Bergman Kernels and Local Holomorphic Morse Inequalities
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Abstract

This thesis consists of a slightly extended version of the submitted paper "Bergman Kernels and Local Holomorphic Morse Inequalities" (Robert Berman). For a given complex manifold X, equipped with a Hermitian metric ω, let L^k be a high power of a hermitian holomorphic line bundle over X. In the paper local versions of Demailly's holomorphic Morse inequalities (that give bounds on the Dolbeault cohomology groups associated to L^k) are presented. They are local in the sense of local index theorems - after integration they yield the usual holomorphic Morse inequalities. The local *weak* inequalities hold on any hermitian manifold (X, ω), regardless of compactness and completeness. The proofs, which are elementary, are based on a new approach to pointwise Bergman kernel estimates, where the kernels are estimated by a model kernel in $\mathbb{C}^n$. The results are used to gain a certain pointwise control of holomorphic sections of L^k, for k large, when L is only assumed to be semi-positive and strictly positive at some point of X.

Key words: Line bundles, Cohomology, Harmonic forms, Holomorphic sections, Bergman kernel, Extremal function.

MSC (2000): 32A25, 32L10, 32L20

"Think globally, act locally." (John Lennon)

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1. Introduction

Let X be an $n-$dimensional complex manifold equipped with a hermitian metric two-form, denoted by ω. The manifold X is allowed to be non-compact. Furthermore, let L be a holomorphic line bundle over X. The hermitian fiber metric on L will be denoted by ϕ. In practice, ϕ is considered as a collection of local functions. Namely, let s be a local holomorphic trivializing section of L, then locally, $\left|s(z)\right|_{\phi}^{2}=e^{-\phi(z)}$, and the canonical curvature two-form of L can be expressed as $\partial\overline{\partial}\phi$. The normalized curvature two-form $\frac{1}{\pi}\frac{i}{2}\partial\overline{\partial}\phi$ represents $c^{1}(L)$, the first Chern class of L, in real cohomology.

1.1. **Background.**

It is well-known that if the manifold X is compact and the holomorphic line bundle L admits a hermitian metric with *positive* curvature two-form, then there is a large supply of global holomorphic sections of L^{k}, for k large. For example, there are enough global holomorphic sections to separate points on X and for a given holomorphic vector bundle E, they generate any fiber of $L^{k}\otimes E$. This is intimately related to the vanishing of the associated Dolbeault cohomology groups:

$$\dim_{\mathbb{C}}H^{0,q}(X,L^{k}\otimes E)=0,\, q>0,$$

for large k. Moreover, the natural map[1]

$$X\rightarrow\mathbb{P}H^{0}(X,L^{k})^{*},$$

embeds X as a (non-singular) projective variety, for large k, which is the content of the famous Kodaira embedding theorem. In fact, the positivity of L corresponds to the notion of *ampleness* in algebraic geometry. The main *analytical* reason behind these properties is that when (L,ϕ) is positive, one can solve the inhomogeneous $\overline{\partial}-$ equation with an $L^{2}-$estimate, which depends on the fiber metric ϕ and its curvature: Hörmander's celebrated L^{2} estimates [11]. But if the curvature of L is only assumed to be positive in some point of X, while semi-positive on all of X, the situation is more complicated. This is the subject of Siu's survey article from 1984 [14]. However, Siu was able to show that

$$\dim_{\mathbb{C}}H^{0,q}(X,L^{k}\otimes E)=o(k^{n}),\, q>0, \tag{1.1}$$

if L is semi-positive [14], [15]. Combing this fact with the classical Riemann-Roch formula for the Euler characteristic of the corresponding complex,

$$\sum_{q=0}^{n}(-1)^{q}\dim_{\mathbb{C}}H^{q}(X,L^{k})=k^{n}\frac{1}{\pi^{n}}\frac{1}{n!}\int_{X}(\frac{i}{2}\partial\overline{\partial}\phi)^{n}+o(k^{n})$$

shows that

$$\dim_{\mathbb{C}}H^{0}(X,L^{k})=k^{n}\frac{1}{\pi^{n}}\frac{1}{n!}\int_{X}(\frac{i}{2}\partial\overline{\partial}\phi)^{n}+o(k^{n}).$$

Hence the dimension of $H^{0}(X,L^{k})$ is of the order k^{n}, if L is positive somewhere. By taking quotients of holomorphic sections, to obtain meromorphic functions on X, one can then show that the transcendental degree of the field of meromorphic functions on X equals n. In other words this says that X is Moishezon. In this

[1] The map sends a point x to the functional that evaluates a holomorphic section at x.

way Siu gave a positive answer to the Grauert-Riemenschneider conjecture, which is related to the Kodaira embedding theorem:

- Any compact complex manifold X admitting a holomorphic line bundle with a hermitian metric whose canonical curvature two-form is positive at some point of X, while semi-positive on all of X, is necesserly Moishezon.

Siu conjectured [14] that for *any* line bundle L there are estimates of the form 1.1, with leading constants expressed as integrals of the curvature of L. This was confirmed by Demailly, when he showed his *holomorphic* Morse inequalities [5]. Namely, for *any* holomorphic line bundle L, the following asymptotic estimate holds:

$$\dim_{\mathbb{C}} H^{0,q}(X, L^k \otimes E) \leq k^n r \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(q)} (\frac{i}{2} \partial\overline{\partial}\phi)^n + o(k^n), \tag{1.2}$$

where $X(q)$ is the subset of X where the curvature-two form $\partial\overline{\partial}\phi$ has exactly q negative eigenvalues and $n-q$ positive ones and r is the rank of E. These are the *weak* holomorphic Morse inequalities - they also have *strong* counterparts. The inspiration came from Witten's analytical proof of the classical Morse inequalities for the Betti numbers of a *real* manifold [17]. Subsequently, proofs based on asymptotic estimates of the heat kernel of the $\overline{\partial}-$ Laplacian, were given by Demailly, Bouche and Bismut ([6], [4] and [2]). All of these proofs use quite delicate analytical arguments, while the proofs presented in this paper are very elementary from the analytical point of view. The main inspiration comes from a recent paper by Berndtsson [1], where the following asymptotic estimate is shown:

$$\dim_{\mathbb{C}} H^q(X, L^k) \leq Ck^{n-q}, \ q \geq 1, \tag{1.3}$$

if L is semi-positive. The starting point in Berndtsson's proof is the Bergman kernel function $B_X^{q,k}(x)$ of $\mathcal{H}^q(X, L^k)$, the space of $\overline{\partial}-$harmonic $(0,q)-$forms with values in L^k :

$$B_X^{q,k}(x) := \sum_i |\Psi_i(x)|^2,$$

where $\{\Psi_i\}$ is an orthonormal base in $\mathcal{H}^q(X, L^k)$. Integrating, and adding up the 1:s yields the evident formula

$$\int_X B_X^{q,k}(x) Vol_{(X,\omega)} = dim_{\mathbb{C}} \mathcal{H}^q(X, L^k),$$

which by the well-known Hodge isomorphism equals $\dim_{\mathbb{C}} H^q(X, L^k)$. In Bergman's classical theory of reproducing kernels there is an *extremal* characterization of the Bergman kernel function which in the present context of vector bundles is generalized by the inequalities

$$S_X^{q,k}(x) \leq B_X^{q,k}(x) \leq \binom{n}{q} S_X^{q,k}(x),$$

where

$$S_X^{q,k}(x) := \sup \frac{|\alpha(x)|^2}{\|\alpha\|_X^2},$$

and the supremum is taken over all $\overline{\partial}-$harmonic $(0,q)-$forms with values in L^k.[2]The estimate 1.3 now follows from the following pointwise inequality for $\overline{\partial}-$harmonic $(0,q)-$forms with values in L^k, proved in [1],

$$|\alpha(x)|^2 \leq Ck^{n-q} \|\alpha\|_X^2 ,$$

The proof is based on Siu's $\partial\overline{\partial}-$technique à la Bochner (this is where the curvature condition enters), a theorem of Skoda concerning submean inequalities for certain semi-positive currents and the Sobolev embedding theorem. However, it is not evident how to determine the optimal constant C appearing in the inequalities.

1.2. **Outline of the paper.** In this paper a simple localization argument is used to estimate the global Bergman kernel function $B_X^{q,k}$ and the related extremal function $S_X^{q,k}$ of $\mathcal{H}^q(X,L^k)$[3] at a fixed point x in X, by the corresponding "model" functions in $\mathbb{C}^n$, when k tends to infinity. The model situation, in turn, reduces to the one variable case. In particular, there will be no need for global identities, such as the Bochner-Kodaira-Nakano identity for non-Kähler manifolds (see [7]), that appear in other approaches.The plan of the paper is as follows.

In section 2 the main results are presented. In section 3 the proof of the local holomorphic Morse inequalities is motivated and sketched. The details are worked out in the following sections along with related results. Section 4 establishes the notation. In section 5 the global Bergman kernel function is related to the global extremal function and the dimension formula. Section 6 states some localization results that are proved in the appendix. In section 7, the "model" functions are computed. In section 8 the proof of the holomorphic Morse inequalities is given and the proof of their strong counterparts is given in section 9 . In section 10 the previous results are applied to line bundles with special curvature conditions, such as semi-positivity. In particular, a certain control of the pointwise behaviour of holomorphic sections is gained. Furthermore, a conjecture is presented.

2. Statement of the main results

Let X be a complex manifold (possibly non-compact) with a hermitian metric ω and let L be a hermitian holomorphic line bundle over X. The hermitian fiber metric on L will be denoted by ϕ. Furthermore, let E be a hermitian holomorphic vector bundle of rank r, over X. The bundle E will be called the *twisting bundle*. However, *in the proofs of the theorems we will assume that there is no twisting bundle E*, since the generalizations of the proofs are straight forward. Denote by $\Omega^{0,q}(X,L^k\otimes E)$ the space of $(0,q)$ forms on X with values in $L^k\otimes E$ (simply referred to as the $(0,q)-$forms). The holomorphic structures on L and E determine the $\overline{\partial}-$operator

$$\overline{\partial}:\, \Omega^{0,q}(X,L^k\otimes E) \longrightarrow \Omega^{0,q+1}(X,L^k\otimes E)$$

and the hermitian metrics determine its formal adjoint $\overline{\partial}^*$. The cohomology groups of the complex associated to $\overline{\partial}$, are called the Dolbeault groups of $L^k\otimes E$, and

[2]The factor $\begin{pmatrix} n \\ q \end{pmatrix}$ appears as the rank of the vector bundle of all $(0,q)-$forms with values in L^k.

[3]where all sections are assumed to be in L^2 in case X is non-compact.

denoted by $H^{0,q}(X, L^k \otimes E)$. The $\overline{\partial}-$Laplacian is defined as $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, so that

$$\Delta_{\overline{\partial}} : \Omega^{0,q}(X, L^k \otimes E) \longrightarrow \Omega^{0,q}(X, L^k \otimes E).$$

Its L^2-kernel is the space of harmonic $(0,q)$-forms with values in L^k, denoted by $\mathcal{H}^{0,q}(X, L^k \otimes E)$. By the well-known Hodge theorem

$$H^{0,q}(X, L^k \otimes E) \cong \mathcal{H}^{0,q}(X, L^k \otimes E),$$

if X is compact.

Now take an orthonormal base $\{\Psi_i\}$ in $\mathcal{H}^q(X, L^k \otimes E)$ and define the *Bergman kernel function* of $\mathcal{H}^q(X, L^k \otimes E)$ by

$$B_X^{q,k}(x) := \sum_i |\Psi_i(x)|^2 .$$

The related *extremal function* of $\mathcal{H}^q(X, L^k \otimes E)$ is defined as

$$S_X^{q,k}(x) := \sup \frac{|\alpha(x)|^2}{\|\alpha\|_X^2},$$

where the supremum is taken over all forms in $\mathcal{H}^q(X, L^k)$. To define the corresponding "model" functions at a point x in X, $B^q_{x,\mathbb{C}^n}$ and $S^q_{x,\mathbb{C}^n}$, proceed as before, replacing the manifold X with $\mathbb{C}^n$, the base metric in X with the Euclidean metric in $\mathbb{C}^n$ and the fiber metric ϕ on L with the fiber metric ϕ_0 on the trivial line bundle over $\mathbb{C}^n$, where

$$\phi_0(z) = \sum_{i=1}^n \lambda_{i,x} |z_i|^2 ,$$

and where the curvature two-form $\partial\overline{\partial}\phi_0$ in $\mathbb{C}^n$ is obtained by "freezing" the curvature two-form $\partial\overline{\partial}\phi$ on X, at the point x (with respect to on orthonormal frame at x).[4] There is no twisting bundle E in the model situation.

Note that if X is compact, then $\mathcal{H}^q(X, L^k \otimes E)$ is finite dimensional and

$$\dim_{\mathbb{C}} H^{0,q}(X, L^k \otimes E) = \dim_{\mathbb{C}} \mathcal{H}^q(X, L^k \otimes E) = \int_X B_X^{q,k}(x) Vol_{(X,\omega)}.$$

Finally, denote by $X(q)$ the subset of X where the curvature-two form $\partial\overline{\partial}\phi$ has exactly q negative eigenvalues and $n-q$ positive ones. Its characteristic function is denoted by $1_{X(q)}$. Moreover, let $X(\leq q) := \bigcup_{0 \leq i \leq q} X(i)$.

The first theorem we shall prove is a local version of Demailly's weak holomorphic Morse inequalities. Note that the manifold X is not assumed to be compact, neither is there any assumption, e.g. completeness, on the hermitian metric ω.

Theorem 2.1. *Let (X, ω) be a hermitian manifold. Then, up to terms of order $o(k^n)$,*

$$B_X^{q,k}(x) \leq rk^n B^q_{x,\mathbb{C}^n}(0), \quad S_X^{q,k}(x) \leq k^n S^q_{x,\mathbb{C}^n}(0) \ ,$$

[4] Equivalently: the $\lambda_{i,x}$ are the eigenvalues of the curvature two-form of L with respect to on orthonormal frame at x.

and

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

Moreover, $\lim_k \frac{1}{k^n} B^{q,k}_X(x) = \lim_k \frac{1}{k^n} S^{q,k}_X(x)$ *if* $r = 1$.

