a_i]} + \psi_{b_i}\, \chi_{[b_i,\infty)}, \tag{16}$$

where χ_A denotes the indicator function of the interval A, and

$$\phi_i = v_i \chi_i + g_i. \tag{17}$$

Hence

$$\phi_i(x) = \begin{cases} \psi_{a_i}(x) & \text{if } x \leq a_i, \\ v_i(x) & \text{if } a_i < x < b_i, \\ \psi_{b_i}(x) & \text{if } x \geq b_i. \end{cases}$$

The function ϕ_i is obviously d_i times differentiable and, furthermore, since the d_i:th derivative $\phi_i^{(d_i)}$ is differentiable of all orders on the set $\mathbb{R} \setminus \{a_i, b_i\}$, the function $\phi_i^{(d_i)}$ satisfies a local Zygmund condition.

If, in addition, we assume that $\gamma_{a_i} > 0$ and $\gamma_{b_i} < 0$ then $\phi_i^{(d_i)} \in C_0$ and thus $\phi_i \in B_\infty^{d_i+1}$. Since $U_{\tau_i} g_i$ can be evaluated using the normal distribution and elementary functions (see Appendix B), we have obtained the desired decomposition

$$v_i \chi_i = \phi_i - g_i$$

as in equation (13). Moreover, note in particular that, if $d_i \geq \mu_i - 1$ with $\mu_i \in \mathbb{N}_+$ then $\phi_i \in B_\infty^{\mu_i}$. Hence, if $d_i = \mu_i - 1$ and $\zeta^{(i)}$ is consistent of order μ_i then equation (9) yields that $|(U_{\tau_i}^{(\zeta^{(i)},h_n)}\phi_i)(x) - (U_{\tau_i}\phi_i)(x)| = O(n^{-\frac{1}{2}\mu_i})$ as $n \to \infty$.

To sum up, we propose

Algorithm 1. *Let $h > 0$ and suppose that $\{\zeta^{(i)}, 1 \leq i \leq m-1\}$ is a sequence of lattice random variables with $Var(\zeta^{(i)}) = \lambda_i$, where λ_i and h must satisfy the condition*

$$\tau_i \in \lambda_i h^2 \mathbb{N} \tag{18}$$

for each $i = 1, \dots, m-1$. The theoretical price v of a discrete barrier option is then approximately equal to $\tilde{v}_0(0)$, where the function $\tilde{v}_0$ is determined by the following recursion scheme

$$\begin{cases} \tilde{v}_{m-1} = e^{-rT}\, U_{\tau_m}\Big(\Psi_K\big(S_0 e^{\nu T + \sigma(\cdot)}\big)\chi_m(\cdot)\Big), \\ \tilde{v}_{i-1} = U_{\tau_i}^{(\zeta^{(i)},h)}\,(\tilde{v}_i\chi_i + g_i) - U_{\tau_i} g_i, \qquad \text{for } 1 \leq i \leq m-1. \end{cases} \tag{19}$$

Here the functions g_i, $i = 1, \dots, m-1$, are defined as in equation (16). The functions $U_{\tau_i} g_i$ can be computed using the receipt in Appendix B. The coefficients $\{\alpha_k^{(i)}\}_{k=0}^{d_i}$ and $\{\beta_k^{(i)}\}_{k=0}^{d_i}$ in the definition of ψ_{a_i} and ψ_{b_i}, cf. equations (14)-(15), are chosen such that

$$\frac{d^k\psi_{a_i}}{dx^k}(a_i) = \frac{d^k\tilde{v}_i}{dx^k}(a_i) \quad \text{and} \quad \frac{d^k\psi_{b_i}}{dx^k}(b_i) = \frac{d^k\tilde{v}_i}{dx^k}(b_i)$$

for $k = 0, 1, \dots d_i$, see Appendix A for further details concerning the estimation of the coefficients $\{\alpha_k^{(i)}\}_{k=0}^{d_i}$ and $\{\beta_k^{(i)}\}_{k=0}^{d_i}$.

Before proceeding, some comments about Algorithm 1 are in order. So far we have assumed that $a_i > -\infty$ and $b_i < \infty$. If $a_i = -\infty$ or $b_i = \infty$ for some i, then we simply let $g_i(x) = \psi_{b_i}(x)\chi_{[b_i,\infty)}$ or $g_i(x) = \psi_{a_i}(x)\chi_{(-\infty,a_i]}$, respectively, in equation (19).

Note that the quantity h is constant, that is, independent of i. This restriction is imposed so that the lattice recombines between the monitoring dates. Needless to say, the functions $\tilde{v}_i$, $i = 1, 2, \dots, m-1$, in equation (19) shall only be calculated at the lattice points.

