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Estimates for some first-order Riesz operators on the complex affine group

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ESTIMATES FOR SOME FIRST-ORDER RIESZ OPERATORS ON THE COMPLEX AFFINE GROUP

ANDERS ÖHGREN

Abstract. On the affine group of the complex plane, which is a Lie group of exponential growth, we consider a right-invariant Laplacian Δ. For certain right-invariant fields Z, we prove that the first-order Riesz operators $Z\Delta^{-1/2}$ and $\Delta^{-1/2}Z$ are of weak type $(1,1)$ with respect to the left Haar measure. It follows that these operators are bounded on L^p, $1 < p < \infty$.

The convolution kernels of these operators behave locally like standard Calderón-Zygmund kernels. The main difficulty concerns the behaviour at infinity, where the standard technique does not apply because of the exponential growth. Instead, we derive explicit expressions for the kernels and use more direct methods. The approaches are quite different for different operators, but in most cases the estimate involves cancellation between positive and negative parts of the kernel.

1. Introduction

Consider the complex affine group $G = \mathbb{C} \rtimes \mathbb{C}^*$, where $\mathbb{C}^*$ denotes $\mathbb{C} \setminus \{0\}$ with multiplication. The group product in G is given by

$$(z, w)(z', w') \quad = \quad (z + wz', ww').$$

This operation corresponds to the composition of affine mappings $\zeta \mapsto w\zeta + z$ in the complex plane.

The left and right Haar measures on G are given by

$$d\mu_l(z,w) = \frac{dzdw}{|w|^4} \qquad \text{and} \qquad d\mu_r(z,w) = \frac{dzdw}{|w|^2}, \tag{1}$$

respectively, where dz and dw denote Lebesgue measure on $\mathbb{C}$. Note that the lack of unimodularity implies that G is of exponential growth. We will usually use the left-invariant measure. In particular, the L^p-spaces we consider on G will be taken with respect to μ_l.

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The Lie algebra $\mathfrak{g}$ of G can be naturally identified with $\mathbb{C} \oplus \mathbb{C}$. As base elements in $\mathfrak{g}$, we take $X = (1, 0)$, $Y = (i, 0)$, $U = (0, 1)$ and $V = (0, i)$. We will regard the elements in $\mathfrak{g}$ as right-invariant vector fields on G.

The right-invariant Laplacian corresponding to our choice of basis is the operator

$$\Delta = -(X^2 + Y^2 + U^2 + V^2)$$

defined on $C_0^\infty(G)$, i. e. the space of infinitely differentiable functions on G with compact support. It has a positive and self-adjoint closure on $L^2(\mu_l)$, which we also denote by Δ.

The operator Δ is positive, self-adjoint and one-to-one. It is therefore possible to define negative powers of Δ, using functional calculus. We will consider first-order Riesz operators on G. These consist of the left or right product of $\Delta^{-1/2}$ with a vector field $Z \in \mathfrak{g}$. The main result is the following theorem.

Theorem 1. *For $Z = X, Y, V$, the Riesz operators $Z\Delta^{-1/2}$ and $\Delta^{-1/2}Z$ are bounded on $L^p(\mu_l)$, for all $p \in (1, \infty)$, and of weak type $(1, 1)$.*

It seems to us that no results have yet been established about the corresponding boundedness properties for the operators $U\Delta^{-1/2}$ and $\Delta^{-1/2}U$, except for $p = 2$.

For any first-order Riesz operator on G, the boundedness on L^2 is elementary. Thus, it is enough to establish that the operators in the theorem are of weak type $(1, 1)$. The boundedness on L^p follows by interpolation for $1 < p < 2$, and then by duality for $p > 2$.

A Riesz operator on G is given by convolution with some kernel K, which in the general case is singular at the identity element e and at infinity. By introducing a smooth cut-off function $\psi \in C_0^\infty(G)$, which equals 1 on a neighbourhood of e, we can separate these singularities according to $K = \psi K + (1 - \psi)K$, and treat them separately.

Convolution with ψK will be called the *local* part of the the operator. Since this kernel has compact support and any Lie group is locally Euclidean, the local part can be treated with classical methods, such as Calderón-Zygmund theory for singular integral operators. For completeness, we have nevertheless included a quite detailed study also of the local parts of the operators. This is found in Section 3.

The singularity at infinity is carried by the kernel $(1 - \psi)K$. This corresponds to long range interactions and we will call this the *global* part of the operator. Since G has exponential volume growth, classical Calderón-Zygmund theory does not apply to this part[1]. Instead, we rely on more direct methods.

[1]However, Hebisch and Steger have recently presented a method to adjust the classical Calderón-Zygmund theory to groups with exponential growth; see [12].

First, we find explicit expressions for the kernels of the global parts of the operators in Theorem 1. We use the heat kernels associated with the Laplacian Δ, to find the convolution kernel of $\Delta^{-1/2}$. By applying an appropriate vector field and neglecting integrable terms, we get suitable expressions for the kernels of the Riesz operators. This is the content of Section 4. The operators $V\Delta^{-1/2}$ and $\Delta^{-1/2}V$ turn out to have kernels which are integrable at infinity, which immediately implies the required boundedness. Furthermore, there is a complete symmetry between the fields X and Y, so it is enough to study the operators $X\Delta^{-1/2}$ and $\Delta^{-1/2}X$.

In Section 5, we prove that the global part of $X\Delta^{-1/2}$ is of weak type $(1,1)$. The proof is based on that in [18], where a corresponding operator for the real affine group is treated, and it involves cancellation between positive and negative parts of the kernel. Indeed, we give a counterexample showing that the kernel cannot be replaced by its absolute value.

The proof that the global part of $\Delta^{-1/2}X$ is of weak type $(1,1)$ is given in section 6, and is also largely based on a proof for a corresponding result on the real affine group, given in [11]. The approach is quite different from the case $X\Delta^{-1/2}$, but again, cancellation between different parts of the kernel is what saves the estimate. The main technique is to expand the kernel into Haar-like functions. The calculations are then reduced to convolving such a Haar-like function with a sum of point measures.

1.1. **Background.**

Riesz operators have been intensely studied in many different contexts during the past two decades. We shall here only give a few references, and for a more detailed background, we refer to [18] or [24].

Lohoué and Varopoulos [16] proved L^p boundedness for first-order Riesz operators on nilpotent groups. This was extended to all Lie groups of polynomial growth by Saloff-Coste [17] and Alexopoulos [1]. Nonamenable groups were treated by Lohoué in [15], and results by Burns, ter Elst and Robinson [6] imply boundedness for Riesz operators on compact Lie groups.

Anker [2] studied L^p properties for Riesz operators in the setting of symmetric spaces, where it is natural to consider Riesz operators related to the Laplace-Beltrami operator. For some other Riemannian manifolds, results have been presented by Lohoué [14], Bakry [3] and Coulhon and Duong [7].

Second-order Riesz operators on the real affine group were studied by Gaudry, Qian and Sjögren in [8], where they proved that[2] the operator $Z_1\Delta^{-1}Z_2$ is bounded on L^p, $1 < p < \infty$, and of weak type $(1,1)$. They also showed that $Z_1Z_2\Delta^{-1}$ and

[2]Here, $Z_1, Z_2 \neq 0$ denote right-invariant fields on the real affine group and Δ is the distinguished Laplacian.

$\Delta^{-1}Z_1Z_2$ has none of these boundedness properties. These results were extended in [9] by Gaudry and Sjögren to any solvable NA group coming from the Iwasawa decomposition of a rank 1 semisimple group[3].

First-order Riesz operators on the real affine group have been studied in [18], [11] and [12]. In [18], Sjögren proved that $Z\Delta^{-1/2}$ is bounded on L^p, $1 < p < 2$, and of weak type $(1,1)$, for $Z \in \mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of the nilpotent group N in the NA decomposition of the real affine group. The corresponding result for $\Delta^{-1/2}Z$, $Z \in \mathfrak{n}$, was given by Gaudry and Sjögren in [11]. The proofs in [18] and [11] are based on rather explicit estimates and use of the kernels. In [12], Hebisch and Steger have given a more abstract approach to the case $Z\Delta^{-1/2}$, which also applies to $Z \in \mathfrak{a}$.

In [23], Wängefors generalized the results in [18] to any NA group arising from the Iwasawa decomposition of a rank 1 semisimple group.

On the complex affine group G, second-order Riesz operators were treated by Gaudry and Sjögren in [10], and have essentially the same boundedness properties as for the real affine group, the exception being when Z_1 or Z_2 equals V. In this case, all the second order Riesz operators are bounded on L^p and of weak type $(1,1)$.

2. Preliminaries

2.1. **Notations.** We will use a few different parametrisations of G. For $(z,w) \in G$, we will on different occasions write

$$z = x + iy = re^{i\theta} \qquad \text{and} \qquad w = u + iv = \rho e^{i\phi} = e^s e^{i\phi} = 2^t e^{i\phi},$$

for $z \in \mathbb{C}$ and $w \in \mathbb{C}^*$. Note that the identity element is $e = (0,1)$ and that the inverse element of $g = (z,w) \in G$ is $g^{-1} = \left(-\frac{z}{w}, \frac{1}{w}\right)$. For a function f on G, we will denote the composition of f with the inverse map $g \mapsto g^{-1}$ by $\check{f}$, i. e. $\check{f}(g) = f(g^{-1})$.

We will often identify $\mathbb{C}$ with $\mathbb{R}^2$ in the natural way, and when f is a function on $\mathbb{C}$, we write e. g. $f(z) = f(x,y)$, when there can be no confusion.

The left- and right-invariant Haar measures on G will be denoted by μ_l and μ_r, respectively, and are given by (1). The modular function is the Radon-Nikodym derivative $d\mu_l/d\mu_r$ and will be denoted by δ. Thus, we have $\delta(z,w) = |w|^{-2}$.

Lebesgue measure will generally be denoted by m, although the domain of m will vary at different occurrences. We usually specify on which set m is considered, although it is often clear from the context.

[3] Recall that the real affine group is the simplest nontrivial example of such an NA group.

There is a natural Riemannian structure on G, induced by our choice of basis in G. We will denote the associated left-invariant distance function by d, and an open ball of radius $r > 0$ around an element $g \in G$ will be denoted by $B(g, r)$. We will mostly work with balls centered at the identity element e, which we denote simply by $B(r)$. Note that $B(g, r) = gB(r)$, by the left-invariance of d.

We will make a lot of estimates where we are only interested in the order of magnitude. Throughout the text, C will denote a positive constant, but the value of C may vary from line to line. When it is necessary, we specify which parameters C may or may not depend on.

2.2. **Invariant vector fields.** The Lie algebra $\mathfrak{g}$ of G is the tangent space at the identity element e, and can be identified with $\mathbb{C} \oplus \mathbb{C}$. The exponential map is then given by

$$\exp(\zeta, \tau) = \left(\frac{e^\tau - 1}{\tau}\zeta, e^\tau\right).$$

As a basis in $\mathfrak{g}$ (over $\mathbb{R}$) we take $X = (1, 0)$, $Y = (i, 0)$, $U = (0, 1)$ and $V = (0, i)$. We will regard the elements in $\mathfrak{g}$ as right-invariant vector fields on G, i. e. we extend $Z \in \mathfrak{g}$ according to

$$Zf(g) = \frac{d}{dt}\bigg|_{t=0} f\big(\exp(tZ)g\big), \qquad g \in G.$$

The right-invariant fields corresponding to our choice of basis are then

$$\begin{aligned}
X &= \frac{\partial}{\partial x}, \\
Y &= \frac{\partial}{\partial y}, \\
U &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\
V &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.
\end{aligned}$$

The corresponding left-invariant fields are given by

$$\begin{aligned}
X^l &= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \\
Y^l &= -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}, \\
U^l &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \\
V^l &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.
\end{aligned}$$

It will be convenient to express some of these fields in terms of polar coordinates (r,θ) and (ρ,θ). We therefore put

$$
\begin{aligned}
R_z &= r\frac{\partial}{\partial r} &= x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\\
T_z &= \frac{\partial}{\partial \theta} &= -y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\\
R_w &= \rho\frac{\partial}{\partial \rho} &= u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\\
T_w &= \frac{\partial}{\partial \phi} &= -v\frac{\partial}{\partial u}+u\frac{\partial}{\partial v}.
\end{aligned}
$$

With this notation, we get

$$
\begin{aligned}
U &= R_z + R_w, & U^l &= R_w,\\
V &= T_z + T_w, & V^l &= T_w.
\end{aligned}
$$

The operators we study are convolution operators. For example, we shall see that $\Delta^{-1/2}$ is given by left-convolution with a locally integrable kernel, which we denote by M. That is, $\Delta^{-1/2}f = M*f$, where

$$
\begin{aligned}
M*f(g) &= \int_G M(h)f(h^{-1}g)\,d\mu_l(h)\\
&= \int_G M(gh^{-1})f(h)\,d\mu_r(h),
\end{aligned}
$$

and $f: G\to\mathbb{C}$ is some suitable function. For our purposes, it is enough to consider $f\in C_0^\infty(G)$. Since M is locally integrable, the integral on the right hand side is well defined.

If the kernel M were a smooth function, the operator $Z\Delta^{-1/2}$ would be given by left convolution with ZM, as is easily verified. However, since M is not smooth, and ZM is not even in L^1_{loc}, we need to be a bit more careful to handle the singularity at e.

Let us study the case $Z = X$. First, note that

$$
\begin{aligned}
X_g f(h^{-1}g) &= \frac{d}{dt}\Big|_{t=0} f\big(h^{-1}\exp(tX)g\big)\\
&= \frac{d}{dt}\Big|_{t=0} f\big((\exp(-tX)h)^{-1}g\big)\\
&= -X_h f(h^{-1}g), \qquad h,g\in G, f\in C_0^\infty(G),
\end{aligned}
$$

so that

$$\begin{aligned} X\Delta^{-1/2}f(g) &= X(M*f)(g) \\ &= \int_G M(h)X_g f(h^{-1}g)d\mu_l(h) \\ &= -\int_G M(h)X_h f(h^{-1}g)d\mu_l(h) \\ &= -\lim_{\epsilon\to 0}\int_{|z|^2+|w-1|^2>\epsilon^2} M(z,w)\frac{\partial}{\partial x}f\big((z,w)^{-1}g\big)|w|^{-4}dzdw, \end{aligned}$$

where $z = x+iy$. Denoting $h_\pm = h_\pm(y,w) = \Big(\pm\sqrt{\epsilon^2-(y^2+|w-1|^2)}+iy, w\Big)$ for $y^2+|w-1|^2<\epsilon^2$, and integrating by parts with respect to x, we get

$$\begin{aligned} X\Delta^{-1/2}f(g) = \lim_{\epsilon\to 0}\Bigg\{ &\int_{|z|^2+|w-1|^2>\epsilon^2}\frac{\partial M}{\partial x}(z,w)f\big((z,w)^{-1}g\big)|w|^{-4}dzdw \\ &+\int_{y^2+|w-1|^2<\epsilon^2}\Big(M(h_+)f(h_+^{-1}g)-M(h_-)f(h_-^{-1}g)\Big)|w|^{-4}dydw\Bigg\}. \end{aligned}$$

By Taylor expansion, it is easy to see that $f(h_\pm^{-1}g) = f(g)+O(\epsilon)$. The kernel M is even in x and satisfies $M(h_\pm) = O(\epsilon^{-3})$ (see Section 3). It follows that the last integral above tends to 0 as $\epsilon\to 0$. The first integral tends to $\mathrm{pv}(XM)*f(g)$ and thus we have that

$$X\Delta^{-1/2}f = \mathrm{pv}(XM)*f, \qquad f\in C_0^\infty(G).$$

By similar calculations, it follows that $Z\Delta^{-1/2}$ is given by left-convolution with the distribution $\mathrm{pv}(ZM)$ for any $Z\in\mathfrak{g}$.

For the operators in the reverse order, we have $\Delta^{-1/2}Zf = M*Zf$. It would be preferable to express also this operator as a convolution with some distribution. For this purpose, we introduce the *transposed* vector field Z^{t}, defined by

$$Z^{\mathrm{t}} = Z^l + Z\delta(e)I,$$

where Z^l is the left-invariant field corresponding to $Z\in\mathfrak{g}$ and I is the identity operator. With this notation, the following lemma[4] gives the desired representation of an operator $\Delta^{-1/2}Z$.

Lemma 1. *For $Z\in\mathfrak{g}$, we have*

$$\Delta^{-1/2}Zf = (Z^{\mathrm{t}}M)*f, \tag{2}$$

for all functions $f\in C_0^\infty(G)$.

Note that the convolution on the right hand side of (2) is also to be interpreted as a singular integral.

[4]Lemma 1 and the notion of transposed vector fields have been adopted from [9]. For completeness, we reproduce the short proof.

Proof. We first recall what it means to take the derivative of a distribution. If k is a distribution on G, the right-invariant derivative Zk of k is defined by

$$\langle Zk, f\rangle \;=\; \langle k, -Zf\rangle, \qquad f \in C_0^\infty(G).$$

If Z^l is the left-invariant field corresponding to $Z \in \mathfrak{g}$, we have for $f, g \in C_0^\infty(G)$, that

$$\begin{aligned}
\langle Z^l f, g\rangle &= \frac{d}{dt}\Big|_{t=0} \int_G f\big(h\exp(tZ)\big) g(h)\delta(h)\,d\mu_r(h) \\
&= \frac{d}{dt}\Big|_{t=0} \int_G f(h) g\big(h\exp(-tZ)\big)\delta\big(h\exp(-tZ)\big)\,d\mu_r(h) \\
&= \langle f, -Z^l g\rangle - Z\delta(e)\langle f, g\rangle.
\end{aligned}$$

The left-invariant derivative of a distribution k is defined accordingly, by

$$\langle Z^l k, f\rangle \;=\; \langle k, -Z^l f\rangle - Z\delta(e)\langle k, f\rangle, \qquad f \in C_0^\infty(G).$$

Using the facts that $\langle k * f, g\rangle = \langle k, g * \check{f}\rangle$ and $(Zf)\check{} = -Z^l\check{f}$, we get

$$\begin{aligned}
\langle \Delta^{-1/2} Zf, g\rangle &= \langle M * (Zf), g\rangle \\
&= -\langle M, g * Z^l \check{f}\rangle \\
&= -\langle M, Z^l(g * \check{f})\rangle \\
&= \langle Z^l M + Z\delta(e)M, g * \check{f}\rangle \\
&= \langle (Z^l M + Z\delta(e)M) * f, g\rangle, \qquad f, g \in C_0^\infty(G).
\end{aligned}$$

□

For the base fields in $\mathfrak{g}$, the corresponding transposed versions are given by

$$\begin{aligned}
X^{\mathrm{t}} &= X^l = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \\
Y^{\mathrm{t}} &= Y^l = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}, \\
U^{\mathrm{t}} &= U^l - 2 = R_w - 2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - 2, \\
V^{\mathrm{t}} &= V^l = T_w = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.
\end{aligned}$$

2.3. **Spectral resolution and boundedness on L^2.** By the following simple observation, we see that the operators $Z \in \mathfrak{g}$ are skew-adjoint:

$$\begin{aligned}\langle Zf, g\rangle &= \frac{d}{dt}\Big|_{t=0} \int_G f\big(\exp(tZ)h\big)g(h)d\mu_l(h) \\ &= \frac{d}{dt}\Big|_{t=0} \int_G f(h')g\big(\exp(-tZ)h'\big)d\mu_l(h') \\ &= \langle f, -Zg\rangle,\end{aligned}$$

where we made the change of variable $h' = \exp(tZ)h$ and used the left-invariance of μ_l. It follows immediately that the Laplacian $\Delta = -(X^2+Y^2+U^2+V^2)$ is self-adjoint and positive, i. e. $\langle \Delta f, f\rangle \geq 0$ for all $f \in C_0^\infty(G)$. Hence it has a closure on $L^2(\mu_l)$, which is also self-adjoint and positive. We will denote this closure also by Δ. Generally, we do not bother to keep track of where the operators we consider are defined. It is enough to think of them as acting on some dense subspace of $L^p(\mu_l)$.

