A NOTE ON ASYMPTOTIC BEHAVIOUR OF EXTINCTION PROBABILITY IN SUPERCRITICAL POPULATION-SIZE-DEPENDENT BRANCHING PROCESS WITH INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM ENVIRONMENTS

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ABSTRACT. In supercritical population-size-dependent branching processes with independent and identically distributed random environments, it is shown that under certain regularity conditions there exist parameters $0 < \alpha_1 \leq \alpha_0 < +\infty$ such that the extinction probability starting with k individuals, q_k, has lower and upper bounds $C_1 k^{-\alpha_0}$ and $C_2 k^{-\alpha_1}$ for sufficiently large k, where $0 < C_1, C_2 < +\infty$ are constants. Therefore, the asymptotic behaviour of q_k is similar to $k^{-\alpha}$ for some α as $k \to \infty$.

1. INTRODUCTION

Suppose $\bar{Z} = \{Z_n\}$ is a population-size-dependent branching process with random environments $\bar{\zeta} = \{\zeta_n\}$ (see [1],[2]). That is, $\{Z_n\}$ defines a temporally non-homogeneous branching process, the evolution of which is depicted as follows. At $n = 0$, there are Z_0 initial individuals, which make up the 0th generation, then given the population number Z_n of the nth generation, the $(n+1)$th population number Z_{n+1} is composed by the number of progeny of Z_n individuals of nth generation each reproducing independently according to the pgf (probability generating function) $\phi_{Z_n,\zeta_n}(s) = \sum_{i=0}^{\infty} p_{Z_n,\zeta_n}(i)s^i$, $(s \in [0,1])$, and so forth. By mathematical terminology, the structure of the model can be delineated below. Let $(\Omega, \mathcal{F}, P)$ be a given probability space, and (Θ, Σ) be a measurable space. Let $\{\{\phi_{k,\theta}(s) = \sum_{i=0}^{\infty} p_{k,\theta}(i)s^i, k = 0,1,2,\cdots\}, \theta \in \Theta\}$, $(s \in [0,1])$ be a family of sequences of pgf for non-negative integer value random variables, satisfying the non-trivial conditions of

$$\sum_{i=0}^{\infty} i p_{k,\theta}(i) < \infty; \quad p_{0,\theta}(0) = 1, \quad p_{0,\theta}(k) = 0;$$

$$0 < p_{k,\theta}(0) + p_{k,\theta}(1) < 1, \quad \text{for all} \quad k = 1,2,\cdots \text{and} \quad \theta \in \Theta.$$

The random environments $\bar{\zeta} = \{\zeta_n\}$ can be regarded as a sequence of random mappings from $(\Omega, \mathcal{F}, P)$ to (Θ, Σ). With any $\zeta \in \Theta$ and any $k \geq 0$ associate the

1991 *Mathematics Subject Classification.* 60J80, 60J10.

Key words and phrases. population-size-dependent branching process; branching process in random environments; population-size-dependent branching process in random environments.

Lu Zhunwei thanks China Scholarship Council very much for her support, also thanks Dr. D.R. Grey very much for his beneficial guidance.

pgf

$$\phi_{k,\zeta}(s) = \sum_{i=0}^{\infty} p_{k,\zeta}(i)s^i, \quad (s \in [0,1]).$$

Without loss of generality, we can assume that $Z_0 = k$.
Write $\mathcal{F}_{n,Z}(\bar{\zeta}) = \sigma(Z_0, \cdots, Z_n, \zeta_0, \cdots, \zeta_n)$ and $\mathcal{F}(\bar{\zeta}) = \sigma(\zeta_0, \cdots, \zeta_n, \cdots)$. Then we have

$$\mathbb{E}[s^{Z_{n+1}}|\mathcal{F}_{n,Z}(\bar{\zeta})] = [\phi_{Z_n,\zeta_n}(s)]^{Z_n} \text{ and } \mathbb{E}[s^{Z_{n+1}}] = \mathbb{E}[\phi_{Z_n,\zeta_n}(s)]^{Z_n}, \quad (n \geq 0). \tag{1.1}$$