Before we conclude this section we will make some comments about the computational complexity. Suppose, for some fixed $i = 1, 2, \dots, m-1$, that there are j numbers of nodes between a_i and b_i. It then requires $O(j)$, $j \to \infty$, computations to calculate the function $U_{\tau_i} g_i$ in equation (19). On the other hand, the number of computations to evaluate the functions $U_{\tau_i}^{(\zeta,h)}(\tilde{v}_i\,\chi_i + g_i)$ or $U_{\tau_i}^{(\zeta,h)}(\tilde{v}_i\,\chi_i)$ is of order $O(j^2)$, $j \to \infty$. Thus, the correction term $U_{\tau_i} g_i$ added to the lattice method will not change the computational complexity. However, since the algorithm in equation (19) requires several evaluations of polynomials it is possible to improve the performance of the algorithm by using Horner's scheme (see [DB], p.14-15), which is an efficient way to evaluate a polynomial.

5 Numerical Examples; Choice of Parameters and Lattice Random Variables

This section gives some numerical examples and study the performance of the algorithm for different choices of random variables $\zeta^{(i)}$ and parameters d_i, γ_{a_i} and γ_{b_i}, $i = 1, \dots, m-1$. The section is divided into two subsections. Subsection 5.1 focuses on trinomial trees and equidistant monitoring times, that is, $\zeta^{(i)} = \eta$ (cf. equation (8)) and $\tau_i = T/m$ for each $i = 1, \dots, m-1$. Subsection 5.2 discusses other choices of lattice random variables and arbitrary monitoring times.

5.1 The Trinomial Tree and Equidistant Monitoring Times

Thus, we assume that $\zeta^{(i)} = \eta$ and $\tau_i = T/m$ for each $i = 1, \dots, m-1$. Furthermore, let for simplicity the parameters d_i, γ_{a_i} and γ_{b_i} be independent of i and put $d = d_i$, $\gamma_b = \gamma_{b_i}$ and $\gamma_a = \gamma_{a_i}$ for each $i = 1, \dots, m-1$.

Figure 2 presents the value of $\tilde{v}_0(0)$, given by Algorithm 1, as a function of n, the number of terms in the random walk between the monitoring dates (cf. equation 6). In this first example we have picked $\lambda = 2/3$ and $\gamma_a = \gamma_b = 0$. The option price is approximately 1.2624 (cf. the straight line in Figure 2). Consider first the case when we use just the basic trinomial tree and do not add (or withdraw) any polynomial. The corresponding price is denoted $d = -1$ in Figure 2. We see that the convergence is slow and oscillating. If we add a polynomial of degree $d = 0$ the convergence is more regular but the rate of convergence seems to be more or less the same. In contrast to these examples, when d is equal to 1, which corresponds to differentiable initial values, we get a faster and smoother convergence. When $d = 2$ or 3 the convergence rate does not increase. In fact, it becomes slower.

Before we proceed, let us just mention that this so called 'zigzag convergence' that can be observed in the cases $d = -1$ and $d = 0$ have been analysed more carefully in [DD],[G], and [WW].

Figure 2 reflects very well the convergence behaviour for the proposed method for all choices of $\lambda \in (0, 1]$ *except* $\lambda = 1/3$, that is, when η is consistent of order 4.

The next two figures present the convergence rate for $d = 1, 2, 3$, and $\lambda = 1$ (the binomial tree) and $\lambda = 1/3$, respectively. The option parameters are the same as in the previous example. In the special case $\lambda = 1$, Figure 3 displays that the convergence pattern is roughly the same as in the case $\lambda = 2/3$. On the other hand, if $\lambda = 1/3$ Figure 4 shows that the method