This seems to be a new result. The inequality for $S^{q,k}_X(x)$ generalizes results of Bouche [4] and Gillet, Soulé, Gromov [8], that are non-local and concern *compact* manifolds X.

If X is compact we can integrate the inequality for $B^{q,k}_X$ in the previous theorem to obtain a new proof of Demailly's weak holomorphic Morse inequalities[5] :

Corollary 2.2. *Suppose that X is compact. Then*

$$\dim_{\mathbb{C}} H^{0,q}_{\overline{\partial}}(X, L^k \otimes E) \leq k^n r \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(q)} (\frac{i}{2}\partial\overline{\partial}\phi)^n + o(k^n).$$

When X is compact the local Morse inequalities can be extended to the asymptotic *equality*

$$B^{q,k}_{\leq \mu_k k}(x) = \frac{k^n}{\pi^n} r 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| + o(k^n),$$

where $B^{q,k}_{\leq \mu_k k}$ is the Bergman kernel function of the space spanned by all the eigenforms of the $\overline{\partial}-$Laplacian, whose eigenvalues are bounded by $\mu_k k$ and μ_k is a certain sequence tending to zero. Again, integrating this yields, with the help of a well-known homological algebra argument[6], a new proof of Demailly's strong holomorphic Morse inequalities.

Theorem 2.3. *Suppose that X is compact. Then*

$$\sum_{i=0}^{q} (-1)^{q-i} \dim_{\mathbb{C}} H^i(X, L^k \otimes E) \leq k^n r \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(\leq q)} (\frac{i}{2}\partial\overline{\partial}\phi)^n + o(k^n).$$

Combining the local weak Morse inequalities for $q = 0$ and the strong Morse inequalities for $q = 1$ yields the following theorem:

Theorem 2.4. *Let X be compact and assume that there is no point in X, where the curvature two-form of L has one negative eigenvalue and $n-1$ positive ones. Then*

$$\limsup B^{0,k}_X(x) = \frac{k^n}{\pi^n} r \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| + o(k^n)$$

almost every where on the part of X where the curvature of L is positive.

The most interesting case when the previous theorem applies is when L is semi-positive and strictly positive at some point.[7] The following corollary allows a certain pointwise control of global holomorphic sections, when the curvature conditions in the previous theorem hold. It says, essentially, that for almost all points of X, where

[5] this also uses Fatou's lemma to interchange the integration and the taking of the limit.

[6] Just as in the papers of Demailly, Bouche and Bismut ([6],[4] and [2]). In fact, the idea of using the "low-energy spectrum" was introduced in Witten's seminal paper [17]. A similar technique is used in the heat equation proof of Atiyah-Singer's index theorem.

[7] For example, if X is projective this curvature assumption implies that L is nef and big, in algebraic geometric terms. The converse is "almost true" (see [7]).

the curvature of L is positive, there is a sequence of global holomorphic sections that become increasingly peaked at x, when k tends to infinity. First some notation: around each point x in X we fix "normal" local holomorphic coordinates, i.e. such that the local expression for the base metric ω becomes

$$\omega(z) = \frac{i}{2}\sum_{i,j} h_{ij}(z)dz_i \wedge \overline{dz_j},\ h_{ij}(0) = \delta_{ij}$$

Let $B_R := \{z : |z| < R\}$ in $\mathbb{C}^n$. Using the local chart around x, B_R is identified with a subset of X (when R is sufficiently small).

Corollary 2.5. *Suppose that X and L satisfy the conditions of the previous theorem. Then for almost all points x in the part of X where the curvature of L is positive, there is a unit norm sequence $\{\alpha_{k_j}\}$ (depending on the point x) of holomorphic section with values in L^{k_j} with the following properties:*

$$\lim \int_{B_{R_{k_j}}(x)} \left|\alpha_{k_j}\right|^2 = 1,$$

for any sequence $B_{R_{k_j}}(x)$ as long as $R_{k_j} \to 0$, $\sqrt{k_j}R_{k_j} \to \infty$ and $\sqrt{k_j}R_{k_j}^3 \to 0$.

When L is positive the next theorem yields a refinement of the previous theorem.

Theorem 2.6. *Let (X, ω) be a complete Kähler manifold and let (L, ϕ) be a hermitian holomorphic line bundle over X, with $\partial\overline{\partial}\phi > \epsilon\omega$ for some positive number ϵ. Then, up to terms of order $o(k^n)$,*

$$B_X^{0,k}(x) = k^n r \frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

The previous theorem is a simple generalization to the non-compact situation of earlier results starting with Tian's work [18],[4],[19]. See also [3] where the complex structure is non-integrable. The proof given here is a straight forward extension of the proof given by Lindholm [12] in $\mathbb{C}^n$ with the Euclidean metric.

Finally, we shall give a local version of Serre's classical theorem A.

Theorem 2.7. *Let X be compact and assume that there is no point in X, where the curvature two-form of L has one negative eigenvalue and $n-1$ positive ones. If Ω is a relatively compact subset of the part of X where L is positive, then for any k_0 there is a k bigger than k_0 such that $H^0(X, L^k \otimes E)$ generates the fibers of $L^k \otimes E$ over almost all points in Ω.*

Theorems of this type are usually proved using homological methods, by showing that the first cohomology group of a certain vector bundle over a blow-up of X (or the first cohomology group of a certain coherent sheaf over X) vanish. In the proof presented here one could say that the substitute for the homological algebra is the strong Morse inequalities for $q = 1$.

3. Overview of the proof of the local weak holomorphic Morse inequalities

3.0.1. *Motivation:* $q=0$. Let us first consider the case when $q=0$, $r=1$ and $B_X^0(x)$ is the Bergman kernel function of the space of *holomorphic* sections[8]. A classical way to study the Bergman kernel function is by means of the following extremal characterization:

$$B_X^0(x)=\sup_{\overline{\partial}f=0}\frac{|f(x)|^2}{\|f\|_X^2} \tag{3.1}$$

We first consider the model case, where $X=\mathbb{C}^n$, $x=0$ and $\phi(z)=\phi_0(z):=\sum_{i=1}^n\lambda_i|z_i|^2$. Suppose that f is holomorphic. Then $|f|^2$ is subharmonic, so integrating $|f|^2$over a polydisc of radius r and then over all non-negative r up to R gives, since ϕ is radial in each variable, the submean inequality

$$\int_{D_R}|f(0)|^2e^{-k\phi_0(z)}\leq\int_{D_R}|f(z)|^2e^{-k\phi_0(z)}. \tag{3.2}$$

By letting R tend to infinity, and performing the change of variables $w=\sqrt{k}z$ in the integral, we get

$$\frac{|f(0)|^2}{\|f(z)\|_{k\phi_0}^2}\leq\left(\int_{\mathbb{C}^n}e^{-k\phi_0}\right)^{-1}=k^n\left(\int_{\mathbb{C}^n}e^{-\sum_{i=1}^n\lambda_i|w_i|^2}\right)^{-1}=\begin{cases}k^n\frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}\text{if all }\lambda_i>0\\ 0\text{ otherwise}\end{cases}$$

which proves theorem 2.1 in the model case - even without letting k tend to infinity. Now we move on to the case where $X\subseteq\mathbb{C}^n$ and $\phi=\phi_0$ as before. The point is that we are really integrating over $\sqrt{k}\mathbb{C}^n$ in the last step, so if we take a small ball $B_\varepsilon(0)$ around 0, we get

$$\frac{|f(0)|^2}{\|f(z)\|_{X,k\phi_0}^2}\leq\left(\int_{B_\varepsilon(0)}e^{-k\phi}\right)^{-1}=k^n\left(\int_{\sqrt{k}B_\varepsilon(0)}e^{-\sum_{i=1}^n\lambda_i|w_i|^2}\right)^{-1}$$
$$=\begin{cases}k^n\frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}+o(k^n)\text{ if all }\lambda_i>0\\ 0\ +o(k^n)\text{ otherwise}\end{cases}$$

In fact, we would have obtained the same result by picking balls of radius R_k with $R_k\to0$ as long as $\sqrt{k}R_k\to\infty$. It is this localization property that allows us to extend the result to any holomorphic Hermitian line bundle (L,ϕ) over a Hermitian manifold (X,ω). Indeed, fix a point x in X and choose complex coordinates and a holomorphic trivialization s of L around x, such that

$$\omega(z)=\tfrac{i}{2}\textstyle\sum_{i,j}h_{ij}(z)\overline{dz_i}\wedge\overline{dz_j},\ h_{ij}(0)=\delta_{ij}\text{ and }\ |s(z)|^2=e^{-\phi(z)},\ \phi(z)=\phi_0(z)+O(|z|^3), \tag{3.3}$$

where ϕ_0 is of the quadratic form, studied above, with coefficients λ_i depending on the point x. Pick any sequence α_k, with α_k a holomorphic section of L^k, and where

[8]If (X,ω) is non-complete then this space may very well be smaller than the space of $\overline{\partial}-$harmonic sections.

locally $\alpha_k = f_k s^{\otimes k}$. Moreover choose R_k so that also $kR_k^3 \to 0$, in order to achieve $\sup_{z\in R_k} |k\phi - k\phi_0| \to 0$. In fact, $R_k := \frac{lnk}{\sqrt{k}}$ will do. Then

$$\frac{|\alpha_k(x)|^2}{\|\alpha_k\|_X^2} \leq \frac{|\alpha_k(x)|^2}{\|\alpha_k\|_{R_k}^2} = \rho_k k^n \frac{|f_k(0)|^2}{\left\| f_k(\frac{w}{\sqrt{k}}) \right\|_{B_{\sqrt{k}R_k},\phi_0}^2},$$

where $\rho_k \to 1$. The factor ρ_k comes from the base manifold metric and the terms of order $O(|z|^3)$ in the fiber metric on L. Hence

$$\frac{|\alpha_k(x)|^2}{\|\alpha_k\|_X^2} \leq \begin{cases} k^n \frac{|\lambda_1|(x)|\lambda_2|(x)\cdots|\lambda_n|(x)}{\pi^n} + o(k^n) = k^n \frac{\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|}{\pi^n} + o(k^n) \text{ if all } \lambda_i(x) > 0 \\ 0 + o(k^n) \text{ otherwise} \end{cases}$$

as before.

The case $q > 0$. We will now try to adapt this method to the case when $q > 0$. We also assume that there is no vector bundle E (i.e. we are tensoring L^k with the trivial line bundle). It turns out that there is an appropriate generalization of 3.1, so we focus on estimating

$$S_X^{q,k}(x) = \sup \frac{|\alpha(x)|^2}{\|\alpha\|_X^2}$$

where α is a harmonic global $(0,q)-$ form with values in L^k, i.e. $\Delta_{\overline{\partial}}\alpha = 0$. The new difficulty that arises is that α *lies in different harmonic spaces for different metrics,* because of the metric dependence of $\overline{\partial}^*$. Therefor there is no obvious generalization of the submean inequality 3.2. However, fixing a point x in X and choosing local coordinates and a trivialization of L as in 3.3, the previous localization argument at least gives that there is sequence ρ_k, tending to 1, such that

$$\|\alpha_k\|_{B_{R_k}}^2 = \rho_k k^{-n} \left\|\alpha^{(k)}\right\|_{\phi_0, B_{\sqrt{k}R_k}}^2 \tag{3.4}$$

where $\alpha^{(k)}(w) = \alpha_k(\frac{w}{\sqrt{k}})$ (scaling componentwise) is defined on the scaled ball $B_{\sqrt{k}R_k}$. Moreover, one can check that the scaled form $\alpha^{(k)}$ satisfies

$$\Delta_{\overline{\partial}}^{(k)}\alpha^{(k)} = 0,$$

where $\Delta_{\overline{\partial}}^{(k)}$ denotes the Laplacian on the ball $B_{\sqrt{k}R_k}$, with respect to the scaled metrics $\omega^{(k)}(w) := \omega(\frac{w}{\sqrt{k}})$ and $(k\phi)^{(k)}(w) := (k\phi)(\frac{w}{\sqrt{k}})$. By our localization argument

$$\Delta_{\overline{\partial}}^{(k)} \to \Delta_{\overline{\partial},\phi_0} \text{over the ball of radius } \sqrt{k}R_k \text{ when } k \to \infty, \tag{3.5}$$

when interpreted in an appropriate way. Let us now show how to obtain the pointwise estimate

$$S_X^{q,k}(x) \leq k^n S_{x,\mathbb{C}^n}^q(0) + o(k^n), \tag{3.6}$$

where $S_{x,\mathbb{C}^n}^q(z)$, is the model extremal function defined in section 2. According to the definition of $S_X^{q,k}(x)$ it is enough to show that, given a *unit norm* sequence α_k with $\Delta_{\overline{\partial}}\alpha_k = 0$, there is a subsequence such that

$$\limsup_k k^{-n} |\alpha_k(x)|^2 \leq S_{x,\mathbb{C}^n}^q(0).$$

To this end, observe that with the usual identifications

$$k^{-n}\left|\alpha_k(x)\right| = \left|k^{-\frac{1}{2}n}\alpha^{(k)}(0)\right|^2,$$

The localization of the norms 3.4 shows that the sequence $k^{-\frac{1}{2}n}\alpha^{(k)}$ is bounded with respect to the "model" norm on the ball $B_{\sqrt{k}R_k}$:

$$\limsup\left\|k^{-\frac{1}{2}n}\alpha^{(k)}\right\|^2_{\phi_0,B_{\sqrt{k}R_k}} = \limsup\left\|\alpha_k\right\|^2_{B_{R_k}} \leq 1$$

This means that the sequence $k^{-\frac{1}{2}n}\alpha^{(k)}$ satisfies

$$\left\|k^{-\frac{1}{2}n}\alpha^{(k)}\right\|^2_{\phi_0} \leq C \text{ and } \Delta^{(k)}_{\overline{\partial}}(k^{-\frac{1}{2}n}\alpha^{(k)}) = 0.$$

Standard techniques for elliptic operators now produce a subsequence that converges *uniformly* on compact subsets of $\mathbb{C}^n$ to a global form β in $\mathbb{C}^n$, with $\left\|\beta\right\|^2_{\phi_0} \leq 1$. Moreover, from the convergence property 3.5 it follows that $\Delta_{\overline{\partial},\phi_0}\beta = 0$ in $\mathbb{C}^n$. Finally, after replacing the former sequence with its subsequence we get that

$$\lim_k k^{-n}\left|\alpha_k(x)\right| = \lim_k k^{-n}\left|k^{-\frac{n}{2}}\alpha^{(k)}(0)\right| = \left|\beta(0)\right|^2 \leq \frac{\left|\beta(0)\right|^2}{\left\|\beta\right\|^2_{\phi_0}} \leq S^q_{x,\mathbb{C}^n}(0),$$

which proves 3.6.