Denote the extinction probability of $\{Z_n\}$ conditionally upon environments and unconditionally respectively by

$$q_k(\bar{\zeta}) := P\{Z_n = 0, \text{for some } n|\mathcal{F}(\bar{\zeta}), Z_0 = k\};$$

$$q_k := P\{Z_n = 0, \text{for some } n|Z_0 = k\}.$$

We know that

$$q_k = \mathbb{E}[q_k(\bar{\zeta})] = \mathbb{E}[q(\bar{\zeta})]^k,$$

where we write $q(\bar{\zeta}) = \sqrt[k]{q_k(\bar{\zeta})}$, not necessarily independent of k.
The random environment $\bar{\zeta} = \{\zeta_n\}$ is often assumed to form a stationary ergodic process. In particularly, the random envrionment may be a sequence of iid (independent identically distributed) random variables. In this paper, we consider only the iid environment model, although subseqent work may generalize the results to certain types of stationary ergodic envrionments.
The most interesting case is the so-called *supercritical* case, in which $q_k < 1$, for all $k \geq 1$. We know that sufficient conditions for supercriticality are

$$\text{(i)} \quad 0 < \mathbb{E}[\log \inf_k \phi'_{k,\zeta_n}(1)] < +\infty; \quad \text{(ii)} \quad \mathbb{E}[-\log(1 - \sup_k \phi_{k,\zeta_n}(0))] < +\infty$$

from [1] and we shall adopt these conditions. In this paper we will show that under some certain regularity conditions, as $k \to \infty$, the asymptotic behaviour of q_k is similar to $k^{-\alpha}$ for some $\alpha > 0$.

2. Conditions, results and proofs

For each $\zeta \in \Theta$, we write

$$f_\zeta(s) := \inf_k \phi_{k,\zeta}(s), \quad F_\zeta(s) := \sup_k \phi_{k,\zeta}(s), \quad (s \in [0,1]). \tag{2.1}$$

Clearly, a thus defined $f_\zeta(s)$ or $F_\zeta(s)$ need no longer be a pgf, but each of them still is an increasing function of s. Noting that the sequences $\{f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_n}(s))\cdots))\}$ and $\{F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_n}(s))\cdots))\}$, $n = 0, 1, 2, \cdots$ both are non-decreasing for fixed $s = 0$, we can see that

$$r_{\bar{\zeta}}(0) := \lim_{n\to\infty} f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_n}(0))\cdots)) \text{ and } R_{\bar{\zeta}}(0) := \lim_{n\to\infty} F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_n}(0))\cdots))$$

are well defined.

Theorem 1. $r_{\bar{\zeta}}(0) \leq q(\bar{\zeta}) = \sqrt[k]{q_k(\bar{\zeta})} \leq R_{\bar{\zeta}}(0)$.

Proof. By (2.1), the definitions of f_ζ and F_ζ, using (1.1) with $n=0$, we have

$$[f_{\zeta_0}(s)]^k \le \mathbb{E}[s^{Z_1}|\mathcal{F}(\bar\zeta), Z_0=k] = [\phi_{k,\zeta_0}(s)]^k \le [F_{\zeta_0}(s)]^k.$$

Now we suppose that

$$[f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_{n-1}}(s))\cdots))]^k \le \mathbb{E}[s^{Z_n}|\mathcal{F}(\bar\zeta), Z_0=k] \le [F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_{n-1}}(s))\cdots))]^k.$$

Noting that

$$\begin{aligned} &\mathbb{E}[s^{Z_{n+1}}|\mathcal{F}(\bar\zeta), Z_0=k] = \mathbb{E}[\mathbb{E}[[s^{Z_{n+1}}|\mathcal{F}(\bar\zeta), Z_0=k]|Z_n]] \\ = \quad &\mathbb{E}[[\phi_{Z_n,\zeta_n}(s)]^{Z_n}|\mathcal{F}(\bar\zeta), Z_0=k] \ge \mathbb{E}[[f_{\zeta_n}(s)]^{Z_n}|\mathcal{F}(\bar\zeta), Z_0=k] \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}[s^{Z_{n+1}}|\mathcal{F}(\bar\zeta), Z_0=k] = \mathbb{E}[\mathbb{E}[[s^{Z_{n+1}}|\mathcal{F}(\bar\zeta), Z_0=k]|Z_n]] \\ = \quad &\mathbb{E}[[\phi_{Z_n,\zeta_n}(s)]^{Z_n}|\mathcal{F}(\bar\zeta), Z_0=k] \le \mathbb{E}[[F_{\zeta_n}(s)]^{Z_n}|\mathcal{F}(\bar\zeta), Z_0=k], \end{aligned}$$