As a positive and self-adjoint operator on $L^2(\mu_l)$, the Laplacian Δ has a spectral resolution

$$\Delta = \int_0^\infty s dE_s.$$

If we first verify that the spectral measure dE_s has no point mass at 0, which amounts to that 0 is not an L^2 eigenvalue of Δ, negative powers of Δ can be defined as

$$\Delta^{-\beta} = \int_0^\infty s^{-\beta} dE_s, \qquad \beta > 0,$$

with a domain that is dense in $L^p(\mu_l)$.

Note that the differential operator Δ is elliptic. Hence, there is a theory for harmonic functions associated with Δ, including a maximum principle (see [4] and [5]). By the following argument, which can be found in [9], we can use this maximum principle to verify that 0 is not an eigenvalue of the Laplacian, i. e. that the operator Δ is one-to-one.

Let $f \in L^2(\mu_l)$ be a harmonic function, i. e. $\Delta f \equiv 0$. We need to prove that $f \equiv 0$. Let

$$f_r(x) = \frac{1}{\mu_l(B(r))} \int_{B(r)} f(gh)d\mu_l(h), \qquad r > 0.$$

Note that f_r is a harmonic function and that $f_r \to f$ pointwise as $r \to 0$. By the Hölder inequality, we have

$$\begin{aligned}|f_r(g)| &\leq \frac{1}{\mu_l(B(r))}\left(\int_G |f(gh)|^2 \chi_{B(r)}(h)d\mu_l(h)\right)^{1/2}\left(\int_G \chi_{B(r)}(h)d\mu_l(h)\right)^{1/2}\\ &= \left(\frac{1}{\mu_l(B(r))}\int_{B(r)} |f(gh)|^2 d\mu_l(h)\right)^{1/2}.\end{aligned}$$

Since $f \in L^2(\mu_l)$, the last integral tends to 0 as $d(g,e) \to \infty$. Hence f_r vanishes at infinity and thus everywhere by the maximum principle. We conclude that $f \equiv 0$.

The L^2 boundedness of the first-order Riesz operators follows easily from the skew-adjointness of $Z \in \mathfrak{g}$.

Lemma 2. *For any $Z \in \mathfrak{g}$, the operators $\Delta^{-1/2}Z$ and $Z\Delta^{-1/2}$ are bounded on $L^2(\mu_l)$.*

Proof. Since X, Y, U and V are skew-adjoint, we get

$$\begin{aligned}\langle Xf, Xf\rangle + \langle Yf, Yf\rangle + \langle Uf, Uf\rangle + \langle Vf, Vf\rangle &= \langle -(X^2+Y^2+U^2+V^2)f, f\rangle\\ &= \langle \Delta f, f\rangle\\ &= \langle \Delta^{1/2}f, \Delta^{1/2}f\rangle,\end{aligned}$$

for any $f \in C_0^\infty(G)$. This shows that $Z\Delta^{-1/2}$ is bounded on $L^2(\mu_l)$. The boundedness of $\Delta^{-1/2}Z$ follows by duality, since $-\Delta^{-1/2}Z$ is the adjoint of $Z\Delta^{-1/2}$. □

2.4. **Left-invariant metric.** We want to derive a natural left-invariant metric d on the group G. To do this, we first consider the real affine group $H = \mathbb{R} \rtimes \mathbb{R}_+$, with group product given by

$$(b,a)(b',a') = (b + ab', aa'), \qquad b \in \mathbb{R}, a > 0.$$

This is the group of composition of affine mappings on $\mathbb{R}$ and is sometimes called the "$ax + b$" group.

It is natural to identify H with the complex upper half-plane, via the map $(b,a) \mapsto b + ia = z$. With this identification, the action of H on itself (i. e. the group product) is given by $(b,a)z = az + b$. Alternatively, H can be regarded as a subgroup of $GL(2,\mathbb{R})$ via the map $(b,a) \mapsto \begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}$. This is consistent with the previous identification if $GL(2,\mathbb{R})$ acts on the complex plane via Möbius transformations, i. e.

$$gz = \frac{az+b}{cz+d}, \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL(2,\mathbb{R}).$$

We want to introduce a metric on H which is left-invariant. The hyperbolic metric in the upper half plane is invariant under the action of $GL(2,\mathbb{R})$ via Möbius transformations (see e. g. [22]). Hence, the hyperbolic metric, given by

$$ds^2 \;=\; \frac{da^2+db^2}{a^2},$$

is invariant with respect to left translations in H. This implies[5]

$$d_H\big((b,a),(0,1)\big) \;=\; \operatorname{arcosh}\left(\frac{b^2+1+a^2}{2a}\right).$$

This expression for a left-invariant distance extends to $\mathbb{R}^n \rtimes \mathbb{R}_+$ for any $n \in \mathbb{N}$, i. e. $b \in \mathbb{R}^n$. We only need to change b^2 to $|b|^2$.

Turning to the complex affine group $G = \mathbb{C} \rtimes \mathbb{C}^*$, we first notice that it can be regarded as a subgroup of $GL(2,\mathbb{C})$, analogously to the real affine group. By the representation

$$\begin{pmatrix} w & z \\ 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix},$$

where $w = \rho e^{i\phi}$, it follows that G can be viewed as a two-stage semidirect product $(\mathbb{C} \rtimes \mathbb{R}_+) \rtimes \mathbb{T}$, and we write $(z,w) = (z,\rho,e^{i\phi})$. Denoting $G' = \mathbb{C} \rtimes \mathbb{R}_+$, a left-invariant metric on G' is determined by

$$d_{G'}\big((z,\rho),(0,1)\big) \;=\; \operatorname{arcosh}\left(\frac{|z|^2+1+\rho^2}{2\rho}\right), \qquad (z,\rho) \in G',$$

which is derived from

$$ds'^2 \;=\; \frac{dx^2+dy^2+d\rho^2}{\rho^2}.$$

Since the product in $G = (\mathbb{C} \rtimes \mathbb{R}_+) \rtimes \mathbb{T}$ is

$$(z,\rho,e^{i\phi})(z',\rho',e^{i\phi'}) \;=\; (z+\rho e^{i\phi}z', \rho\rho', e^{i(\phi+\phi')}),$$

the action of $\mathbb{T}$ on G' is given by

$$\tau_\phi(z,\rho) \;=\; (e^{i\phi}z,\rho),$$

which in the matrix representation corresponds to conjugation with $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}$. Hence, $d_{G'}$ is invariant under the action of $\mathbb{T}$, and it follows that the metric induced by

$$ds^2 \;=\; ds'^2 + d\phi^2,$$

[5] We can find $d_H(h,e)$ for an arbitrary $h = (b,a) = b+ia \in H$ by first taking $b = 0$ and then act with a rotation around $e = (0,1) = i$, i. e. a $g \in SO(2,\mathbb{R}) \subset GL(2,\mathbb{R})$.

is left-invariant on G. Here, we assume that $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is represented by $[-\pi, \pi)$, so that $\phi \in [-\pi, \pi)$. Thus,

$$d\big((z,w),(0,1)\big)^2 = A(z,w)^2 + \phi^2,$$

where we have introduced the notation

$$\begin{aligned} A(z,w) &= d_{G'}\big((z,|w|),(0,1)\big) \\ &= \operatorname{arcosh}\left(\frac{|z|^2+1+|w|^2}{2|w|}\right). \end{aligned}$$

The distance between any two elements (z,w) and (z',w') in G is now determined by the left-invariance, and we have

$$\begin{aligned} d\big((z',w'),(z,w)\big)^2 &= d\big((z,w)^{-1}(z',w'),(0,1)\big)^2 \\ &= A\left(\frac{z'-z}{w},\frac{w'}{w}\right)^2 + \varphi^2 \\ &= \operatorname{arcosh}^2\left(\frac{|z'-z|^2+|w'|^2+|w|^2}{2|w'||w|}\right) + \varphi^2, \end{aligned}$$

where $\varphi = \arg(w/w') \in [-\pi,\pi)$ and we used that $(z,w)^{-1} = (-z/w, 1/w)$.

2.5. **Exponential volume growth.** In the introduction, we mentioned that the lack of unimodularity implies that G has exponential growth, i. e. that the volume of the balls $B(N)$, measured with μ_l, grows exponentially with the radius N as $N \to \infty$. This can be seen from the following nice argument. Since the modular function δ is multiplicative and not identically 1, there must be some $g \in B(1)$ with $\delta(g) > 1$. We then have

$$B(1)g^n \subseteq B(n+1),$$

and it follows that

$$\begin{aligned} \mu_l\big(B(n+1)\big) &\geq \mu_l\big(B(1)g^n\big) \\ &= \int_{B(1)g^n} \delta(h)\,d\mu_r(h) \\ &= \int_{B(1)} \delta(hg^n)\,d\mu_r(h) \\ &= \big(\delta(g)\big)^n \mu_l\big(B(1)\big), \end{aligned}$$

where the right-hand side grows exponentially with n.

2.6. **Heat kernels.** The semigroup $\{e^{-t\Delta}\}_{t>0}$ is given by left convolution with the heat kernels h_t, $t > 0$ (see e. g. Theorem 3.4 in [13]). From [10], we have the following proposition for the heat kernels on G.

Proposition 1. *The heat semigroup kernels on G are given by the functions*

$$h_t(z,w) = \frac{1}{16\pi^2 t^2} \frac{|w|A}{\sinh(A)} \sum_{k\in\mathbb{Z}} \exp\left[-\frac{A^2 + (\phi - 2k\pi)^2}{4t}\right],$$

in the sense that $e^{-t\Delta}f = h_t * f$, $t > 0$, for all $f \in C_0^\infty(G)$.

We will not go into the details of the proof, but merely state that the argument is based on the decomposition $G = G' \rtimes \mathbb{T}$. In fact, h_t is the tensor product of the heat kernels associated with a distinguished Laplacian on G', given by

$$\frac{1}{8\pi^{3/2}t^{3/2}} \frac{\rho A}{\sinh A} e^{-A^2/(4t)}, \quad \text{for } t > 0,$$

and the heat kernels on the circle group $\mathbb{T}$, for the Laplacian $\frac{d^2}{d\phi^2}$, given by

$$\frac{1}{2\sqrt{\pi t}} \sum_{k\in\mathbb{Z}} e^{-(\phi-2k\pi)^2/(4t)}, \quad \text{for } t > 0.$$

For details on the proof of Proposition 1, see [10] and further references.

3. The local parts

Recall that we define the local part of a first-order Riesz operator on G as the convolution with ψK, where K is the convolution kernel of the operator and ψ is a smooth function with compact support in G which equals 1 on some neighbourhood of e.

Our treatment of the local parts of the first-order Riesz transforms on G is essentially the same as in [8], where the local parts of second order Riesz operators on the real affine group is treated in the same way. Another approach, based on pseudodifferential operators, can be found in [9]. The result is the following proposition.

Proposition 2. *The local parts of the operators $\Delta^{-1/2}Z$ and $Z\Delta^{-1/2}$, $Z \in \mathfrak{g}$, are of weak type $(1,1)$ with respect to μ_l.*

The proof of Proposition 2 will be carried out through a series of lemmas. First, we determine the convolution kernel for the local part of any first-order Riesz operator $\Delta^{-1/2}Z$ or $Z\Delta^{-1/2}$, $Z \in \mathfrak{g}$, up to integrable terms. This is given in Lemma 3 below.

The basic idea is then that since the kernels have compact support and the Haar measure μ_l and Riemannian distance d are locally equivalent to Lebesgue measure m and Euclidean distance $|\cdot|$, we can replace μ_l and d with m and $|\cdot|$. The whole computation is therefore essentially the same as on $\mathbb{R}^4$ and classical Calderón-Zygmund theory can be applied. This is the content of Lemma 6.

For the application of Calderón-Zygmund theory, we first need to prove that the L^2-boundedness in Lemma 2 holds for the local and global parts separately. This is taken care of in Lemma 5.

Before we go into the details of these lemmas, we derive the convolution kernel M of $\Delta^{-1/2}$. For any $s > 0$, we have

$$s^{-1/2} = \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-ts} dt.$$

Since the Laplacian Δ is a self-adjoint, positive and one-to-one operator, there is no problem in exchanging s for Δ, to get

$$\Delta^{-1/2} = \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-t\Delta} dt.$$

Here, the integral on the right hand side should be interpreted using spectral theory, which means that

$$\langle \Delta^{-1/2} f, g \rangle = \pi^{-1/2} \int_0^\infty t^{-1/2} \langle e^{-t\Delta} f, g \rangle dt,$$

for f in the domain of $\Delta^{-1/2}$ and $g \in C_0^\infty(G)$. Here, $e^{-t\Delta} f = h_t * f$. If $\int_0^\infty t^{-1/2} h_t dt$ is in L^1_{loc}, we can change the order of integration to conclude that $\Delta^{-1/2}$ is given by left-convolution with the kernel

$$\begin{aligned} M(z,w) &= \pi^{-1/2} \int_0^\infty t^{-1/2} h_t(z,w) dt \\ &= \frac{1}{16\pi^{5/2}} \frac{|w|A}{\sinh(A)} \sum_{k\in\mathbb{Z}} \int_0^\infty t^{-5/2} \exp\left[-\frac{A^2 + (\phi - 2k\pi)^2}{4t} \right] dt \\ &= \frac{3}{4\pi^2} \frac{|w|A}{\sinh(A)} \sum_{k\in\mathbb{Z}} \frac{1}{(A^2 + (\phi - 2k\pi)^2)^{3/2}}, \end{aligned} \tag{3}$$

where we used, in the last step, the fact that $\int_0^\infty t^{-5/2} e^{-\frac{c}{t}} dt = \frac{3}{2}\sqrt{\pi} c^{-3/2}$. From the last expression, it is easy to see that M is locally integrable. First of all, the factor $\frac{|w|A}{\sinh(A)}$ is a smooth function. Also, if the term with $k = 0$ is left out, the remaining sum is nicely convergent to a smooth function. Hence, the only need for concern is the term $k = 0$, i. e. $(A^2 + \phi^2)^{-3/2}$, which has a singularity at the identity element $e = (0, 1)$. Recall that

$$A(z,w) = \operatorname{arcosh}\left(\frac{|z|^2 + 1 + |w|^2}{2|w|} \right).$$

By Taylor expansion, we have

$$\operatorname{arcosh}(1+x) \;=\; \sqrt{2x}+O(x),$$

and

$$\frac{|z|^2+1+|w|^2}{2|w|} \;=\; 1+\frac{|z|^2+\epsilon^2}{2}+O\big((|z|^2+\epsilon^2)^{3/2}\big),$$

where we have written $|w| = 1+\epsilon$. If B denotes some small neighbourhood around e, we get

$$\int_B \frac{d\mu_l(z,w)}{(A^2+\phi^2)^{3/2}} \;\leq\; C\int_B \frac{dzdw}{(|z|^2+\epsilon^2+\phi^2)^{3/2}} \;<\; \infty.$$

Hence, M is locally integrable.

Lemma 3. *The local part of a first-order Riesz operator on G is given by left convolution with some kernel K, which, up to integrable terms, is given by*

$$K(z,w) \;=\; \frac{\kappa(z,w)}{(A^2+\phi^2)^2},$$

for some smooth function κ with compact support.

Proof. Similarly to the above, we only need to consider the behaviour close to $A = 0$, $\phi = 0$ of the term with $k = 0$ in the right hand side of (3). Also, the smooth function $\frac{|w|A}{\sinh(A)}$ has no relevance for the current estimates. We are left with studying the kernel

$$N(z,w) \;=\; \frac{1}{(A^2+\phi^2)^{3/2}}.$$

Applying the vector field X to N, i. e. differentiating with respect to x, and multiplying with the cut-off function ψ, we now get that the local part of $X\Delta^{-1/2}$ is given by left convolution with a kernel, which, up to integrable terms, is given by

$$\begin{aligned} K(z,w) &= \psi(z,w)XN(z,w) \\ &= \psi(z,w)\frac{\partial}{\partial x}(A^2+\phi^2)^{-3/2} \\ &= \psi(z,w)\frac{-3A}{(A^2+\phi^2)^{5/2}}\frac{\partial A}{\partial x}. \end{aligned}$$

Since

$$\frac{\partial A}{\partial x} \;=\; \frac{x}{|w|\sinh(A)}$$

is smooth, we can write

$$K(z,w) \;=\; \kappa(z,w)\frac{1}{(A^2+\phi^2)^2}, \tag{4}$$

where κ is a smooth and bounded function with compact support. By similar calculations, we get that the local part of any of the kernels ψZN and $\psi Z^t N$ can be obtained as left convolution with a kernel written in this form, for any Z among the basis fields in $\mathfrak{g}$. $\square$

Before we proceed, we state and prove a neat covering lemma, which holds for any Lie group G.

Lemma 4. *Given any $\epsilon, \delta > 0$, there exists a sequence $\{g_i\}_{i\in\mathbb{N}}$ in G and an $n > 0$, such that $G = \cup_i B(g_i, \epsilon)$ and no point $g \in G$ belongs to more than n of the sets $B(g_i, \delta)$.*

Proof. Since any second countable metric space is separable, there is a countable dense subset of G, say $\{g_j : j \in \mathbb{N}\}$. Then, $\cup_{j\in\mathbb{N}} B(g_j, \epsilon/3)$ is an open covering of G. Let $I \subseteq \mathbb{N}$ be a set of indices which is maximal in the sense that the balls $B(g_i, \epsilon/3)$, $i \in I$, are pairwise disjoint and this disjointness is broken if any $j \in \mathbb{N} \setminus I$ is added to I.

For any $h \in G$, we now have, by the maximality of I, that

$$d(h, B(g_i, \epsilon/3)) \le \epsilon/3,$$

for some $i \in I$. Hence, $d(h, g_i) < \epsilon$ and we conclude that

$$G = \cup_{i\in I} B(g_i, \epsilon).$$

To prove that the balls $B(g_i, \delta)$, $i \in I$, have finite overlap, suppose that some $h \in G$ belongs to k of the balls $B(g_i, \delta)$, $i \in I$. We can assume without loss of generality that these have indices $i = 1, \ldots, k$. Then, $g_1, \ldots, g_k \in B(h, \delta)$, which implies

$$\cup_{i=1}^{k} B(g_i, \epsilon/3) \subseteq B(h, \delta + \epsilon/3).$$

Taking the left-invariant Haar measure and using the fact that the balls $B(g_i, \epsilon/3)$, $i \in I$, are pairwise disjoint, we get

$$k\mu_l(B(\epsilon/3)) \quad \le \quad \mu_l(B(\delta + \epsilon/3)),$$

and k is therefore bounded above by a constant only depending on ϵ and δ. $\square$

The next lemma implies that the L^2-boundedness in Lemma 2 holds for the local and global parts separately.