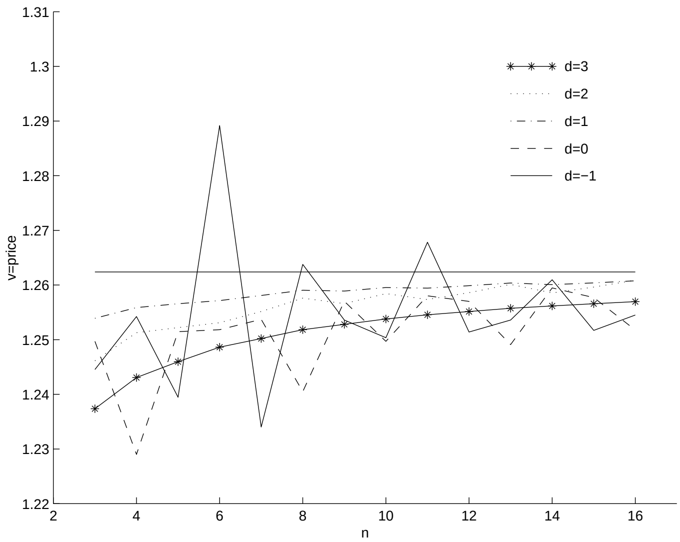

Figure 2: Convergence rate for the proposed method with $\zeta = \eta$ (trinomial tree), $\lambda = 2/3$ and $\gamma_a = \gamma_b = 0$. The option parameters are $S_0 = 100$, $K = 90$, $H_l \equiv 80$, $H_u \equiv 120$, $\sigma = 0.3$, $r = 0.1$, $T = 1$ year, and $m = 50$ (number of monitoring times, corresponds to weekly monitoring). The monitoring times are equally spaced in time.

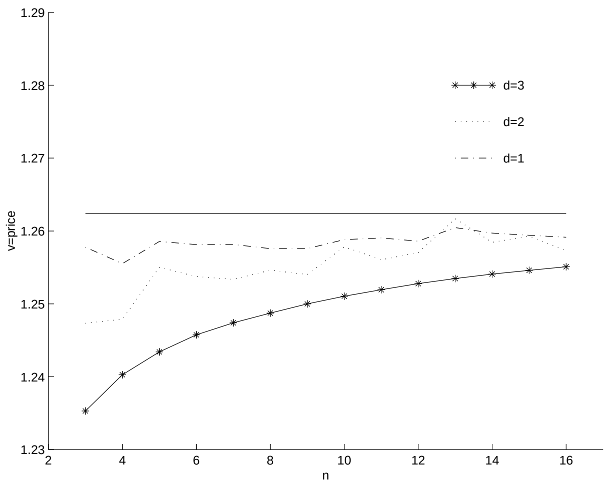

Figure 3: Convergence rate for the proposed method with $\lambda = 1$. The other parameters are as in Figure 2.

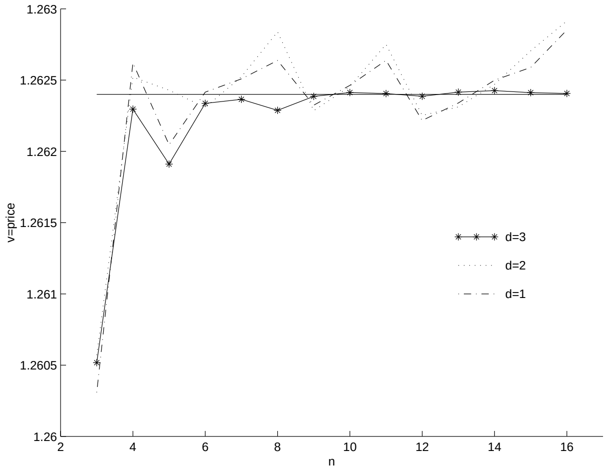

Figure 4: Convergence rate for the proposed method with $\lambda = 1/3$. The other parameters are as in Figure 2.

obtains the best convergence rate when $d = 3$. Table 1 collects the prices obtained for different values of λ and d. The table clearly illustrates that the fastest convergence occurs when $d = 3$ and $\lambda = 1/3$. Finally, Figure 5 shows how the smoothing of the initial value improves the convergence rate.

The next example, Table 2, investigates how the values of γ_a and γ_b influence the error, or rather, if there is a difference in the convergence rate in the two cases $\gamma_a = \gamma_b = 0$ and $\gamma_a > 0$, $\gamma_b < 0$. From a theoretical point of view there is a distinct difference between these cases. If $\gamma_a > 0$ and $\gamma_b < 0$, then the function g (cf. Section 4) is bounded, whereas if $\gamma_a = \gamma_b = 0$ then $g(x) = O(x^d)$ as x tends to infinity.

Recall that the density function for the standard normal distribution decreases as $O(e^{-x^2/2})$ as x tends to infinity. Thus we believe that the growth in g has hardly any greater impact on the rate of convergence, as the example in Table 2 indicates. Unfortunately, we have not been able to prove this (see Section 6 for a further discussion). But still, we suggest that Algorithm (1) should be used with the parameter values $d = 3$, $\lambda = 1/3$ and $\gamma_a = \gamma_b = 0$. Setting $\gamma_a = \gamma_b = 0$ has one practical advantage, the algorithm is easier to implement.