Then by the inductive assumption we get

$$[f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_n}(s))\cdots))]^k \le \mathbb{E}[s^{Z_{n+1}}|\mathcal{F}(\bar\zeta), Z_0=k] \le [F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_n}(s))\cdots))]^k,$$

for all $s \in [0,1]$. Therefore, by induction, we have shown that the formula above holds for all $n \ge 0$. Furthermore, since

$$q_k(\bar\zeta) = \lim_{n\to\infty} \mathbb{E}[s^{Z_n}|\mathcal{F}(\bar\zeta), Z_0=k]_{s=0},$$

it follows that

$$\lim_{n\to\infty}[f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_n}(0))\cdots))]^k \le q_k(\bar\zeta) \le \lim_{n\to\infty}[F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_n}(0))\cdots))]^k,$$

which is equivalent to

$$[\lim_{n\to\infty} f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_n}(0))\cdots))]^k \le q_k(\bar\zeta) \le [\lim_{n\to\infty} F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_n}(0))\cdots))]^k,$$

or

$$r_{\bar\zeta}(0)^k \le q_k(\bar\zeta) \le R_{\bar\zeta}(0)^k.$$

Taking kth root for each sides of the inequalities above, we obtain

$$r_{\bar\zeta}(0) \le q(\bar\zeta) = \sqrt[k]{q_k(\bar\zeta)} \le R_{\bar\zeta}(0)$$

and the proof is complete. □

Denote $\bar\xi_n := -\log\sup_k \phi'_{k,\zeta_n}(1)$ and $G(\alpha) := \mathbb{E}[\exp(\alpha\bar\xi_n)]$. We know that $\{\bar\xi_n\}$ is an iid random variables sequence from the definition of $\bar\zeta$. By the supercriticality assumption (i) above, we have

$$G'(0) = \mathbb{E}[\bar\xi_n] = \mathbb{E}[-\log\sup_k \phi'_{k,\zeta_n}(1)] \le \mathbb{E}[-\log\inf_k \phi'_{k,\zeta_n}(1)] < 0,$$

and since G is continuous and strictly convex there may exist $\alpha_0 > 0$ such that $G(\alpha_0) = 1$, $G'(\alpha_0) < +\infty$. If α_0 exists, it is unique. Under this condition and following the ideas of Grey and Lu(see[3], [4]), we have the results below.

Theorem 2. *Let $\alpha_0 > 0$ exist such that $G(\alpha_0) = 1$, $G'(\alpha_0) < +\infty$, then there exists a constant $c_1 > 0$ such that $P\{q(\bar\zeta) < x\} \le 1 - c_1(1-x)^{\alpha_0}$ for all $x \in [0,1]$.*

Proof. Write $X_0 = 0$, $X_{n+1} = f_{\zeta_n}(X_n)$, $(n \geq 0)$. We have

$$X_{n+1} = f_{\zeta_n}(f_{\zeta_{n-1}}(\cdots(f_{\zeta_0}(0))\cdots)) = f_{\zeta_0}(f_{\zeta_1}(\cdots(f_{\zeta_n}(0))\cdots)),$$

since $\bar{\zeta}$ is a sequence of iid random variables. Therefore, $\lim_{n\to\infty} X_n = r_{\bar{\zeta}}(0)$. Make the transformations $Y = -\log(1 - r_{\bar{\zeta}}(0))$ and $Y_n = -\log(1 - X_n)$ for all $n \geq 0$. Then we know that Y is the limiting random variable of $\{Y_n\}$ and

$$Y_{n+1} = -\log(1 - f_{\zeta_n}(X_n)) = -\log(1 - X_n) + \left(-\log \frac{1 - f_{\zeta_n}(X_n)}{1 - X_n}\right).$$

