Extension of the Corrected Barrier Approximation by Broadie, Glasserman and Kou

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Abstract. This paper considers the problem of pricing discrete barrier options. A discrete barrier option is a barrier option where the barrier is monitored only at specific dates. This paper continues the work initiated by Broadie et al. in [BGK] and determine formulas to estimate the price of discrete up-and-out/in calls, down-and-out/in puts and double barrier options. Numerical examples presented in this paper show that the formulas yield good results.

Key words: Option pricing, discrete barrier options, heavy traffic approximation.

JEL Classification: G13

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1 Introduction

A barrier option is activated (knocked in) or extinguished (knocked out) when a specified asset price, index or rate reaches a specified level, or barrier. If a barrier option has two barriers then it is usually referred to as a double barrier option.

It is common to assume that the underlying asset is continuously monitored against the barrier. However, for many traded barrier options the barrier is monitored only at specific dates. These options are usually referred to as *discrete* barrier options.

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In contrast to continuous barrier options the price of a discrete barrier option does not in general possess a closed form price formula. The price can be expressed in terms of the multivariate normal distribution. Here the dimension of the relevant multivariate normal distribution is equal to the number of price fixing dates, which, in most cases, is too large for numerical evaluation.

Methods to price discrete barrier options have been discussed earlier in the literature. Methods based on the trinomial method have been investigated by, among many others, Broadie et al. [BGK2], Hörfelt [H2] and Ahn et al. [AFG]. In [BBG] Monte Carlo methods were employed to price discrete barrier options. Another technique which has given remarkably good results was first proposed by Chuang, [Ch], and, independently, by Broadie, Glasserman, and Kou, [BGK]. They proposed that one should use a result from sequential analysis and queue theory, namely "Siegmund's corrected heavy traffic approximation", which is useful to estimate the joint distribution of a random walk and its maximum.

Chuang *only* suggested the possibility of using Siegmund's result for pricing discrete barrier options. However, Chuang never pursued the idea. Broadie et al. derived pricing formulas for some discrete (single) barrier options, but not all.

The purpose of this paper is to continue the work initiated by Broadie et al. and, by using approximations developed by Siegmund, estimate the price of all discrete single and double barrier options. The work presented in this article has similarities with an independent work by Kou, see [K][2]. This work will be further discussed in the next section.

We make the same assumptions as in the papers mentioned above by Broadie et al. That is, we assume throughout that the price of the underlying asset $\{S_t\}_{t\geq 0}$ evolves under the risk-neutral martingale measure according to

$$S_t = S_0 e^{(r-q-\sigma^2/2)\,t+\sigma W_t}, \quad t \geq 0,$$

where $\{W_t\}_{t\geq 0}$ is a Brownian motion and where the risk free rate r, the dividend yield q, the volatility σ, and the initial price S_0 are assumed to be constants. Moreover, it will be assumed that the monitoring dates, or the price fixing dates, are equally spaced in time.

This paper is structured as follows. In Section 2 we will present the results by Broadie et al. as well as our main result about single barrier options. The latter result will be proved in Sections 3 and 4. Moreover, Section 4 discusses Siegmund's corrected heavy traffic approximation. In Section 5 we will present some numerical examples. Section 6 deals with approximations for the price of discrete *double* barrier options. Section 7 contains numerical examples for double barrier options.

[2]We would like to thank an anonymous referee for having brought to our attention Kou's work

2 Discrete Single Barrier Options

The first part of this section specifies the payoff functions for those *single* barrier options that will be treated in this paper. In Section 6 we return to double barrier options.

In what follows, let K, T, and H denote the strike price, time of maturity and barrier level, respectively. Moreover, let A denote the set of all monitoring dates, i.e.

$$A = \{\Delta t,\, 2\Delta t, \dots , m\Delta t\}, \quad \Delta t = T/m,$$

where m is the number of monitoring times.

Assume $S_0 < H$. If $\chi = 1$ then the payoff of a discrete up-and-out call is given by

$$\max\big(\chi(S_T - K), 0\big) 1_{\{\max_{t\in A} S_t < H\}}. \tag{1}$$

If instead $\chi = -1$ then we get the payoff of a discrete up-and-out *put*. Up-and-out calls/puts are sometimes also referred to as knock-out (up) options. By replacing the event $\{\max_{t\in A} S_t < H\}$ in equation (1) with the complement $\{\max_{t\in A} S_t \geq H\}$ we get the pay out function of discrete up-and-*in* calls/puts, or *knock-in* (up) option. By replacing the set A in equation (1) with the interval $[0, T]$ we get the payoff of a *continuous* up-and-out call/put.