Before we proceed and show some further examples of the performance of the algorithm, one can add that there is a great difference in price between a discrete and a corresponding continuous barrier option. There are known pricing formulas for continuous barrier options with constant barriers (e.g. [KI]). If we apply these formulas to the continuous counterpart to the double barrier option presented in the caption of Figure 2, then we get that the value of this continuous barrier option is 0.7022, which is only 56 percent of the

n	$d=1,\ \lambda=2/3$	$d=1,\ \lambda=1$	$d=3,\ \lambda=1/3$
3	1.2539	1.2578	1.2605
4	1.2559	1.2555	1.2623
5	1.2566	1.2586	1.2619
6	1.2572	1.2581	1.2623
7	1.2581	1.2581	1.2624
8	1.2591	1.2576	1.2623
9	1.2589	1.2576	1.2624
10	1.2595	1.2588	1.2624
11	1.2594	1.2590	1.2624

Table 1: Convergence rate for the proposed method for different values on λ and d. The other parameters are as in Figure 2.

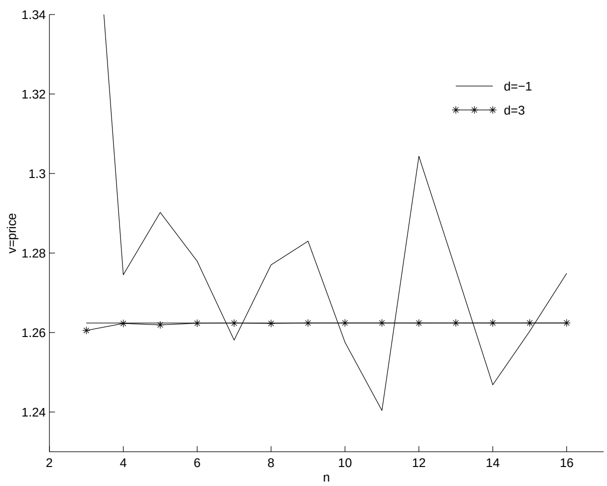

Figure 5: Convergence rate for the proposed method when $\lambda = 1/3$. The other parameters are as in Figure 2.

n	$\gamma_a = 0.0$	$\gamma_a = 0.1$	$\gamma_a = 1$	$\gamma_a = 10$
3	1.2605	1.2605	1.2606	1.2608
4	1.2623	1.2623	1.2623	1.2625
5	1.2619	1.2619	1.2619	1.2620
6	1.2623	1.2623	1.2624	1.2624
7	1.2624	1.2624	1.2624	1.2624
8	1.2623	1.2623	1.2623	1.2623
9	1.2624	1.2624	1.2624	1.2624
10	1.2624	1.2624	1.2624	1.2624
11	1.2624	1.2624	1.2624	1.2624

Table 2: Convergence rate for the proposed method for different values on γ_a, and γ_b. The value on d, λ, and γ_b are 3, 1/3, and $-\gamma_a$, respectively. The other parameters are as in Figure 2.

value for the discrete barrier option. So it is worth to emphasize that one should not neglect the fact that some barrier options are discretely and not continuously monitored.

In the next example we have compared our method with an algorithm developed by Broadie et al. (see [BGK]) which is designed to estimate the value of discrete single barrier options. For simplicity we henceforth call this method the BGK method. The idea behind the BGK method is as follows. A discrete barrier at place H is replaced by a discrete barrier at place $H\exp(\pm 0.5h_n)$, with $+$ for an upper barrier, $-$ for a lower barrier and where h_n is the stepsize for the log price. The factor $0.5h_n$ is the expected overshoot of a trinomial random walk. Subsequently the theoretical value is computed using the trinomial method on a mesh with the property that certain nodes on the mesh coincide with the new barrier at place $H\exp(\pm 0.5h_n)$. Numerical experiments in that paper indicate that the convergence rate for the method is $O(1/n)$. In order to increase the convergence rate a Richardson interpolation is used. For further details, see [BGK].

Table 3 shows results from the different methods. The values in the second and third column are taken from a numerical example in [BGK]. The BGK method has been used with as well as without Richardson extrapolation. In the final column we have the values from the method presented in this work. As we can see, the example indicates that the method presented in this paper outperforms the BGK method.

The final example in this subsection computes the value of a discrete *moving* barrier option. The results are presented in Table 4. The price of a moving barrier option with continuous barriers can be estimated using a method described in [RZ].