Since

$$\frac{1 - f_{\zeta_n}(X_n)}{1 - X_n} = \sup_k \frac{1 - \phi_{k,\zeta_n}(X_n)}{1 - X_n} \leq \sup_k \phi'_{k,\zeta_n}(1),$$

we get

$$Y_{n+1} \geq Y_n + \bar{\xi_n}.$$

Noting that $Y_n \geq 0$, we know that $Y_n \geq \max\{0, Y_n + \bar{\xi_n}\}$. Define $W_0 = 0$ and $W_{n+1} = \max\{0, W_n + \bar{\xi_n}\}$ for $n \geq 0$. Then $\{W_n\}$ is a random walk with left reflecting barrier at 0 and jumps $\{\bar{\xi_n}\}$. Since

$$\mathbb{E}[\bar{\xi}_n] = \mathbb{E}[-\log \sup_k \phi'_{k,\zeta_n}(1)] \leq \mathbb{E}[-\log \inf_k \phi'_{k,\zeta_n}(1)] < 0,$$

by our supercritical assumption (i) together with a result on random walk with one reflecting barrier (see [5]), we can infer that there exists W, which is a random variable with the limting/equilibrium distribution of W_n. It is easy to prove by induction on n that $Y_n \geq W_n$ for all $n \geq 0$ and thence that

$$P\{Y < y\} \leq P\{W < y\}, \text{ for all } y \geq 0.$$

By the familiar duality argument

$$\begin{aligned} W_n &= \max\{0, \bar{\xi}_{n-1}, \bar{\xi}_{n-1} + \bar{\xi}_{n-2}, \cdots, \bar{\xi}_{n-1} + \bar{\xi}_{n-2} + \cdots + \bar{\xi}_0\} \\ &= -\min\{0, -\bar{\xi}_{n-1}, (-\bar{\xi}_{n-1} - \bar{\xi}_{n-2}), \cdots, (-\bar{\xi}_{n-1} - \bar{\xi}_{n-2} - \cdots - \bar{\xi}_0)\}, \end{aligned}$$

in distribution, we know that $-W$ has the same distribution as the all-time M of an unrestricted random walk with jumps $\{-\bar{\xi_n}\}$. Then by the assumption of the existence of α_0 and the result of Feller (see [6]), we know that

$$P\{M < -t\} \sim ce^{-\alpha_0 t}, \text{ for some constant } c > 0, \text{ as } t \to \infty.$$

Hence $P\{Y < y\} \leq P\{W < y\} \leq 1 - c_1 e^{-\alpha_0 y}$ for all $y \geq 0$ for some constant $c_1 > 0$. In terms of $r_{\bar{\zeta}}(0)$, this becomes

$$P\{r_{\bar{\zeta}}(0) < x\} \leq 1 - c_1(1 - x)^{\alpha_0}, \text{ for all } x \in [0, 1].$$

Note that $q(\bar{\zeta}) \geq r_{\bar{\zeta}}(0)$. Therefore

$$P\{q(\bar{\zeta}) < x\} \leq P\{r_{\bar{\zeta}}(0) < x\} \leq 1 - c_1(1 - x)^{\alpha_0}, \text{ for all } x \in [0, 1].$$

□

Now, we denote $u_0 = -\log(1 - s_0)$, $\hat{\xi}_n = -\log \frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}$, $H(\alpha) = \mathbb{E}[\exp(\alpha \hat{\xi}_n]$, where $s_0 \in [0, 1)$ and its value will be chosen later.

Theorem 3. *If when s_0 is sufficiently close to* 1 *there exists $\alpha = \alpha(s_0) > 0$ such that $H(\alpha) = 1$, $H'(\alpha) < +\infty$, then there exists a constant $c_2 = c_2(s_0) > 0$ such that*

$$P\{q(\bar{\zeta}) < x\} \geq 1 - c_2(1-x)^\alpha, \text{ for all } x \in [0,1].$$

Proof. Write $X'_0 = 0$, $X'_{n+1} = F_{\zeta_n}(X'_n)$, $(n \geq 0)$. We have

$$X'_{n+1} = F_{\zeta_n}(F_{\zeta_{n-1}}(\cdots(F_{\zeta_0}(0))\cdots)) = F_{\zeta_0}(F_{\zeta_1}(\cdots(F_{\zeta_n}(0))\cdots)),$$

also since $\bar{\zeta}$ is a sequence of iid random variables. Therefore, $\lim_{n\to\infty} X'_n = R_{\bar{\zeta}}(0) \geq q(\bar{\zeta})$. Make the transformations $Y' = -\log(1 - R_{\bar{\zeta}}(0))$ and $Y'_n = -\log(1 - X'_n)$ for all $n \geq 0$. Then we know that Y' is the limiting random variable of $\{Y'_n\}$. Denote $U_0 = u_0$, $U_{n+1} = \max\{u_0, U_n + \hat{\xi}_n\}$ for $n \geq 0$, so that $\{U_n\}$ is a random walk with left reflecting barrier at u_0 and with jumps $\{\hat{\xi}_n\}$. Now we show that $Y'_n \leq U_n$ for all $n \geq 0$. Obviously, $Y'_0 = 0 \leq u_0 = U_0$. If $Y'_n \leq U_n$ then either $Y'_n \leq u_0$ in which case