The model case. All that remains is to determine the model extremal function $S^q_{x,\mathbb{C}^n}$. When $q = 0$ and the operator is $\overline{\partial}$ this is easy. Indeed, the upper bound $1_{X(0)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$ on $S^0_{x,\mathbb{C}^n}$ was obtained in the beginning of section 3 and equality is achieved for constant functions. In section 7 we reduce the general case, when $q > 0$, and the operator is the $\overline{\partial}-$Laplacian to the previous case. It is shown that

$$S^q_{x,\mathbb{C}^n}(0) = B^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|. \tag{3.7}$$

All together this means that

$$S^{q,k}_X(x) \leq \frac{k^n}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + o(k^n).$$

The main point in the calculation of the model extremal function $S^q_{x,\mathbb{C}^n}$ is the following explicit expression for the $\overline{\partial}-$Laplacian:

$$\Delta_{\overline{\partial}}(fd\overline{z}^I) = (\sum_{i\in I}\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial\overline{z}_i}^{*\phi_0} + \sum_{i\in I^c}\frac{\partial}{\partial\overline{z}_i}^{*\phi_0}\frac{\partial}{\partial\overline{z}_i})fd\overline{z}^I =: (\Delta_{\overline{\partial},I}f)d\overline{z}^I),$$

where

$$\phi_0(z) := \sum_{i=1}^n\lambda_i\left|z_i\right|^2$$

In particular $\Delta_{\overline{\partial}}\alpha = 0$ precisely when $\Delta_{\overline{\partial},I}f_I = 0$ for all components f_I of the $q-$form α. Now for each I we make a change of frames (and conjugation of coordinates) so that

$$\begin{array}{ccc} L^2_{\phi_0} & \leftrightarrow & L^2_{\Phi_I} \\ f_I & \leftrightarrow & F_I \\ \Delta_{\overline{\partial},I} & \leftrightarrow & \sum_{i=1}^n \frac{\partial}{\partial \overline{z_i}}^{*\Phi_I} \frac{\partial}{\partial \overline{z_i}} \end{array},$$

where

$$\Phi_I(z) := \sum_{i \in I} -\lambda_i \left|z_i\right|^2 + \sum_{i \in I^c} \lambda_i \left|z_i\right|^2 .$$

By choosing a sequence of cut-off functions judicially one can integrate "without getting boundary terms" and infer that $\Delta_{\overline{\partial},I} f_I = 0$ precisely when $\frac{\partial}{\partial \overline{z_i}} F_I = 0$ for all I, i.e. when F_I is holomorphic. Now assume that it is not the case that q of the numbers λ_i are negative and the rest are positive. Then at least one of the coefficients of the quadratic form $\Phi_I(z)$ is negative. As in the beginning of section 3 one then sees that $F_I(0) = 0$, showing that the model extremal function $S^q_{x,\mathbb{C}^n}(z)$ is zero at 0. Next, if q of the numbers λ_i are negative and the rest are positive, we may assume that it is the first q numbers λ_i that are negative. Unless $I = (1, 2, ...q)$ the quadratic form $\Phi_I(z)$ will still have a negative coefficient, which shows that the only contribution to the extremal function comes from the component indexed by $(1, 2, ...q)$. In this case all coefficients are positive and we see, as in the beginning of section 3, that

$$S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} \left|\lambda_1\right| \left|\lambda_2\right| \cdots \left|\lambda_n\right| = \frac{1}{\pi^n} \left| det(\frac{i}{2} \partial\overline{\partial}\phi_0)_x \right| .$$

Remark 3.1. The correct relation between the Bergman kernel function and the extremal function is given in lemma 5.1:

$$S^{q,k}_X(x) \leq B^{q,k}_X(x) \leq \sum_I S^{q,k}_{X,I}(x),$$

where

$$S^{q,k}_{X,I}(x) := \sup \frac{\left|\alpha_I(x)\right|^2}{\left\|\alpha\right\|^2_X},$$

for α a harmonic $(0, q)-$form. In the model case we saw that $\sum_I S^q_{x,\mathbb{C}^n,I}(0) = 0+0+...+0+\frac{1}{\pi^n} \left| det(\frac{i}{2}\partial\overline{\partial}\phi_0)_x \right|$if the curvature-two form $\partial\overline{\partial}\phi$ has exactly q negative eigenvalues and $n-q$ positive ones and that the sum vanishes in the remaining cases. For $S^{q,k}_X(x)$ this happens in the large k limit (modulo terms of order $o(k^n)$), showing that

$$B^{q,k}_X(x) \leq 0 + 0 + ... + 0 + \frac{k^n}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right| + o(k^n),$$

which finishes the sketch of the proof of the local weak holomorphic Morse inequalities.

4. Notation etc.

We assume given a hermitian holomorphic line bundle (L, ϕ) over (X, ω). The statements of the results of this paper concern $L^k \otimes E$, where E is a given hermitian holomorphic line bundle. However, in the proofs we assume that there is no twisting bundle E. Hence, this section will be concerned with L^k. The space of all global

$(0,q)-$ forms with values in L^k, is denoted by $\Omega^{0,q}(X,L^k)$ and the induced norm on this space is denoted by $\left\|\cdot\right\|_X$.

Local coordinates and frames. Around each point x in X we choose local complex coordinates $\{z_i\}$ and a holomorphic trivializing section s of L such that[9]

$$\omega(z) = \tfrac{i}{2}\textstyle\sum_{i,j} h_{ij}(z)dz_i \wedge \overline{dz_j},\ h_{ij}(0)=\delta_{ij} \quad \text{and} \quad |s(z)|^2 = e^{-\phi(z)},\ \phi(z)=\phi_0(z)+O(|z|^3),$$

where

$$\phi_0(z) := \sum_{i=1}^{n} \lambda_{i,x} |z_i|^2 .$$

Any $a^{0,q}$ in $\Omega^{0,q}(X,L^k)$ can be written, locally, as

$$\alpha^{0,q} = \sum_I f_I d\overline{z}^I \otimes s^{\otimes k}.$$

Note that in the local coordinates the base metric ω coincides with the Euclidean metric at the origin. Moreover, we have the identities,

$$\lambda_{1,x}\lambda_{2,x}\cdots\lambda_{n,x}Vol_{(X,\omega)} = det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x Vol_{(X,\omega)} = \frac{1}{n!}(\frac{i}{2}\partial\overline{\partial}\phi)^n.$$

In the following we fix a point x in X and consider small neighbourhoods of x.

The balls B_{R_k} and $B_{\sqrt{k}R_k}$. Let $B_R := \{z : |z| < R\}$ in $\mathbb{C}^n$. Using the local chart around x, B_R is identified with a subset of X (when R is sufficiently small).

Let $R_k := \frac{\ln k}{\sqrt{k}}$ so that $\sqrt{k}R_k = \ln k$, then

$$R_k \to 0,\ \sqrt{k}R_k \ \to \infty \text{ and } \ \tfrac{(\sqrt{k}R_k)^p}{\sqrt{k}} \to 0,$$

for all p.

Local scalings. Given a function f on the ball B_{R_k}, let

$$f^{(k)}(w) := f(\frac{w}{\sqrt{k}})$$

be defined on $B_{\sqrt{k}R_k}$. Forms are now scaled by scaling the components. E.g.

$$\omega^{(k)}(w) = \tfrac{i}{2}\textstyle\sum_{i,j} h_{ij}(\tfrac{w}{\sqrt{k}})dw_i \wedge \overline{dw_j},$$

Observe that scaling the fiber metric on L^k gives

$$(k\phi)^{(k)}(w) = \sum_{i=1}^{n} \lambda_i |w_i|^2 + \frac{1}{\sqrt{k}}O(|w|^3)$$

which motivates the choice of scaling.

[9]this is always possible; see [16].

Local norms. Using the trivialization we locally identify L with the trivial line bundle. The following notation for norms will be used:

- $\|\cdot\|_{\psi}$ is the norm determined by $|1|_{\psi}^2 = e^{-\psi}$.
- $\|\cdot\|_{(k)}$ is the norm induced by the scaled base metric $\omega^{(k)}$ and the scaled fiber metric on L^k, $(k\phi)^{(k)}$, restricted to the scaled ball $B_{\sqrt{k}R_k}$ in $\mathbb{C}^n$.
- $\|\cdot\|_{,R}$ denotes the restriction of a norm to the ball B_R.
- $\|\cdot\|_{,2,m}$ denotes the L^2-Sobolev norm with m derivatives: $\|\cdot\|_{,2,m}^2 = \sum_{|\alpha|\leq m} \|\partial^{\alpha}\cdot\|^2$, determined by a choice of L^2- norm.

Operators. We denote by $\overline{\partial}^{*\phi_0}$ and $\overline{\partial}^{*(k)}$ the formal adjoints of $\overline{\partial}$ with respect to $\|\cdot\|_{\phi_0}$ and $\|\cdot\|_{(k)}$, respectively. The latter operator in only defined on $B_{R'_k}$. $\overline{\partial}^*$ usually refers to the global operator, which is the formal adjoint of

$$\overline{\partial} : \Omega^{0,q}(X, L^k) \longrightarrow \Omega^{0,q+1}(X, L^k).$$

Note that $(\overline{\partial} + \overline{\partial}^*)^2 = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ is the $\overline{\partial}-$ Laplacian.

Cut-off functions. If χ is a smooth function, equal to 1 on $B_{1/2}$ and supported on B_1, we let

$$\chi_R(w) := \chi(\frac{w}{R}),$$

so that $\chi_R(w)$ is equal to 1 on $B_{1/2R}$ with support in B_{R_k}. Also, $|\nabla\chi| \leq C\frac{1}{R}$.

Estimates. We write $a_k =_k b_k$ if there is a sequence ρ_k such that $\rho_k \to 1$, when $k \to \infty$ and $a_k = \rho_k b_k$. We define $a_k \leq_k b_k$ in a similar way. Furthermore, the letter C in an estimate stands for a positive number, and means that the estimate is valid up to a constant, which is independent of k.

5. The Bergman kernel function $B(x)$, the extremal function $S(x)$ and dimension.

Assume given a hermitian manifold (X, ω) with a hermitian holomorphic line bundle (L, ϕ) and a rank r hermitian holomorphic line bundle E. In section 2 the Bergman kernel function $B_X^{q,k}$ and the extremal function $S_X^{q,k}$ of the space of global $\overline{\partial}-$harmonic $(0,q)-$ forms (with values in $L^k \otimes E$), were defined. We will also have use for component versions of $S_X^{q,k}$. These will be defined when there is no twisting vector bundle E, in order to simplify the notation (the generalization being obvious). For a given orthonormal frame e_x^I in $\bigwedge_x^{0,q}(X, L^k)$ let

$$S_{X,I}^{q,k}(x) := \sup \frac{|\alpha_I(x)|^2}{\|\alpha\|_X^2},$$

where $\alpha_I(x)$ denotes the component of α along e_x^I. It will be clear from the context what frame is being used. These functions are closely related according to the following, simple, yet important lemma. The first statement generalizes a lemma used in [1] (see also [12]) and the second statement is used in [4].

Lemma 5.1. *Let L be a hermitian holomorphic line bundle over X. With notation as above*

$$(i)\ S_X^{q,k}(x) \leq B_X^{q,k}(x) \leq \sum_I S_{X,I}^{q,k}(x)$$

Moreover, consider the vector bundle $L^k \otimes E$, where E is a given hermitian holomorphic vector bundle of rank r. Denote by $M^{q,k}(x)$ the maximal rank of the subspace of the fiber $L^k \otimes E$ at the point x, consisting of sections that may be extended to global sections in $\mathcal{H}^{0,q}(X, L^k \otimes E)$. Then

$$(ii)\ S_X^{q,k}(x) \leq B_X^{q,k}(x) \leq M^{q,k}(x) S_X^{q,k}(x)$$

Proof. We first prove (i). To prove the first inequality in the statement, take any α in $\mathcal{H}^q(X, L^k)$ of unit norm. Since α is contained in an orthonormal base, obviously $|\alpha(x)|^2 \leq B(x)$, which proves the first inequality. For the second inequality, let $\{\Psi^i\}$ be an orthonormal base for $\mathcal{H}^q(X, L^k)$, so that $B_X^{q,k}(x) := \sum_i |\Psi^i(x)|^2 = \sum_{i,J} |\Psi_J^i(x)|^2$. Fix a J and let $c_i := \overline{\Psi_J^i(x)}$. Then, summing only over i, gives

$$\sum_i \left|\Psi_J^i(x)\right|^2 = \sum_i c_i \Psi_J^i(x) = \alpha_J(x) \left(\sum_i |c_i|^2,\right)^{1/2}$$

where $\alpha = \sum_i \frac{c_i}{(\sum_i |c_i|^2,)_i^{1/2}} \Psi^i$ lies in $\mathcal{H}^q(X, L^k)$ and is of unit norm. Thus,

$$\left(\sum_i \left|\Psi_J^i(x)\right|^2\right)^{1/2} = \alpha_J(x) = \frac{|\alpha_J(x)|}{\|\alpha\|} \leq S_J(x)^{1/2}.$$

Finally, squaring the last relation and summing over J proves the second inequality, in the first statement.