Assume $S_0 > H$. The payoff of a discrete down-and-out call/put, or knock-out (down) option, can be written as

$$\max\big(\chi(S_T - K), 0\big) 1_{\{\min_{t\in A} S_t > H\}}$$

with χ equal to 1 or -1, respectively. The payoff of a discrete/continuous down-and-in call/put, or knock-in (down) option, are straightforward.

The most naive approach to approximate the value of a discrete barrier option would be to ignore the fact that the barrier is discrete and price the option as a continuous barrier option with the same barrier. For continuous barrier options there are known formulas, see for instance [R]. However, numerical examples show that this method can lead to substantial mispricings (see [BGK]) even in the case of daily monitoring. In a paper by Broadie et al. [BGK] it was shown that this simple approximation can be improved for some barrier options just by shifting the barrier. The next theorem is taken from [BGK].

Theorem 1. *Let $v^d(H)$ be the price of a discretely monitored knock-in or knock-out down call or up put with barrier H. Let $v(H)$ be the price of the corresponding continuously monitored barrier option. Then, as $m \to \infty$,*

$$v^d(H) = v(He^{\pm\beta\sigma\sqrt{T/m}}) + o(\frac{1}{\sqrt{m}})$$

where $+$ applies if $H > S_0$,$-$ applies if $H < S_0$, and $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$, with ζ the Riemann zeta function.

Numerical results presented in the same paper indicate that the shift of the barrier gives surprisingly good approximations for moderate to large values of m if the initial asset price is not too close to the barrier.

The proof of Theorem 1 in [BGK] was based on certain results developed in a series of articles written by Sigmund and, to some extend, Yuh. In this paper we will use the same results to determine approximation formulas for discrete down-and-out/in puts and up-and-out/in calls. Note that none of these options are included in Theorem 1.

Before we present our main result of this section we will introduce some functions. Let N denote the standard normal distribution function and let the function F_+ be defined in the following way

$$F_+(a,b;\theta) = N(a-\theta) - e^{2b\theta} N(a-2b-\theta)$$

for $a \le b$, $b > 0$ and $\theta \in \mathbb{R}$. Moreover, if $a \ge b$, $b < 0$ and $\theta \in \mathbb{R}$ then we put

$$F_-(a,b;\theta) = F_+(-a,-b;-\theta).$$

It is well-known that

$$F_+(a,b;\theta) = P(\,\theta + W_1 \le a,\ \max_{0\le t\le 1}(\theta t + W_t) \le b\,)$$

and

$$F_-(a,b;\theta) = P(\,\theta + W_1 \ge a,\ \min_{0\le t\le 1}(\theta t + W_t) \ge b\,).$$

These facts will be used later on in this paper.

We are now in the position to present our main result about discrete single barrier options.

Theorem 2. *Suppose that*

$$\phi(x) = \frac{\ln(x/S_0)}{\sigma\sqrt{T}}, \qquad x > 0,$$

and let $c = \phi(K)$ *and* $d = \phi(H)$. *Set moreover*

$$\theta_0 = \frac{(r-q-\sigma^2/2)\sqrt{T}}{\sigma} \quad \text{and} \quad \theta_1 = \theta_0 + \sigma\sqrt{T}.$$

Let v_{uoc} *denote the theoretical value at time* $t = 0$ *of a discretely monitored up-and-out call. If* $K < H$ *then, as* $m \to \infty$,

$$\begin{aligned} v_{uoc} =& S_0 e^{-qT}\big(F_+(d, d+\beta/\sqrt{m};\theta_1) - F_+(c, d+\beta/\sqrt{m};\theta_1)\big) \\ &- Ke^{-rT}\big(F_+(d, d+\beta/\sqrt{m};\theta_0) - F_+(c, d+\beta/\sqrt{m};\theta_0)\big) \\ &+ o(\frac{1}{\sqrt{m}}), \end{aligned} \tag{2}$$

where β is defined as in Theorem 1. If $K \geq H$, $v_{uoc} = 0$.