$$\begin{aligned} Y'_{n+1} &= -\log(1 - F_{\zeta_n}(X'_n)) \leq -\log(1 - F_{\zeta_n}(s_0)) \text{ (since } X'_n \leq s_0) \\ &= u_0 + \left(-\log\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right) \leq U_n + \hat{\xi}_n \leq U_{n+1} \end{aligned}$$

or alternatively $u_0 \leq Y'_n \leq U_n$ in which case

$$\begin{aligned} Y'_{n+1} &= -\log(1 - F_{\zeta_n}(X'_n)) = -\log(1 - X'_n) + \left(-\log\frac{1 - F_{\zeta_n}(X'_n)}{1 - X'_n}\right) \\ &= Y'_n + \left(-\log\frac{1 - F_{\zeta_n}(X'_n)}{1 - X'_n}\right) \leq Y'_n + \left(-\log\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right) \\ &\leq U_n + \hat{\xi}_n \leq U_{n+1}, \end{aligned}$$

where the first inequality holds since

$$\frac{1 - F_{\zeta_n}(X'_n)}{1 - X'_n} = \inf_k \frac{1 - \phi_{k,\zeta_n}(X'_n)}{1 - X'_n} \geq \inf_k \frac{1 - \phi_{k,\zeta_n}(s_0)}{1 - s_0} = \frac{1 - F_{\zeta_n}(s_0)}{1 - s_0},$$

as $X'_n \geq s_0$. Now let us show that $\mathbb{E}[\hat{\xi}_n] < 0$ for all s_0 sufficiently close to 1, which obviously implies that if, for each s_0 chosen sufficiently close to 1, the corresponding $\alpha = \alpha(s_0)$ can be found such that $\{U_n\}$ has an equilibrium distribution random variable, say U. In fact, by the supercriticality assumption (ii) and $\frac{1-\phi_{k,\zeta_n}(s_0)}{1-s_0} \geq 1 - \phi_{k,\zeta_n}(0)$, we have

$$\begin{aligned} \mathbb{E}[\hat{\xi}_n] &= \mathbb{E}\left[-\log\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right] = \mathbb{E}\left[-\log\inf_k \frac{1 - \phi_{k,\zeta_n}(s_0)}{1 - s_0}\right] \\ &\leq \mathbb{E}[-\log\inf_k(1 - \phi_{k,\zeta_n}(0))] = \mathbb{E}[-\log(1 - \sup_k \phi_{k,\zeta_n}(0))] < +\infty. \end{aligned}$$

Moreover, when $s_0 \uparrow 1$, $-\log\inf_k \frac{1-\phi_{k,\zeta_n}(s_0)}{1-s_0} \downarrow -\log\inf_k \phi'_{k,\zeta_n}(1)$, hence, by dominated convergence theorem and the supercriticality assumption (i), we have

$$\mathbb{E}\left[-\log\inf_k \frac{1 - \phi_{k,\zeta_n}(s_0)}{1 - s_0}\right] \downarrow \mathbb{E}[-\log\inf_k \phi'_{k,\zeta_n}(1)] = -\mathbb{E}[\log\inf_k \phi'_{k,\zeta_n}(1)] < 0,$$

Hence, s_0 can be chosen sufficiently close to 1 so that $\mathbb{E}[\hat{\xi}_n] < 0$. The other part of the proof follows using similar arguments as in the proof of Theorem 2 but with the inequalities reversed and taking place Y, W, $r_{\bar{\zeta}}(0)$ by Y', U, $R_{\bar{\zeta}}(0)$ respectively. □

Remark. In Theorem 3 we write $\alpha = \alpha(s_0)$, $c_2 = c_2(s_0)$ by the reason of that u_0 and $\hat{\xi}_n$ are dependent on the choice value of s_0 in their definitions.