To prove (ii), take $M^{q,k}(x)$ global sections of $\mathcal{H}^q(X, L^k \otimes E)$ that are linearly independent at x, and complete this set in order to obtain a base for the fiber of $L^k \otimes E$ at x. By the definition of $M^{q,k}(x)$, the component of any global section α along a "completing" base element vanishes (otherwise we could add α to the list of $M^{q,k}(x)$ global sections, to obtain $M^{q,k}(x) + 1$ global sections that are linearly independent at x). Hence, if we now let the index I run over all the elements in the base at x, we see that $S_{X,I}^{q,k}(x) = 0$ for all I corresponding to the "completing" base elements. The second statement now follows from the first one, since $S_{X,I}^{q,k}(x) \leq S_X^{q,k}(x)$ for all I. □

Note that the Bergman kernel function can be considered as a certain dimensional density. Indeed,

$$\int_X B_X^{q,k} Vol_\omega = \dim_{\mathbb{C}} \mathcal{H}^{0,q}(X, L^k \otimes E),$$

if X is compact, since we just get a one from each base element.

6. Localization

In this section we collect some results that will allow us to make precise the localization of the norms and $\overline{\partial}-$Laplacians, for large k, that was referred to in section 3. The proofs are straight forward and given in the appendix. The main point is to choose the sequence of decreasing radii R_k, such that lemma 6.1 holds.

Fix a point x in X and take local coordinates and a trivializing frame around x, as in section 4. Consider the balls centred at x, B_{R_k} and $B_{\sqrt{k}R_k}$, defined in section 4, where $R_k \to 0$, $\sqrt{k}R_k \to \infty$ and $\frac{(\sqrt{k}R_k)^p}{\sqrt{k}} \to 0$, for all p.

Localization of the norms.

Lemma 6.1. *When k tends to infinity,*

$$\sup_{|w|\leq\sqrt{k}R_k} \left|\partial^{\alpha}((k\phi)^{(k)} - \phi_0)(w)\right| \to 0,$$

for any $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{2n}) \in \mathbb{N}^{2n}$, where ∂^{α} denotes partial differentiaton with respect to the real *coordinates corresponding to w.*

The following proposition says, essentially, that we may replace the scaled norms with the model norm when k tends to infinity.

Proposition 6.2. *Fix a point x in X. Given a sequence β_k of $(0,q)$-forms on the balls $B_{\sqrt{k}R_k}$ in $\mathbb{C}^n$ there is sequence ρ_k tending to 1, such that*

$$\|\beta_k\|_{(k)} = \rho_k \|\beta_k\|_{\phi_0,\sqrt{k}R_k}.$$

With notation as in section 4 this means that

$$\|\beta_k\|_{(k)} =_k \|\beta_k\|_{\phi_0,\sqrt{k}R_k}.$$

Localization and properties of the operator $\overline{\partial} + \overline{\partial}^*$. The following proposition shows that $\overline{\partial}^*$ transforms as $\overline{\partial}$ under scalings.

Proposition 6.3. *Let α be $(0,q)$-form on B_{R_k}. Then the following identity holds on the scaled ball $B_{\sqrt{k}R_k}$:*

$$\overline{\partial}^{*(k)}\alpha^{(k)} = \frac{1}{\sqrt{k}}(\overline{\partial}^*\alpha)^{(k)} \tag{6.1}$$

In order to compare powers of the scaled $\overline{\partial}-$Laplacian with powers of the model $\overline{\partial}-$Laplacian, the following proposition is used.

Proposition 6.4. *There is a sequence ϵ_k, tending to zero, such that*

$$\left\|(\overline{\partial} + \overline{\partial}^{*(k)})^m\alpha^{0,q} - (\overline{\partial} + \overline{\partial}^{*\phi_0})^m\alpha^{0,q}(w)\right\|^2_{2,m} \leq \epsilon_k \left\|\alpha^{0,q}\right\|^2_{2,m},$$

where the norms are the $L^2_{\phi_0}$ Sobolev norms with m derivatives restricted to the ball of radius $\sqrt{k}R_k$.

Let χ be a real-valued smooth function on X. We let forms act by wedge multiplication from the left. The usual Leibniz rule for $\overline{\partial}$ gives that

$$\overline{\partial}(\chi\cdot) - (\chi\cdot)\overline{\partial} = \pm(\overline{\partial}\chi)$$

By taking formal adjoints on both sides we obtain a similar relation involving $\overline{\partial}^*$. Adding these two identities yields a Leibniz rule for $\overline{\partial}+\overline{\partial}^*$:

$$\overline{\partial}+\overline{\partial}^*(\chi\cdot)-(\chi\cdot)(\overline{\partial}+\overline{\partial}^*)=\pm(\overline{\partial}\chi)\mp(\overline{\partial}\chi)^*. \tag{6.2}$$

Note that the algebraic adjoint $(\overline{\partial}\chi)^*$ is independent of the fiber metric; it only depends on the base metric ω.

7. The model case

In this section we will be concerned with $\mathbb{C}^n$ with its standard metric. Any smooth function ϕ defines a hermitian metric on the trivial line bundle and associated bundles, via $|1|^2_\phi(z)=e^{-\phi(z)}$. Explicitly, this means that if $\alpha^{0,q}=\sum_I f_I d\overline{z}^I$ is a $(0,q)-$form on $\mathbb{C}^n$, then

$$\left|\alpha^{0,q}\right|^2_\phi(z)=\sum_I |f_I(z)|^2 e^{-\phi(z)}.$$

The standard differential operators on smooth functions are extended to operators on forms, by letting them act componentwise. We denote by $\frac{\partial}{\partial\overline{z_i}}^{*\phi}$ the formal adjoint of $\frac{\partial}{\partial\overline{z_i}}$ with respect to the norm induced by ϕ. A partial integration shows that

$$\frac{\partial}{\partial\overline{z_i}}^{*\phi}=e^{\phi}(-\frac{\partial}{\partial z_i})e^{-\phi}=-\frac{\partial}{\partial z_i}+\phi_z \tag{7.1}$$

The following classical commutation relations are essential for what follows:

$$\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial\overline{z_j}}^{*\phi}-\frac{\partial}{\partial\overline{z}_j}^{*\phi}\frac{\partial}{\partial\overline{z}_i}=\phi_{\overline{z_i}z_j} \tag{7.2}$$

When $\phi=\phi_0:=\sum_{i=1}^n\lambda_i\left|z_i\right|^2$ this means that

$$\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial\overline{z_j}}^{*\phi_0}-\frac{\partial}{\partial\overline{z}_j}^{*\phi_0}\frac{\partial}{\partial\overline{z}_i}=\delta_{ij}\lambda_{ii} \tag{7.3}$$

The next lemma gives an explicit expression for $\Delta_{\overline{\partial},\phi_0}$, that will enable us to compute the model Bergman kernel function.

Lemma 7.1. *We have that*

$$\Delta_{\overline{\partial},\phi_0}(fd\overline{z}^I)=(\sum_{i\in I}\frac{\partial}{\partial\overline{z}_i}\frac{\partial}{\partial\overline{z_i}}^{*\phi_0}+\sum_{i\in I^c}\frac{\partial}{\partial\overline{z_i}}^{*\phi_0}\frac{\partial}{\partial\overline{z}_i})fd\overline{z}^I=:(\Delta_{\overline{\partial},I}f)d\overline{z}^I)$$

Proof. All adjoints and formal adjoints are taken with respect to ϕ_0. The following notation will be used. We let $d\overline{z}^i$ act on forms by wedge multiplication, and denote the adjoint by $d\overline{z}^{i*}$. Then we have the anti-commutation relations

$$\overline{dz}^i\overline{dz}^{j*}+\overline{dz}^{j*}\overline{dz}^i=0 \text{ if } i\neq j \tag{7.4}$$

Also,

$$(\overline{dz}^{i*}d\overline{z}^i)d\overline{z}^J=\begin{cases}1, & i\notin J\\ 0, & i\in J\end{cases}\quad(\overline{dz}^i\overline{dz}^{i*})d\overline{z}^J=\begin{cases}0, & i\notin J\\ 1, & i\in J\end{cases} \tag{7.5}$$

$\overline{\partial}$ can now be expressed as

$$\overline{\partial}=\sum_i \frac{\partial}{\partial \overline{z_i}}\overline{dz}^i=\sum_i \overline{dz}^i\frac{\partial}{\partial \overline{z_i}.}$$

Applying $*$ to this relation immediately yields

$$\overline{\partial}^*=\sum_i \frac{\partial^*}{\partial \overline{z_i}}\overline{dz}^{i*}=\sum_i \overline{dz}^{i*}\frac{\partial^*}{\partial \overline{z_i}.}.$$

Now we compute $\Delta_{\overline{\partial}}:=\overline{\partial\partial}^*+\overline{\partial}^*\overline{\partial}:$

$$\overline{\partial\partial}^*+\overline{\partial}^*\overline{\partial}=\sum_{i,j}\frac{\partial}{\partial \overline{z_i}}\frac{\partial^*}{\partial \overline{z_j}}\overline{dz}^i\overline{dz}^{j*}+\sum_{i,j}\frac{\partial^*}{\partial \overline{z_j}}\frac{\partial}{\partial \overline{z_i}}\overline{dz}^{j*}\overline{dz}^i.$$

By applying 7.4 to second term and splitting the sum this equals

$$\sum_i \frac{\partial}{\partial \overline{z_i}}\frac{\partial^*}{\partial \overline{z_i}}\overline{dz}^i\overline{dz}^{i*}+\sum_i \frac{\partial^*}{\partial \overline{z_i}}\frac{\partial}{\partial \overline{z_i}}\overline{dz}^{i*}\overline{dz}^i+\sum_{i\neq j}(\frac{\partial}{\partial \overline{z_i}}\frac{\partial^*}{\partial \overline{z_j}}-\frac{\partial^*}{\partial \overline{z_j}}\frac{\partial}{\partial \overline{z_i}})\overline{dz}^i\overline{dz}^{j*}.$$

According to the commutation relation 7.3 the second sum vanishes and 7.5 finally gives that

$$(\overline{\partial\partial}^*+\overline{\partial}^*\overline{\partial})(fd\overline{z}^I)=(\sum_{i\in I}\frac{\partial}{\partial \overline{z}_i}\frac{\partial}{\partial \overline{z_i}}^{*\phi_0}+\sum_{i\in I^c}\frac{\partial}{\partial \overline{z_i}}^{*\phi_0}\frac{\partial}{\partial \overline{z}_i})fd\overline{z}^I.$$

□

Remark 7.2. For general fiber metrics ϕ, $\Delta_{\overline{\partial}}$ does not respect the module decomposition of $\bigwedge^{0,q}(\mathbb{C}^n)$ that has $d\overline{z}^I$ as base elements.

We will not have any direct use of the following proposition. However, it fits well into the context and explains the appearance of the characteristic function $1_{X(q)}$ in the main formulas. Moreover, it can be used to give a direct proof of the fact that the global Bergman kernel function $B_X^{q,k}$ vanishes outside $X(q)$, modulo terms of order $o(k^n)$. The result appears already in Hörmander's seminal paper [11].

Proposition 7.3. *Suppose that $\partial\overline{\partial}\phi_0$ is non-degenerate and that it is* not *the case that q of the numbers λ_i are negative and $n-q$ of the numbers are positive. Then there is an a priori estimate:*

$$\|\alpha^q\|_{\phi_0}^2\leq C_q\left\langle\Delta_{\overline{\partial}}\alpha^q,\alpha^q\right\rangle_{\phi_0}=C_q(\left\|\overline{\partial}\alpha^q\right\|_{\phi_0}^2+\left\|\overline{\partial}^{*\phi_0}\alpha^q\right\|_{\phi_0}^2),$$

for all smooth $(0,q)-$ forms in $\mathbb{C}^n$ with compact support.

Proof. In what follows, all norms, adjoints and formal adjoints are taken with respect to ϕ_0. Suppose that $\alpha^{0,q}=\sum_I f_Id\overline{z}^I$ has compact support. According to proposition 7.3

$$\left\|\overline{\partial}\alpha^q\right\|^2+\left\|\overline{\partial}^*\alpha^q\right\|^2=\langle\Delta_{\overline{\partial}}\alpha^q,\alpha^q\rangle=\sum_I\langle\Delta_If_I,f_I\rangle, \tag{7.6}$$

where

$$\Delta_If=\sum_{i\in I}\frac{\partial}{\partial \overline{z}_i}\frac{\partial}{\partial \overline{z_i}}^*+\sum_{i\in I^c}\frac{\partial}{\partial \overline{z_i}}^*\frac{\partial}{\partial \overline{z}_i})f.$$

Now, by assumption, for all I there is some $\lambda_i > 0$ with $i \in I$ or some $\lambda_i < 0$ with $i \in I^c$. For a given I first assume that i_I is in I and $\lambda_{I_i} > 0$.Then, the commutation identity 7.3 gives that

$$\frac{\partial}{\partial \overline{z}_{i_I}} \frac{\partial}{\partial \overline{z}_{i_I}}^* = \lambda_{i_I} + \frac{\partial}{\partial \overline{z}_{i_I}}^* \frac{\partial}{\partial \overline{z}_{i_I}}, \text{ where } \lambda_{i_I} > 0.$$

Thus,

$$\langle \Delta_I f_I, f_I \rangle = \lambda_{i_I} \|f_I\|^2 + \left\| \frac{\partial}{\partial \overline{z}_{i_I}} f_I \right\|^2 + \sum_{i \neq i_I, i \in I} \left\| \frac{\partial}{\partial \overline{z}_i}^* f_I \right\|^2 + \sum_{i \in I^c} \left\| \frac{\partial}{\partial \overline{z}_i} f_I \right\|^2,$$

which implies that

$$\langle \Delta_I f_I, f_I \rangle \geq |\lambda_{i_I}| \|f_I\|^2 . \tag{7.7}$$

Similarly, the second case, when i_I is in I^c and $\lambda_{I_i} < 0$ also yields the inequality 7.7. Finally, by 7.6 and 7.7

$$\langle \Delta_{\overline{\partial}} \alpha^q, \alpha^q \rangle \geq \sum_I |\lambda_{i_I}| \|f_I\|^2 \geq \min_I |\lambda_{i_I}| \|\alpha^q\|^2_{\phi_0},$$

which shows that C_q may be taken to be $\min_I |\lambda_{i_I}|$. □

In order to be able to integrate partially in $\mathbb{C}^n$ without getting boundary terms, the following lemma is useful. See [10] for a slightly more general proof. The point is too choose $\chi_R(z) = (\chi(\frac{z}{R}))^2$, where χ is a smooth compact supported function, that equals 1 on the unit ball, say.