The theoretical value v_{dop} at time $t = 0$ for a discretely monitored down-and-out put is given by

$$\begin{aligned} v_{dop} =& Ke^{-rT}\big(F_-(d, d-\beta/\sqrt{m};\theta_0) - F_-(c, d-\beta/\sqrt{m};\theta_0)\big) \\ &- S_0 e^{-qT}\big(F_-(d, d-\beta/\sqrt{m};\theta_1) - F_-(c, d-\beta/\sqrt{m};\theta_1)\big) \\ &+ o(\frac{1}{\sqrt{m}}), \quad m \to \infty, \end{aligned} \tag{3}$$

provided $K > H$. If $K \leq H$, $v_{dop} = 0$.

Remarkably enough, one will *not* get the above approximations by simply shifting the barrier as in Theorem 1. The pricing formulas for the corresponding continuous barrier options are of course obtained by letting $m \to \infty$ in equation (2) or (3).

The value of the corresponding knock-in options now follow by the fact that the sum of two otherwise identical in- and out- call (put) options is a plain call (put) option.

Before we proceed and prove Theorem 2 we will make some comments about the results in [K]. In that paper it is shown that Theorem 1 actually can be extended to cover down-and-out/in puts and up-and-out/in calls as well. However, numerical examples presented in Section 5 indicate that the approximation formulas in Theorem 2 in most cases yield better results.

3 Pricing Discrete Barrier Options

The next lemma will be useful to price both discrete single and double barrier options. The proof is based on a technique often referred to as change of numeraire. The technique is also employed in [K].

Lemma 1. *Let the function ϕ and the constants c, θ_1, and θ_2 be defined as in Theorem 2. Suppose that H_1 and H_2 are real numbers such that $0 < H_1 \leq S_0 \leq H_2$ and set $d_1 = \phi(H_1)$ and $d_2 = \phi(H_2)$. If χ is a constant equal to 1 or -1 then*

$$\begin{aligned} e^{-rT}E\big[\max\big(\chi(S_T-K),0\big)1_{\{\min_{t\in A} S_t > H_1,\ \max_{t\in A} S_t < H_2\}}\big] \\ =\chi S_o e^{-qT} P(\,\chi\,(\theta_1 + W_1)\ >\ \chi c,\ \tau_m^{\theta_1}(d_1,d_2) > 1) \\ -\,\chi K e^{-rT} P(\,\chi\,(\theta_0 + W_1)\ >\ \chi c,\ \tau_m^{\theta_0}(d_1,d_2) > 1) \end{aligned}$$

where, for all $\theta \in \mathbb{R}$ and $b_1 < 0 < b_2$,

$$\tau_m^{\theta}(b_1,b_2) = \inf\big\{\,t \in \{1/m,\,2/m,\dots\}\,;\ (\theta t + W_t) \notin (b_1,b_2)\,\big\}.$$

Proof. Set $m_T = \min_{t\in A} S_t$ and $M_T = \max_{t\in A} S_t$. Firstly, note that

$$
\begin{aligned}
&e^{-rT}E\big[\max\big(\chi(S_T-K),0\big)1_{\{m_T>H_1,\ M_T<H_2\}}\big] \\
&\quad=\chi S_o e^{-qT}E\big[\exp\big(-\frac{\sigma^2}{2}T+\sigma W_T\big)1_{\{\chi S_T>\chi K,\ m_T>H_1,\ M_T<H_2\}}\big] \\
&\qquad-\chi K e^{-rT}P\big(\chi S_T>\chi K,\ m_T>H_1,\ M_T<H_2\big).
\end{aligned}
$$

Define a probability measure $\tilde{P}$ by $d\tilde{P} = \exp(-(\sigma^2/2)T+\sigma W_T)dP$. We now get

$$
\begin{aligned}
&e^{-rT}E\big[\max\big(\chi(S_T-K),0\big)1_{\{m_T>H_1,\ M_T<H_2\}}\big] \\
&\quad=\chi S_0 e^{-qT}\tilde{P}\big(\chi S_T>\chi K,\ m_T>H_1,\ M_T<H_2\big) \\
&\qquad-\chi K e^{-rT}P\big(\chi S_T>\chi K,\ m_T>H_1,\ M_T<H_2\big).
\end{aligned}
$$