Lemma 5. *Let K be a convolution kernel on G which is bounded outside any neighbourhood of e, and let $\psi \in C_0^\infty(G)$ be equal to 1 on some neighbourhood of e. If left convolution with $\operatorname{pv} K$ defines an operator which is bounded on $L^2(\mu_l)$, then left convolution with $\operatorname{pv}(\psi K)$ is also bounded on $L^2(\mu_l)$.*

Proof. Fix an $\epsilon > 0$ small enough so that $\psi = 1$ on $B(3\epsilon)$. According to the previous covering lemma, we can pick a sequence $\{g_i\}_{i\in\mathbb{N}}$ in G such that $G = \cup_i B(g_i, \epsilon)$ and no point belongs to more than n of the sets $B(g_i, 2\epsilon)$, $i \in \mathbb{N}$. By defining

$$\begin{cases} U_1 &= B(g_1, \epsilon), \\ U_{k+1} &= B(g_{k+1}, \epsilon) \setminus \cup_{j=1}^{k} U_j, \quad k = 1, 2, \dots, \end{cases}$$

we get a sequence of pairwise disjoint sets $\{U_i\}_{i\in\mathbb{N}}$ with $U_i \subset B(g_i, \epsilon)$ and $G = \cup_i U_i$. Note that $U_i^{-1} \subset B(g_i, \epsilon)^{-1}$ and $G = \cup_i U_i^{-1}$.

For $f \in C_0^\infty(G)$, we put $Tf = \mathrm{pv}(\psi K) * f$. To separate the singularity of the kernel, we also define

$$T_1 f(g) = \mathrm{pv} \int_{B(g_i, 2\epsilon)} \psi(gh) K(gh) f(h^{-1}) d\mu_l(h) \qquad \text{for } g \in U_i^{-1}.$$

We then have

$$Tf(g) - T_1 f(g) = \int_{G \setminus B(g_i, 2\epsilon)} \psi(gh) K(gh) f(h^{-1}) d\mu_l(h) \qquad \text{for } g \in U_i^{-1}.$$

Here, the singularity has been cut away from the integral, since $g^{-1} \in U_i \subset B(g_i, \epsilon)$ and $h \notin B(g_i, 2\epsilon)$ implies $d(gh, e) = d(h, g^{-1}) > \epsilon$. Hence, the kernel K is bounded by some constant, and we get

$$\begin{aligned} |Tf(g) - T_1 f(g)| &\leq C \int_G \psi(gh) |f(h^{-1})| d\mu_l(h) \\ &= C\psi * |f|(g). \end{aligned}$$

Hence, by Young's inequality, $T - T_1$ is bounded on $L^2(\mu_l)$.

It remains to see that T_1 is bounded on $L^2(\mu_l)$. Since $g^{-1} \in U_i \subset B(g_i, \epsilon)$ and $h \in B(g_i, 2\epsilon)$ implies $d(gh, e) < 3\epsilon$, we have

$$\begin{aligned} T_1 f(g) &= \mathrm{pv} \int_{B(g_i, 2\epsilon)} K(gh) f(h^{-1}) d\mu_l(h) \\ &= \mathrm{pv}\, K * (f \chi_{B(g_i, 2\epsilon)^{-1}})(g) \qquad \text{for } g \in U_i^{-1}. \end{aligned}$$

Denoting $f_i = f\chi_{B(g_i, 2\epsilon)^{-1}}$, we get

$$\begin{aligned} \int_{U_i^{-1}} |T_1 f(g)|^2 d\mu_l(g) &= \int_{U_i^{-1}} |\,\mathrm{pv}\, K * f_i(g)|^2 d\mu_l(g) \\ &\leq \|\,\mathrm{pv}\, K * f_i\|_{L^2(\mu_l)}^2 \\ &\leq C \int_{B(g_i, 2\epsilon)^{-1}} |f(g)|^2 d\mu_l(g) \end{aligned}$$

and

$$\begin{aligned}\int_G |T_1 f(g)|^2 d\mu_l(g) &= \sum_i \int_{U_i^{-1}} |T_1 f(g)|^2 d\mu_l(g)\\ &\leq C \sum_i \int_{B(g_i,2\epsilon)^{-1}} |f(g)|^2 d\mu_l(g)\\ &\leq Cn \int_G |f(g)|^2 d\mu_l(g).\end{aligned}$$

□

The following lemma is the main part in the proof of Proposition 2. To state it, we first introduce some notation. Let $\varrho : \mathbb{R}^3 \times \mathbb{T} \to G$ be the diffeomorphism given by

$$(x, y, s, \phi) \mapsto (x + iy, e^{s+i\phi}).$$

Lemma 6. *Let T be an operator given by $Tf = \mathrm{pv}\, K * f$, which is bounded on $L^2(\mu_l)$ and whose kernel $K \in L^1_{\mathrm{loc}}(G \setminus \{e\})$ has compact support in G. If there is a constant $C > 0$ such that*

$$\begin{aligned}|K(\varrho(\omega))| &\leq C|\omega|^{-4},\\ |\nabla_\omega K(\varrho(\omega))| &\leq C|\omega|^{-5},\end{aligned}$$

for all $\omega \neq 0$ in $\mathbb{R}^3 \times \mathbb{T}$, then T is of weak type $(1,1)$ with respect to μ_l on G.

Proof. For a function f on G, we write $\tilde{f} = f \circ \varrho$. We have

$$\begin{aligned}Tf(x + iy, e^{s+i\phi}) &= \int_G K\big((x + iy, e^{s+i\phi})h^{-1}\big) f(h) d\mu_r(h)\\ &= \int_{\mathbb{R}^3\times\mathbb{T}} K\big((x + iy, e^{s+i\phi})(x' + iy', e^{s'+i\phi'})^{-1}\big) \tilde{f}(x', y', s', \phi') dx'dy'ds'd\phi'\\ &= \widetilde{T}\tilde{f}(x, y, s, \phi),\end{aligned}$$

where the last equality defines the operator $\widetilde{T}$. Since K is compactly supported and Lebesgue measure m on $\mathbb{R}^3 \times \mathbb{T}$ is (via ϱ) locally equivalent with μ_l on G, it follows from the L^2-boundedness of T that

$$\|\widetilde{T}\tilde{f}\|_{L^2(\mathbb{R}^3\times\mathbb{T},m)} \leq C\|\tilde{f}\|_{L^2(\mathbb{R}^3\times\mathbb{T},m)}. \tag{5}$$

for all $f \in L^2(\mu_l)$ supported in some fixed compact set.

Let $\kappa \in C_0^\infty(G)$ with $0 \leq \kappa \leq 1$ and $\kappa \equiv 1$ on the unit ball $B(1)$ in G. We put

$$T_\kappa f = T(\kappa f) \qquad \text{and} \qquad \widetilde{T}_\kappa \tilde{f} = \widetilde{T}(\tilde{\kappa}\tilde{f}).$$

By (5), we have that $\widetilde{T}_\kappa$ is bounded on $L^2(\mathbb{R}^3 \times \mathbb{T}, m)$, since κ is compactly supported. We will prove that $\widetilde{T}_\kappa$ is of weak type $(1,1)$ with respect to m, by showing that its integration kernel satisfies the standard Calderón-Zygmund estimates.

The integration kernel of $\widetilde{T}_\kappa$ is

$$L(\omega;\omega') = K\big(\varrho(\omega)\varrho(\omega')^{-1}\big)\kappa(\varrho(\omega)).$$

By the assumed estimate of K, we have

$$|L(\omega,\omega')| \leq C\Big|\varrho^{-1}\big(\varrho(\omega)\varrho(\omega')^{-1}\big)\Big|^{-4}.$$

for all $\omega,\omega' \in \mathbb{R}^3\times\mathbb{T}$ with $\omega\neq\omega'$. Since K and κ are compactly supported, there is a constant $c>0$ such that

$$\Big|\varrho^{-1}\big(\varrho(\omega)\varrho(\omega')^{-1}\big)\Big| \leq c|\omega-\omega'|$$

for all $\omega,\omega' \in \mathbb{R}^3\times\mathbb{T}$ such that $\varrho(\omega)\varrho(\omega')^{-1}\in \operatorname{supp} K$ and $\varrho(\omega)\in\operatorname{supp}\kappa$. Hence, we have found that

$$|L(\omega,\omega')| \leq C|\omega-\omega'|^{-4}$$

for $\omega\neq\omega'$. Similarly, it follows from the assumed estimate of $\nabla_\omega K(\varrho(\omega))$ that L satisfies

$$|\nabla_\omega L(\omega,\omega')| + |\nabla_{\omega'} L(\omega,\omega')| \leq C|\omega-\omega'|^{-5}.$$

By Calderón-Zygmund theory, we now have that $\widetilde{T}_\kappa$ is of weak type $(1,1)$, i. e. that

$$m\big(\{(x,y,s,\phi)\in\mathbb{R}^3\times\mathbb{T} : |T_\kappa f(x+iy,e^{s+i\phi})|>\lambda\}\big) \leq \frac{C}{\lambda}\|\tilde{f}\|_{L^1(\mathbb{R}^3\times\mathbb{T},m)}.$$

Note that $T_\kappa f = Tf$ if $\operatorname{supp} f\subseteq B(1)$. We can once again use the local equivalence of μ_l and m and the fact that the kernel of T_κ is compactly supported, to get

$$\mu_l\big(\{g\in G : |Tf(g)|>\lambda\}\big) \leq \frac{C}{\lambda}\|f\|_{L^1(\mu_l)}, \tag{6}$$

for all functions $f\in L^1(\mu_l)$ with $\operatorname{supp} f\subseteq B(1)$. To complete the proof, it remains to extend this result to all functions $f\in L^1(\mu_l)$. For this, we will use Lemma 4, with $\epsilon=1$.

First, fix an $N>1$ such that

$$\operatorname{supp} K\cdot B(1)\subseteq B(N),$$

so that $\operatorname{supp} f\subseteq B(1)$ implies $\operatorname{supp} Tf\subseteq B(N)$. Let τ_h denote right translation with $h\in G$, i. e. $\tau_h f(g) = f(gh)$. Since T is a left-convolution operator, we have $T=\tau_h T\tau_{h^{-1}}$ for any $h\in G$. It follows that

$$\operatorname{supp} f\subseteq B(1)h \quad\Rightarrow\quad \operatorname{supp} Tf\subseteq B(N)h, \qquad h\in G.$$

Let $\{g_i\}_{i\in\mathbb{N}}$ be a sequence in G such that $G=\cup_i B(g_i,1)$ and no $g\in G$ belongs to more than n of the sets $B(g_i,N)$. Let $\{\psi_i\}_{i\in\mathbb{N}}$ be a sequence of L^∞ functions on G with $0\le\psi_i\le 1$, $\psi_i=0$ outside $B(g_i,1)^{-1}$ and $\sum_i\psi_i=1$ everywhere[6]. Since $B(g_i,1)^{-1}=B(1)g_i^{-1}$ and $B(g_i,N)^{-1}=B(N)g_i^{-1}$, we get

$$\operatorname{supp} T(\psi_i f)\subseteq B(g_i,N)^{-1}$$

and we conclude that the sets $\operatorname{supp} T(\psi_i f)$, $i\in\mathbb{N}$, have finite overlap bounded by n. Thus

$$\begin{aligned}\mu_l\big(\{g\in G:|Tf(g)|>\lambda\}\big) &\le \mu_l\big(\{g\in G:\sum_i|T(\psi_i f)(g)|>\lambda\}\big)\\ &\le \sum_i\mu_l\big(\{g\in G:|T(\psi_i f)(g)|>\lambda/n\}\big)\\ &= \sum_i\mu_l\big(\{gg_i^{-1}\in G:|T\tau_{g_i^{-1}}(\psi_i f)(g)|>\lambda/n\}\big)\\ &= \sum_i\delta\left(g_i^{-1}\right)\mu_l\big(\{g\in G:|T\tau_{g_i^{-1}}(\psi_i f)(g)|>\lambda/n\}\big),\end{aligned}$$

where we used, in the third step, the identity $T=\tau_{g_i}T\tau_{g_i^{-1}}$. Noting that

$$\operatorname{supp}\tau_{g_i^{-1}}(\psi_i f)\quad\subseteq\quad B(1),$$

we can now use (6) to conclude that

$$\begin{aligned}\mu_l\big(\{g\in G:|Tf(g)|>\lambda\}\big) &\le \frac{Cn}{\lambda}\sum_i\delta\left(g_i^{-1}\right)\int_G|\tau_{g_i^{-1}}(\psi_i f)(g)|d\mu_l(g)\\ &= \frac{Cn}{\lambda}\sum_i\|\psi_i f\|_{L^1(\mu_l)}\\ &= \frac{Cn}{\lambda}\|f\|_{L^1(\mu_l)},\end{aligned}$$

which finishes the proof. □

To see that the local parts of the first-order Riesz operators are of weak type $(1,1)$, it remains only to check that the corresponding kernels satisfy the estimates in Lemma 6.

By Taylor expansion, we find that

$$\tilde{A}^2=x^2+y^2+s^2+O\big((x^2+y^2+s^2)^{3/2}\big),\qquad x^2+y^2+s^2\to 0,$$

where $z=x+iy$ and $w=e^{s+i\phi}$. Inserting this in the expression given in Lemma 3 for the kernel of an operator of the form $Z\Delta^{-1/2}$ or $\Delta^{-1/2}Z$, $Z\in\mathfrak{g}$, it is easy to see that the kernels satify the required estimates in Lemma 6. This concludes the proof of Proposition 2.

[6]For example, we can take $\psi_i=\chi_{B(g_i,1)^{-1}}/\sum_j\chi_{B(g_j,1)^{-1}}$.

4. The kernels

We need to find the convolution kernels for the global parts of the operators in Theorem 1. The outline will be to rewrite the expression (3) for the kernel M of $\Delta^{-1/2}$, and then apply an appropriate vector field. We then try to simplify the result as much as possible, keeping only the parts essential to our purposes. We can therefore neglect any terms of the kernels, which are integrable at infinity, since these correspond to strong type $(1,1)$ operators, according to Young's inequality.

Let

$$R(z) \quad = \quad \frac{x}{(1+|z|^2)^2}, \qquad z \in \mathbb{C},$$

and define $R_w(z) = |w|^{-2}R(w^{-1}z)$, for $w \in \mathbb{C} \setminus \{0\}$. The results in this section are the following three propositions.

Proposition 3. *The global parts of the operators $V\Delta^{-1/2}$ and $\Delta^{-1/2}V$ are given by left convolution with integrable kernels.*

Proposition 4. *The global part of the operator $X\Delta^{-1/2}$ is given by left convolution with a kernel which, up to integrable terms and a constant factor, is given by*

$$P(z,w) \quad = \quad R(z)\frac{|w|^2}{\log|w|}\chi_{(0,\epsilon)}(|w|),$$

where $\epsilon > 0$ can be chosen arbitrarily small.

Proposition 5. *The global part of $\Delta^{-1/2}X$ is given by left convolution with a kernel which, up to integrable terms and a constant factor, is given by*

$$Q(z,w) \quad = \quad R\left(\frac{z}{w}\right)\frac{1}{\log|w|}\chi_{(N,\infty)}(|w|),$$

where $N > 0$ can be chosen arbitrarily large.

Recall that the operator $\Delta^{-1/2}$ is given by left convolution with the kernel

$$M(z,w) \quad = \quad \frac{3}{4\pi^2}\frac{|w|A}{\sinh(A)}\sum_{k\in\mathbb{Z}}\frac{1}{(A^2+(\phi-2k\pi)^2)^{3/2}}.$$

We need to find a more suitable expression for M. First, we will use the Poisson summation formula $\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(2\pi k)$, where $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx$. We take $f(x) = (1+x^2)^{-3/2}$ and we need to find $\hat{f}$. Since

$$\alpha^{-\gamma} = \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-\alpha s}s^{\gamma-1}ds, \qquad \alpha,\gamma > 0,$$

we can write

$$f(x) = \frac{1}{8\Gamma(3/2)} \int_0^\infty e^{-\frac{s}{4}} e^{-\frac{sx^2}{4}} s^{1/2} ds.$$

Taking the Fourier transform and interchanging the order of integration now yields

$$\hat{f}(\xi) = \frac{1}{2} \int_0^\infty e^{-\frac{s}{4}} e^{-\frac{\xi^2}{s}} ds.$$

With the proper translation and dilation adjustments, we now see that

$$\begin{aligned} M(z,w) &= \frac{3}{4\pi^2} \frac{|w|}{A^2 \sinh(A)} \sum_{k\in\mathbb{Z}} f\left(\frac{2\pi}{A}\left(k - \frac{\phi}{2\pi}\right)\right) \\ &= \frac{3}{4\pi^2} \frac{|w|}{A^2 \sinh(A)} \sum_{k\in\mathbb{Z}} e^{-ik\phi} \frac{A}{2\pi} \hat{f}(Ak) \\ &= \frac{3}{16\pi^3} \frac{|w|}{A \sinh(A)} \sum_{k\in\mathbb{Z}} e^{-ik\phi} \int_0^\infty e^{-\frac{s}{4}} e^{-\frac{A^2k^2}{s}} ds \\ &= \frac{3}{4\pi^3} \frac{|w|}{A \sinh(A)} + \frac{3}{8\pi^3} \frac{|w|}{A \sinh(A)} \sum_{k=1}^\infty \cos(k\phi) \int_0^\infty e^{-\frac{s}{4}} e^{-\frac{A^2k^2}{s}} ds \\ &= M_0(z,w) + \sum_{k=1}^\infty M_k(z,w). \end{aligned}$$

Next, we will prove that it is only necessary to keep the part M_0, since the remaining sum is integrable over the region $\{A > 1\}$, after a vector field has been applied. We state this as a lemma.