N	BGK	BGK (2-pt Extrapol.)	N	Hö
256	9.4969		40	9.4895
504	9.4935	9.4899	60	9.4907
1240	9.4919	9.4907	80	9.4907
2308	9.4912	9.4905	100	9.4906
4524	9.4909	9.4905	120	9.4905
8632	9.4907	9.4905	140	9.4905

Table 3: The value of a discrete down-and-out call, the option parameters are $S_0 = K = 100$, $H_l \equiv 95$, $H_u \equiv \infty$, $T = 0.2$ year, $\sigma = 0.6$, and $r = 0.1$. There are 4 monitoring dates which are equally spaced in time, i.e. the monitoring dates are given by $\{\tau, 2\tau, 3\tau, 4\tau\}$ where $\tau = T/4$. The quantity N denotes the total number of iterations, i.e. $N = 4n$. The random variables/parameters in Hö are $\zeta = \eta$ (trinomial tree), $\gamma_a = \gamma_b = 0$, $d = 3$, and $\lambda = 1/3$.

m	n	$k_l = -k_u = -5$	$k_l = -k_u = 5$
	10	6.9918	5.0999
12	20	6.9918	5.1001
	30	6.9918	5.1001
	10	5.7039	3.8642
50	15	5.7040	3.8641
	20	5.7040	3.8641
	6	4.9418	3.1920
250	8	4.9420	3.1921
	10	4.9421	3.1920

Table 4: The value of a discrete moving double barrier knock-out call for different number of monitoring dates (monthly, weekly and daily). The monitoring times are equally spaced in time. The option parameters are $S_0 = 95$, $K = 100$, $H_l(t) = 90 + k_l t$, $H_u(t) = 160 + k_u t$, $T = 1$ year, $\sigma = 0.25$, and $r = 0.1$. The random variables and parameters are $\zeta = \eta$ (trinomial tree), $\gamma_a = \gamma_b = 0$, $d = 3$, and $\lambda = 1/3$. The corresponding continuous prices are, according to [RZ], approximately 4.34 and 2.54, respectively.

5.2 Arbitrary Monitoring Dates and Other Lattice Random Walks

In the previous subsection we assumed that the monitoring dates were equidistant in time. What happens if we drop this assumption? Recall from Algorithm 1 that the monitoring times τ_i must satisfy

$$\tau_i \in \lambda_i h^2 \mathbb{N} \tag{20}$$

for all $i = 1, \ldots, m-1$. Thus, if the monitoring times are arbitrary then the values of the parameters λ_i must in certain cases be dependent of i.

Of course, it is not necessary to assume that the monitoring times are equidistant, as in the previous subsection, to be able to find a λ_i which is independent of i and satisfies equation (20) for all $i = 1, \ldots, m-1$. For instance, if there is a $\Delta > 0$ such that

$$\tau_i \in \Delta \mathbb{N}, \tag{21}$$

for all $i = 1, \ldots, m-1$, and if we let

$$h = \sqrt{\frac{\Delta}{\lambda k}}$$

where $k \in \mathbb{N}_+$ and $\lambda > 0$, then it is evident that equation (20) holds with $\lambda_i = \lambda$.

Now, suppose that we *must* pick λ_i differently depending on i. How should we choose the lattice random walk? Of course, we may apply the trinomial tree for all $\lambda \in (0, 1]$ but this approach has the disadvantage that the random variable η is, unless $\lambda = 1/3$, only consistent of order 2. As we have seen in the previous sections, the proposed algorithm is more efficient if the lattice random variable is consistent of order 4 than of order 2 and hence, it would be of interest to find a class of lattice random variables where the members of this class have different variance but still are consistent of order 4. Next we will construct such a class.

Consider a symmetrical lattice random variable ϑ that have (at the most) five possible outcomes with

$$P(\vartheta = 0) = p_0 \quad \text{and} \quad P(\vartheta = j) = P(\vartheta = -j) = p_j, \quad j = 1, 2.$$

where $p_0 + 2p_1 + 2p_2 = 1$. The Taylor formula gives that the random variable ϑ is exactly consistent of order 4 if

$$p_0 = 1 - \frac{\lambda}{4}(5 - 3\lambda), \quad p_1 = \frac{\lambda}{6}(4 - 3\lambda), \quad \text{and} \quad p_2 = \frac{\lambda}{24}(3\lambda - 1),$$

for any $\lambda = \mathrm{Var}(\vartheta)$ such that

$$\frac{1}{3} \leq \lambda \leq \frac{4}{3}.$$

N	M_1	N	M_2	N	M_3
21	2.6504	26	2.1340	28	2.0236
42	2.3586	52	2.2739	56	2.0219
63	2.3591	78	2.0549	84	2.0216
84	2.3591	104	2.0548	112	2.0216
105	2.3591	130	2.0548	140	2.0216