Set $B = \{1/m, 2/m, \dots, 1\}$, $k = \ln(K/S_0)$, $h_1 = \ln(H_1/S_0)$, $h_2 = \ln(H_2/S_0)$, and $\theta = \eta\sqrt{T}/\sigma$ for $\eta \in \mathbb{R}$. The scaling property for Brownian motion gives

$$
\begin{aligned}
&P\big(\chi(\eta T+\sigma W_T)>\chi k,\ \min_{t\in A}(\eta t+\sigma W_t)>h_1,\ \max_{t\in A}(\eta t+\sigma W_t)<h_2\big) \\
&\quad= P(\chi(\theta+W_1)>\chi c,\ \tau_m^{\theta}(d_1,d_2)>1).
\end{aligned}
\tag{4}
$$

If we replace η by $r-q-\sigma^2/2$ in equation (4) we obtain

$$
\begin{aligned}
&P\big(\chi S_T>\chi K,\ m_T>H_1,\ M_T<H_2\big) \\
&\quad= P(\chi(\theta_0+W_1)>\chi c,\ \tau_m^{\theta_0}(d_1,d_2)>1).
\end{aligned}
$$

The Cameron-Martin theorem states that $\{W_t-\sigma t\}_{0\le t\le T}$ is a $\tilde{P}$-Brownian motion. Thus, by setting $\eta = r-q+\sigma^2/2$ in equation (4) we get

$$
\begin{aligned}
&\tilde{P}\big(\chi S_T>\chi K,\ m_T>H_1,\ M_T<H_2\big) \\
&\quad= P(\chi(\theta_1+W_1)>\chi c,\ \tau_m^{\theta_1}(d_1,d_2)>1),
\end{aligned}
$$

which completes the proof. □

We will now introduce the discrete counterpart to F_+ (cf. Section 2). Set for $b>0$, $a\le b$, and $\theta\in\mathbb{R}$,

$$
F_+^{(m)}(a,b;\theta) = P\big(\theta+W_1\le a,\ \max_{t\in B}(\theta t+W_t)\le b\big),
$$

where $B = \{1/m, 2/m, \ldots, 1\}$. The next aim is to show that the price for both a discrete up-and-out call and a down-and-out put may be expressed in terms of the function $F_+^{(m)}$.

First, let us consider discrete up-and-out calls. Suppose that $K < H$. Let c, d, θ_0, and θ_1 be defined as in Theorem 2. By letting $H_1 \to 0$ and setting $H_2 = H$ and $\chi = 1$ in Lemma 1 we get

$$\begin{aligned} v_{uoc} =& S_0 e^{-qT} P\big(\theta_1 + W_1 > c,\ \max_{t \in B} (\theta_1 t + W_t) < d\big) \\ &- K e^{-rT} P\big(\theta_0 + W_1 > c,\ \max_{t \in B} (\theta_0 t + W_t) < d\big). \end{aligned}$$

From the definition of $F_+^{(m)}$ we have

$$P\big(\,\theta + W_1 > c,\ \max_{t \in B} (\theta t + W_t) < d\,\big) = F_+^{(m)}(d, d; \theta) - F_+^{(m)}(c, d; \theta),$$

for any $\theta \in \mathbb{R}$, which gives the desired representation, namely

$$\begin{aligned} v_{uoc} =& S_0 e^{-qT} \big(F_+^{(m)}(d, d; \theta_1) - F_+^{(m)}(c, d; \theta_1)\big) \\ &- K e^{-rT} \big(F_+^{(m)}(d, d; \theta_0) - F_+^{(m)}(c, d; \theta_0)\big). \end{aligned} \tag{5}$$

In a similar way and by using the symmetry of Brownian motion it can be shown that the value of a discrete down-and-out put is given by

$$\begin{aligned} v_{dop} =& K e^{-rT} \big(F_+^{(m)}(-d, -d; -\theta_0) - F_+^{(m)}(-c, -d; -\theta_0)\big) \\ &- S_0 e^{-qT} \big(F_+^{(m)}(-d, -d; -\theta_1) - F_+^{(m)}(-c, -d; -\theta_1)\big), \end{aligned} \tag{6}$$

provided $K > H$.

The next section discusses an approximation of the function $F_+^{(m)}$.