Lemma 7.4. *There is an exhaustion sequence χ_R such that for any smooth function f and $(0,q)-$ form α in $L^2_{\phi_0}(\mathbb{C}^n)$*

$$\lim_R \langle \Delta_{\overline{\partial}} \alpha, \chi_R \alpha \rangle = \left\| \overline{\partial} + \overline{\partial}^* \alpha \right\|^2_{\phi_0}, \quad \lim_R \left\langle \frac{\partial}{\partial \overline{z}_i}^* \frac{\partial}{\partial \overline{z}_i} f, \chi_R f \right\rangle = \left\| \frac{\partial}{\partial \overline{z}_i} f \right\|^2$$

and similarly for $\left\| \frac{\partial^*}{\partial \overline{z}_i} f \right\|^2$.

Next, we turn to the Bergman kernel function $B^q_{\phi_0,\mathbb{C}^n}(z)$ and the extremal function $S^q_{\phi_0,\mathbb{C}^n}(z)$. They are defined as in section 5 but with respect to the space $\mathcal{H}^{0,q}_{\phi_0}(\mathbb{C}^n)$ of all $(0,q)-$forms in $\mathbb{C}^n$, that are $\overline{\partial}-$harmonic with respect to the fiber metric ϕ_0. Note that with notation as in section 2,

$$B^q_{\phi_0,\mathbb{C}^n}(z) = B^q_{x,\mathbb{C}^n}(z)$$

if $\phi(z) = \phi_0(z) + O(|z|^3)$ is the local expression of the fiber metric of L at the point x in X.

Proposition 7.5. *The following identity holds:*

$$B^q_{\phi_0,\mathbb{C}^n}(0) = S^q_{\phi_0,\mathbb{C}^n}(0) = \frac{1}{\pi^n} |\lambda_1| |\lambda_2| \cdots |\lambda_n|,$$

if q of the numbers λ_i are negative and the rest are positive. Otherwise,

$$B^q_{\mathbb{C}^n}(0) = S^q_{\mathbb{C}^n}(0) = 0.$$

With notation as in section 2 this means that

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} 1_{X(q)}(x) \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

Proof. Suppose that $\alpha^{0,q} = \sum_I f_I d\overline{z}^I, \alpha^{0,q} \in L^2_{\phi_0}(\mathbb{C}^n)$ and $\Delta_{\overline{\partial},\phi_0}\alpha^{0,q} = 0$. Then $\langle \Delta_{\overline{\partial}}\alpha^{0,q}, \chi_R \alpha^{0,q} \rangle_{\phi_0} = 0$ and we get, as in the proof of proposition 7.3, that

$$\sum_I \langle \Delta_I f_I, \chi_R f_I \rangle_{\phi_0} = 0.$$

Letting $R \to \infty$ and using lemma 7.4 shows that

$$\left\| \tfrac{\partial^{*\phi_0}}{\partial \overline{z}_i} f_I \right\|^2_{\phi_0} = 0,\ i \in I \quad \left\| \tfrac{\partial}{\partial \overline{z}_i} f_I \right\|^2_{\phi_0} = 0,\ i \in I^c \ . \tag{7.8}$$

Let $F_I(\zeta) := e^{-\sum_{i \in I} \lambda_i |z_i|^2} f_I$, where $\zeta_i = \overline{z}_i$ if $i \in I$ and $\zeta_i = z_i$ if $i \in I^c$. Then 7.8 and 7.1 implies that F_I is holomorphic:

$$\frac{\partial}{\partial \overline{z}_i} F_I = 0 \text{ for all } i.$$

Moreover,

$$|f_I|^2_{\phi_0} = |F_I|^2_{\Phi_I},\ \Phi_I(z) := \sum_{i \in I} -\lambda_i |z_i|^2 + \sum_{i \in I^c} \lambda_i |z_i|^2 . \tag{7.9}$$

Now assume that it is not the case that q of the numbers λ_i are negative and the rest are positive. Then $\Phi_I(z) := \sum_{i=1}^n \Lambda_i^I |z_i|^2$, with some $\Lambda_i^I := \Lambda_{i_I}^I \leq 0$. By assumption, $\int |f_I|^2 e^{-\phi} < \infty$. Now 7.9 and Fubini-Tonelli's theorem give that

$$\int |F_I(0,..,z_{i_I},0,..)|^2_{\phi_I} e^{-\Lambda_{i_I}^I |z_{i_I}|^2} < \infty.$$

Since $\frac{\partial}{\partial \overline{z}_{i_I}} F_I = 0$ and $\Lambda_{i_I}^I \leq 0$ this forces $F_I(0,..,z_{i_I},0,..) \equiv 0$ and in particular $f_I(0) = 0$,[10]which proves that $B^q_{\mathbb{C}^n}(0) = S^q_{\mathbb{C}^n}(0) = 0$.

Finally, assume that q of the numbers λ_i are negative and the rest are positive. We may assume that λ_1 up to λ_q are the negative ones. Then the same argument as the one above gives that $f_I(0) = 0$ if $I \neq I_0 := (1, 2, ..., q)$. Now since $\Lambda_i^{I_0} > 0$ for all i we get

$$S^q_{\mathbb{C}^n}(0) = \sup \frac{|\alpha(x)|^2}{\|\alpha\|^2_{\phi_0}} = \sup \frac{|f_{I_0}(x)|^2}{\|f_{I_0}\|^2_{\phi_0}} = \sup \frac{|F_{I_0}(x)|^2}{\|F_{I_0}\|^2_{\Phi_I}} = \frac{\Lambda_1 \Lambda_2 \cdots \Lambda_n}{\pi^n} = \frac{|\lambda_1| |\lambda_2| \cdots |\lambda_n|}{\pi^n},$$

where the upper bound in the last step follows from the submean inequality 3.2 and equality is achieved for f such that f is of unit norm and $F_{I_0} \equiv 1$, while the other components are zero. The proof is finished by observing that $S^q_{\mathbb{C}^n}(0) = B^q_{\mathbb{C}^n}(0)$. This follows from 5.1, but it is also easy to see directly in this special case, since all the components F_I vanish if $I \neq (1, 2, ..., q)$. □

[10]this follows for example from the submean inequality 3.2

8. The Weak Holomorphic Morse Inequalities

In this section we prove the local version of Demailly's holomorphic Morse inequalities, over *any* complex manifold X. The usual version is obtained as a corollary.

The proof of the following lemma is based on standard elliptic boot strapping

Lemma 8.1. *For each k, suppose that $\beta^{(k)}$ is a smooth form on the ball $B_{R'_k}$ such that $\Delta^{(k)}_{\overline{\partial}}\beta^{(k)} = 0$. Identify $\beta^{(k)}$ with a form in $L^2_{\phi_0}(\mathbb{C}^n)$ by extending with zero. Then there is constant C independent of k such that*

$$\sup_{z\in B_1} \left|\beta^{(k)}(z)\right|^2_{\phi_0} \leq C \left\|\beta^{(k)}\right\|^2_{\phi_0, B_2}$$

Moreover, if the sequence of norms $\left\|\alpha^{(k)}\right\|^2_{\phi_0,\mathbb{C}^n}$ is bounded, then there is a subsequence of $\left\{\beta^{(k)}\right\}$ which converges uniformly with all derivatives on any ball in $\mathbb{C}^n$ to a smooth form β, where β is in $L^2_{\phi_0}(\mathbb{C}^n)$.

Proof. Fix a ball B_R, of radius R in $\mathbb{C}^n$. By Gårding's inequality for the elliptic operator $(\Delta^{(k)}_{\overline{\partial}})^m$,

$$\left\|\beta^{(k)}\right\|^2_{B_R,2,2m} \leq C_{R,k}\left(\left\|\beta^{(k)}\right\|^2_{B_{2R}} + \left\|(\Delta^{(k)}_{\overline{\partial}})^m\beta^{(k)}\right\|^2_{B_{2R}}\right), \tag{8.1}$$

for all positive integers m. Since $\Delta^{(k)}_{\overline{\partial}}$ converges to $\Delta_{\overline{\partial},\phi_0}$ on the ball B_2 it is straight forward to see that $C_{R,k}$ may be taken to be independent of k. For example, it follows immediately from proposition 6.4. Hence, for all positive integers m,

$$\left\|\beta^{(k)}\right\|^2_{B_1,2,2m} \leq C \left\|\beta^{(k)}\right\|^2_{B_2}$$

and the continuous injection $L^{2,k} \hookrightarrow C^0$, $k > n$, provided by the Sobolev embedding theorem, proves the first statement in the lemma. To prove the second statement assume that $\left\|\alpha^{(k)}\right\|^2_{\phi_0,\mathbb{C}^n}$ is uniformly bounded in k. Then 8.1 shows that

$$\left\|\beta^{(k)}\right\|^2_{B_R,2,2m} \leq D_R$$

Since this holds for any $m \geq 1$, Rellich's compactness theorem yields, for each R, a subsequence of $\left\{\beta^{(k)}\right\}$, which converges in all Sobolev spaces $L^{2,k}(B_R)$ for $k \geq 0$. The compact embedding $L^{2,k} \hookrightarrow C^l$, $k > n + \frac{1}{2}l$, shows that the sequence converges in all $C^l(B_R)$. Choosing a diagonal sequence, with respect to a sequence of balls exhausting $\mathbb{C}^n$, finishes the proof of the lemma. □

Now we can prove the following local weak holomorphic Morse inequalities, theorem 2.1, (see section 2 for notation).

Theorem 8.2. *Let (X, ω) be a hermitian manifold. Then, up to terms of order $o(k^n)$, the Bergman kernel function $B^{q,k}_X$ and the extremal function $S^{q,k}_X$ of the space of global $\overline{\partial}$−harmonic $(0, q)$ forms with values in $L^k \otimes E$, satisfy*

$$B^{q,k}_X(x) \leq rk^n B^q_{x,\mathbb{C}^n}(0), \quad S^{q,k}_X(x) \leq k^n S^q_{x,\mathbb{C}^n}(0) \ ,$$

where

$$B^q_{x,\mathbb{C}^n}(0) = S^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} 1_{X(q)} \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|$$

and $\lim_k \frac{1}{k^n} B^{q,k}_X(x) = \lim_k \frac{1}{k^n} S^{q,k}_X(x)$ *if* $r = 1$.

Proof. First we will prove that

$$\limsup_k k^{-n} S^{q,k}_X(x) \leq S^q_{x,\mathbb{C}^n}(0)$$

To this end, take any convergent subsequence $\left\{k_j^{-n} S^{q,k_j}(x)\right\}$. We have to prove that it is bounded by $S^q_{\mathbb{C}^n}(0)$. By definition, there is a unit norm sequence α_k in $\mathcal{H}^{0,q}(X, L^k)$ such that

$$\limsup_k k^{-n} S^{q,k}_X(x) = \limsup_k k^{-n} \left|\alpha_k(x)\right|^2.$$

Now consider the sequence $\beta^{(k_j)} := k_j^{-\frac{1}{2}n} \alpha^{(k_j)}$, where $\beta^{(k_j)}$ is a form on the ball $B_{\sqrt{k_j} R_{k_j}}$ that we identify with a form in $L^2_{\phi_0}(\mathbb{C}^n)$, by extending with zero. Note that

$$\limsup_j \left\|\beta^{(k_j)}\right\|^2_{\phi_0} = k_j^{-n} \left\|\alpha^{(k_j)}\right\|^2_{\phi_0, B_{\sqrt{k}R_k}} \leq_{k_j} \left\|\alpha_{k_j}\right\|^2_X = 1,$$

where we have used proposition 6.2 ("norm localization"). According to lemma there is a subsequence of $\left\{\beta^{(k_j)}\right\}$ that converges uniformly with all derivatives to β on any ball in $\mathbb{C}^n$, where β is smooth and $\|\beta\|^2_{\phi_0} \leq 1$. Replace the sequence with this subsequence. We now show that $\Delta_{\overline{\partial},\phi_0}\beta = 0$. To this end, we write

$$\Delta_{\overline{\partial},\phi_0}\beta = \Delta_{\overline{\partial},\phi_0}(\beta - \beta^{(k_j)}) + (\Delta_{\overline{\partial},\phi_0} - \Delta^{(k_j)}_{\overline{\partial}})(\beta^{(k_j)}) + 0 \tag{8.2}$$

on $B_{R'_k}$. By proposition 6.4 the sequence $\Delta_{\overline{\partial},\phi_0} - \Delta^{(k_j)}_{\overline{\partial}}$, consists of differential operators with coefficients tending uniformly to zero on any ball. Since $\left\{\beta^{(k_j)}\right\}$ converges uniformly with all derivatives on any ball in $\mathbb{C}^n$ we infer that the right hand side in 8.2 converges uniformly to the zero function on any ball. In particular, $\Delta_{\overline{\partial},\phi_0}\beta = 0$.

Finally, we see that

$$\lim k^{-n} S^{q,k}(x) = \lim k^{-n} \left|\alpha_k(x)\right| = \lim \left|\beta^{(k_j)}(0)\right|^2 = \left|\beta(0)\right|^2 \leq \frac{\left|\beta(0)\right|^2}{\|\beta\|^2_{\phi_0}} \leq S^q_{x,\mathbb{C}^n}(0),$$

which shows that

$$S^{q,k}_X(x) \leq k^n S^q_{x,\mathbb{C}^n}(0) + o(k^n)$$

Moreover, by proposition 7.5,

$$S^q_{x,\mathbb{C}^n}(0) = B^q_{x,\mathbb{C}^n}(0) = \frac{1}{\pi^n} 1_{X(q)}(x) \left| det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x \right|.$$

Lemma 5.1, then shows that $B^{q,k}_X(x) \leq o(k^n)$ outside the set $X(q)$.