Theorem 4. *If* $\mathbb{E}[1 - \sup_k \phi_{k,\zeta_n}(0)]^{-\alpha_0} < +\infty$, *then* $\lim_{s_0 \uparrow 1} \mathbb{E}\left[\frac{1-F_{\zeta_n}(s_0)}{1-s_0}\right]^{-\alpha_0} \geq 1$, *where* α_0 *was defined in Theorem 2 and the convergence is monotone decreasing.*

Proof. Since $\alpha_0 > 0$ and

$$\begin{aligned} \frac{1 - F_{\zeta_n}(s_0)}{1 - s_0} &= \inf_k \frac{1 - \phi_{k,\zeta_n}(s_0)}{1 - s_0} \geq \inf_k (1 - \phi_{k,\zeta_n}(0)) \\ &\geq 1 - \sup_k \phi_{k,\zeta_n}(0), \end{aligned}$$

then

$$\mathbb{E}\left[\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right]^{-\alpha_0} \leq \mathbb{E}[1 - \sup_k \phi_{k,\zeta_n}(0)]^{-\alpha_0} < +\infty.$$

Noting that

$$\left(\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right)^{-\alpha_0} = \left(\inf_k \frac{1 - \phi_{k,\zeta_n}(s_0)}{1 - s_0}\right)^{-\alpha_0} \leq (\inf_k \phi'_{k,\zeta_n}(1))^{-\alpha_0},$$

and

$$\left(\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right)^{-\alpha_0} \downarrow (\inf_k \phi'_{k,\zeta_n}(1))^{-\alpha_0}, \text{ as } s_0 \uparrow 1,$$

and the definition of α_0, by dominated convergence theorem, we know that

$$\lim_{s_0 \uparrow 1} \mathbb{E}\left[\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right]^{-\alpha_0} = \mathbb{E}[\inf_k \phi'_{k,\zeta_n}(1)]^{-\alpha_0} \geq \mathbb{E}[\sup_k \phi'_{k,\zeta_n}(1)]^{-\alpha_0} = G(\alpha_0) = 1,$$

where the convergence is monotone decreasing. □

Theorem 5. *Under the conditions of Theorem 2 and Theorem 4, for any* $s_0 \in [0, 1)$ *sufficiently close to* 1, *there always exists* $\alpha = \alpha(s_0)$, *which is unique for* s_0 *and satisfies the condition of Theorem 3, and* $\lim_{s_0 \uparrow 1} \alpha(s_0) = \alpha_1$ *exists, where the convergence is monotone increasing and* $\alpha_1 \leq \alpha_0$.

Proof. When s_0 is fixed, $H(\alpha) = \mathbb{E}[\exp(\alpha \hat{\xi}_n(s_0))]$ is continuous and strictly convex with respect to α, where we write $\hat{\xi}_n$ by $\hat{\xi}_n(s_0)$ for emphasizing that it depends on the value of s_0. Under the condition of Theorem 4, for every $s_0 \in [0, 1)$, we always have $H(0) = \mathbb{E}[1] = 1$ and

$$1 < \mathbb{E}\left[\frac{1 - F_{\zeta_n}(s_0)}{1 - s_0}\right]^{-\alpha_0} < +\infty, \text{ i.e. } 1 < H(\alpha_0) < +\infty.$$

By the supercriticality assumption (ii), noting that $H'(\alpha) = \mathbb{E}[\hat{\xi}_n(s_0) \exp(\alpha \hat{\xi}_n(s_0))]$, we have $H'(0) = \mathbb{E}[\hat{\xi}_n(s_0)] < +\infty$. Since in the proof of Theorem 3, we have shown that $\lim_{s_0 \uparrow 1} \mathbb{E}[\hat{\xi}_n(s_0)] = \mathbb{E}[-\log \inf_k \phi'_{k,\zeta_n}(1)] < 0$, and where the convergence is monotone decreasing, we have $\lim_{s_0 \uparrow 1} H'(0) < 0$ and the convergence is monotone decreasing too. This implies that when $s_0 < 1$ and is sufficiently close to 1, $H(\alpha)$ decreases at $\alpha = 0$. Hence there exists a constant $\delta > 0$ such that $H(\delta) < H(0) = 1$. As $H(\alpha)$ is continuous and strictly convex, by the intermediate value theorem, there must exist unique $\alpha = \alpha(s_0) \in (0, \alpha_0)$, such that $H(\alpha) = 1$, when the fixed value s_0

is sufficiently close to 1. Furthermore, since $H(\alpha)$ is strictly convex, $H(\alpha_0) < +\infty$ and s_0 is sufficiently close to 1, we have

$$H'(\alpha) \leq \frac{H(\alpha_0) - H(\alpha)}{\alpha_0 - \alpha} < +\infty, \text{ for all } \alpha = \alpha(s_0).$$