4 Siegmund's Corrected Heavy Traffic Approximation

The following result is often referred to as *Siegmund's corrected heavy traffic approximation.*

Theorem 3. *Suppose $b > 0$, $a \leq b$ and $\theta \in \mathbb{R}$. If*

$$\lambda_m^\theta(b) = \inf\big\{n \in \mathbb{N};\ \frac{\theta n}{\sqrt{m}} + W_n > b\sqrt{m}\,\big\},$$

then, as $m \to \infty$,

$$\begin{aligned} &P\big(\,\theta\sqrt{m} + W_m < a\sqrt{m},\ \lambda_m^\theta(b) \leq m\,\big) \\ &\qquad = e^{2\theta(b+\beta/\sqrt{m})} N\big(a - 2(b + \beta/\sqrt{m}) - \theta\big) + o\big(\frac{1}{\sqrt{m}}\big), \end{aligned}$$

where β is defined as in Theorem 1.

For a proof of Theorem 3, see [Si2] p.220-224.

First some comments about the constant β. It can be shown that

$$\beta = \lim_{m\to\infty} E\big[W_{\lambda^\theta_m(b)} - b\sqrt{m}\,;\ \lambda^\theta_m(b) < \infty\big]$$

for any $\theta \in \mathbb{R}$ (see [Si2], p.215). Thus, the constant β may be viewed as an approximation to the average of the amount by which the random walk $\{\theta n/\sqrt{m} + W_n\}_{n\in\mathbb{N}}$ exceeds the boundary $b\sqrt{m}$ the first time the random walk is above the boundary.

Our aim is now to find an approximation of $F_+^{(m)}$ based on Theorem 3. Note that the scaling property yields

$$P(\,\theta\sqrt{m} + W_m < a\sqrt{m},\ \lambda^\theta_m(b) \le m\,) = P\big(\,\theta + W_1 < a,\ \max_{t\in B}(\theta t + W_t) > b\,\big),$$

where, as previous, $B = \{1/m,\ 2/m, \dots, 1\}$. Therefore, according to Theorem 3,

$$\begin{aligned} F_+^{(m)}(a,b;\theta) =& P\big(\,\theta + W_1 < a\,\big) \\ & - P\big(\,\theta + W_1 < a,\ \max_{t\in B}(\theta t + W_t) > b\,\big) \\ =& N(a-\theta) \\ & - e^{2\theta(b+\beta/\sqrt{m})} N(a - 2(b+\beta/\sqrt{m}) - \theta) + o(\frac{1}{\sqrt{m}}) \end{aligned}$$

as $m \to \infty$. If we compare this expression of $F_+^{(m)}$ with F_+ in Section 2, then we see that

$$F_+^{(m)}(a,b;\theta) = F_+(a, b+\beta/\sqrt{m};\theta) + o(\frac{1}{\sqrt{m}}), \quad \text{as } m\to\infty. \tag{7}$$

Thus, to calculate the probability of the event

$$\{\,\theta + W_1 \le a,\ \max_{t\in B}(\theta t + W_t) \le b\,\}$$

using the formula for a continuous barrier, one should first lift the barrier $\beta/\sqrt{m}$ units upwards. This compensates for the fact that when the random walk $\{\theta t + W_t,\ t = 1/m, 2/m, \dots\}$ breaches the barrier, it exceeds it on average with $\beta/\sqrt{m}$ units.

Equation (7) in conjunction with the equations (5) and (6) complete the proof of Theorem 2. In the next section we will show some numerical examples.

5 Numerical Examples

Let us first consider the value of discrete up-and-out calls with different barrier levels but with the other parameters fixed. Table 1 shows the value obtained by using different methods. In the first column in the table we have the level of the barrier. The values of the other option parameters are in the caption. The second column contains the value of the corresponding continuous barrier option. In the third column we have used the formula in Theorem 2, with $o(1/\sqrt{m})$ set to zero.

The values in the fourth column are obtained by using a method proposed in [BGK] and [K], that is, lifting the barrier upwards by a factor $\exp(\beta\sigma\sqrt{T/m})$ and then use the formula for the value of a continuous up-and-out call.

In the fifth column we have collected prices obtained by a so called trinomial method presented in [BGK2] (the errors of the trinomial prices are according to the same article approximately ± 0.001). Finally, in the last three columns we have the relative error measured in percentage for the different approximations.