Table 5: The value of a discrete double barrier knock-out call with randomly chosen monitoring dates. The quantity N denotes the total number of iterations, cf. Table 3. In the examples above the monitoring dates are given by $M_1 = \{0.06, 0.08, 0.15, 0.35, 0.38, 0.44, 0.45, 0.63, 0.67, 0.69, 0.71, 1.00\}$ $M_2 = \{0.12, 0.25, 0.27, 0.45, 0.48, 0.55, 0.69, 0.72, 0.73, 0.87, 0.89, 1.00\}$, and $M_3 = \{0.19, 0.37, 0.57.0.62, 0.63, 0.73, 0.75, 0, 79, 0.84, 0.90, 0.92, 1.00\}$. The other option parameters are as in Figure 2. The proposed algorithm has been used with $\zeta = \vartheta$, $\gamma_a = \gamma_b = 0$ and $d = 3$.

Table 5 presents a numerical example which shows the convergence rate when Algorithm 1 is applied with $\zeta = \vartheta$.

Next we will draw attention to certain other random variables that are consistent of order 6. Recall that according to the discussion that precedes Algorithm 1, we may expect that the convergence rate improves for greater values of consistency number μ_i, provided that $d_i = \mu_i - 1$.

For instance, consider the random variable θ, defined by

$$P(\theta = 0) = p_0, \quad \text{and} \quad P(\theta = j) = P(\theta = -j) = p_j, \quad j = 1, 2, 3,$$

with

$$\begin{cases} p_0 &= 1 - \frac{\lambda}{36}(49 - 42\lambda + 15\lambda^2), \\ p_1 &= \frac{\lambda}{16}(12 - 13\lambda + 5\lambda^2), \\ p_2 &= \frac{\lambda}{40}(-3 + 10\lambda - 5\lambda^2), \\ p_3 &= \frac{\lambda}{720}(4 - 15\lambda + 15\lambda^2), \end{cases}$$

and

$$\frac{1}{5}(5 - \sqrt{10}) \leq \lambda \leq \frac{1}{5}(5 + \sqrt{10}).$$

Cumbersome computations yield that θ is exactly consistent of order 6.

If we apply Algorithm 1 with the random variable θ then we also get a very fast convergence rate. For instance, for the option presented in Figure 2 the algorithm converges to the price 1.2624 already for $n \geq 2$ if $\lambda = 1$ and $d = 5$, see Table 6.

For a further discussion on other lattice random variables that can be useful in lattice methods we refer to [H2].

n	$\lambda = \frac{1}{5}(5-\sqrt{10})$	$\lambda = 1$
1	1.2525	1.2613
2	1.2615	1.2624
3	1.2617	1.2624
4	1.2623	1.2624
5	1.2624	1.2624

Table 6: The value of a discrete double barrier knock-out call. The option parameters are as in Figure 2. The proposed algorithm has been used with $\zeta = \theta$, $\gamma_a = \gamma_b = 0$ and $d = 5$.

6 Conclusions and Suggestions for Future Research

This paper has designed a numerical procedure to price discrete European barrier options and showed that the convergence rate of lattice methods depends on two factors, namely the smoothness of the initial function and the moments of the terms in the random walk. The pricing of discrete barrier options is equivalent to solving a series of initial value problem for the heat equation. The main idea has been to decompose each initial value f_i as a difference

$$f_i = \phi_i - g_i, \tag{22}$$

where ϕ_i is smooth and the expectation of g_i with respect to the gaussian measure can be computed explicit. By applying the lattice method to the smooth part ϕ_i we have obtained a numerical procedure that yields fast and accurate results.

However, the research presented in this article leaves certain questions unanswered. We will now conclude this section by raising some questions that might be of some interest for future research.

- It would be of great value to prove certain modifications of Theorem 1. In our application we are perhaps more interested in point-wise estimates of the error rather than estimates in the supremum norm. It seems plausible that the convergence rate for

$$(U_\tau^{(\zeta,h)} f)(x) - (U_\tau f)(x) \tag{23}$$

  for some fixed x mainly depends on f around some neighbourhood of x. Therefore, it may be possible to derive sharp point-wise bounds for the difference in equation (23) without having to assume that the initial value f is in C_0 (for instance).

- It would be of interest to investigate how the derivatives $\tilde{v}_i^{(k)}$ best should be estimated. This problem is also relevant in the estimation of delta and gamma (the first and second derivative of the option price with respect to the underlying asset price).

- Given an positive integer μ, is it possible to construct a lattice random variable ζ such that ζ is consistent of order μ?