Next, if x is in $X(q)$ we may assume that λ_1 up to λ_q are the negative eigenvalues. By the proof of proposition 7.5 we then have that $\beta^I = 0$ if $I \neq (1,2,...,q)$. We deduce that for $I \neq (1,2,...,q)$:

$$\lim k_j^{-n} S_I^{k_j,q}(0) = \lim k_j^{-n} \left| \alpha_{k_j}^I(0) \right|^2 = \left| \beta^I(0) \right|^2 = 0.$$

This proves that

$$\lim k^{-n} S_I^{k,q}(0) = 0,$$

if $I \neq (1,2,...,q)$. Finally, lemma 5.1 shows that

$$\limsup_k k^{-n} B_X^{k,q}(x) \leq 0 + 0 + ... + S_{x,\mathbb{C}^n}^q(0) = B_{x,\mathbb{C}^n}^q(0),$$

which finishes the proof of the theorem, when $r = 1$. $\square$

As a corollary we obtain Demailly's weak holomorphic Morse inequalities:

Corollary 8.3. *Suppose that X is compact. Then*

$$\dim_{\mathbb{C}} H_{\overline{\partial}}^{0,q}(X, L^k \otimes E) \leq k^n r \frac{(-1)^q}{\pi^n} \frac{1}{n!} \int_{X(q)} (\frac{i}{2} \partial\overline{\partial}\phi)^n + o(k^n).$$

Proof. Consider the sequence of functions $k^{-n} B_X^{q,k}(x)$ on X. First note that $k^{-n} S_X^{q,k}(x)$ is dominated by an L^1-function if X is compact. Indeed, lemma 8.1 implies that for any unit norm sequence a_k in $\mathcal{H}^{0,q}(X, L^k)$

$$\sup_{x \in B_{\frac{1}{\sqrt{k}}}(x_0)} \left| \alpha_k(x) \right|^2 \leq C(x) \left\| \alpha^{(k)} \right\|_{\phi_0, B_1}^2 \leq_k C(x) k^n,$$

where we have used the norm localization (proposition 6.2) as in the proof of the previous theorem. Since X is compact it can, for each k, be covered by a *finite* number of the balls $B_{\frac{1}{\sqrt{k}}}(x_0)$, so that we can choose a constant C, with the property that for all x in X

$$k^{-n} \left| \alpha_k(x) \right|^2 \leq_k C,$$

This shows that $k^{-n} S_X^{q,k}(x) \leq C$. By 5.1 and the fact that X has finite volume, this means that the sequence $k^{-n} B_X^{q,k}(x)$ is dominated by an L^1-function. Finally, the Hodge theorem shows that

$$\limsup_k dim_{\mathbb{C}} k^{-n} H_{\overline{\partial}}^{0,q}(X, L^k) = \limsup_k \int_X k^{-n} B_X^{q,k}$$

and Fatou's lemma yields, since the sequence $k^{-n} B_X^{q,k}$ is L^1- dominated,

$$\int \limsup k^{-n} B_X^{q,k} \leq r \frac{(-1)^q}{\pi^n} \int_{X(q)} (\frac{i}{2} \partial\overline{\partial}\phi)^n,$$

where we have used the previous theorem and the fact that

$$\frac{1}{n!} (\frac{i}{2} \partial\overline{\partial}\phi)^n = (-1)^q \left| det_\omega (\frac{i}{2} \partial\overline{\partial}\phi)_x \right| Vol_X$$

on $X(q)$. $\square$

Remark 8.4. It is illuminating to see what goes wrong on a non-compact manifold, when one tries to go from the local version of the weak Morse inequalities to their global counter part, corollary 8.3. As in section 3, this is already clear in the simplest case, when $q = 0$. Indeed, let $X = D^1$ be the open unit disc in $\mathbb{C}$ and let $\phi = \phi_0 = 0$. Then $\mathcal{H}^{0,0}_{\overline{\partial}+\overline{\partial}^*}(X, L^k)$ is the usual space of square integrable holomorphic function on the unit disc. In particular, it is independent of k. It is well-known that this is an infinite dimensional space; e.g. polynomials are linearly independent. Equivalently,

$$\int_{D^1} B^{0,k}_{D^1} = \infty.$$

But applying theorem 9.5 naively would give

$$\limsup_k k^{-n} \int_{D^1} B^{0,k}_{D^1} = \int_{D^1} \limsup k^{-n} B^{0,k}_{D^1} \leq 0,$$

which clearly yields a contradiction. Hence, something goes wrong when we interchange the integration and the taking of the limit. Well, as in the beginning of section 3, for any function f holomorphic in D^1

$$\frac{|f(x)|^2}{\|f\|^2_{D^1}} \leq \left(\int_{B_{\sqrt{k}R_k}(x)} 1 \right)^{-1} \to 0,$$

but we have to start with smaller and smaller balls, i.e. a larger and larger k, in the right hand side when x is taken closer and closer to the boundary! Otherwise, the ball will not be contained in the unit disc. This means that there is no k, independent of x, for which the sequence $\left(\int_{B_{\sqrt{k}R_k}(x)} 1 \right)^{-1}$ is even defined. Hence, $B^{0,k}_{D^1}$ will not be dominated by an L^1- function, by this argument, and we cannot invoke Fatou's lemma!

9. The Strong Holomorphic Morse inequalities

In this section X is assumed to be compact. While the weak holomorphic Morse inequalities give estimates on individual cohomology groups their strong counter parts yield bounds on the alternating sums:

$$\sum_{i=0}^{q} (-1)^{q-i} \dim_{\mathbb{C}} H^i(X, L^k \otimes E)$$

for each $q \geq 1$. In fact, for $q = n$ this is actually an asymptotic *equality*, giving an asymptotic version of the Riemann-Roch theorem for $L^k \otimes E$, which computes the Euler characteristic of the complex $H^*(X, L^k \otimes E)$. The proof of the estimates is based on an asymptotic expression for the dimension of

$$\mathcal{H}^i_{\leq \nu_k}(X, L^k \otimes E),$$

the space spanned by the eigenforms of $\Delta_{\overline{\partial}}$ whose eigenvalues are bounded by ν_k, where $\{\nu_k\}$ is a judicially chosen sequence. The other ingredient is the following well-known lemma:[11]

[11]see the comments in section 2.

Lemma 9.1. *We have that*

$$\sum_{i=0}^{q}(-1)^{q-i}\dim_{\mathbb{C}}\mathcal{H}^i(X,L^k\otimes E)\leq\sum_{i=0}^{q}(-1)^{q-i}\dim_{\mathbb{C}}\mathcal{H}^i_{\leq\nu_k}(X,L^k\otimes E),$$

with equality if $q=n$.

The point is that for any μ $\left(\mathcal{H}^*_\mu(X,L^k\otimes E),\overline{\partial}\right)$ forms a finite dimensional sub complex of the Dolbeault complex $\left(\Omega^*(X,L^k\otimes E),\overline{\partial}\right)$ which is exact for non-zero μ. The fact that $\overline{\partial}$ commutes with $\Delta_{\overline{\partial}}$ ensures that it is a complex. The lemma follows from this and a simple homological algebra argument (see [6]).

The asymptotic expression

$$\dim_{\mathbb{C}}\mathcal{H}^i_{\leq\nu_k}(X,L^k\otimes E)=rk^n\frac{(-1)^q}{\pi^n}\frac{1}{n!}\int_{X(q)}(\frac{i}{2}\partial\overline{\partial}\phi)^n+o(k^n)$$

will be obtained by an extension of the previous methods (section 8). The basic idea is as follows. Assume, for simplicity, that there is no twisting bundle E. First we obtain an *upper* bound on the corresponding Bergman kernel function, i.e. the spectral function, by a direct generalizaton of the harmonic case: the local weak Morse inequalities. We just have to make sure that terms of the form $\left\|(\Delta^{(k)}_{\overline{\partial}})^m\beta^{(k)}\right\|^2_{\sqrt{k}R_k}$, which are now non-zero, tend to zero with k. Recall, that $\Delta^{(k)}_{\overline{\partial}}$ denotes the $\overline{\partial}$-Laplacian with respect to the scaled metrics on the ball $B_{\sqrt{k}R_k}$. In fact, for scaling reasons they give contributions which are polynomial in $\frac{\nu_k}{k}$. This dictates the choice $\nu_k=\mu_k k$ with μ_k tending to zero with k. The last step is to get a *lower* bound on the spectral density function on $X(q)$, which amounts to proving the existence of a unit norm sequence $\{\alpha_k\}$ in $\mathcal{H}^i_{\leq\nu_k}(X,L^k)$ with

$$|a_k(x)|^2=k^n\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|+o(k^n).$$

To this end we first take a sequence $\{\alpha_k\}$ of $(0,q)-$forms on $\mathbb{C}^n$, which are harmonic with respect to the *flat* metric and the fiber metric $k\phi_0$, having the corresponding property. The point is that the mass of these forms concentrate around 0, when k tends to infinity. Hence, by cutting down their support to small decreasing balls we obtain global forms $\widetilde{\alpha_k}$ on X, that are of unit norm in the limit. Moreover, their Laplacians are "small". By cutting the high frequencies, that is projecting on $\mathcal{H}^i_{\leq\nu_k}(X,L^k)$, we finally get the sought after sequence.

We now proceed to carry out the details of the argument sketched above.

Proposition 9.2. *Assume that $\mu_k\to 0$. Then the following estimate holds:*

$$B^{q,k}_{\leq\mu_k k}(x)\leq k^n r\frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|+o(k^n).$$

Proof. In the proof we shall assume that there is no twisting vector bundle E. The proof is a simple modification of the proof of the local holomorphic Morse inequalities and in what follows these modifications will be presented. The difference is that α_k is in $\mathcal{H}^{0,q}_{\leq\mu_k k}(X,L^k)$ and we have to make sure that the all terms of the form

$(\Delta_{\overline{\partial}}^{(k)})^m(k^{-\frac{n}{2}}\alpha^{(k)}) = (\Delta_{\overline{\partial}}^{(k)})^m(\beta^{(k)})$ vanish in the limit. But for any ball B,

$$\left\|(\Delta_{\overline{\partial}}^{(k)})^m\beta^{(k_j)}\right\|^2_{\phi_0,B} \leq k^{-n}\left\|(\Delta_{\overline{\partial}}^{(k)})^m\alpha^{(k_j)}\right\|^2_{\phi_0,B_{\sqrt{k}R_k}} \leq_k k^{-2m}\left\|(\Delta_{\overline{\partial}})^m\alpha_k\right\|^2_X$$

and the last term is bounded by a sequence tending to zero:

$$k^{-2m}(\mu_k k)^{2m} \to 0,$$

since by assumption α_k is of unit norm and in $\mathcal{H}^{0,q}_{\leq\mu_k k}(X, L^k)$ and $\mu_k \to 0$. Thus

$$\left\|\beta^{(k_j)}\right\|^2_{\phi_0,B,2,m} \leq C\left(\left\|\beta^{(k_j)}\right\|^2_{\phi_0,B} + \left\|(\Delta_{\overline{\partial}}^{(k)})^m\beta^{(k_j)}\right\|^2_{\phi_0,B}\right) \leq_j C1 + (\mu_{k_j})^{2m}) \leq_j C,$$

shows that the conclusion of lemma 8.1 is still valid. Finally, $\Delta_{\overline{\partial}}\beta = 0$ more or less as before. The only difference is that we now get a third term in 8.2, whose norm on any ball converges to zero, by the argument given above. This shows that $\left\|\Delta_{\overline{\partial}}\beta\right\|^2_B = 0$ on any ball, and the rest of the argument goes through word by word. □

The next lemma provides the sequence that takes the right values at a given point x in $X(q)$, with "small" Laplacian, that was referred to in the beginning of the section.

Lemma 9.3. *For any point x_0 in $X(q)$, there is a sequence $\{\alpha_k\}$ such that a_k is in $\Omega^{0,q}(X, L^k)$ with* $\lim_k \|\alpha_k\|^2 = 1$,

$$\begin{array}{rcl}\lim_k \left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|^2 & = & 0 \\ \left|a_k(x_0)\right|^2 & = & k^n\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|\end{array}$$

Moreover, there is a sequence δ_k , independent of x_0 and tending to zero, such that

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle_X \leq \delta_k$$

Proof. We may assume that the first q eigenvalues λ_{i,x_0} are negative, while the remaining eigenvalues are positive. Let $\beta = \left(\frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}\right)^{\frac{1}{2}} e^{+\sum_{i=1}^q \lambda_i|z_i|^2} d\overline{z_1} \wedge d\overline{z_2} \wedge \ldots \wedge d\overline{z_q}$, so that[12] $|\beta|^2_{\phi_0} = \frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}e^{-\sum_{i=1}^n|\lambda_i||z_i|^2}$ and $\|\beta\|_{\phi_0,\mathbb{C}^n} = 1$. Observe that β is in $L^{2,m}_{\phi_0}$ for all m. Now define α_k on X by

$$\alpha^{(k)}(w) := \alpha_k(\frac{w}{\sqrt{k}}) := k^{\frac{n}{2}}\chi_{\sqrt{k}R_k}(w)\beta(w),$$

where $\chi_{\sqrt{k}R_k}$ is the cut-off function defined in section 4, so that $|a_k(x)|^2 = k^n\frac{|\lambda_1||\lambda_2|\cdots|\lambda_n|}{\pi^n}$. To see that $\lim_k \|\alpha_k\|^2_X = 1$, note that

$$\|\alpha_k\|^2_X =_k \left\|\chi_{R_{k'}}\beta\right\|^2_{\phi_0,\mathbb{C}^n} = \|\beta\|^2_{\phi_0,\frac{1}{2}\sqrt{k}R_k} + \left\|\chi_{R_{k'}}\beta\right\|^2_{\phi_0,\geq\frac{1}{2}\sqrt{k}R_k}$$

and the "tail" $\|\beta\|^2_{\phi_0,\geq\frac{1}{2}\sqrt{k}R_k}$ tends to zero, since β is in $L^2_{\phi_0}(\mathbb{C}^n)$ and $\sqrt{k}R_k$ tends to infinity.

[12] compare the proof of proposition 7.5.