Lemma 7. *For any $Z \in \mathfrak{g}$, we have*

$$\int_{\{A>1\}} |Z(M - M_0)| d\mu_l < \infty \quad \text{and} \quad \int_{\{A>1\}} |Z^{\mathrm{t}}(M - M_0)| d\mu_l < \infty.$$

Proof. Before we go into the details, we state some elementary integral estimates, which we will use. It is easy to see that for $\alpha \geq 1$,

$$I_1(\alpha) = \int_0^\infty \exp\left(-\frac{s}{4} - \frac{\alpha^2}{s}\right) ds < C\alpha e^{-\alpha}, \tag{7}$$

$$I_2(\alpha) = \int_0^\infty s^{-1} \exp\left(-\frac{s}{4} - \frac{\alpha^2}{s}\right) ds < Ce^{-\alpha}. \tag{8}$$

We will also repeatedly use that, since $\operatorname{arcosh}(t) > \log(t)$ and therefore $e^{-A} < \frac{2|w|}{|z|^2+1+|w|^2}$, we have

$$\begin{aligned}
(9) \qquad \int_{\{A>1\}} |w| e^{-2A} d\mu_l &< \int_{\mathbb{C}^2} |w| \left(\frac{2|w|}{|z|^2+1+|w|^2} \right)^2 \frac{dzdw}{|w|^4} \\
&< C \int_{\rho=0}^{\infty} \int_{r=0}^{\infty} \rho \left(\frac{2\rho}{r^2+1+\rho^2} \right)^2 rdr \frac{d\rho}{\rho^3} \\
&< \infty.
\end{aligned}$$

We start with $Z = V$. Recall that $V^{\mathrm{t}} = \frac{\partial}{\partial \phi}$ and note that A depends on neither ϕ nor θ. We have

$$\begin{aligned}
|V^{\mathrm{t}}(M - M_0)| &\leq \sum_{k=1}^{\infty} \left| \frac{\partial M_k}{\partial \phi} \right| \\
&\leq C \frac{|w|}{A \sinh(A)} \sum_{k=1}^{\infty} k I_1(Ak).
\end{aligned}$$

By using (7), we get

$$\begin{aligned}
|V^{\mathrm{t}}(M - M_0)| &< C \frac{|w|}{\sinh(A)} \sum_{k=1}^{\infty} k^2 e^{-Ak} \\
&= C \frac{|w|}{\sinh(A)} e^{-A} \sum_{k=1}^{\infty} k^2 e^{-A(k-1)} \\
&\leq C|w| e^{-2A}.
\end{aligned}$$

Hence, we conclude by (9) that

$$\int_{\{A>1\}} |V^{\mathrm{t}}(M - M_0)| d\mu_l < \infty.$$

Since M_k does not depend on θ and $V = \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}$, we have immediately also that

$$\int_{\{A>1\}} |V(M - M_0)| d\mu_l \quad < \quad \infty.$$

Before we proceed with the other base vector fields, we want to estimate $\sum_{k=1}^{\infty} |M_k|$ and $\sum_{k=1}^{\infty} \left| \frac{\partial M_k}{\partial A} \right|$. We have

$$M_k(z, w) \quad = \quad \frac{3}{8\pi^3} \frac{|w|}{A \sinh(A)} \cos(k\phi) I_1(Ak),$$

so that

$$\sum_{k=1}^{\infty} |M_k| \;\leq\; C \frac{|w|}{\sinh(A)} \sum_{k=1}^{\infty} k e^{-Ak} \;\leq\; C|w| e^{-2A}.$$

Furthermore,

$$\begin{aligned}\frac{\partial M_k}{\partial A} &= -\frac{3}{4\pi^3}\frac{|w|\cos(k\phi)}{\sinh(A)}k^2 I_2(Ak)\\ &\quad -\frac{3}{8\pi^3}\frac{|w|\cos(k\phi)}{A\sinh(A)}\frac{A\cosh(A)+\sinh(A)}{A\sinh(A)}I_1(Ak),\end{aligned}$$

so by the estimates (7) and (8), we get

$$\begin{aligned}\left|\frac{\partial M_k}{\partial A}\right| &\leq C\frac{|w|}{\sinh(A)}k^2e^{-Ak}+C\frac{|w|}{A\sinh(A)}Ake^{-Ak}\\ &\leq C\frac{|w|}{\sinh(A)}k^2e^{-Ak}\end{aligned}$$

and therefore

$$\sum_{k=1}^{\infty}\left|\frac{\partial M_k}{\partial A}\right| \leq C|w|e^{-2A}.$$

Now, studying $Z = X = \frac{\partial}{\partial x}$, we have $XM_k = \frac{\partial M_k}{\partial A}\frac{\partial A}{\partial x}$, where

$$\begin{aligned}\frac{\partial A}{\partial x} &= \frac{\partial}{\partial x}\operatorname{arcosh}\left(\frac{x^2+y^2+1+|w|^2}{2|w|}\right)\\ &= \frac{1}{\sinh(A)}\frac{x}{|w|},\end{aligned}$$

and since $A > 1$, we can estimate

$$\left|\frac{\partial A}{\partial x}\right| \leq \frac{|z|}{|w|}e^{-A}.$$

Putting our estimates together, we conclude that

$$\begin{aligned}|X(M-M_0)| &\leq \sum_{k=1}^{\infty}|XM_k|\\ &= \left|\frac{\partial A}{\partial x}\right|\sum_{k=1}^{\infty}\left|\frac{\partial M_k}{\partial A}\right|\\ &\leq C|z|e^{-3A}.\end{aligned}\tag{10}$$

Since $\frac{|z|}{|w|}e^{-A}$ is bounded, it follows from (9) that

$$\int_{\{A>1\}}|X(M-M_0)|d\mu_l < \infty.$$

By symmetry, the same estimates hold for $Z = Y$, i. e.

$$\int_{\{A>1\}}|Y(M-M_0)|d\mu_l < \infty.$$

For the transposed operator X^{t}, recall that $X^{\mathrm{t}} = uX + vY$. Hence, almost the same estimate as (10) holds also for X^{t}. We only need to multiply with an extra $|w|$ on the right-hand side, i. e.

$$|X^{\mathrm{t}}(M - M_0)| \leq C|z||w|e^{-3A},$$

and, since $|z|e^{-A}$ is bounded, it follows again by (9) that

$$\int_{\{A>1\}} |X^{\mathrm{t}}(M - M_0)|d\mu_l < \infty.$$

The case $Y^{\mathrm{t}} = -vX + uY$ is completely analogous.

It remains to study $Z = U$. Recall that $U = r\frac{\partial}{\partial r} + \rho\frac{\partial}{\partial \rho}$ and $U^{\mathrm{t}} = \rho\frac{\partial}{\partial \rho} - 2$. We first note that

$$\frac{\partial A}{\partial r} = \frac{1}{\sinh(A)}\frac{\partial}{\partial r}\left(\frac{r^2+1+\rho^2}{2\rho}\right) = \frac{1}{\sinh(A)}\frac{r}{\rho}$$

and

$$\frac{\partial A}{\partial \rho} = \frac{1}{\sinh(A)}\frac{\partial}{\partial \rho}\left(\frac{r^2+1+\rho^2}{2\rho}\right) = \frac{1}{\sinh(A)}\frac{\rho^2 - r^2 - 1}{2\rho^2}.$$

Furthermore, we have

$$\frac{\partial M_k}{\partial \rho} = \frac{3}{8\pi^3}\frac{\partial}{\partial \rho}\left(\frac{\rho}{A\sinh(A)}\cos(k\phi)I_1(Ak)\right) = \rho^{-1}M_k + \frac{\partial M_k}{\partial A}\frac{\partial A}{\partial \rho}$$

and

$$\frac{\partial M_k}{\partial r} = \frac{\partial M_k}{\partial A}\frac{\partial A}{\partial r}.$$

Collecting terms, we get

$$UM_k = M_k + \frac{1}{\sinh(A)}\frac{\rho^2 + r^2 - 1}{2\rho}\frac{\partial M_k}{\partial A},$$

and we see that

$$\sum_{k=1}^{\infty}|UM_k| \leq \sum_{k=1}^{\infty}\left(|M_k| + C\left|\frac{\partial M_k}{\partial A}\right|\right) \leq C|w|e^{-2A}.$$

Hence, we can once again use (9) to conclude that

$$\int_{\{A>1\}} |U(M - M_0)|d\mu_l < \infty.$$

Finally,

$$\begin{aligned}\sum_{k=1}^{\infty}|U^{\mathrm{t}}M_k| &= \sum_{k=1}^{\infty}\left|M_k+\rho\frac{\partial A}{\partial\rho}\frac{\partial M_k}{\partial A}-2M_k\right| \\ &\leq \sum_{k=1}^{\infty}\left(|M_k|+C\left|\frac{\partial M_k}{\partial A}\right|\right) \\ &\leq C|w|e^{-2A},\end{aligned}$$

so that

$$\int_{\{A>1\}}|U^{\mathrm{t}}(M-M_0)|d\mu_l<\infty,$$

and the proof is finished. □

Proposition 3 follows immediately from Lemma 7, since M_0 only depends on the moduli of z and w, and not on their arguments. Hence, $VM_0=V^{\mathrm{t}}M_0=0$.

Proof of Proposition 4. We will start from $XM_0=\frac{\partial M_0}{\partial x}$ and prove that $|XM_0-P|$ has finite integral over the region where $A>1$. We have $XM_0=\frac{\partial M_0}{\partial A}\frac{\partial A}{\partial x}$, where

$$\frac{\partial M_0}{\partial A}=-\frac{3|w|}{4\pi^3}\frac{\sinh(A)+A\cosh(A)}{A^2\sinh^2(A)}=-\frac{3|w|}{4\pi^3}\frac{1+A\coth(A)}{A^2\sinh(A)}$$

and

$$\frac{\partial A}{\partial x}=\frac{x}{|w|\sinh(A)}.$$

Thus,

$$XM_0=-\frac{3x}{4\pi^3}\frac{1+A\coth(A)}{A^2\sinh^2(A)}.$$

By adding and subtracting the same expressions, we get

$$
\begin{aligned}
(11) \qquad \frac{4\pi^3}{3}|XM_0 - 4P| \le{}& \frac{|x|}{A^2\sinh^2(A)} \\
&+\frac{|x|}{A}\left|\frac{\coth(A)}{\sinh^2(A)} - \frac{1}{\cosh^2(A)}\right| \\
&+\frac{|x|}{\cosh^2(A)}\left|\frac{1}{A} - \frac{1}{\log\left(2\cosh(A)\right)}\right| \\
&+\frac{|x|}{\log\left(2\cosh(A)\right)\cosh^2(A)}\chi_{[\epsilon,\infty)}(|w|) \\
&+\frac{|x|}{\log\left(\frac{|z|^2+1+|w|^2}{|w|}\right)}\left|\frac{1}{\cosh^2(A)} - \left(\frac{2|w|}{1+|z|^2}\right)^2\right|\chi_{(0,\epsilon)}(|w|) \\
&+4|w|^2\frac{|x|}{(1+|z|^2)^2}\left|\frac{1}{\log\left(\frac{|z|^2+1+|w|^2}{|w|}\right)} - \frac{1}{\log\left(\frac{1}{|w|}\right)}\right|\chi_{(0,\epsilon)}(|w|)
\end{aligned}
$$

and we will prove that each term on the right hand side has a finite integral over the region $\{A > 1\}$. First, note that for any sufficiently small $\delta > 0$, we have

(12)

$$
\begin{aligned}
\int_{\{A>1\}} \frac{|x|}{A^2\cosh^2(A)}d\mu_l &\le C\iint_{\{\rho,r>0:A>1\}}\frac{r}{A^2}\left(\frac{2\rho}{1+r^2+\rho^2}\right)^2 r dr\frac{d\rho}{\rho^3} \\
&\le C\int_{\rho=\delta}^{\infty}\int_{r=0}^{\infty}\frac{dr}{1+\rho^2+r^2}\frac{d\rho}{\rho} \\
&\quad + C\int_{\rho=0}^{\delta}\int_{r=0}^{\infty}\frac{1}{\left(\operatorname{arcosh}\left(\frac{1+r^2+\rho^2}{2\rho}\right)\right)^2}\frac{dr}{1+\rho^2+r^2}\frac{d\rho}{\rho} \\
&\le C_\delta + C\int_{\rho=0}^{\delta}\frac{1}{(\log(2\rho))^2}\frac{d\rho}{\rho} \\
&< \infty,
\end{aligned}
$$

where, in the third step, we used that $\operatorname{arcosh}(\frac{1+r^2+\rho^2}{2\rho}) > \log(\frac{1+r^2+\rho^2}{2\rho}) > -\log(2\rho)$. Now, starting with the first term in (11), we have for $A > 1$ that

$$
\frac{|x|}{A^2\sinh^2(A)} \quad \le \quad C\frac{|x|}{A^2\cosh^2(A)}.
$$

Furthermore, for the second term,

$$\begin{aligned}\frac{|x|}{A}\left|\frac{\coth(A)}{\sinh^2(A)}-\frac{1}{\cosh^2(A)}\right| &= A\left|\coth^3(A)-1\right|\frac{|x|}{A^2\cosh^2(A)}\\ &\le C\frac{|x|}{A^2\cosh^2(A)}.\end{aligned}$$

To estimate the third term in (11), consider

$$\begin{aligned}\frac{1}{\log(2\cosh(A))} &= \frac{1}{A+\log(1+e^{-2A})}\\ &= \frac{1}{A}\frac{1}{1+O(A^{-1}e^{-2A})}\\ &= \frac{1}{A}+\frac{1}{A^2}O(e^{-2A}), \qquad A\to\infty,\end{aligned}$$

and thus

$$\frac{|x|}{\cosh^2(A)}\left|\frac{1}{A}-\frac{1}{\log(2\cosh(A))}\right| \le C\frac{|x|}{A^2\cosh^2(A)}.$$

It now follows from (12) that the integrals of the first three terms in the right hand side of (11) are finite. For the fourth term, we have

$$\begin{aligned}&\int_{\{A>1\}}\frac{|x|}{\log\left(2\cosh(A)\right)\cosh^2(A)}\chi_{[\epsilon,\infty)}(|w|)d\mu_l(z,w)\\ &\qquad\le C\int_{\rho=\epsilon}^{\infty}\int_{r=0}^{\infty} r\left(\frac{2\rho}{1+\rho^2+r^2}\right)^2 rdr\frac{d\rho}{\rho^3}\\ &\qquad<\infty.\end{aligned}$$

To estimate the fifth term, we note that

$$\begin{aligned}\left|\frac{1}{\cosh^2(A)}-\left(\frac{2|w|}{1+|z|^2}\right)^2\right| &= \frac{4|w|^2}{(1+|z|^2)^2}\left|\left(\frac{1+|z|^2}{1+|z|^2+|w|^2}\right)^2-1\right|\\ &\le \frac{8|w|^4}{(1+|z|^2)^3},\end{aligned}$$

so that

$$\begin{aligned}&\int_{\{A>1\}}\frac{|x|}{\log(2\cosh(A))}\left|\frac{1}{\cosh^2(A)}-\left(\frac{2|w|}{1+|z|^2}\right)^2\right|\chi_{(0,\epsilon)}(|w|)d\mu_l(z,w)\\ &\qquad\le C\int_{\rho=0}^{\epsilon}\int_{r=0}^{\infty} r\frac{\rho^4}{(1+r^2)^3}rdr\frac{d\rho}{\rho^3}\\ &\qquad<\infty.\end{aligned}$$

Finally, we have for $|w| < 1$ that

$$\begin{aligned} \left| \frac{1}{\log\left(\frac{|z|^2+1+|w|^2}{|w|}\right)} - \frac{1}{\log\left(\frac{1}{|w|}\right)} \right| &= \frac{\log(1+|z|^2+|w|^2)}{(\log|w|)^2} \frac{1}{1+\frac{\log(1+|z|^2+|w|^2)}{-\log|w|}} \\ &\leq \frac{\log(2+|z|^2)}{(\log|w|)^2}, \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\{A>1\}} |w|^4 \frac{|x|}{(1+|z|^2)^2} &\left| \frac{1}{\log\left(\frac{|z|^2+1+|w|^2}{|w|}\right)} - \frac{1}{\log\left(\frac{1}{|w|}\right)} \right| \chi_{(0,\epsilon)}(|w|) d\mu_l(z,w) \\ &\leq \int_{\rho=0}^{\epsilon} \int_{r=0}^{\infty} \rho^4 \frac{r}{(1+r^2)^2} \frac{\log(2+r^2)}{(\log(\rho))^2} r dr \frac{d\rho}{\rho^3} \\ &< \infty. \end{aligned}$$

Hence, we have found that $\int_{\{A>1\}} |XM_0 - 4P| d\mu_l < \infty$. This and Lemma 7 proves the proposition. $\square$

Proof of Proposition 5. By the observations $\langle \check{f}, g\rangle = \langle f, \delta\check{g}\rangle$, $(f*g)\check{} = \check{g}*\check{f}$ and $\langle k*f, g\rangle = \langle k, g*\check{f}\rangle$ for any $f, g, k \in C_0^\infty(G)$, we get

$$\begin{aligned} \langle k*f, g\rangle &= \langle \check{f}*\check{k}, \delta\check{g}\rangle \\ &= \langle \check{f}, (\delta\check{g})*k\rangle \\ &= \langle f, \check{\delta}(\check{k}*(\delta g))\rangle \\ &= \langle f, (\check{\delta}\check{k})*g\rangle. \end{aligned}$$

Since $\langle \Delta^{-1/2}Xf, g\rangle = -\langle f, X\Delta^{-1/2}g\rangle$ for suitable functions f, g on G, we get, with P as in Proposition 4,

$$\begin{aligned} Q(z,w) &= -\check{\delta}(z,w)\check{P}(z,w) \\ &= -\delta\left(-\frac{z}{w}, \frac{1}{w}\right) P\left(-\frac{z}{w}, \frac{1}{w}\right) \\ &= |w|^2 R\left(-\frac{z}{w}\right) \frac{\left|\frac{1}{w}\right|^2}{\log\left|\frac{1}{w}\right|} \chi_{(0,\epsilon)}\left(\left|\frac{1}{w}\right|\right) \\ &= R\left(\frac{z}{w}\right) \frac{1}{\log|w|} \chi_{(\epsilon^{-1},\infty)}(|w|). \end{aligned}$$

$\square$

5. The operator $X\Delta^{-1/2}$

Proposition 6. *The global part of the operator $X\Delta^{-1/2}$ is of weak type $(1,1)$.*

Proof. We want to split the kernel into a sum of terms, where $\log|w|$ for each term can be approximated by a constant, and the total error is integrable. For this, we first introduce the notation

$$I_j = \left]\exp\left(-(j+1)^4\right), \exp\left(-j^4\right)\right],$$

for $j \in \mathbb{N}$. Now, note that we have, for $|w| \in I_j$,

$$\begin{aligned} \left|\frac{1}{\log|w|} + \frac{1}{j^4}\right| &= \left|\frac{j^4 + \log|w|}{j^4 \log|w|}\right| \\ &\leq \frac{\left|j^4 + \log|w|\right|}{j^8} \\ &\leq Cj^{-5} \\ &\leq C\left|\log|w|\right|^{-5/4}. \end{aligned}$$

By putting

$$P_j(z,w) = -j^{-4} R(z)|w|^2 \chi_{I_j}(|w|),$$

we therefore get (by choosing $\epsilon = e^{-1}$)

$$\begin{aligned} \left|P - \sum_{j=1}^{\infty} P_j\right| &= |R(z)||w|^2 \left|\frac{\chi_{[0,\epsilon]}(|w|)}{\log|w|} + \sum_{j=1}^{\infty} \frac{\chi_{I_j}(|w|)}{j^4}\right| \\ &\leq |R(z)||w|^2 \sum_{j=1}^{\infty} \left|\frac{1}{\log|w|} + \frac{1}{j^4}\right| \chi_{I_j}(|w|) \\ &\leq C|R(z)||w|^2 \left|\log|w|\right|^{-5/4} \chi_{[0,\epsilon]}(|w|). \end{aligned}$$

Thus

$$\begin{aligned} \int \left|P - \sum_{j=1}^{\infty} P_j\right| d\mu_l &\leq C \int_{r=0}^{\infty} |R(z)| r dr \int_{\rho=0}^{\epsilon} \rho^2 |\log\rho|^{-5/4} \rho^{-3} d\rho \\ &\leq C \int_{r=0}^{\infty} \frac{r^2}{(1+r^2)^2} dr \int_{\rho=0}^{\epsilon} |\log\rho|^{-5/4} \frac{d\rho}{\rho} \\ &< \infty, \end{aligned} \tag{13}$$

so that $\sum P_j$ gives the desired splitting of the kernel.