Appendix A

The intention with this appendix is to show how the coefficients $\{\alpha_k\}_{k=0}^{d}$ in

$$\psi_a(x) = e^{\gamma(x-a)} \sum_{k=0}^{d} \frac{\alpha_k}{k!}(x-a)^k, \quad \gamma, a \in \mathbb{R},$$

can be estimated so that

$$\frac{d^k \psi_a}{dx^k}(a) = \frac{d^k \tilde{v}}{dx^k}(a) \quad \text{for each } k = 0, 1, \ldots, d,$$

where $\tilde{v}$ is a function only known at discrete points, say at $x \in h\mathbb{Z}$ with $h > 0$.

Firstly, the Leibnitz rule implies

$$\frac{d^k \psi_a}{dx^k}(a) = \sum_{j=0}^{k} \binom{k}{j} \gamma^{k-j} \alpha_j, \quad k = 0, 1, \ldots, d.$$

Thus, α_k is defined recursively by

$$\alpha_k = \frac{d^k \tilde{v}}{dx^k}(a) - \sum_{j=0}^{k-1} \binom{k}{j} \gamma^{k-j} \alpha_j, \quad k = 0, 1, \ldots, d.$$

It remains to estimate the derivatives of $\tilde{v}(x)$ at the point $x = a$. A natural approach to this problem is to differentiate an appropriate interpolations polynomial. Suppose that $[r]$ stands for the smallest integer $\geq r$ and $j^* = [a/h]$. In addition, assume $d \leq \delta \in \mathbb{N}$ and let $\kappa = [\delta/2]$. If p denotes the (interpolation) polynomial of degree δ which satisfies

$$p\big((j^* + j)h\big) = \tilde{v}\big((j^* + j)h\big), \quad j = -\kappa, -\kappa + 1, \ldots, -\kappa + \delta,$$

then the first d derivatives at the point a can be estimated by

$$\frac{d^k \tilde{v}}{dx^k}(a) \approx \frac{d^k p}{dx^k}(a), \quad k = 0, 1, \ldots, d.$$

In the numerical examples presented in Section 5 the above procedure have been used with $\delta = 3$ with exception of Table 6 where $\delta = 5$.

Appendix B

Using the following lemma, the functions $U_\tau g_i$, $i = 1, 2, \dots, m-1$ can be evaluated in an efficient way.

Lemma 1. *Let* $\varphi(x) = \frac{d}{dx}\Phi(x)$. *If*

$$g = \psi_a \, \chi_{(-\infty,a]} + \psi_b \, \chi_{[\,b,\infty)}$$

where ψ_a *and* ψ_b *are defined by*

$$\psi_a(x) = e^{\gamma_a\,(x-a)} \sum_{k=0}^{d} \frac{\alpha_k}{k!}\,(x-a)^k$$

and

$$\psi_b(x) = e^{\gamma_b\,(x-b)} \sum_{k=0}^{d} \frac{\beta_k}{k!}\,(x-b)^k$$

then

$$\begin{aligned}(U_\tau g)(x) =& e^{\gamma_a (x-a)+\gamma_a^2\,\tau/2} \sum_{k=0}^{d} \Big(\frac{\hat{\alpha}_k}{k!}\, M_k\big(\frac{a-x}{\sqrt{\tau}} - \gamma_a\sqrt{\tau}\,\big)\Big) \\ &+ e^{\gamma_b (x-b)+\gamma_b^2\,\tau/2} \sum_{k=0}^{d} \Big(\frac{\hat{\beta}_k}{k!}\,(-1)^k M_k\big(\frac{x-b}{\sqrt{\tau}} + \gamma_b\sqrt{\tau}\,\big)\Big)\end{aligned}$$

where

$$\hat{\alpha}_k = \tau^{k/2} \sum_{i=0}^{d-k} \frac{\alpha_{i+k}}{i!}(x-a+\gamma_a\tau)^i$$

and

$$\hat{\beta}_k = \tau^{k/2} \sum_{i=0}^{d-k} \frac{\beta_{i+k}}{i!}(x-b+\gamma_b\tau)^i$$

and where the functions M_k *are defined recursively by*

$$M_k(y) = \begin{cases} \Phi(y) & \text{if } k = 0, \\ -\varphi(y) & \text{if } k = 1, \\ y^{k-1}\, M_1(y) + (k-1)\, M_{k-2}(y) & \text{if } k = 2, 3, \dots, d. \end{cases} \tag{24}$$