Next, we show that $\lim_k \|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\|_X^2 = 0$. Changing variables gives (proposition 6.3)

$$\left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|_X^2 \leq_k \left\|(\Delta_{\overline{\partial}}^{(k)})^m(\chi_{\sqrt{k}R_k}\beta)\right\|_{\phi_0,\sqrt{k}R_k}^2$$

and using Leibniz' rule (6.2) repeatedly yields

$$\left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|_X^2 \leq_k \left\|(\Delta_{\overline{\partial}}^{(k)})^m\beta\right\|_{\phi_0,\sqrt{k}R_k}^2$$

Since $\Delta_{\overline{\partial},\phi_0}\beta = 0$ we conclude, using proposition 3.5, that

$$\left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|_X^2 \leq_k \left\|(\Delta_{\overline{\partial}}^{(k)})^m\beta - (\Delta_{\overline{\partial},\phi_0})^m\beta\right\|_{\phi_0,\sqrt{k}R_k}^2 \leq \epsilon_k \|\beta\|_{\phi_0,\sqrt{k}R_k,2,m}^2 \to 0,$$

since β is in $L_{\phi_0}^{2,m}$ for all m. To prove the last statement, observe that, as above,

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle = \left\|\frac{1}{\sqrt{k}}(\overline{\partial} + \overline{\partial}^*)\alpha_k\right\|_X^2 =_k \left\|(\overline{\partial} + \overline{\partial}^{*(k)})(\chi_{\sqrt{k}R_k}\beta)\right\|_{\sqrt{k}R_k}^2.$$

Hence, by Leibniz' rule

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle \leq_k \left\|(\chi_{\sqrt{k}R_k}(\overline{\partial} + \overline{\partial}^{*(k)})\beta\right\|_{\sqrt{k}R_k}^2 + \frac{C}{(\sqrt{k}R_k)^2}\|\beta\|_{\mathbb{C}^n}^2.$$

Using proposition 6.4 we see that

$$\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle \leq_k \varepsilon_k \|\beta\|_{2,1}^2 + \frac{C}{(\sqrt{k}R_k)^2}\|\beta\|^2.$$

Note that even if $\|\beta\|^2$ is independent of the eigenvalues λ_{i,x_0}, the norms $\left\|\partial^l\beta\right\|^2$depend on the eigenvalues, and hence on the point x_0. But the dependence amounts to a factor of eigenvalues and since X is compact, we deduce that $\left\|\partial^l\beta\right\|^2$is bounded by a constant independent of the point x_0. This shows that $\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\rangle_X \leq \delta_k$. Note that ϵ_k also can be taken to be independent of the point x_0, by a similar argument. $\square$

Proposition 9.4. *Assume that $\frac{\delta_k}{\mu_k} \to 0$ where δ_k is the sequence appearing in lemma 9.3. Then, for any point x in $X(q)$, the following holds*

$$\liminf k^{-n}B_{\leq\mu_k k}^{q,k}(x) \geq r\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

Proof. Let $\{\alpha_k\}$ be the sequence that lemma 9.3 provides and decompose it with respect to the orthogonal decomposition $\Omega^{0,q}(X, L^k) = \mathcal{H}_{\leq\mu_k k}^q(X, L^k) \oplus \mathcal{H}_{>\mu_k k}^q(X, L^k)$, induced by the spectral decomposition of the elliptic operator $\Delta_{\overline{\partial}}$:

$$\alpha_k = \alpha_{1,k} + \alpha_{2,k}$$

First, we prove that

$$\lim_k k^{-n}\left|\alpha_2^{(k)}(0)\right|^2 = 0.$$

As in the proof of lemma 8.1 we have that

$$k^{-n}\left|\alpha_2^{(k)}(0)\right|^2 \leq C(x)\left(k^{-n}\left\|\alpha_2^{(k)}\right\|_{(k),B_1}^2 + k^{-n}\left\|(\Delta_{\overline{\partial}}^{(k)})^m\alpha_2^{(k)}\right\|_{(k),B_1}^2\right).$$

To see that the first term tends to zero, observe that by the spectral decomposition of $\Delta_{\overline{\partial}}$:

$$\|\alpha_{2,k}\|_X^2 \leq \frac{1}{\mu_k k}\left\langle \Delta_{\overline{\partial}}\alpha_{2,k}, \alpha_{2,k}\right\rangle_X \leq \frac{1}{\mu_k}\left\langle k^{-1}\Delta_{\overline{\partial}}\alpha_k, \alpha_k\right\rangle_X \leq \frac{\delta_k}{\mu_k}$$

Furthermore, the second term also tends to zero:

$$k^{-n}\left\|(\Delta_{\overline{\partial}}^{(k)})^m\alpha_2^{(k)}\right\|_{(k),B_1}^2 \leq \left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_{2,k}\right\|_X^2 \leq \left\|k^{-m}(\Delta_{\overline{\partial}})^m\alpha_k\right\|_X^2 \to 0.$$

by the choice of the sequence $\{\alpha_k\}$. Finally,

$$k^{-n}S_{\mu_k k}^{q,k}(x) \geq \frac{k^{-n}\left|\alpha_{k,1}(0)\right|^2}{\|\alpha_{k,1}\|_X^2} \geq k^{-n}\left|\alpha_{1,k}(0)\right|^2 = k^{-n}\left|\alpha_k(0)-\alpha_{2,k}(0)\right|^2$$

and

$$k^{-n}\left|\alpha_k(0)-\alpha_{2,k}(0)\right|^2 \geq_k \frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + 0,$$

which finishes the proof according to lemma 5.1 when there is now twisting bundle E (or $r=1$).

The generalization to the case when there is a twisting bundle E is not quite as direct as in the other results. Take a local frame $\{e_\gamma\}_1^r$ of the twisting bundle E, that is orthonormal on some neighbourhood of x. Then we get r global $(0,q)-$forms with values in $L^k \otimes E$.

$$\alpha_k \otimes e_1,\ \alpha_k \otimes e_2,\ \ \alpha_k \otimes e_r,$$

where α_k is as above. Proceeding as above, we obtain r forms in $\mathcal{H}^q_{\leq \mu_k k}(X, L^k)$. If these forms were mutually orthogonal we would be done, since we could obtain an orthonormal base of $\mathcal{H}^q_{\leq \mu_k k}(X, L^k)$ containing the r forms. However, the forms are not far from being mutually orthogonal in the following sense: they can be written as $\alpha_k \otimes e_1$ plus an L^2- small form that is "small" at x for large k, as in the case when there is no twisting bundle, above. This is enough to make the argument work. The details are omitted. □

Now we can prove the following asymptotic equality:

Theorem 9.5. *Assume that $\mu_k \to 0$ and that $\frac{\delta_k}{\mu_k} \to 0$, where δ_k is the sequence appearing in lemma. Then*

$$B_{\leq \mu_k k}^{q,k}(x) = rk^n\frac{1}{\pi^n}1_{X(q)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + o(k^n)$$

Proof. This follows immediately from proposition 9.2 and proposition 9.4 if x is in $X(q)$. If x is outside of $X(q)$ then the upper bound given by 9.2 shows that $B_{\leq \mu_k k}^{q,k}(x) = 0 + o(k^n)$, which finishes the proof of the theorem. □

Remark 9.6. A slightly more involved argument shows that the conclusion is valid for $B_{\leq \mu_0 k^\epsilon}^{q,k}(x)$, for any μ_0, as long as $0 < \epsilon < 1$.

Combining the previous theorem with lemma 9.1 yields a proof of the strong holomorphic Morse inequalities:

Theorem 9.7. *Assume that X is compact. Then*

$$\sum_{i=0}^{q}(-1)^{q-i}\dim_{\mathbb{C}} H^i(X, L^k\otimes E)\leq k^n r\frac{(-1)^q}{\pi^n}\frac{1}{n!}\int_{X(\leq q)}(\frac{i}{2}\partial\overline{\partial}\phi)^n+o(k^n).$$

Proof. By lemma 9.1

$$k^{-n}\sum_{i=1}^{q}(-1)^{q-i}\dim_{\mathbb{C}}\mathcal{H}^i(X,L^k)\leq\sum_i(-1)^{q-i}k^{-n}\dim_{\mathbb{C}}\mathcal{H}^i_{\leq\nu_k}(X,L^k),$$

with equality for $n=1$. Moreover, $k^{-n}\dim_{\mathbb{C}}\mathcal{H}^i_{\leq\nu_k}(X,L^k)=\int_{X(i)}k^{-n}B^{q,k}_{\leq\nu_k}(x)Vol_X.$ Now set $\nu_k:=(\delta_k)^{\frac{1}{2}}k$, where δ_k is the sequence appearing in lemma 9.3. By theorem 9.5

$$\lim_k k^{-n}B^{q,k}_{\leq\nu_k}(x)=\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

Since X is compact, a similar argument to the one given in the beginning of the proof of corollary 8.3 shows that the sequence $B^{q,k}_{\leq\sqrt{k}}(x)$ is bounded by an L^1- function. Finally, Fatou's lemma yields

$$\limsup_k k^{-n}\sum_{i=1}^{q}(-1)^{q-i}\dim_{\mathbb{C}}\mathcal{H}^i(X,L^k)\leq\sum_i(-1)^{q-i}\frac{1}{\pi^n}\int_{X(i)}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|Vol_X,$$

with equality for $n=1$.Since $\frac{1}{n!}(\partial\overline{\partial}\phi)^n=(-1)^i\left|det_\omega(\partial\overline{\partial}\phi)_x\right|Vol_X$ on $X(i)$, this finishes the proof of the theorem. □

Remark 9.8. In the case when there is at least one point x_0 where the curvature two-form of L is positive, and no point where it has one negative eigenvalue and $n-1$ positive ones, the weak and strong Morse inequalities together imply that there are "many" holomorphic sections of L :

$$\dim_{\mathbb{C}} H^0(X,L^k)=Ck^n.$$

This fact, in turn, shows that X is Moishezon, which proves the Grauert-Riemenschneider conjecture, as was observed by Demailly [6]. However, it is interesting to see that one does not really have to use the *strong* Morse inequalities at all, as is shown by Berndtsson in [1]. In rough terms the argument goes as follows. The space of all polynomials of degree less than $\frac{1}{2}k$ forms a space of local holomorphic sections with norms concentrated on a small ball around x_0. The dimension of this space is clearly of the order k^n. Now, the integrated version of proposition 9.2.[13], $\dim_{\mathbb{C}}\mathcal{H}^1_{\leq\mu_k k}(X,L^k)=o(k^n)$, says that one can solve many $\overline{\partial}-$equations with a good L^2 estimate, the space of right hand sides for which this cannot be done, being of dimension $o(k^n)$. This means that "almost all" the local candidates above, may be extended to global holomorphic sections, showing that $\dim_{\mathbb{C}} H^0(X,L^k)=Ck^n$. It would be interesting to see if this argument could be refined so as to allow a local control of the modified sections around x_0.

[13][1] uses a similar estimate instead - based on different methods - under the stronger assumption that L is semi-positive.

10. Line bundles with curvature conditions and existence of peak sections

Combining the local weak Morse inequalities for $q=0$, the weak Morse inequalities for $q=1$ and the strong Morse inequalities for $q=1$ yields the following theorem:

Theorem 10.1. *Let X be compact and assume that there is no point in X, where the curvature two-form of L has one negative eigenvalue and $n-1$ positive ones. Then*

$$\limsup B_X^{0,k}(x) = r\frac{k^n}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + o(k^n)$$

almost everywhere on the part of X where the curvature of L is positive.

Proof. Since $\mathcal{H}^0(X,L^k) = H^0(X,L^k)$, the weak local Morse inequalities (theorem 2.1) show that

$$\limsup \frac{1}{k^n}B_X^{0,k}(x) \leq 1_{X(0)}\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|, \tag{10.1}$$

for *any* line bundle L (in fact, this is the easiest case that was dealt with in the beginning of section 3). Furthermore, since $X(1)$ is empty, the weak Morse inequalities for $q=1$ (corollary 8.3) say that

$$\dim_{\mathbb{C}} H^1(X,L^k) \leq o(k^n).$$

Combining this fact with the strong Morse inequalities for $q=1$ (theorem 9.7), shows that

$$\dim_{\mathbb{C}} \frac{1}{k^n}H^0(X,L^k) \geq \int_X 1_{X(0)}\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| Vol_{(X,\omega)}.$$

Now, since the sequence $\limsup \frac{1}{k^n}B_X^{0,k}(x)$ is bounded by an L^1- function (compare the proof of corollary 8.3), Fatou's lemma together with the latter inequality, give that

$$\int_X \limsup \frac{1}{k^n}B_X^{0,k}(x)Vol_{(X,\omega)} \geq \int 1_{X(0)}\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| Vol_{(X,\omega)}. \tag{10.2}$$

Finally, combining 10.1 and 10.2, proves the theorem. □

The most interesting case when the previous theorem applies is when L is semi-positive and strictly positive at some point.[14] The following corollary allows a certain pointwise control of global holomorphic sections, when the curvature conditions in the previous theorem hold.

Corollary 10.2. *Suppose that X and L satisfy the conditions of the previous theorem. Then for almost all points x in $X(0)$ (the part of X where the curvature of*

[14]For example, if X is projective this curvature assumption implies that L is nef and big, in algebraic geometric terms. The converse is "almost true" (see [7]).

L is positive) there is a unit norm sequence $\{\alpha_{k_j}\}$ (depending on the point x) of holomorphic section with values in L^{k_j} with the following properties:

$$\lim \int_{B_{R_{k_j}}(x)} \left|\alpha_{k_j}\right|^2 = 1,$$

for any sequence $B_{R_{k_j}}(x)$ as long as $R_{k_j} \to 0$, $\sqrt{k_j}R_{k_j} \to \infty$ and $\sqrt{k_j}R_{k_j}^3 \to 0$.

Proof. For any point x in X the argument that leads to the local weak Morse inequalities for holomorphic sections, given in section 3, shows that

$$\left|\alpha_{k_j}(x)\right| \leq \rho_k \frac{k^n}{\pi^n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| \left\|\alpha_{k_j}\right\|^2_{B_{R_k}}$$

for *any* sequence of global holomorphic sections $\{\alpha_k\}$, where ρ_k is a sequence tending to one. By the previous theorem we can, for almost all points x in the part of X where the curvature of L is positive, choose such a *unit norm* sequence $\{\alpha_{k_j}\}$ with the additional property that

$$\left|\alpha_{k_j}(x)\right|^2 = \frac{k_j^n}{\pi^n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + o(k_j^n),$$

Hence,

$$\frac{k_j^n}{\pi^n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + o(k_j^n) \leq \rho_k \frac{k^n}{\pi^n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| \left\|\alpha_{k_j}\right\|^2_{B_{R_k}}$$

Dividing by k_j^n on both sides and letting k_j tend to infinity shows that

$$\liminf \left\|\alpha_{k_j}\right\|^2_{B_{R_k}} = 1,$$

which proves the corollary. □

In fact, it seems reasonable to conjecture that the previous equality actually holds *everywhere* on X, replacing $\limsup$ with $\lim$.