The next step will be to fix $j \in \mathbb{N}$ and a function $f \in L^1(\mu_l)$, and estimate $\|P_j * f\|_{L^{1,\infty}(\mu_l)}$. First, we will need some more notation. For each $j \in \mathbb{N}$, which is now fixed, we make a partition of the positive real axis, by putting

$$J_i =]\exp(ij^2), \exp((i+1)j^2)], \qquad i \in \mathbb{Z},$$

and we also define

$$f_i(z,w) = f(z,w)\chi_{J_i}(|w^{-1}|), \qquad i \in \mathbb{Z}.$$

Note that $\check{f}_i(z,w) = \check{f}(z,w)\chi_{J_i}(|w|)$. We now get

$$\begin{aligned}
P_j * f_i(z,w) &= \int P_j((z,w)(z',w'))f_i((z',w')^{-1})d\mu_l(z',w') \\
&= -j^{-4}\iint R(z+wz')|ww'|^2\chi_{I_j}(|ww'|)\check{f}(z',w')\chi_{J_i}(|w'|)|w'|^{-4}dw'dz' \\
&= -j^{-4}\iint_{\{(z',w'):|w'|\in J_i\cap(|w|^{-1}I_j)\}} R_{w^{-1}}(z'+w^{-1}z)\check{f}(z',w')|w'|^{-2}dw'dz'.
\end{aligned}$$

The estimates can now be divided into three cases, depending on the set of integration on the right-hand side. For each w, exactly one of the following occurs,

1. $J_i \cap (|w|^{-1}I_j) = J_i$,
2. $J_i \cap (|w|^{-1}I_j) = \emptyset$,
3. $J_i \cap (|w|^{-1}I_j)$ is a proper subset of J_i.

The second case obviously gives no contribution, so writing $\chi^{(1)}$ for the characteristic function of the set where the first case appears, and similarly writing $\chi^{(3)}$ for the third case, we have

$$P_j * f_i = (P_j * f_i)\chi^{(1)} + (P_j * f_i)\chi^{(3)}. \tag{14}$$

Note that $\chi^{(1)}$ and $\chi^{(3)}$ do not depend on z. Thus we write in the following $\chi^{(k)}(z,w) = \chi^{(k)}(w)$, $k = 1, 3$.

We begin by studying the first case. We introduce

$$F_i(z') = \int \check{f}_i(-z',w')|w'|^{-2}dw', \qquad i \in \mathbb{Z},$$

and note that $\|F_i\|_{L^1(\mathbb{C})} \leq \|f_i\|_{L^1(\mu_l)}$, since

$$\begin{aligned}
\int_{\mathbb{C}} |F_i(z')|dz' &\leq \int_G \left|\check{f}_i(-z',w')\right| |w'|^{-2}dw'dz' \\
&= \int_G \left|\check{f}_i\right| d\mu_r \\
&= \int_G |f_i|\, d\mu_l.
\end{aligned}$$

In particular, $F_i \in L^1(\mathbb{C})$ for each $i \in \mathbb{Z}$. We now have

(15)

$$\begin{aligned}
(P_j * f_i)\chi^{(1)} &= -j^{-4}\chi^{(1)}(w)\iint_{\{(z',w'):|w'|\in J_i\}} R_{w^{-1}}(z'+w^{-1}z)\check{f}(z',w')|w'|^{-2}dw'dz' \\
&= -j^{-4}\chi^{(1)}(w)\int R_{w^{-1}}(z'+w^{-1}z)\left(\int \check{f}_i(z',w')|w'|^{-2}dw'\right)dz' \\
&= -j^{-4}\chi^{(1)}(w)\int R_{w^{-1}}(z'+w^{-1}z)F_i(-z')dz' \\
&= -j^{-4}\chi^{(1)}(w)R_{w^{-1}} * F_i(w^{-1}z),
\end{aligned}$$

where the last convolution is in $\mathbb{C}$. Our aim is to estimate $\mu_l(E_\lambda)$ in terms of $\|F_i\|_{L^1(\mathbb{C})}$, where

$$\begin{aligned}
E_\lambda &= \left\{(z,w) : J_i \cap (|w|^{-1}I_j) = J_i,\ j^{-4}\left|R_{w^{-1}} * F_i(w^{-1}z)\right| > \lambda\right\} \\
&= \left\{(z,w) : \log|w| + \log(J_i) \subseteq \log(I_j),\ \left|R_{w^{-1}} * F_i(w^{-1}z)\right| > j^4\lambda\right\} \\
&\subseteq \left\{(z,w) : \log|w| \in \log(I_j) - \log(J_i),\ \left|R_{w^{-1}} * F_i(w^{-1}z)\right| > j^4\lambda\right\}.
\end{aligned}$$

Changing variables according to $\zeta = w^{-1}z$ and $\eta = w^{-1}$ yields $d\mu_l(z,w) = d\mu_r(\zeta,\eta)$, and thus

$$\mu_l(E_\lambda) \leq \mu_r\left(\left\{(\zeta,\eta) : \log|\eta| \in \log(J_i) - \log(I_j),\ |R_\eta * F_i(\zeta)| > j^4\lambda\right\}\right).$$

Note that

$$\log(I_j) - \log(J_i) =]-(j+1)^4, -j^4] -]ij^2, (i+1)j^2]$$

will be contained in some interval with length bounded by Cj^3. We can therefore use Lemma 9 below, to get

$$\begin{aligned}
\mu_l(E_\lambda) &\leq \frac{C\sqrt{j^3}}{j^4\lambda}\|F_i\|_{L^1(\mathbb{C})} \\
&\leq \frac{C}{\lambda}j^{-5/2}\|f_i\|_{L^1(\mu_l)}.
\end{aligned}$$

Hence

$$\|(P_j * f_i)\chi^{(1)}\|_{L^{1,\infty}(\mu_l)} \leq Cj^{-5/2}\|f_i\|_{L^1(\mu_l)}.$$

We now sum in $i \in \mathbb{Z}$, while still keeping j fixed. Since $\log(I_j)$ is an interval of length bounded by Cj^3 and $\log(J_i)$ is an interval of length j^2, no point belongs to more than Cj of the sets $J_i \cap (|w|^{-1}I_j)$, when $i \in \mathbb{Z}$ varies.

We need the following simple result on the subadditivity of the $L^{1,\infty}$ quasinorm. If ϕ_k are functions in weak L^1, for which no point belongs to more than N of the supports of the ϕ_k, we have by trivial means that

$$\left\|\sum \phi_k\right\|_{L^{1,\infty}} \leq N\sum \|\phi_k\|_{L^{1,\infty}}. \tag{16}$$

Thus

$$\left\| \sum_{i\in\mathbb{Z}} (P_j * f_i)\, \chi^{(1)} \right\|_{L^{1,\infty}(\mu_l)} \le Cj \sum_{i\in\mathbb{Z}} \left\| (P_j * f_i)\, \chi^{(1)} \right\|_{L^{1,\infty}(\mu_l)} \tag{17}$$
$$\le Cj^{-3/2} \sum_{i\in\mathbb{Z}} \|f_i\|_{L^1(\mu_l)}$$
$$= Cj^{-3/2} \|f\|_{L^1(\mu_l)}.$$

To study Case 3, we first observe that by making the same calculations as in (15), we get the estimate

$$|P_j * f_i| \chi^{(3)} \;\le\; j^{-4} \left| R_{w^{-1}} * F_i(w^{-1}z) \right|$$

and hence

$$\begin{aligned} \int |P_j * f_i| \chi^{(3)} dz &\le j^{-4} \int \left| R_{w^{-1}} * F_i(w^{-1}z) \right| dz \\ &= j^{-4} |w|^2 \left\| R_{w^{-1}} * F_i \right\|_{L^1(\mathbb{C})} \\ &\le j^{-4} |w|^2 \left\| R \right\|_{L^1(\mathbb{C})} \left\| f_i \right\|_{L^1(\mu_l)} . \end{aligned}$$

So by Chebyshev's inequality

$$m\left(\left\{ z : |P_j * f_i| \chi^{(3)} > \lambda \right\} \right) \;\le\; Cj^{-4} |w|^2 \left\| f_i \right\|_1 / \lambda,$$

where m denotes Lebesgue measure on $\mathbb{C}$. Thus we get

$$\begin{aligned} \mu_l\left(\left\{ (z,w) : |P_j * f_i| \chi^{(3)} > \lambda \right\} \right) &\le \int \chi^{(3)}(w) \frac{Cj^{-4} |w|^2 \left\| f_i \right\|_1}{\lambda} |w|^{-4} dw \\ &= \frac{Cj^{-4} \|f_i\|_1}{\lambda} \int \chi^{(3)}(w) |w|^{-2} dw \\ &= \frac{Cj^{-4} \|f_i\|_1}{\lambda} \int_{\{w : \log|w| \in \log(\partial I_j) - \log(J_i)\}} |w|^{-2} dw \\ &= \frac{Cj^{-4} \|f_i\|_1}{\lambda} \int_{\log(\partial I_j) - \log(J_i)} dt, \end{aligned}$$

where, in the third step, we used the fact that $J_i \cap (|w|^{-1} I_j)$ is a proper subset of J_i exactly when $\log|w| \in \log(\partial I_j) - \log(J_i)$, and the last equality followed from introducing polar coordinates and changing variables according to $t = \log|w|$. Now, $m\left(\log(\partial I_j) - \log(J_i)\right) = 2j^2$, so we conclude that

$$\left\| (P_j * f_i) \chi^{(3)} \right\|_{L^{1,\infty}(\mu_l)} \;\le\; Cj^{-2} \|f_i\|_{L^1(\mu_l)}.$$

We shall now sum in $i \in \mathbb{Z}$. For a fixed j, no point belongs to more than 2 of the sets $\log(\partial I_j) - \log(J_i)$, when i varies. Hence, by (16), we conclude that

$$\left\| \sum_{i \in \mathbb{Z}} (P_j * f_i) \chi^{(3)} \right\|_{L^{1,\infty}(\mu_l)} \leq C j^{-2} \sum_{i \in \mathbb{Z}} \|f_i\|_{L^1(\mu_l)} \leq C j^{-3/2} \|f\|_{L^1(\mu_l)}. \tag{18}$$

Since $P_j * f = \sum_i P_j * f_i$, we now get from (14), (17) and (18) that

$$\|P_j * f\|_{L^{1,\infty}(\mu_l)} \leq C j^{-3/2} \|f\|_{L^1(\mu_l)}.$$

To finish the proof, we sum in j. Note that the sequence $j^{-3/2}$, $j \in \mathbb{N}$, is in $\ell \log \ell$. We can therefore use a summation theorem for weak L^1 in [21] (Lemma 2.3), to get

$$\left\| \sum_{j \in \mathbb{N}} P_j * f \right\|_{L^{1,\infty}(\mu_l)} \leq C \|f\|_{L^1(\mu_l)}.$$

Because of (13), we therefore have

$$\|P * f\|_{L^{1,\infty}(\mu_l)} \leq C \|f\|_{L^1(\mu_l)}.$$

and the proof is finished. □

Lemma 8. *The g-function*

$$g(f)(z) = \left(\int_{\mathbb{C}} |R_w * f(z)|^2 \, |w|^{-2} dw \right)^{1/2},$$

where $f \in L^1(\mathbb{C})$, defines an operator of weak type $(1,1)$ with respect to Lebesgue measure on $\mathbb{C}$.

Proof. Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{C}, |w|^{-2} dw)$ and consider the operator B which takes a function $f \in C_0^\infty(\mathbb{C})$ to an $\mathcal{H}$-valued function on $\mathbb{C}$, by the map

$$Bf(z) = (R_\cdot * f)(z).$$

Note that $g(f)(z) = \|Bf(z)\|_{\mathcal{H}}$. The operator B can be seen as a convolution of f with an $\mathcal{H}$-valued kernel K, given by $(K(z))(w) = R_w(z)$ for $z \in \mathbb{C}$, $w \in \mathbb{C} \setminus \{0\}$, i. e. $Bf(z) = K * f(z)$.

We will prove that K is a vector-valued singular integral kernel of Calderón-Zygmund type (see [19], II.5). Writing $(K(z))(w) = (K(x, y))(w)$, we need to prove

$$
\begin{array}{ll}
(i) & \displaystyle\int_{r_1<|z|<r_2} K(z)dz = 0, \qquad 0<r_1<r_2<\infty, \\
(ii) & \displaystyle\|K(x,y)\|_{\mathcal{H}} < \frac{C}{x^2+y^2}, \\
(iii) & \displaystyle\|\nabla K(x,y)\|_{\mathcal{H}} < \frac{C}{(x^2+y^2)^{3/2}}.
\end{array}
$$

Here, (i) follows directly from the fact that $R(-z) = -R(z)$ for any $z \in \mathbb{C}$. For (ii), we have

$$
\begin{aligned}
\|K(z)\|_{\mathcal{H}}^2 &= \int_{\mathbb{C}} \left||w|^{-2}R(w^{-1}z)\right|^2 \frac{dw}{|w|^2} \\
&\leq \int_{\mathbb{C}} |w|^{-4} \frac{|w^{-1}z|^2}{(1+|w^{-1}z|^2)^4} \frac{dw}{|w|^2} \\
&= 2\pi \int_0^\infty \frac{(\frac{r}{\rho})^2}{\left(1+(\frac{r}{\rho})^2\right)^4} \rho^{-5} d\rho \\
&= 2\pi r^{-4} \int_0^\infty \frac{t^5}{(1+t^2)^4} dt = Cr^{-4}.
\end{aligned}
$$

To prove (iii), note that

$$
(K(x,y))(w) = |w|^{-2}R\left(w^{-1}z\right) = \frac{xu+yv}{(u^2+v^2+x^2+y^2)^2} = \frac{xu+yv}{(|w|^2+|z|^2)^2},
$$

where $w = u+iv$. We get

$$
\left(\frac{\partial K}{\partial x}(x,y)\right)(w) = \frac{u(|w|^2+|z|^2) - 2x(xu+yv)}{(|w|^2+|z|^2)^3}.
$$

Thus, we have the estimate

$$
\left|\left(\frac{\partial K}{\partial x}(x,y)\right)(w)\right| \leq C\frac{|w|}{(|w|^2+|z|^2)^2}
$$

and, by symmetry, the same holds for $\frac{\partial K}{\partial y}$. We now have

$$
\begin{aligned}
\|\nabla K(x,y)\|_{\mathcal{H}}^2 &= \int_{\mathbb{C}} |(\nabla K(x,y))(w)|^2 \, |w|^{-2} dw \\
&\leq C \int_0^\infty \frac{\rho^2}{(\rho^2+r^2)^4} \frac{d\rho}{\rho} \\
&= Cr^{-6}.
\end{aligned}
$$

Hence, K is a singular integral of Calderón-Zygmund type, and the operator B can be continuously extended to an operator of weak type (1,1), i. e., there is a constant C such that for all $f \in L^1(\mathbb{C})$ and $\lambda > 0$, we have

$$m\left(\{z \in \mathbb{C} : \|Bf(z)\|_{\mathcal{H}} > \lambda\}\right) < \frac{C}{\lambda}\|f\|_{L^1(\mathbb{C})},$$

but here $\|Bf(z)\|_{\mathcal{H}} = g(f)(z)$, which finishes the proof. $\square$

Lemma 9. *Let $a > 0$, $N \in \mathbb{N}$ and the measure μ be given by $d\mu(z,w) = |w|^{-2}dwdz$. For each $f \in L^1(\mathbb{C})$, the function*

$$(z,w) \longmapsto R_w * f(z)$$

is in weak L^1 with respect to the measure μ in the set $\mathbb{C}\times\left\{w \in \mathbb{C} : |w| \in [a, ae^N]\right\}$, with quasinorm bounded by $C\sqrt{N}\|f\|_1$.

Proof. We begin by defining

$$J_\lambda(z) = \int_{\{w:|w|\in[a,ae^N],|R_w*f(z)|>\lambda\}} |w|^{-2}dw$$

and noting that

$$\begin{aligned} J_\lambda(z) &\le \int_{\{w:|w|\in[a,ae^N]\}} |w|^{-2}dw \\ &= \int_a^{ae^N} \rho^{-2}\ 2\pi\rho\ d\rho \ = \ 2\pi(\log(ae^N) - \log(a)) \ = \ 2\pi N. \end{aligned}$$

Using the notation from the previous lemma, we also have the estimate

$$\begin{aligned} J_\lambda(z) &= \int_{\{w:|w|\in[a,ae^N],(\lambda^{-1}|R_w*f(z)|)^2>1\}} |w|^{-2}dw \\ &< \int_{\{w:|w|\in[a,ae^N],(\lambda^{-1}|R_w*f(z)|)^2>1\}} (\lambda^{-1}|R_w * f(z)|)^2|w|^{-2}dw \\ &\le \lambda^{-2}\int_{\mathbb{C}} |R_w * f(z)|^2|w|^{-2}dw \\ &= \left(\frac{g(f)(z)}{\lambda}\right)^2. \end{aligned}$$

Hence, we get

$$\begin{aligned}
\mu\left(\left\{(z,w)\in\mathbb{C}\times\mathbb{C} : |w|\in[a,ae^N], |R_w * f(z)|>\lambda\right\}\right)& \\
&= \int_{\{(z,w):|w|\in[a,ae^N],|R_w*f(z)|>\lambda\}} |w|^{-2}dwdz \\
&= \|J_\lambda\|_{L^1(\mathbb{C})} \\
&= \int_0^{2\pi N} m\left(\{z : J_\lambda(z)>\alpha\}\right)d\alpha \\
&\le \int_0^{2\pi N} m\left(\left\{z : \left(\frac{g(f)(z)}{\lambda}\right)^2>\alpha\right\}\right)d\alpha \\
&= \int_0^{2\pi N} m\left(\left\{z : g(f)(z)>\sqrt{\alpha}\lambda\right\}\right)d\alpha \\
&\le \int_0^{2\pi N} \frac{C}{\sqrt{\alpha}\lambda}\|f\|_{L^1(\mathbb{C})}d\alpha \\
&= \frac{C\sqrt{N}}{\lambda}\|f\|_{L^1(\mathbb{C})}.
\end{aligned}$$

□

In the proof of Lemma 8, we used the cancellation between positive and negative parts of the kernel P. That this cancellation is crucial is shown by the following example, which proves that left convolution with $|P|$ defines an operator which is not of weak type (p,p) for any $p\in[1,\infty)$. Hence, it is not possible to prove the L^p estimates for $X\Delta^{-1/2}$ without taking into account the sign of P.