Proof. Let $\hat{\alpha}_k$ be defined as above and set $\hat{\gamma}_a = \gamma_a \sqrt{\tau}$. If $\hat{\psi}_a(\xi) = \psi_a(x + \sqrt{\tau}\xi)$ then

$$\hat{\psi}_a(\xi) = e^{\gamma_a(x-a)+\hat{\gamma}_a\xi} \sum_{k=0}^{d} \frac{\hat{\alpha}_k}{k!}(\xi - \hat{\gamma}_a)^k,$$

since for any $k = 0, 1, \ldots, d$,

$$\begin{aligned}\Big(\frac{d^k}{d\xi^k} \sum_{i=0}^{d} \frac{\alpha_i}{i!}(x + \sqrt{\tau}\xi - a)^i\Big)\Big|_{\xi=\hat{\gamma}_a} &= T^{k/2} \sum_{i=0}^{d-k} \frac{\alpha_{i+k}}{i!}(x + \sqrt{\tau}\hat{\gamma}_a - a)^i \\ &= \hat{\alpha}_k.\end{aligned}$$

Let $c_a = (a - x)/\sqrt{\tau}$. The scaling property for Brownian motion and the definition of $\hat{\psi}_a$ give

$$\begin{aligned}\big(U_\tau\,(\psi_a\,\chi_{(-\infty,a]}\,)\big)(x) =& E\big[\psi_a(x + W_\tau)\,\chi_{(-\infty,a]}(x + W_\tau)\,\big] \\ =& E\big[\hat{\psi}_a(W_1)\,\chi_{(-\infty,c_a]}(W_1)\,\big] \\ =& e^{\gamma_a(x-a)} \sum_{k=0}^{d} \frac{\hat{\alpha}_k}{k!} \int_{-\infty}^{c_a} e^{\hat{\gamma}_a\xi}\,(\xi - \hat{\gamma}_a)^k \varphi(\xi) d\xi.\end{aligned}$$

Note moreover that

$$\begin{aligned}\int_{-\infty}^{c_a} e^{\hat{\gamma}_a\xi}\,(\xi - \hat{\gamma}_a)^k \varphi(\xi) d\xi &= \int_{-\infty}^{c_a} (\xi - \hat{\gamma}_a)^k\, e^{\hat{\gamma}_a^2/2}\,\varphi(\xi - \hat{\gamma}_a) d\xi \\ &= e^{\hat{\gamma}_a^2/2} \int_{-\infty}^{c_a - \hat{\gamma}_a} \xi^k\,\varphi(\xi)\,d\xi.\end{aligned}$$

Thus, if we set

$$M_k(y) = \int_{-\infty}^{y} \xi^k \varphi(\xi) d\xi \tag{25}$$

for each integer $k \geq 0$, then

$$\big(U_\tau\,(\psi_a\,\chi_{(-\infty,a]}\,)\big)(x) = e^{\gamma_a(x-a)+\hat{\gamma}_a^2/2} \sum_{k=0}^{d} \frac{\hat{\alpha}_k}{k!}\, M_k(c_a - \hat{\gamma}_a).$$

By using a similar argument we get

$$\big(U_\tau\,(\psi_b\,\chi_{[b,\infty)}\,)\big)(x) = e^{\gamma_b(x-b)+\hat{\gamma}_b^2/2} \sum_{k=0}^{d} \frac{\hat{\beta}_k}{k!} \int_{c_b - \hat{\gamma}_b}^{\infty} \xi^k \varphi(\xi) d\xi,$$

where $c_b = (b-x)/\sqrt{\tau}$, $\hat{\gamma}_b = \gamma_b \sqrt{\tau}$ and where the $\hat{\beta}_k$:s are defined as in the proposition. From the symmetry of the normal density we now conclude

$$\big(U_\tau\,(\,\psi_b\,\chi_{[b,\infty)}\,)\big)(x) = e^{\gamma_b(x-b)+\hat{\gamma}_b^2/2} \sum_{k=0}^{d} \frac{\hat{\beta}_k}{k!}(-1)^k\, M_k(\hat{\gamma}_b - c_b).$$

It remains to show that the functions M_k, $k = 0, 1, \ldots, d$, satisfy equation (24). It is evident that $M_0(y) = \Phi(y)$. Since $\frac{d}{d\xi}\varphi(\xi) = -\xi\varphi(\xi)$ we also have $M_1(y) = -\varphi(y)$. Partial integration now yields for $k \geq 2$

$$\begin{aligned} M_k(y) &= -\xi^{k-1}\,\varphi(\xi)\Big|_{\xi=y} + (k-1)\int_{-\infty}^{y} \xi^{k-2}\,\varphi(\xi)d\xi \\ &= y^{k-1}\,M_1(y) + (k-1)\,M_{k-2}(y)\,. \end{aligned}$$

□

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