Now, let $s_0' < s_0''$ be two points sufficiently close to 1. Then

$$\begin{aligned} \hat{\xi}_n(s_0') &= -\log \frac{1 - F_{\zeta_n}(s_0')}{1 - s_0'} = -\log \inf_k \frac{1 - \phi_{k,\zeta_n}(s_0')}{1 - s_0'} \\ &> -\log \inf_k \frac{1 - \phi_{k,\zeta_n}(s_0'')}{1 - s_0''} = -\log \frac{1 - F_{\zeta_n}(s_0'')}{1 - s_0''} = \hat{\xi}_n(s_0''). \end{aligned}$$

Hence $\mathbb{E}[\exp(\alpha(s_0')\hat{\xi}_n(s_0''))] < \mathbb{E}[\exp(\alpha(s_0')\hat{\xi}_n(s_0'))] = 1$. By the strict convexity of $H(\alpha)$ corresponding to s_0'', we know that $\alpha(s_0'') > \alpha(s_0')$. Therefore, $\alpha(s_0)$ is a strict increasing function for s_0 and bounded by α_0. This implies that $\lim_{s_0 \uparrow 1} \alpha(s_0) = \alpha_1$ exists, where the convergence is monotone increasing and $\alpha_1 \leq \alpha_0$. □

Remark. 1. Under the conditions of Theorem 2 and Theorem 3, we have

$$1 - c_2(s_0)(1-x)^{\alpha(s_0)} \leq P\{q(\bar{\zeta}) < x\} \leq 1 - c_1(1-x)^{\alpha_0},$$

for all $x \in [0, 1]$.
2. The condition of Theorem 4 is stronger than the supercriticality assumption (ii).
3. Theorem 5 implies that when s_0 is chosen sufficiently close to 1, the condition of Theorem 3 can be replaced by the conditions of Theorem 2 and Theorem 4, i.e. the codition of Theorem 5 is stronger than the condition of Theorem 3.
4. From Theorem 5, we know that

$$\alpha_1 = \max\{\alpha(s_0) | \alpha(s_0) \text{ satisfies the condition of Theorem 3}\},$$

and then we get a more precise lower boundary of the distribution function of $q(\bar{\zeta})$, which is no longer dependent on s_0, i.e.

$$1 - c_2(1-x)^{\alpha_1} \leq P\{q(\bar{\zeta}) < x\},$$

for all $x \in [0, 1]$.

Theorem 6. *Under the condition of Theorem 5, there exist constants* $0 < \alpha_1 \leq \alpha_0 < +\infty$ *and* $0 < C_1, C_2 < +\infty$ *such that*

$$C_1 k^{-\alpha_0} \leq q_k \leq C_2 k^{-\alpha_1}, \text{ for all large enough } k.$$

Proof. From Theorem 5 we have

$$1 - c_2(1-x)^{\alpha_1} \leq P\{q(\bar{\zeta}) < x\} \leq 1 - c_1(1-x)^{\alpha_0}, \text{ for all } x \in [0, 1]. \tag{2.2}$$

By Feller(1971)V.6(p.150)Lemma 1(see [5]), We know that

$$\begin{aligned} q_k &= \mathbb{E}[q(\bar{\zeta})]^k = \int_0^1 x^k dP\{q(\bar{\zeta}) < x\} \\ &= k \int_0^1 x^{k-1} P\{q(\bar{\zeta}) \geq x\} dx. \end{aligned}$$

Using the right inequality of (2.3), we have

$$q_k \geq c_1 k \int_0^1 x^{k-1}(1-x)^{\alpha_0} dx = c_1 k B(k, \alpha_0 + 1).$$

Noting that

$$B(k, \alpha_0 + 1) \sim \frac{\Gamma(\alpha_0 + 1)}{k^{\alpha_0+1}}, \text{ as } k \to \infty,$$

we have $q_k \geq c_1 \Gamma(\alpha_0 + 1)k^{-\alpha_0}$ for all sufficiently large k. Write $C_1 = c_1\Gamma(\alpha_0 + 1)$. Then we get $q_k \geq C_1 k^{-\alpha_0}$. Similarly, using the left inequality of (2.3), we can infer that $q_k \leq C_2 k^{-\alpha_1}$. Thus the proof is complete. □

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