We fix $p\in[1,\infty)$ and a large $N\in\mathbb{N}$. We will construct a function $f\in L^p(\mu_l)$ with implicit dependence on N and see that the weak type (p,p) inequality is violated for N large enough. Let the functions $\chi_1,\chi_2:\mathbb{C}\to\mathbb{R}$ be given by

$$\chi_1(z)=\begin{cases}1, & |z|\le N,\\ 0, & |z|>N,\end{cases}$$

and

$$\chi_2(w)=\begin{cases}1, & |w|\in[N^{-1},N],\\ 0, & |w|\notin[N^{-1},N].\end{cases}$$

We define $f : G \to \mathbb{R}$ by $\check{f}(z,w) = \chi_1(z)\chi_2(w)$, which yields

$$\begin{aligned}\|f\|^p_{L^p(\mu_l)} &= \|\check{f}\|^p_{L^p(\mu_r)} \\ &= \int_{|z|\le N} dz \int_{|w|\in[N^{-1},N]} \frac{dw}{|w|^2} \\ &= \pi N^2 \cdot 2\pi \int_{\rho=N^{-1}}^{N} \frac{d\rho}{\rho} \\ &= 4\pi^2 N^2 \log N.\end{aligned}$$

Convolving f with $|P|$, we get

$$\begin{aligned}|P| * f(z,w) &= \int_{(z',w')\in G} \big|P((z,w)(z',w'))\big|\check{f}(z',w')d\mu_l(z',w') \\ &= \int_{|z'|\le N}\int_{|w'|\in[N^{-1},N]} \big|R(z+wz')\big| \frac{|ww'|^2}{\big|\log|ww'|\big|}\chi_{(0,\epsilon)}\big(|ww'|\big)\frac{dz'dw'}{|w'|^4} \\ &= \int_{|z'|\le N} |w|^2\big|R(z+wz')\big|dz' \int_{|w'|\in[N^{-1},N]\cap(0,|w|^{-1}\epsilon)} \frac{1}{\big|\log|ww'|\big|}\frac{dw'}{|w'|^2}.\end{aligned}$$

We will estimate these two integrals from below. In the first one, we change integration variable to $\zeta = wz'$ and for $1 < |z| < |w|N$, we get

$$\begin{aligned}\int_{|z'|\le N} |w|^2\big|R(z+wz')\big|dz' &= \int_{|\zeta|\le|w|N} \big|R(z+\zeta)\big|d\zeta \\ &\ge \int_{|\zeta|\le|z|} \big|R(z+\zeta)\big|d\zeta \\ &\ge c,\end{aligned}$$

for some constant $c > 0$. For the second integral, we note that if $|w| \in (N^{-1}, 1)$, we have $[N^{-1}, N] \cap (0, |w|^{-1}\epsilon) = [N^{-1}, |w|^{-1}\epsilon)$ and

$$\begin{aligned}\int_{|w'|\in[N^{-1},|w|^{-1}\epsilon)} \frac{1}{\big|\log|ww'|\big|}\frac{dw'}{|w'|^2} &= 2\pi\int_{\rho'=N^{-1}}^{|w|^{-1}\epsilon} \frac{-1}{\log(|w|\rho')}\frac{d\rho'}{\rho'} \\ &= 2\pi\Big[-\log\big|\log(|w|\rho')\big|\Big]_{\rho'=N^{-1}}^{|w|^{-1}\epsilon} \\ &\ge 2\pi\log(\log N),\end{aligned}$$

where we chose $\epsilon = e^{-1}$. Putting

$$E_N = \{(z,w) \in G : 1 < |z| < |w|N, N^{-1} < |w| < 1\},$$

we conclude that

$$|P| * f(z,w) \ge 2\pi c\log(\log N) \qquad \text{for } (z,w) \in E_N.$$

It follows that

$$\big\||P| * f\big\|_{L^{p,\infty}(\mu_l)} \geq 2\pi c \log(\log N)\big(\mu_l(E_N)\big)^{1/p}$$

and

$$\begin{aligned}\mu_l(E_N) &= \int_{N^{-1}<|w|<1}\left(\int_{1<|z|<|w|N} dz\right)\frac{dw}{|w|^4}\\ &= 2\pi^2\int_{N^{-1}}^{1}(N^2\rho^2-1)\frac{d\rho}{\rho^3}\\ &= 2\pi^2 N^2\log N + O(N^2)\\ &= \frac{1}{2}\Big(1+O\big((\log N)^{-1}\big)\Big)\|f\|^p_{L^p(\mu_l)}.\end{aligned}$$

Thus

$$\big\||P| * f\big\|_{L^{p,\infty}(\mu_l)} \geq 2\pi c\log(\log N)2^{-1/p}\Big(1+O\big((\log N)^{-1}\big)\Big)^{1/p}\|f\|_{L^p(\mu_l)},$$

and by taking N large enough we would violate any weak type (p,p) estimate.

6. The operator $\Delta^{-1/2}X$

Proposition 7. *The global part of the operator* $\Delta^{-1/2}X$ *is of weak type* $(1,1)$.

We need some preparations before we proceed with the proof.

Definition 1. A *scale of partitions* is a sequence $\{\mathcal{P}_j\}_{j\in\mathbb{Z}}$, such that every $\mathcal{P}_j$ is a partition of $\mathbb{C}=\mathbb{R}^2$ into rectangles $\{I_j^p\times I_j^q\}_{p,q\in\mathbb{Z}}$, which have side lengths $\alpha 2^{-j}\times\beta 2^{-j}$ for some fixed $\alpha,\beta>0$.

By I_j^{k-} and I_j^{k+} we denote the left and right half of the interval I_j^k, respectively.

Definition 2. A sequence $\{\Delta_j\}_{j\in\mathbb{Z}}$ of step functions on $\mathbb{C}=\mathbb{R}^2$, is called a *scale of Haar-like functions* if there exists a scale of partitions such that for every $j\in\mathbb{Z}$, we have

$$\Delta_j = \sum_{p,q\in\mathbb{Z}}\kappa_j^{pq},$$

where κ_j^{pq} is of one of the following forms;

$$\kappa_j^{pq}(x,y) = a_j^{pq}\left(\chi_{I_j^{p-}}(x)-\chi_{I_j^{p+}}(x)\right)\left(\chi_{I_j^{q-}}(y)-\chi_{I_j^{q+}}(y)\right),\qquad p,q\in\mathbb{Z},\tag{19}$$

or

$$\kappa_j^{pq}(x,y) = a_j^{pq}\left(\chi_{I_j^{p-}}(x)-\chi_{I_j^{p+}}(x)\right)\chi_{I_j^q}(y),\qquad p,q\in\mathbb{Z},\tag{20}$$

or

$$\kappa_j^{pq}(x,y) = a_j^{pq}\chi_{I_j^p}(x)\left(\chi_{I_j^{q-}}(y)-\chi_{I_j^{q+}}(y)\right),\qquad p,q\in\mathbb{Z},\tag{21}$$

where a_j^{pq} are scalars.

In the proof of Proposition 7, we will expand the function R into Haar-like functions, which are supported in rectangles with dyadic side lengths. For $m, n, p, q \in \mathbb{Z}$, we define

$$\tilde{J}_m^p = \left[\left(p + \frac{(-1)^m - 3}{6}\right) 2^{-m}, \left(p + \frac{(-1)^m + 3}{6}\right) 2^{-m}\right)$$

and

$$J_n^q = [(q-1)2^{-n}, q2^{-n}).$$

Note that for each m and n, $\{\tilde{J}_m^p\}_{p\in\mathbb{Z}}$ and $\{J_n^q\}_{q\in\mathbb{Z}}$ are partitions of $\mathbb{R}$, which are dyadic in m and n respectively, in the sense that e. g. each $\tilde{J}_m^p$ is the union of two $\tilde{J}_{m+1}^{p'}$.

The intervals $\tilde{J}_m^p$ are translates of J_m^q. The reason for introducing these translated intervals is that 0 is an interior point of $\tilde{J}_m^0$ for each m, and that $\tilde{J}_m^0$ increases to $\mathbb{R}$ as $m \to -\infty$.

We now define our Haar-like functions. For $m, p, q \in \mathbb{Z}$ and $n \geq 1$, we put

$$h_{mn}^{pq}(x,y) = 2^{m+n-1}\left(\chi_{\tilde{J}_m^{p-}}(x) - \chi_{\tilde{J}_m^{p+}}(x)\right)\left(\chi_{J_{n-1}^{q-}}(y) - \chi_{\tilde{J}_{n-1}^{q+}}(y)\right) \tag{22}$$

and

$$h_{m0}^{pq}(x,y) = 2^m\left(\chi_{\tilde{J}_m^{p-}}(x) - \chi_{\tilde{J}_m^{p+}}(x)\right)\chi_{J_0^q}(y). \tag{23}$$

For a suitably nice function Γ with $\int_{x\in\mathbb{R}} \Gamma(x,y)dx = 0$ for every $y \in \mathbb{R}$, we can expand Γ with respect to the functions h_{mn}^{pq}, with the coefficients

$$c_{mn}^{pq} = 2^{-m-n}\int_{\mathbb{R}^2} \Gamma(x,y)h_{mn}^{pq}(x,y)dxdy.$$

This is the content of the following lemma, which also gives an estimate for the coefficients. This estimate would not hold unless the intervals $\tilde{J}_m^p$ were translated away from the ordinary Haar intervals J_m^q.

Lemma 10. *Let $\Gamma \in C^2(\mathbb{R}^2)$ satisfy*

$$\begin{aligned} |\Gamma(x,y)| &\leq C(1+|x|+|y|)^{-3}, \\ \left|\frac{\partial \Gamma}{\partial y}(x,y)\right| &\leq C(1+|x|+|y|)^{-4}, \\ \left|\frac{\partial^2 \Gamma}{\partial x \partial y}(x,y)\right| &\leq C(1+|x|+|y|)^{-5}, \end{aligned}$$

for some constant $C > 0$, and assume that we also have $\int_{x\in\mathbb{R}} \Gamma(x,y)dx = 0$ for every $y \in \mathbb{R}$. Then for every $m, p, q \in \mathbb{Z}$ and $n \geq 0$, we have

$$|c_{mn}^{pq}| \leq C2^{m/2}(1+2^m+|p|)^{-5/2}(2^n+|q|)^{-3/2}, \tag{24}$$

and

$$\Gamma = \sum_{m,n,p,q} c_{mn}^{pq} h_{mn}^{pq}, \tag{25}$$

with uniform convergence in $\mathbb{R}^2$.

Proof. By applying the mean value theorem twice, we find that for $(x,y) \in \tilde{J}_m^p \times J_n^q$,

$$\Gamma(x,y) = (x-x_0)(y-y_0)\frac{\partial^2\Gamma}{\partial x \partial y}(\xi,\eta),$$

for some $(\xi,\eta) \in \tilde{J}_m^p \times J_n^q$. Here, x_0 and y_0 denote the left end points of $\tilde{J}_m^p$ and J_n^q, respectively. Hence, we conclude that

$$\begin{aligned}
|c_{mn}^{pq}| &\le \int_{\tilde{J}_m^p \times J_n^q} |\Gamma(x,y)| dxdy \\
&\le 2^{-2m}2^{-2n} \sup_{\tilde{J}_m^p \times J_n^q} \left| \frac{\partial^2 \Gamma}{\partial x \partial y} \right| \\
&\le 2^{-2m}2^{-2n}(1+\operatorname{dist}((0,0), \tilde{J}_m^p \times J_n^q))^{-5}.
\end{aligned}$$

Now, $1+\operatorname{dist}((0,0), \tilde{J}_m^p \times J_n^q) \le C(1+|p|2^{-m}+|q|2^{-n})$, and thus

$$\begin{aligned}
|c_{mn}^{pq}| &\le C2^{-2m}2^{-2n}(1+|p|2^{-m})^{-5/2}(1+|q|2^{-n})^{-5/2} \\
&= C2^{m/2}2^{n/2}(2^m+|p|)^{-5/2}(2^n+|q|)^{-5/2} \\
&\le C2^{m/2}(2^m+|p|)^{-5/2}(2^n+|q|)^{-2} \\
&\le C2^{m/2}(1+2^m+|p|)^{-5/2}(2^n+|q|)^{-3/2},
\end{aligned}$$

if $|p| \ge 1$ or $m \ge 0$.

It remains only to study the case $p=0$ and $m<0$. We first consider $n \ge 1$,

$$\begin{aligned}
|c_{mn}^{0q}| &= \frac{1}{2}\left| \int_{x \in \tilde{J}_m^{0-}} - \int_{x \in \tilde{J}_m^{0+}} \left(\int_{y \in J_n^{q-}} - \int_{y \in J_n^{q+}} \Gamma(x,y)dy \right) dx \right| \\
&\le 2^{-2n} \left| \int_{x \in \tilde{J}_m^{0-}} \frac{\partial \Gamma}{\partial y}(x,\tilde{y})dx \right| + 2^{-2n} \left| \int_{x \in \tilde{J}_m^{0+}} \frac{\partial \Gamma}{\partial y}(x,\bar{y})dx \right|,
\end{aligned}$$

for some $\tilde{y} \in J_n^{q-}$ and $\bar{y} \in J_n^{q+}$. Since $\int_{x\in\mathbb{R}} \Gamma(x,y)dx = 0$, we have

$$\int_{x \in I} \frac{\partial \Gamma}{\partial y}(x,y)dx = -\int_{x \in \mathbb{R}\setminus I} \frac{\partial \Gamma}{\partial y}(x,y)dx,$$

for any interval $I \subset \mathbb{R}$. Therefore, since either $\tilde{J}_m^{0\pm}$ or $\mathbb{R} \setminus \tilde{J}_m^{0\pm}$ is contained in the set $\{x \in \mathbb{R} : |x| \geq 2^{-m}/6\}$, we obtain

$$\begin{aligned}
|c_{mn}^{0q}| &\leq C2^{-2n} \max_{y=\tilde{y},\bar{y}} \int_{|x|\geq 2^{-m}/6} \left| \frac{\partial \Gamma}{\partial y}(x,y) \right| dx \\
&\leq C2^{-2n} \max_{y=\tilde{y},\bar{y}} \int_{x=2^{-m}/6}^{\infty} \frac{dx}{(1+|x|+|y|)^4} \\
&\leq C2^{-2n} \max_{y=\tilde{y},\bar{y}} \frac{1}{(1+2^{-m}/6+|y|)^3} \\
&\leq C2^{-2n}2^{m/2} \frac{1}{(1+|q|2^{-n})^{5/2}} \\
&\leq C2^{m/2}(2^n+|q|)^{-3/2}.
\end{aligned}$$

Finally, for $p=0$, $m<0$ and $n=0$, we get

$$\begin{aligned}
|c_{m0}^{0q}| &\leq \left| \int_{x\in\tilde{J}_m^{0-}} - \int_{x\in\tilde{J}_m^{0+}} \left(\int_{y\in J_0^q} \Gamma(x,y)dy \right) dx \right| \\
&\leq \left| \int_{x\in\tilde{J}_m^{0-}} \Gamma(x,\tilde{y})dx \right| + \left| \int_{x\in\tilde{J}_m^{0+}} \Gamma(x,\tilde{y})dx \right|,
\end{aligned}$$

for some $\tilde{y} \in J_0^q$. Hence, similarly to the case $n \geq 1$, we get

$$\begin{aligned}
|c_{m0}^{0q}| &\leq 2\int_{|x|\geq 2^{-m}/6} |\Gamma(x,\tilde{y})|dx \\
&\leq C\int_{x=2^{-m}/6}^{\infty} \frac{dx}{(1+|x|+|\tilde{y}|)^3} \\
&\leq C\frac{1}{(1+2^{-m}/6+|\tilde{y}|)^2} \\
&\leq C2^{m/2}(1+|\tilde{y}|)^{-3/2} \\
&\leq C2^{m/2}(1+|q|)^{-3/2},
\end{aligned}$$

and we have proved the inequality (24).

It remains to prove the uniform convergence and the equality (25). First, note that for fixed $(x,y) \in \mathbb{R}^2$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ there is only one pair (p,q) with $h_{mn}^{pq}(x,y) \neq 0$. So by (24) and the definition of h_{mn}^{pq}, we get

$$\begin{aligned}
\left| \sum_{m,n,p,q} c_{mn}^{pq} h_{mn}^{pq} \right| &\leq C \sum_{m\in\mathbb{Z},n\in\mathbb{N}} 2^{m/2}(1+2^m)^{-3/2}2^{-n/2} \\
&\leq C \sum_{m\in\mathbb{Z},n\in\mathbb{N}} 2^{-|m|/2}2^{-n/2}.
\end{aligned}$$

Hence, the series converges uniformly. We denote the limit by S, and we put $E = \Gamma - S$.

It remains to prove that $E = 0$. To do this, we fix $m_0, q_0 \in \mathbb{Z}$ and we let $\mathfrak{I}^{q_0}_{m_0}$ be those h^{pq}_{mn} which are supported in $\tilde{J}^0_{m_0} \times J^{q_0}_0$. Note that

$$S\chi_{\mathbb{R}\times J^{q_0}_0} \quad = \quad \sum_{f\in\mathfrak{I}^{q_0}_{m_0}} \frac{\langle \Gamma, f\rangle}{\|f\|^2} f + D,$$

where the term D is a sum of functions h^{pq}_{mn} with $m < m_0$. We can extend $\mathfrak{I}^{q_0}_{m_0}$ to an orthogonal basis $\mathfrak{J}^{q_0}_{m_0}$ for $L^2(\tilde{J}^0_{m_0} \times J^{q_0}_0)$, by including functions of the type

$$2^{n-1}\chi_{\tilde{J}^0_{m_0}}(x)\left(\chi_{J^{q-}_{n-1}}(y) - \chi_{J^{q+}_{n-1}}(y)\right), \qquad n \geq 1, \tag{26}$$

supported in $\tilde{J}^0_{m_0} \times J^{q_0}_0$, and the constant function 1. Putting

$$g_{m_0,q_0} \quad = \quad \sum_{f\in\mathfrak{J}^{q_0}_{m_0}\setminus\mathfrak{I}^{q_0}_{m_0}} \frac{\langle \Gamma, f\rangle}{\|f\|^2} f,$$

we get

$$(E + D)\chi_{\tilde{J}^0_{m_0}\times J^{q_0}_0} - g_{m_0,q_0} = \Gamma\chi_{\tilde{J}^0_{m_0}\times J^{q_0}_0} - \sum_{f\in\mathfrak{J}^{q_0}_{m_0}} \frac{\langle \Gamma, f\rangle}{\|f\|^2} f.$$

Here, the right-hand side equals zero almost everywhere in $\tilde{J}^0_{m_0} \times J^{q_0}_0$, since the functions in $\mathfrak{J}^{q_0}_{m_0}$ constitute an orthogonal L^2-basis. Furthermore, since each $f \in \mathfrak{J}^{q_0}_{m_0}$ is right-continuous in both x and y, the same holds for the sum in the right-hand side. It follows that

$$E + D \quad = \quad g_{m_0,q_0} \qquad \text{everywhere in } \tilde{J}^0_{m_0} \times J^{q_0}_0.$$

Integrating with respect to x yields

$$\int_{x\in\tilde{J}^0_{m_0}} \Gamma(x,y)dx \; - \int_{x\in\tilde{J}^0_{m_0}} \Big(S(x,y) \; - \; D(x,y)\Big)dx \quad = \quad \int_{x\in\tilde{J}^0_{m_0}} g_{m_0,q_0}(x,y)dx,$$

for any fixed $y \in J^{q_0}_0$. Here, $S - D$ is a sum of functions in $\mathfrak{I}^{q_0}_{m_0}$ and hence its integral vanishes. As $m_0 \to -\infty$, the set $\tilde{J}^0_{m_0}$ increases to $\mathbb{R}$, and the integral of Γ tends to 0. For the right-hand side, note that g_{m_0,q_0} is constant with respect to x in $\tilde{J}^0_{m_0}$, since it is a sum of functions of the type (26). Hence, the integral on the right-hand side equals $2^{-m_0}g_{m_0,q_0}(x,y)$ for some arbitrary $x \in \tilde{J}^0_{m_0}$. It follows that $g_{m_0,q_0} \to 0$ uniformly as $m_0 \to -\infty$.