Note the surprisingly great differences in price between the discrete and the corresponding continuous barrier option. So it is worth to emphasise that one should not neglect the fact that some barrier options are discretely and not continuously monitored. We also see that the approximation derived in this chapter yields good results, and that the accuracy of the result is dependent of how close the barrier is to the initial price. This method also performs better than the approximation proposed in [BGK] and [K].

In Table 2 we have varied the number of price fixing dates as well. As is to be expected, the approximation developed in this paper degrade as the number of monitoring times decreases. In the extreme case with the barrier very close to the initial asset price, the method even performs worse than the approximation proposed by [BGK] and [K]. However, one may remark that in the extreme case none of the methods work especially well.

In the final example, presented in Table 3, we have examined how the other parameters influence the accuracy of the approximation.

It is of course not possible to draw any certain conclusions from just numerical examples. But the results presented here indicate that the approximation gives good results for small values of T/m and if the barrier is not too close to the initial asset price.

6 Pricing Discrete Double Barrier Options

The purpose of this section is to determine approximations for the value of discrete double barrier options. The payoff of a discrete double barrier knock-out call/put, with barriers H_1 and H_2, is given by

$$\max\big(\chi(S_T - K), 0\big) 1_{\{\min_{t\in A} S_t > H_1,\ \max_{t\in A} S_t < H_2\}}$$

with χ equal to 1 and -1, respectively. By replacing the event $\{\min_{t\in A} S_t > H_1,\ \max_{t\in A} S_t < H_2\}$ with its complement we get the pay out function of

H	Continuous Barrier (1)	Hö (2)	BGK and K (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
155	12.775	12.891	12.905	12.894	0.9	0.0	0.1
150	12.240	12.426	12.448	12.431	1.5	0.0	0.1
145	11.395	11.676	11.707	11.684	2.5	0.1	0.2
140	10.144	10.541	10.581	10.551	3.9	0.1	0.3
135	8.433	8.947	8.994	8.959	5.9	0.1	0.4
130	6.314	6.909	6.959	6.922	8.8	0.2	0.5
125	4.012	4.605	4.649	4.616	13.0	0.2	0.7
120	1.938	2.410	2.442	2.418	19.8	0.3	1.0
115	0.545	0.803	0.819	0.807	32.5	0.5	1.5
112	0.127	0.257	0.264	0.260	51.1	1.2	1.6

Table 1: Up-and-out call options price results, varying H. The option parameters are $S_0 = 110$, $K = 100$, $\sigma = 0.3$, $r = 0.10$, $q = 0.0$, $T = 0.2$ and $m = 50$. If one assumes that there are 250 trading days per year, then $m = 50$ corresponds to daily monitoring.

m	H	Continuous Barrier (1)	Hö (2)	BGK and K (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
25	130	6.314	7.124	7.221	7.148	11.7	0.3	1.0
	125	4.012	4.829	4.918	4.851	17.3	0.5	1.4
	120	1.938	2.600	2.669	2.616	25.9	0.6	1.9
	115	0.545	0.916	0.950	0.925	41.1	0.9	2.8
	112	0.127	0.320	0.336	0.329	61.4	3.0	2.0
5	130	6.314	7.837	8.286	7.934	20.4	1.2	4.4
	125	4.012	5.622	6.062	5.721	29.9	1.7	5.9
	120	1.938	3.326	3.683	3.409	43.1	2.5	8.0
	115	0.545	1.404	1.624	1.481	63.2	5.2	9.6
	112	0.127	0.622	0.751	0.708	82.1	12.3	6.0

Table 2: Up-and-out call options price results, varying H and m. The option parameters are $S_0 = 110$, $K = 100$, $\sigma = 0.3$, $r = 0.10$, $q = 0.0$ and $T = 0.2$.