Conjecture 10.3. *Under the assumptions of the previous theorem*

$$\lim_k B_X^{0,k}(x) = r\frac{k^n}{\pi^n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right| + o(k^n),$$

on the part of X where L is positive.

When L is positive the next theorem yields a refinement of the previous theorem.

Theorem 10.4. *Assume that (X,ω) is a complete Kähler manifold and that (L,ϕ) is a positive hermitian holomorphic line bundle over X with $\partial\overline{\partial}\phi > \epsilon\omega$ for some positive number ϵ. Then, up to terms of order $o(k^n)$,*

$$B_X^{0,k}(x) = k^n r \frac{1}{\pi^n} \left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

Proof. First observe that by lemma 5.1 $B_X^{0,k}(x) = S_X^{0,k}(x)$. In this special case, where $q = 0$, the upper bound on $S_X^{0,k}(x)$, which in general follows from the local weak holomorphic Morse inequalities, is considerably easier to obtain than in the general case, as was explained in section 3.

To prove the lower bound, we have to find a sequence γ_k in $H^{0,0}(X,L^k)$ with $\frac{|\gamma_k(x)|^2}{\|\gamma_k\|_X^2}=k^n\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|+o(k^n)$. To this end, fix a point x and first consider the following sequence of functions on X

$$\alpha_k(z)=\frac{k^n}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|\chi_{R_k}(z),$$

where χ_{R_k} is the cut-off function defined in section 4. The sought after sequence will be obtained as α_k+u_k, where u_k is determined as follows. By a well-known generalization of Hörmander's L^2- method (see [11],[7]) the fact that $\partial\overline{\partial}\phi>\epsilon\omega$ implies that, for any k, we can find a u_k in $H^{0,0}(X,L^k)$ such that

$$\overline{\partial}u_k(z)=\overline{\partial}(-\alpha_k),\text{ and }\|u_k\|_X^2\leq C\frac{1}{k}\left\|\overline{\partial}\alpha_k\right\|_X^2.$$

Thus $u_k+\alpha_k$ is holomorphic and

$$\|u_k\|_X^2\leq C\frac{1}{k}\left\|\overline{\partial}\alpha_k\right\|_X^2\leq_k C\frac{1}{k}\frac{1}{(R_k)^2}e^{-\frac{1}{4}k(R_k)^2}\rightarrow 0,\ k\rightarrow\infty,$$

since $\sqrt{k}R_k\rightarrow\infty$. This also implies that $u_k(x)\rightarrow 0$. Indeed, since u_k is holomorphic in the ball $B_{1/2R_k}(x)$ this follows from the argument that gave the upper bound on $S_X^{0,k}(x)$. Finally,

$$\limsup_k k^{-n}\frac{|\gamma_k(x)|^2}{\|\gamma_k\|_X^2}=\limsup_k k^{-n}\frac{|a_k(x)+u_k(x)|^2}{\|a_k+u_k\|_X^2}=\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|,$$

which finishes the proof of the theorem. □

Finally, we shall give a local version of Serre's classical theorem A. A similar argument has been used by Bouche to give a proof of the Kodaira embedding theorem, when L is (strictly) positive [4].

Theorem 10.5. *Let X be compact and assume that there is no point in X, where the curvature two-form of L has one negative eigenvalue and $n-1$ positive ones. If Ω is a relatively compact subset of the part of X where L is positive, then for any k_0 there is a k bigger than k_0 such that $H^0(X,L^k\otimes E)$ generates the fibers of $L^k\otimes E$ over almost all points in Ω.*

Proof. According to 5.1,

$$B_X^{0,k}(x)\leq M^{0,k}(x)S_E^0(x)$$

Using theorem 10.1 and the local Morse inequalities, theorem 2.1 for $q=0$, we infer that

$$r\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|\leq(\limsup M_k^0)\frac{1}{\pi^n}\left|det_\omega(\frac{i}{2}\partial\overline{\partial}\phi)_x\right|$$

almost everywhere on the set where L is positive. This means that for all such x we can for any given k_0 find a k bigger than k_0 such that $M^{0,k}(x)=r$. The theorem now follows from a simple compactness argument, since $M^{0,k}(x)=r$ is an open condition with respect to x. □

It should be possible to prove the previous theorem by first showing that X is Moishezon, as in the papers by Siu and Demailly, and then use Kodaira's "blow-up trick" as in the usual proof of the Kodaira embedding theorem [9]. In fact, this would prove a slightly stronger result, saying that the statement holds on all of Ω for sufficiently large k. However, it is interesting to see that theorem 10.5 may be obtained *directly* from a local asymptotic control of the Bergman kernel function. Moreover, if conjecture 10.3 is correct the same method would prove the stronger statement referred to above.

11. Appendix

In the following proofs we will assume that there is no twisting bundle E. The generalizations of the proofs are straight forward. Fix a point x in X. Recall (section4) that we have chosen complex coordinates $\{z_i\}$ and a holomorphic trivializing section s of L around x such that

$$\omega(z) = \tfrac{i}{2}\textstyle\sum_{i,j} h_{ij}(z)dz_i\wedge\overline{dz_j},\ h_{ij}(0)=\delta_{ij} \quad \text{and} \quad |s(z)|^2 = e^{-\phi(z)},\ \phi(z)=\phi_0(z)+O(|z|^3),$$

where

$$\phi_0(z) := \sum_{i=1}^n \lambda_{i,x}\,|z_i|^2 .$$

The corresponding norms will be denoted by $\|\cdot\|$. Furthermore, recall that the scaled base metric $\omega^{(k)}(w)=\omega(\frac{w}{\sqrt{k}})$ and the scaled fiber metric $(k\phi)^{(k)}(w)=k\phi(\frac{w}{\sqrt{k}})$ on L are defined on the ball $B_{\sqrt{k}R_k}$ in $\mathbb{C}^n$ and that the corresponding norms on $B_{\sqrt{k}R_k}$ are denoted by $\|\cdot\|_{(k)}$. We denote by $\overline{\partial}^{*(k)}$ the formal adjoint of $\overline{\partial}$, with respect to the latter norm.

Proof of lemma 6.1. Let $s=(s_1,...,s_{2n})$ parametrize $\mathbb{R}^{2n}\cong\mathbb{C}^n$ and let $t=\sqrt{k}s$. By Taylor's theorem there are smooth g_α, bounded by C on B_{R_k}, such that

$$\phi(s)=\phi_0(s)+\sum_{|\alpha|=3} g_\alpha(s)s^\alpha,\ s\in B_{R_k} \quad (k\phi)^{(k)}(t)=\phi_0(t)+\sum_{|\alpha|=3} g_\alpha(\frac{t}{\sqrt{k}})\frac{1}{\sqrt{k}}t^\alpha,\ t\in B_{\sqrt{k}R_k},$$

Thus

$$\sup_{t\in B_{\sqrt{k}R_k}} \left|(k\phi)^{(k)}(t)-\phi_0(t)\right| \le C\frac{1}{\sqrt{k}}(\sqrt{k}R_k)^3,$$

which tends to zero, by the choice of balls. The derivatives are estimated in a similar fashion. The point is that there is a factor $\frac{1}{\sqrt{k}}$ in the remainder term, and that $\frac{1}{\sqrt{k}}(\sqrt{k}R_k)^3$ tends to zero.

We can now show how to localize the norms.

Proof of proposition 6.2. Observe that

$$\|\beta_k\|^2_{R_k} = \sum_{I,J}\int_{B_{R_k}} |f_{IJ}(z)|^2\, h_{IJ}(z)e^{-k\phi(z)}\det(h_{ij})(z)dm(z), \tag{11.1}$$

where $dm(z)$ is the Lebesgue measure and $h_{IJ}(z)$ is a certain polynomial in the metric coefficients h_{ij} (see [16]). Moreover, by definition, $\left\|\beta^{(k)}\right\|^2_{(k)}$ is the norm with respect to the scaled metrics on the scaled ball:

$$\left\|\beta^{(k)}\right\|_{(k)}^2:=\sum_{I,J}\int_{\mathbf{B}_{\sqrt{k}R_k}}\left|f_{IJ}(\frac{w}{\sqrt{k}})\right|^2h_{IJ}(\frac{w}{\sqrt{k}})e^{-k\phi(\frac{w}{\sqrt{k}})}\det(h_{ij})(\frac{w}{\sqrt{k}})dm(w).$$

Lemma 6.1 shows that we may replace the scaled fiber metric $k\phi(\frac{w}{\sqrt{k}})$ with the model fiber metric ϕ_0 :

$$\left\|\beta^{(k)}\right\|_{(k)}^2=_k\sum_{I,J}\int_{\mathbf{B}_{\sqrt{k}R_k}}\left|f_{IJ}(\frac{w}{\sqrt{k}})\right|^2h_{IJ}(\frac{w}{\sqrt{k}})e^{-\phi_0(w)}\det(h_{ij})(\frac{w}{\sqrt{k}})dm(w).$$

Finally, observe that by continuity the scaled metric coefficients, satisfy

$$\sup_{w\in B_{\sqrt{k}R_k}}\left|h_{ij}(\frac{w}{\sqrt{k}})-\delta_{ij}\right|=\sup_{w\in B_{\sqrt{k}R_k}}\left|h_{ij}(z)-\delta_{ij}\right|\to 0.$$

Using this fact it is not hard to see that

$$\left\|\beta^{(k)}\right\|_{(k)}^2=_k\sum_{I}\int_{B_{\sqrt{k}R_k}}\left|f_{IJ}(\frac{w}{\sqrt{k}})\right|^2e^{-\phi_0(w)}\mathrm{dm}(w)=_k\left\|\beta^{(k)}\right\|_{\phi_0,\sqrt{k}R_k}^2$$

which finishes the proof of the proposition.

Next we show how $\overline{\partial}^*$ transforms under scalings.

Proof of proposition 6.1. First note that

$$\langle\alpha,\gamma\rangle=\left\langle\alpha^{(k)},\gamma^{(k)}\right\rangle_{(k)}k^{-n},\tag{11.2}$$

which follows from the local expression 11.1 for the norm, in the proof of proposition 6.2 and the change of variables $z=\frac{w}{\sqrt{k}}$ in the integral. Now, observe that, by the definition of the formal adjoint $\overline{\partial}^{*(k)}$, and the chain rule for $\overline{\partial}$,

$$\left\langle\overline{\partial}^{*(k)}\alpha^{(k)},\beta^{(k)}\right\rangle_{(k)}=\left\langle\alpha^{(k)},\overline{\partial}(\beta^{(k)})\right\rangle_{(k)}=\frac{1}{\sqrt{k}}\left\langle\alpha^{(k)},(\overline{\partial}\beta)^{(k)}\right\rangle_{(k)},$$

for all smooth forms β with support in the ball B_{R_k}. Thus, by 11.2,

$$\left\langle\alpha^{(k)},(\overline{\partial}\beta)^{(k)}\right\rangle_{(k)}=k^{-n}\left\langle\alpha,\overline{\partial}\beta\right\rangle=k^{-n}\left\langle\overline{\partial}^*\alpha,\beta\right\rangle=\left\langle(\overline{\partial}^*\alpha)^{(k)},\beta^{(k)}\right\rangle_{(k)}.$$

This means that,

$$\left\langle\overline{\partial}^{*(k)}\alpha^{(k)},\beta^{(k)}\right\rangle_{(k)}=\left\langle\frac{1}{\sqrt{k}}(\overline{\partial}^*\alpha)^{(k)},\beta^{(k)}\right\rangle_{(k)},$$

for all smooth forms β with support in the ball B_{R_k}, which proves the proposition.

Finally, to prove proposition 6.4 we first show a special case. By lemma 6.1 $(k\phi)^{(k)}$ tends to ϕ_0 uniformly with all derivatives on $B_{\sqrt{k}R_k}$, when k tends to infinity, and it is clear that $\omega^{(k)}$ tends to to ω^0 uniformly with all derivatives on $B_{\sqrt{k}R_k}$ (in the component sense).

Lemma 11.1. *There is a sequence ϵ_k tending to zero, such that*

$$\left\|\overline{\partial}^{*(k)}\alpha-\overline{\partial}^{*\phi_0}\alpha\right\|_{2,1}^2\leq\epsilon_k\left\|\alpha\right\|_{2,1}^2$$

on $B_{\sqrt{k}R_k}$.

Proof. For given base and fiber metrics the following identity holds:

$$\overline{\partial}^* = \pm * \partial * \tag{11.3}$$

(see [16]), where $*$ is the Hodge star operator determined by the metrics, i.e.

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta \rangle \frac{1}{n!} \omega^n$$

Observe that $*$ is an algebraic operator that depends smoothly on the metrics, i.e. on h_{ij} and ϕ. If we express $\overline{\partial}$ with respect to the standard module basis of $\bigwedge^{0,q}(\mathbb{C}^n)$, we see that

$$(\overline{\partial}^* \alpha)_I = \sum_{I,J} \mathcal{D}_{IJ} f_J$$

where $\mathcal{D}_{\mathcal{I}\mathcal{J}}$ is a first order scalar PDE, with coefficients depending smoothly on h_{ij} and ϕ and their first derivatives, according to 11.3. This fact, combined with the convergence of the metrics, shows that

$$\left\| (\overline{\partial}^{*(k)} \alpha)_I - (\overline{\partial}^{*(\phi_0)} \alpha)_I \right\|^2 \leq \epsilon_k \left\| \alpha \right\|_{2,1}^2 \text{ on } B_{\sqrt{k} R_k},$$

which proves the lemma. □

To prove proposition 6.4 use that

$$(\overline{\partial}^* + \overline{\partial})^m = \overline{\partial}^* \overline{\partial} \cdots \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^* \cdots \overline{\partial}\overline{\partial}^*,$$

and proceed as in the proof of the previous lemma.

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