Finally, we conclude that

$$E\chi_{\mathbb{R}\times J^{q_0}_0} = \lim_{m_0\to-\infty}(E + D)\chi_{\tilde{J}^0_{m_0}\times J^{q_0}_0} = \lim_{m_0\to-\infty} g_{m_0,q_0} = 0,$$

and since $q_0 \in \mathbb{Z}$ is arbitrary, we have $E = 0$. $\square$

Proof of Theorem 7. We wish to prove that for any $\lambda > 0$, we have

$$\mu_l\left(\left\{(z,w) \in G : |Q * \tilde{f}(z,w)| > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_G |\tilde{f}(z,w)| d\mu_l(z,w) \tag{27}$$

for every $\tilde{f} \in L^1(\mu_l)$. The first part of the proof will consist of step by step simplifications of this statement. We will first change variables and then see that we can leave out some integrable terms, which correspond to strong type (1,1) operators.

In the second and main part of the proof, we use Lemma 10 to expand the function R with respect to the functions h_{mn}^{pq} given in (22) and (23), and we approximate $\tilde{f}$ by a sum of point measures. Thus, the problem is essentially reduced to analysing a sum of convolutions of functions h_{mn}^{pq} with point measures. We use Lemma 11 to estimate each term in this sum. The proof is then concluded by collecting all details.

First, we will change coordinates in (27) according to

$$w = 2^t e^{i\phi}, \qquad z = \zeta 2^t e^{i\phi},$$

with $t \in \mathbb{R}$, $\phi \in [-\pi, \pi)$ and $\zeta \in \mathbb{C}$. Note that this yields

$$d\mu_l(z,w) \;=\; C d\eta(\zeta, t, \phi),$$

where η denotes Lebesgue measure[7] on $\mathbb{C} \times \mathbb{R} \times [-\pi, \pi)$. Hence, (27) takes the form

$$\eta\left(\left\{(\zeta,t,\phi) : |Q * \tilde{f}(\zeta 2^t e^{i\phi}, 2^t e^{i\phi})| > \lambda\right\}\right) \leq \frac{C}{\lambda}\int |\tilde{f}(\zeta 2^t e^{i\phi}, 2^t e^{i\phi})| d\eta(\zeta,t,\phi).$$

We define $f(\zeta,t,\phi) = \tilde{f}(\zeta 2^t e^{i\phi}, 2^t e^{i\phi})$ and note that $\|\tilde{f}\|_{L^1(\mu_l)} = C\|f\|_{L^1(\eta)}$. Putting $\Phi(\zeta,t,\phi) = Q * \tilde{f}(\zeta 2^t e^{i\phi}, 2^t e^{i\phi})$, we have

$$\begin{aligned}
\Phi(\zeta,t,\phi) &= \int Q((\zeta 2^t e^{i\phi}, 2^t e^{i\phi})(z',w'))\tilde{f}((z',w')^{-1}) d\mu_l(z',w') \\
&= \iint_{\{2^t|w'|>N\}} R\left(\frac{\zeta + z'}{w'}\right) \frac{1}{t\log 2 + \log|w'|}\tilde{f}\left(-\frac{z'}{w'}, \frac{1}{w'}\right)|w'|^{-4} dz' dw' \\
&= \iint_{\{2^t|w'|>N\}} \frac{1}{t\log 2 + \log|w'|} R_{w'}\left(\zeta + z'\right)\tilde{f}\left(-\frac{z'}{w'}, \frac{1}{w'}\right)|w'|^{-2} dz' dw' \\
&= C\iiint_{\{t-v>1\}} \frac{1}{t-v} R_{2^{-v}e^{-i\psi}}\left(\zeta + z'\right)\tilde{f}\left(-z' 2^v e^{i\psi}, 2^v e^{i\psi}\right) dz' dv d\psi \\
&= C\iint_{\{t-v>1\}} \frac{1}{t-v} R_{2^{-v}e^{-i\psi}} * f\left(\zeta, v, \psi\right) dv d\psi,
\end{aligned}$$

[7]Here, we avoid our convention of denoting Lebesgue measure by m, since we want to use m to denote an integer index.

where we made in the fourth step the change of variable $w' = 2^{-v}e^{-i\psi}$, $v \in \mathbb{R}$, $\psi \in [-\pi, \pi)$, and we also chose $N = 2$. In the last expression, the convolution is with respect to the first variable of f. Note that $\Phi(\zeta, t, \phi)$ is actually independent of ϕ.

We can rewrite (27) as

$$\eta\left(\{(\zeta, t, \phi) : |\Phi(\zeta, t, \phi)| > \lambda\}\right) \leq \frac{C}{\lambda} \int |f| d\eta.$$

What we need to prove is therefore that the map $f \mapsto \Phi$ is of weak type (1,1) with respect to the measure η.

For the variable t, we wish to use the counting measure on $\mathbb{Z}$, instead of the Lebesgue measure on $\mathbb{R}$, when we take the $L^{1,\infty}$ quasinorm of the map $f \mapsto \Phi$. Therefore, we need to pass to a discrete dependence on t. Next, we see that this can be done with an error corresponding to a strong type $(1,1)$ operator. Hence, this error is negligible for our purposes.

We denote $k = [t]$ and $j = [v]$, with square brackets denoting the integer part. Note that $t - v > 1$ implies $k - j \geq 1$. Changing $t - v$ to $k - j$ in the denominator inside the integral in the last expression for Φ yields an error bounded by

$$\begin{aligned}\iint_{\{t-v>1\}} \left| \frac{1}{t-v} - \frac{1}{k-j} \right| &\left| R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) \right| dv d\psi \\ &\leq C \iint_{\{t-v>1\}} \frac{1}{(t-v)^2} \left| R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) \right| dv d\psi,\end{aligned}$$

and since

$$\begin{aligned}\int d\zeta \int dt \int d\phi \iint_{\{t-v>1\}} &\frac{1}{(t-v)^2} \left| R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) \right| dv d\psi \\ &\leq \int dv \int d\phi \int d\psi \int d\zeta \left| R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) \right| \int_{\{t-v>1\}} \frac{dt}{(t-v)^2} \\ &\leq C \int dv \int d\phi \int d\psi \| R_{2^{-v}e^{-i\psi}} \|_{L^1(\mathbb{C})} \| f(\cdot, v, \phi) \|_{L^1(\mathbb{C})} \\ &\leq C \| f \|_{L^1(\eta)},\end{aligned}$$

this error corresponds to a strong type $(1,1)$ operator. We also need to change the limit of the integral. Changing $\{t - v > 1\}$ to $\{k - v > 1\}$ gives the error

$$\int_{\psi=-\pi}^{\pi} \int_{v=[t]-1}^{t-1} \frac{1}{k-j} R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) dv d\psi,$$

and since

$$\begin{aligned}\int d\zeta \int dt \int d\phi \left| \int_{\psi=-\pi}^{\pi} \int_{v=[t]-1}^{t-1} \frac{1}{k-j} R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) dv d\psi \right| \\ \leq \int dv \int d\phi \int d\psi \int d\zeta \left| R_{2^{-v}e^{-i\psi}} * f(\zeta, v, \phi) \right| \int_{t=v}^{v+1} dt \\ \leq C\|f\|_{L^1(\eta)},\end{aligned}$$

this error also corresponds to an operator of strong type (1,1).

Hence, we have found that it remains to prove that the operator which maps $f \in L^1(\eta)$ to the function

$$(z, t, \phi) \mapsto \int_{\psi=-\pi}^{\pi} \int_{v=-\infty}^{k} \frac{1}{k-j} R_{2^{-v}e^{-i\psi}} * f\,(z, v, \psi)\, dv d\psi,$$

where $k = [t]$ and $j = [v]$, is of weak type (1,1) with respect to η. (Here, we have also changed the notation ζ to z, merely for convenience.) Since the dependence of t here is only via k, we can take the counting measure in k instead of Lebesgue measure in t, when we take the $L^{1,\infty}(\eta)$ quasinorm.

We have $R_{2^{-v}e^{-i\psi}} = (R_{e^{-i\psi}})_{2^{-v}}$ and

$$\begin{aligned} R_{e^{-i\psi}}(z) &= R(e^{i\psi} z) = \frac{\mathrm{Re}(e^{i\psi} z)}{(1+|z|^2)^2} = \frac{\cos(\psi)x - \sin(\psi)y}{(1+x^2+y^2)^2} \\ &= \cos(\psi)R(x,y) - \sin(\psi)R(y,x). \end{aligned}$$

By symmetry, it is therefore enough to study the function

$$(z, t, \phi) \mapsto \int_{\psi=-\pi}^{\pi} \int_{v=-\infty}^{k} \frac{1}{k-j} \cos(\psi) R_{2^{-v}} * f\,(z, v, \psi)\, dv d\psi. \tag{28}$$

Now write $f(z, v, \psi) = f^{v,\psi}(z)$. We will approximate each $f^{v,\psi}$ with a sequence of point measures. Let $\gamma_l^{v,\psi}$ be the measure consisting of point masses at $(p2^{-j-l}, q2^{-j-l}) \in \mathbb{R}^2 = \mathbb{C}$ equal to

$$\int_{x=p2^{-j-l}}^{(p+1)2^{-j-l}} \int_{y=q2^{-j-l}}^{(q+1)2^{-j-l}} f^{v,\psi}(x,y) dy dx, \qquad p, q \in \mathbb{Z},$$

where $j = [v]$ and we have, as usual, written $f^{v,\psi}(z) = f^{v,\psi}(x+iy) = f^{v,\psi}(x,y)$. Then

$$f^{v,\psi} = \gamma_0^{v,\psi} + \sum_{l=1}^{\infty} (\gamma_l^{v,\psi} - \gamma_{l-1}^{v,\psi}), \tag{29}$$

with the sum converging in the weak-$*$ topology.

We will treat each of the terms in the right-hand side of (29) separately, and we start with $\gamma_0^{v,\psi}$. For each $0 \le \xi < 1$, we expand

$$R_{2^{-\xi}} = \sum_{m,n,p,q} c_{mn}^{pq}(\xi) h_{mn}^{pq}$$

according to Lemma 10, and we get

$$|c_{mn}^{pq}(\xi)| \le C 2^{m/2}(1 + 2^m + |p|)^{-5/2}(2^n + |q|)^{-3/2}. \tag{30}$$

Writing $v = j + \xi$, we get

$$\begin{aligned}
&\int_{-\infty}^{k} \int_{-\pi}^{\pi} \frac{1}{k-j} \cos(\psi) R_{2^{-v}} * \gamma_0^{v,\psi}(z) d\psi dv \\
&= \sum_{m,n,p,q} \int_{-\infty}^{k} \int_{-\pi}^{\pi} \frac{1}{k-j} \cos(\psi) c_{mn}^{pq}(\xi)\, (h_{mn}^{pq})_{2^{-j}} * \gamma_0^{v,\psi}(z) d\psi dv \\
&= \sum_{m,n,p,q} \sum_{j<k} \frac{1}{k-j} \left((h_{mn}^{pq})_{2^{-j}} * \int_0^1 \int_{-\pi}^{\pi} \cos(\psi) c_{mn}^{pq}(\xi) \gamma_0^{j+\xi,\psi} d\psi d\xi \right)(z) \\
&= \sum_{m,n,p,q} F_{mnpq}^0(z,k),
\end{aligned}$$

where the last step defines the functions F_{mnpq}^0. We now fix m, n, p and q and write

$$\nu_0^j = \int_0^1 \int_{-\pi}^{\pi} \cos(\psi) c_{mn}^{pq}(\xi) \gamma_0^{j+\xi,\psi} d\psi d\xi, \qquad j \in \mathbb{Z}, \tag{31}$$

so that

$$F_{mnpq}^0 = \sum_{j<k} \frac{1}{k-j} (h_{mn}^{pq})_{2^{-j}} * \nu_0^j.$$

We need to estimate the $L^{1,\infty}$ quasinorm of F_{mnpq}^0.

We consider first the case $m, n \ge 0$. The measure ν_0^j is supported in the set $2^{-j}\mathbb{Z} \times 2^{-j}\mathbb{Z}$ and the supports of the functions $(h_{mn}^{pq})_{2^{-j}}$ are rectangles with side lengths $2^{-j-m} \times 2^{-j-n}$. It follows that

$$\left\{ (h_{mn}^{pq})_{2^{-j}} * \nu_0^j \right\}_{j \in \mathbb{Z}}$$

is a scale of Haar-like functions, with respect to a scale of partitions with $\alpha = 2^{-m}$ and $\beta = 2^{-n}$. Lemma 11 below implies that

$$\left\| F_{mnpq}^0 \right\|_{L^{1,\infty}} \le C \sum_{j\in\mathbb{Z}} \int \left| (h_{mn}^{pq})_{2^{-j}} * \nu_0^j(z) \right| dz = C \sum_{j \in \mathbb{Z}} \|\nu_0^j\|$$

We next consider the case $m < 0$, $n \ge 0$. In the x-direction, the support of a function $(h_{mn}^{pq})_{2^{-j}}$ is now wider than the distance between two point masses of ν_0^j

with a factor $2^{|m|}$. We therefore split ν_0^j into a sum of measures η_i^j,

$$\nu_0^j = \sum_{i=1}^{2^{|m|}} \eta_i^j,$$

where each η_i^j is supported in $2^{-j}\mathbb{Z} \times 2^{-j}(i+2^{|m|})\mathbb{Z}$ and $\|\nu_0^j\| = \sum \|\eta_i^j\|$. In words, this means that each η_i^j consists of a sum of y-rows of ν_0^j, $2^{-j+|m|}$ apart in the x-direction (cf. Fig. 1). Then for each i,

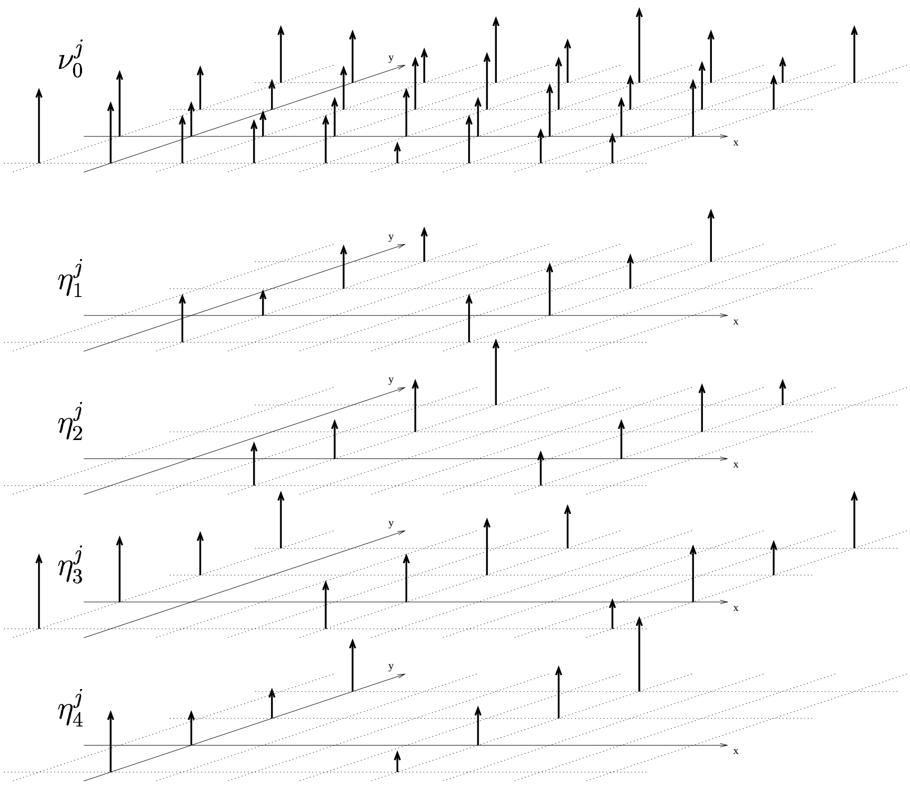

FIGURE 1. For the case $m < 0$, $n \geq 0$, the measure ν_0^j is split into a sum $\sum \eta_i^j$. In this illustration $m = -2$.

$$\left\{ (h_{mn}^{pq})_{2^{-j}} * \eta_i^j \right\}_{j\in\mathbb{Z}},$$

is a scale of Haar-like functions. We get from Lemma 11 that

$$\left\| \sum_{j<k} \frac{1}{k-j} (h_{mn}^{pq})_{2^{-j}} * \eta_i^j \right\|_{L^{1,\infty}} \leq C \sum_{j\in\mathbb{Z}} \int \left| (h_{mn}^{pq})_{2^{-j}} * \eta_i^j(z) \right| dz = C \sum_{j\in\mathbb{Z}} \|\eta_i^j\|.$$

To get the corresponding result for ν_0^j, we wish to sum in i. However, since $L^{1,\infty}$ is not a normed space, we must introduce a factor on the right hand side which is proportional to the logarithm of the number of terms (see [20], Proposition 3). Thus, we conclude that

$$\left\|F^0_{mnpq}\right\|_{L^{1,\infty}} \leq C|m| \sum_{j\in\mathbb{Z}} \|\nu_0^j\|, \qquad m<0, n\geq 0.$$

The cases $m \geq 0$, $n<0$ and $m,n<0$ can be handled analoguously, by splitting the measure ν_0^j instead along the y-direction and in both directions, respectively. It is then evident that we, for any m and n, have the estimate

$$\left\|F^0_{mnpq}\right\|_{L^{1,\infty}} \leq C(1+|m|+|n|) \sum_{j\in\mathbb{Z}} \|\nu_0^j\|.$$

Next, we turn to the terms with $l \geq 1$ in the decomposition (29). By making the refinement $l-1 \to l$ in two steps, we can write

$$\gamma_l^{v,\psi} - \gamma_{l-1}^{v,\psi} = (\gamma_l^{v,\psi} - \theta_l^{v,\psi}) + (\theta_l^{v,\psi} - \gamma_{l-1}^{v,\psi}),$$

where $\theta_l^{v,\psi}$ is the measure consisting of point masses at $(p2^{-j-l}, q2^{-j-l+1})$ equal to

$$\int_{x=p2^{-j-l}}^{(p+1)2^{-j-l}} \int_{y=q2^{-j-l+1}}^{(q+1)2^{-j-l+1}} f^{v,\psi}(x,y)\,dy\,dx, \qquad p,q\in\mathbb{Z}.$$

Note that for any $p,q\in\mathbb{Z}$, the point masses of $\gamma_l^{v,\psi} - \theta_l^{v,\psi}$ located at $(p2^{-j-l}, 2q2^{-j-l})$ and $(p2^{-j-l}, (2q+1)2^{-j-l})$have the same magnitude but opposite signs. Letting τ_h denote the translation operator in $\mathbb{R}^2$, we can therefore write

$$\gamma_l^{v,\psi} - \theta_l^{v,\psi} = \sigma_l^{v,\psi} - \tau_{(0,2^{-l-j})}\sigma_l^{v,\psi},$$

for some measure $\sigma_l^{v,\psi}$ supported in $2^{-j-l}\mathbb{Z}\times 2^{-j-l+1}\mathbb{Z}$. Similarly, we can write

$$\theta_l^{v,\psi} - \gamma_{l-1}^{v,\psi} = \tilde{\sigma}_l^{v,\psi} - \tau_{(2^{-l-j},0)}\tilde{\sigma}_l^{v,\psi},$$

where $\tilde{\sigma}_l^{v,\psi}$ is supported in $2^{-j-l+1}\mathbb{Z}\times 2^{-j-l+1}\mathbb{Z}$. Note that

$$\|\sigma_l^{v,\psi}\| \leq \|\gamma_l^{v,\psi}\| \qquad \text{and} \qquad \|\tilde{\sigma}_l^{v,\psi}\| \leq \|\gamma_l^{v,\psi}\|.$$