Panel	H	Continuous Barrier (1)	Hö (2)	BGK and K (3)	Trinomial Method	Relative error (in percent) (1)	(2)	(3)
A	155	6.798	7.270	7.290	7.274	6.6	0.1	0.2
	140	2.916	3.251	3.265	3.254	10.4	0.1	0.3
	125	0.566	0.693	0.699	0.695	18.6	0.2	0.6
B	140	3.766	4.516	4.578	4.531	16.9	0.3	1.0
	130	1.576	2.086	2.130	2.097	24.9	0.5	1.6
	120	0.331	0.541	0.561	0.546	39.4	0.9	2.8
C	140	7.171	8.277	8.354	8.296	13.6	0.2	0.7
	130	3.653	4.550	4.608	4.565	20.0	0.3	0.9
	120	1.110	1.629	1.659	1.637	32.2	0.5	1.4

Table 3: Up-and-out call options price results, varying K, σ and T. The option parameters are $S_0 = 110$, $r = 0.1$ and $q = 0.0$ for all panels. Panel A has $K = 100$, $\sigma = 0.3$, $T = 1$ and $m = 250$ (daily monitoring). Panel B has $K = 100$, $\sigma = 0.6$, $T = 0.2$ and $m = 50$ (daily monitoring). Panel C has $K = 90$, $\sigma = 0.6$, $T = 0.2$ and $m = 50$ (daily monitoring)

a discrete double barrier knock-*in* call/put.

Let the stopping time $\tau_m^\theta(b_1, b_2)$ and the constants c, d_1, d_2, θ_0, and θ_1 be defined as in Theorem 2 and Lemma 1. Define moreover the function

$$G^{(m)}(a_1, a_2, b_1, b_2; \theta) = P\big(\, a_1 < \theta + W_1 \leq a_2,\ \tau_m^\theta(b_1, b_2) > 1\big)$$

for $b_1 < 0 < b_2$ and $b_1 \leq a_1 \leq a_2 \leq b_2$.

As a direct consequence of Lemma 1 we now get that if $K < H_2$ then the theoretical value v_{koc} at time $t = 0$ of a discrete double-barrier knock-out call equals

$$\begin{aligned} v_{koc} =& S_0 e^{-qT} G^{(m)}(\max(c, d_1), d_2, d_1, d_2; \theta_1) \\ & - K e^{-rT} G^{(m)}(\max(c, d_1), d_2, d_1, d_2; \theta_0)\,. \end{aligned}$$

Similary, if $K > H_1$ then the theoretical value v_{kop} of a discrete double barrier knock-out put at time $t = 0$ is given by

$$\begin{aligned} v_{kop} =& K e^{-rT} G^{(m)}(d_1, \min(c, d_2), d_1, d_2; \theta_0) \\ & - S_0 e^{-qT} G^{(m)}(d_1, \min(c, d_2), d_1, d_2; \theta_1)\,. \end{aligned}$$

The price of a double barrier *knock-in* option can be computed using a similar argument as in Section 2. Thus, in order to estimate the price of a discrete double barrier option we need to find an approximation of the function $G^{(m)}$.

If we let G be the continuous analogue to $G^{(m)}$, i.e. for given $b_1 < 0 < b_2$ and $b_1 \leq a_1 \leq a_2 \leq b_2$ set

$$G(a_1, a_2, b_1, b_2; \theta) = P\big(\, a_1 < \theta + W_1 \leq a_2,\ \tau^\theta(b_1, b_2) > 1\,\big)$$

with

$$\tau^\theta(b_1, b_2) = \inf\{\, t \geq 0\,;\ (\theta t + W_t) \notin (b_1, b_2)\,\}.$$

Then Siegmund suggests, see [Si] p.716, that one approximate the function $G^{(m)}$ by computing the continuous analogue G with the lower barrier replaced $\beta/\sqrt{m}$ units downwards and the upper barrier moved $\beta/\sqrt{m}$ units upwards (cf. equation (7)). In other words, Siegmund suggests the following approximation

$$G^{(m)}(a_1, a_2, b_1, b_2; \theta) \approx G(a_1, a_2, b_1 - \beta/\sqrt{m}, b_2 + \beta/\sqrt{m}; \theta).$$

However, in this case there are no estimates of the approximation error but numerical examples presented in [Si] indicate that the approximation yields good results.