We now get

$$\begin{aligned} R_{2^{-v}} * (\gamma_l^{v,\psi} - \gamma_{l-1}^{v,\psi}) &= (R_{2^{-v}} - \tau_{(0,2^{-l-j})}R_{2^{-v}}) * \sigma_l^{v,\psi} \\ &\quad + (R_{2^{-v}} - \tau_{(2^{-l-j},0)}R_{2^{-v}}) * \tilde{\sigma}_l^{v,\psi} \\ &= (R_{2^{-\xi}} - \tau_{(0,2^{-l})}R_{2^{-\xi}})_{2^{-j}} * \sigma_l^{v,\psi} \\ &\quad + (R_{2^{-\xi}} - \tau_{(2^{-l},0)}R_{2^{-\xi}})_{2^{-j}} * \tilde{\sigma}_l^{v,\psi}. \end{aligned}$$

By using Lemma 10 and the mean value theorem, we can expand

$$R_{2^{-\xi}} - \tau_{(0,2^{-l})}R_{2^{-\xi}} = \sum_{m,n,p,q} c_{mn}^{pq}(\xi,l)h_{mn}^{pq}$$

and

$$R_{2^{-\xi}} - \tau_{(2^{-l},0)} R_{2^{-\xi}} = \sum_{m,n,p,q} \tilde{c}^{pq}_{mn}(\xi,l) h^{pq}_{mn},$$

with the estimate

$$|c^{pq}_{mn}(\xi,l)| \le C 2^{-l} 2^{m/2} (1+2^m+|p|)^{-5/2} (2^n+|q|)^{-3/2} \tag{32}$$

and the same for $\tilde{c}^{pq}_{mn}(\xi,l)$. In analogy with our previous notation, we put

$$\nu^j_l = \int_0^1 \int_{-\pi}^{\pi} \cos(\psi) \left(c^{pq}_{mn}(\xi,l) \sigma^{v,\psi}_l + \tilde{c}^{pq}_{mn}(\xi,l) \tilde{\sigma}^{v,\psi}_l \right) d\psi d\xi, \tag{33}$$

and we get

$$\begin{aligned}
\int_{-\infty}^{k} \int_{-\pi}^{\pi} \frac{1}{k-j} \cos(\psi) R_{2^{-v}} * (\gamma^{v,\psi}_l - \gamma^{v,\psi}_{l-1})(z) d\psi dv \\
&= \sum_{m,n,p,q} \sum_{j<k} \frac{1}{k-j} (h^{pq}_{mn})_{2^{-j}} * \nu^j_l(z) \\
&= \sum_{m,n,p,q} F^l_{mnpq}(z).
\end{aligned}$$

To estimate the $L^{1,\infty}$ quasinorms of the functions F^l_{mnpq}, we again consider different cases. Note that the support of ν^j_l is contained in $2^{-l-j}\mathbb{Z} \times 2^{-l-j}\mathbb{Z}$. Thus, for $m,n \ge l$,

$$\{h^{pq}_{mn} * \nu^j_l\}_{j\in\mathbb{Z}}$$

is a scale of Haar-like functions. Hence, Lemma 11 yields

$$\|F^l_{mnpq}\|_{L^{1,\infty}} \le C \sum_{j\in\mathbb{Z}} \|\nu^j_l\|.$$

For the other cases, i. e. if $m < l$ or $n < l$, we once again split ν^j_l into a sum of measures with sparser point masses. The calculations will be completely similar to the case with $\gamma^{v,\psi}_0$. We can summarize the resulting estimates in

$$\left\|F^l_{mnpq}\right\|_{1,\infty} \le C(1+|l-m|+|l-n|) \sum_{j\in\mathbb{Z}} \|\nu^j_l\|,$$

which also includes the case $l=0$.

Next, we need to estimate $\|\nu^j_l\|$. For $l=0$, we get, from the defining expression (31) and the estimate (30) of the coefficients $c^{pq}_{mn}(\xi)$, that

$$\begin{aligned}
\|\nu^j_0\| &\le \int_0^1 \int_{-\pi}^{\pi} |c^{pq}_{mn}(\xi)| \|\gamma^{j+\xi,\psi}_0\| d\psi d\xi \\
&\le C 2^{m/2} (1+2^m+|p|)^{-5/2} (2^n+|q|)^{-3/2} \int_0^1 \int_{-\pi}^{\pi} \|\gamma^{j+\xi,\psi}_0\| d\psi d\xi.
\end{aligned}$$

For $l \geq 1$, we get, from (33) and (32), that

$$\begin{aligned} \|\nu_l^j\| &\leq \int_0^1 \int_{-\pi}^{\pi} \left(|c_{mn}^{pq}(\xi,l)| \|\sigma_l^{j+\xi,\psi}\| + |\tilde{c}_{mn}^{pq}(\xi,l)| \|\tilde{\sigma}_l^{j+\xi,\psi}\| \right) d\psi d\xi \\ &\leq C 2^{-l} 2^{m/2} (1+2^m+|p|)^{-5/2} (2^n+|q|)^{-3/2} \int_0^1 \int_{-\pi}^{\pi} \|\gamma_l^{j+\xi,\psi}\| d\psi d\xi. \end{aligned}$$

Since $\|\gamma_l^{v,\psi}\| \leq \int_{\mathbb{C}} |f^{v,\psi}(z)| dz$, we therefore get

$$\|\nu_l^j\| \leq C \frac{2^{-l} 2^{m/2}}{(1+2^m+|p|)^{5/2}(2^n+|q|)^{3/2}} \int_{\{0<\xi<1\}} |f(z, j+\xi, \psi)| dz d\psi d\xi,$$

for $l \geq 0$. Summing in j now yields

$$\left\| F_{mnpq}^l \right\|_{L^{1,\infty}} \leq C \frac{(1+|l-m|+|l-n|) 2^{-l} 2^{m/2}}{(1+2^m+|p|)^{5/2}(2^n+|q|)^{3/2}} \|f\|_{L^1}. \tag{34}$$

To finish the proof, we need to estimate the level sets of the integral in (28). Because of the decomposition (29) and the expansion of R into Haar functions, this integral equals $\sum_{l,m,n,p,q} F_{mnpq}^l$. By (34) and the addition theorem for $L^{1,\infty}$ (Lemma 2.3 in [21]), it is therefore enough to prove that the five-fold sequence

$$\left\{ \frac{(1+|l-m|+|l-n|) 2^{-l} 2^{m/2}}{(1+2^m+|p|)^{5/2}(2^n+|q|)^{3/2}} \right\}_{l,m,n,p,q}$$

is in $l \log l$. Note here that $p, q, m \in \mathbb{Z}$, while l and n only range over $\mathbb{N}$. It is easily seen that this sequence is even in l^γ for any $\gamma > 2/3$, by summing in the order q, p, n, m and l. Thus, the proof of Theorem 7 is finished. □

Lemma 11. *For any scale of Haar-like functions $\{\Delta_j\}_{j\in\mathbb{Z}}$, and any $\lambda > 0$, we have*

$$m \times c\left(\left\{(x,y,k) \in \mathbb{R}^2 \times \mathbb{Z} : \left|\sum_{j<k} \frac{\Delta_j(x,y)}{k-j}\right| > \lambda\right\}\right) \leq \frac{C}{\lambda} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} |\Delta_j| dx dy,$$

where m denotes Lebesgue measure on $\mathbb{R}^2$ and c is the counting measure on $\mathbb{Z}$. The constant C does not depend on the parameters α and β from Definition 1.

Proof. Fix $\lambda > 0$. For $j < k$, we denote by χ_j^k the characteristic function of the set

$$\left\{(x,y) \in \mathbb{R}^2 : \frac{|\Delta_j(x,y)|}{k-j} > \lambda\right\}.$$

We will divide the sum $\sum_{j<k} \frac{\Delta_j(x,y)}{k-j}$ into two parts, one with large terms and one with small. We then estimate the level sets of each of these two parts separately.

We have

$$
\begin{aligned}
m\times c&\left(\left\{(x,y,k):\left|\sum_{j<k}\frac{\Delta_j(x,y)}{k-j}\right|>\lambda\right\}\right)\\
&\le m\times c\left(\left\{(x,y,k):\left|\sum_{j<k}\frac{\Delta_j(x,y)}{k-j}\chi_j^k(x,y)\right|>\lambda/2\right\}\right)\\
&\quad+m\times c\left(\left\{(x,y,k):\left|\sum_{j<k}\frac{\Delta_j(x,y)}{k-j}(1-\chi_j^k(x,y))\right|>\lambda/2\right\}\right)\\
&=\mathrm{I}+\mathrm{II}.
\end{aligned}
$$

We first study I, which is easier to estimate. If $\left|\sum_{j<k}\frac{\Delta(x,y)}{k-j}\chi_j^k(x,y)\right|>\lambda/2$, there must be some $j<k$ such that $\chi_j^k(x,y)\neq 0$, and hence

$$
\begin{aligned}
&\left\{(x,y,k):\left|\sum_{j<k}\frac{\Delta_j(x,y)}{k-j}\chi_j^k(x,y)\right|>\lambda/2\right\}\\
&\qquad\subset\left\{(x,y,k):\exists j<k\text{ with }\chi_j^k(x,y)\neq 0\right\}=\bigcup_{j\in\mathbb{Z}}A_j,
\end{aligned}
$$

where $A_j=\{(x,y,k):k>j\text{ and }\chi_j^k(x,y)\neq 0\}$. It now follows that

$$
\begin{aligned}
\mathrm{I}&\le\sum_{j\in\mathbb{Z}}m\times c(A_j)\\
&=\sum_{j\in\mathbb{Z}}m\times c\left(\left\{(x,y,k):j<k<j+\frac{|\Delta_j(x,y)|}{\lambda}\right\}\right)\\
&\le\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^2}\frac{|\Delta_j(x,y)|}{\lambda}dxdy,
\end{aligned}
$$

where we used Fubini's theorem for the final step.

We now turn to estimating II. For every fixed k, we have that $(1-\chi_j^k)\Delta_j$ is of the form

$$
(1-\chi_j^k)\Delta_j=\sum_{p,q\in\mathbb{Z}}\kappa_j^{pq},
$$

with all κ_j^{pq} in one of the forms in Definition 2.

With the Chebyshev inequality, we get

$$
\begin{aligned}
(35)\qquad \mathrm{II} &\leq \left(\frac{2}{\lambda}\right)^2 \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} \left| \sum_{j<k} \frac{\Delta_j(x,y)}{k-j}(1-\chi_j^k(x,y)) \right|^2 dxdy \\
&= \left(\frac{2}{\lambda}\right)^2 \sum_{k\in\mathbb{Z}} \sum_{j,j'<k} \frac{1}{k-j}\frac{1}{k-j'} \langle (1-\chi_j^k)\Delta_j, (1-\chi_{j'}^k)\Delta_{j'} \rangle
\end{aligned}
$$

so we need to estimate the scalar product

$$
\langle (1-\chi_j^k)\Delta_j, (1-\chi_{j'}^k)\Delta_{j'} \rangle \quad = \sum_{p,q,p',q'\in\mathbb{Z}} \langle \kappa_j^{pq}, \kappa_{j'}^{p'q'} \rangle.
$$

The crucial point for the estimate of this scalar product is that the products on the right hand side equal zero, except for a very sparse set $\mathcal{I}$ of indices $(p,q,p',q') \in \mathbb{Z}^4$. For example, if the functions κ_j^{pq}, $j,p,q \in \mathbb{Z}$, are of the form (19), then $\langle \kappa_j^{pq}, \kappa_{j'}^{p'q'} \rangle = 0$ unless the support of $\kappa_{j'}^{p'q'}$ contains one of the corners, the middle point or one of the middle points of the sides of the support of κ_j^{pq} or vice versa (cf. Figure 2). We now proceed with the details of this argument.

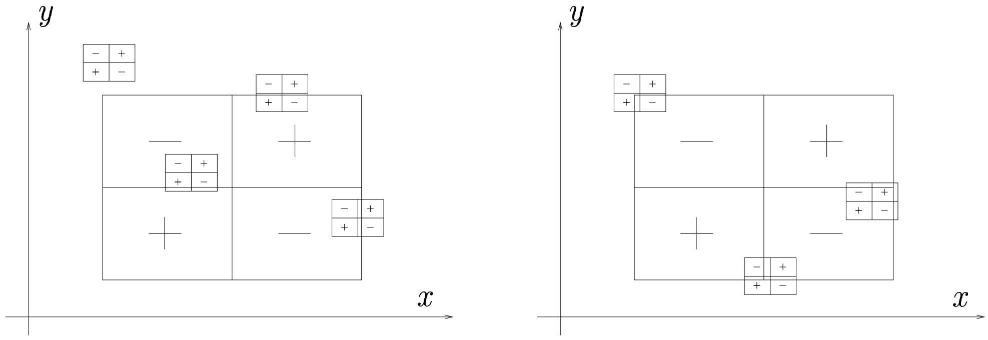

FIGURE 2. The supports of κ_j^{pq} and $\kappa_{j'}^{p'q'}$ are illustrated as rectangles in the xy-plane, with plus and minus signs indicating areas where the functions have opposite signs, according to (19). In this illustration $j < j'$. On the left are shown four different positions of the support of $\kappa_{j'}^{p'q'}$ relative to the support of κ_j^{pq}, which all give $\langle \kappa_j^{pq}, \kappa_{j'}^{p'q'} \rangle = 0$ by cancellation. On the right, three positions are shown, which give $\langle \kappa_j^{pq}, \kappa_{j'}^{p'q'} \rangle \neq 0$.

For the moment, we assume that $j \leq j'$. When taking the scalar product $\langle \kappa_j^{pq}, \kappa_{j'}^{p'q'} \rangle$, we can obviously multiply κ_j^{pq} with the characteristic function of the support of $\kappa_{j'}^{p'q'}$, which reduces its L^2 norm by a factor $2^{j-j'}$. Hence, we get by the Cauchy-Schwarz inequality that

$$\left|\langle(1-\chi_j^k)\Delta_j,(1-\chi_{j'}^k)\Delta_{j'}\rangle\right| \;\leq\; 2^{-|j-j'|}\sum_{(p,q,p',q')\in\mathfrak{I}}\|\kappa_j^{pq}\|_2\|\kappa_{j'}^{p'q'}\|_2.$$

Now, if the functions κ_j^{pq}, $j,p,q\in\mathbb{Z}$, are of the form (19), we have for each pair (p,q) that there are at most C pairs (p',q') such that $(p,q,p',q')\in\mathfrak{I}$, and vice versa. Hence,

$$\begin{aligned}\left|\langle(1-\chi_j^k)\Delta_j,(1-\chi_{j'}^k)\Delta_{j'}\rangle\right| &\leq 2^{-|j-j'|}\sum_{(p,q,p',q')\in\mathfrak{I}}\left(\|\kappa_j^{pq}\|_2^2+\|\kappa_{j'}^{p'q'}\|_2^2\right)\\ &\leq C2^{-|j-j'|}\left(\sum_{p,q\in\mathbb{Z}}\|\kappa_j^{pq}\|_2^2+\sum_{p',q'\in\mathbb{Z}}\|\kappa_{j'}^{p'q'}\|_2^2\right)\\ &= C2^{-|j-j'|}\left(\|(1-\chi_j^k)\Delta_j\|_2^2+\|(1-\chi_{j'}^k)\Delta_{j'}\|_2^2\right).\end{aligned}$$

For the cases with κ_j^{pq}, $j,p,q\in\mathbb{Z}$, of the form (20) or (21), we need to be a bit more careful. For every (p,q), there are at most $C2^{j'-j}$ pairs (p',q') such that $(p,q,p',q')\in\mathfrak{I}$, and for every (p',q') there are only C pairs (p,q) such that $(p,q,p',q')\in\mathfrak{I}$. (Note that we still assume $j\leq j'$.) We therefore get

$$\begin{aligned}\left|\langle(1-\chi_j^k)\Delta_j,(1-\chi_{j'}^k)\Delta_{j'}\rangle\right| &\leq \sum_{(p,q,p',q')\in\mathfrak{I}}2^{-\frac{3}{4}|j-j'|}\|\kappa_j^{pq}\|_2 2^{-\frac{1}{4}|j-j'|}\|\kappa_{j'}^{p'q'}\|_2\\ &\leq \sum_{(p,q,p',q')\in\mathfrak{I}}\left(2^{-\frac{3}{2}|j-j'|}\|\kappa_j^{pq}\|_2^2+2^{-\frac{1}{2}|j-j'|}\|\kappa_{j'}^{p'q'}\|_2^2\right)\\ &\leq C2^{|j-j'|}\sum_{p,q\in\mathbb{Z}}2^{-\frac{3}{2}|j-j'|}\|\kappa_j^{pq}\|_2^2+\sum_{p',q'\in\mathbb{Z}}2^{-\frac{1}{2}|j-j'|}\|\kappa_{j'}^{p'q'}\|_2^2\\ &= C2^{-|j-j'|/2}\left(\|(1-\chi_j^k)\Delta_j\|_2^2+\|(1-\chi_{j'}^k)\Delta_{j'}\|_2^2\right).\end{aligned}$$

Thus, we can conclude that in all cases, we have

$$\left|\langle(1-\chi_j^k)\Delta_j,(1-\chi_{j'}^k)\Delta_{j'}\rangle\right| \leq C2^{-|j-j'|/2}\left(\|\Delta_j(1-\chi_j^k)\|_2^2+\|\Delta_{j'}(1-\chi_{j'}^k)\|_2^2\right),$$

which by symmetry holds also for $j>j'$. We now find that

$$\begin{aligned}&\sum_{j,j'<k}\frac{1}{k-j}\frac{1}{k-j'}\langle(1-\chi_j^k)\Delta_j,(1-\chi_{j'}^k)\Delta_{j'}\rangle\\ &\qquad\leq C\sum_{j,j'<k}\frac{2^{-|j-j'|/2}}{(k-j)(k-j')}\left(\|(1-\chi_j^k)\Delta_j\|_2^2+\|(1-\chi_{j'}^k)\Delta_{j'}\|_2^2\right)\\ &\qquad\leq C\sum_{j<k}\frac{1}{(k-j)^2}\|(1-\chi_j^k)\Delta_j\|_2^2.\end{aligned}$$

Inserting this into (35), we finally get

$$\begin{aligned}
\mathrm{II} &\leq C\left(\frac{2}{\lambda}\right)^2 \sum_{k\in\mathbb{Z}} \sum_{j<k} \frac{1}{(k-j)^2} \|(1-\chi_j^k)\Delta_j\|_2^2 \\
&= C\left(\frac{2}{\lambda}\right)^2 \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} |\Delta_j(x,y)|^2 \sum_{k>j} \frac{1}{(k-j)^2}(1-\chi_j^k(x,y))dxdy \\
&\leq \frac{C}{\lambda} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^2} |\Delta_j(x,y)|dxdy,
\end{aligned}$$

where the last step followed from the observation

$$\sum_{k>j} \frac{1}{(k-j)^2}(1-\chi_j^k(x,y)) = \sum_{k\geq j+\frac{|\Delta_j(x,y)|}{\lambda}} \frac{1}{(k-j)^2} \leq C\frac{\lambda}{|\Delta_j(x,y)|},$$

which ends the proof. □

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