Even for G the exact formulas are complicated. The function G may be written as

$$\begin{aligned} G(a_1, a_2, b_1, b_2; \theta) = {} & N(a_2 - \theta) - N(a_1 - \theta) \\ & - G_+(a_2, b_1, b_2; \theta) + G_+(a_1, b_1, b_2; \theta) \\ & - G_-(a_2, b_1, b_2; \theta) + G_-(a_1, b_1, b_2; \theta), \end{aligned} \tag{8}$$

where $G_-(a, b_1, b_2; \theta) = G_+(-a, -b_1, -b_2; -\theta)$ and

$$G_+(a, b_1, b_2; \theta) = \sum_{i=1}^{\infty} \big(e^{2\alpha_1^{(i)}\theta} N(a - 2\alpha_1^{(i)} - \theta) - e^{2\alpha_2^{(i)}\theta} N(a - 2\alpha_2^{(i)} - \theta) \big)$$

with $\alpha_1^{(i)} = i(b_2 - b_1) + b_1$ and $\alpha_2^{(i)} = i(b_2 - b_1)$, see [H].

As we can see, G_+ is expressed as an infinite series. Fortunately, the convergence rate for this series is exponential, see [H]. In most cases only the first leading terms are necessary to obtain an accuracy which is sufficient for our purposes. For further details concerning the truncation error, see [H] and [L].

There are other ways to represent the function G, for instance as Fourier series, see [IM] p.31. For a further discussion concerning various representations and corresponding numerical properties of the function G, see [DL], [H], [HLY] or [Sc].

To see how Siegmund's approximation performs, we will now present some numerical examples.

7 Numerical examples

Table 4 and 5 give the price of a discrete double barrier knock-out call. The prices in the column called "Trinomial method" are determined from a trinomial method developed in [H2]. This numerical procedure is based on certain results from the theory of Besov spaces. These results imply that the convergence rate of the trinomial method is dependent on the smoothness of the payoff function. By iteratively applying a certain smoothing operator to the value function at each monitoring date, one obtains an efficient numerical method to price discrete barrier options. For further details, see [H2]. The errors for the prices in "Trinomial method" are approximately ± 0.001.

If we consider the prices in Table 4 and 5 we see that the accuracy in the pricing formulas for double barrier options seems to be slightly worse than in the single barrier case. However, still the approximation gives good results for small values of T/m and if the barriers are not too close to the initial asset price.

8 Conclusions

We have addressed the problem of pricing discrete barrier options. We have continued the work of Broadie et al. in [BGK] and derived approximation formulas for discrete up-and-out/in calls, down-and-out/in puts and double barrier options. Numerical examples presented in this paper show that the accuracy of the approximation is good, especially when the underlying asset is monitored against the barrier(s) with short time intervals and when the initial asset price is not too close to (any of) the barrier(s).

H_1	H_2	Continuous Barrier (1)	Hö (2)	Trinomial Method (3)	Relative error (in percent) (1)	(2)
70	130	4.5651	4.7784	4.7842	4.6	0.1
75	125	3.5614	3.8375	3.8446	7.3	0.2
80	120	2.3499	2.6524	2.6601	11.7	0.3
85	115	1.1408	1.4055	1.4120	19.2	0.5
90	110	0.2284	0.3791	0.3826	40.3	0.9
75	110	0.3423	0.4799	0.4841	29.3	0.8
90	125	3.2292	3.6074	3.6143	10.7	0.2

Table 4: Double barrier knock-out calls price results, varying H_1 and H_2. The option parameters are $S_0 = 100,\ K = 100,\ \sigma = 0.3,\ r = 0.10,\ q = 0.0,\ T = 0.2$ and $m = 50$ (daily monitoring).

m	H_1	H_2	Continuous Barrier (1)	Hö (2)	Trinomial Method (3)	Relative error (in percent) (1)	(2)
25	80	120	2.3499	2.7606	2.7752	15.3	0.5
	85	115	1.1408	1.5052	1.5180	24.9	0.9
	90	110	0.2284	0.4441	0.4514	49.3	1.6
	75	110	0.3423	0.5445	0.5362	37.1	1.5
	90	125	3.2292	3.7363	3.7491	13.9	0.3
5	80	120	2.3499	3.1157	3.1726	25.9	1.8
	85	115	1.1408	1.8563	1.9115	40.3	2.9
	90	110	0.2284	0.7035	0.7401	69.1	4.9
	75	110	0.3423	0.7570	0.7962	57.0	4.9
	90	125	3.2292	4.1294	4.1724	22.6	1.0

Table 5: Double barrier knock-out calls price results, varying H_1, H_2 and m. The option parameters are $S_0 = 100,\ K = 100,\ \sigma = 0.3,\ r = 0.10,\ q = 0.0$ and $T = 0.2$.

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