Position and height of the Global Maximum of a twice differentiable Stochastic Process

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Abstract

For a stochastic process ω with absolutely continuous sample path derivative, a formula for the joint density of (T, Z), the position and height of the global maximum of ω in a closed interval, is given. The formula is derived using the Generalized Rice's formula. The presented result can be applied both to stationary and non-stationary processes under mild assumptions on the process. The formula for the density is explicit but involves integrals that have to be computed using numerical integration. The computation of the density is discussed and some numerical examples are given.

Keywords: Stochastic process, global maximum, supremum, generalized Rice's formula, extremes

*Research supported in part by the Knut and Alice Wallenberg Foundation through the Gothenburg Stochastic Centre

1 Introduction

Finding the distribution of global maximum is a classical problem in probability. Most of the research on the properties of the global maximum of a stochastic process concentrates on the distribution for extreme heights; see e.g. Leadbetter et al. (1983). Often the studied process is Gaussian, and the distribution is given by means of upper and lower bounds; see Diebolt and Posse (1996) and references therein. Introducing the location of the global maximum makes the analysis more complicated. As far as we know the joint density of position and height of the global maximum has not been studied before. We turn now to some definitions and preliminary results.

Let $C = C([0, L], \mathbb{R})$ be the space of continuous functions $\omega : [0, L] \mapsto \mathbb{R}$. Denote by $\mathcal{F}$ the completed σ-algebra of uniform convergence and let P be the probability measure defined on sets in $\mathcal{F}$. In what follows we assume that the sample paths of the process have absolutely continuous derivatives with probability one; see Cramér and Leadbetter (1967) for suitable conditions.

In the following definition and lemma we assume that $\omega(t)$ is deterministic. The derivative and the second derivative of a function $\omega(t)$ will be denoted $\dot{\omega}(t)$ and $\ddot{\omega}(t)$, respectively.

Denote by $Z(\omega)$ and $T(\omega)$ the height and position of the global maximum of $\omega(t)$

$$Z(\omega) = \sup\{\omega(t) : t \in [0, L]\}, \qquad T(\omega) = \inf\{t \in [0, L] : \omega(t) = Z(\omega)\}. \tag{1}$$

The position of the maximum T can be located at the end points of the interval, i.e. $T \in \{0, L\}$, or in the interior of the interval, then $\dot{\omega}(T) = 0$. Analysis of the (T, Z)-distribution for the case when the global maximum is also a local maximum is a non-trivial problem. Here, we shall derive the distribution of T, Z as the limit distribution of a family of random variables, T_ϵ, Z_ϵ, $\epsilon \geq 0$, defined below:

Definition 1 *For a fixed level $\epsilon \geq 0$, define $A_\epsilon = \{0, L\} \cup \{t \in [0, L] : \dot{\omega}(t) = \epsilon\}$, then the position and height of the global ϵ-maximum $T_\epsilon(\omega), Z_\epsilon(\omega)$ say, are given by*

$$Z_\epsilon(\omega) = \sup\{\omega(t) : t \in A_\epsilon\}, \quad T_\epsilon(\omega) = \inf\{t \in A_\epsilon : \omega(t) = Z_\epsilon(\omega)\}.$$

Obviously, for continuously differentiable ω, $(T_0, Z_0) = (T, Z)$. The following lemma gives conditions when T_ϵ, Z_ϵ converges to T_0, Z_0 as ϵ goes to zero. The simple proof is given in the appendix.

Lemma 2 *If $\omega(t)$, $t \in [0, L]$, is continuously differentiable then $0 \leq Z(\omega) - Z_\epsilon(\omega) \leq L \cdot \epsilon$. Furthermore, if there is only one $t \in [0, L]$ such that $\omega(t) = Z(\omega)$ then $T_\epsilon \to T$ as ϵ goes to zero.*

A simple example, in which T_ϵ does not converges to T, is as follows. Consider ω having only one local maximum at $t \in (0, L)$. Suppose that t is also a global maximum, i.e. $Z(\omega) = \omega(t)$. Further, let $\omega(0) < \omega(t) = \omega(L)$ then $T_\epsilon(\omega) = L$ for all $\epsilon > 0$ while $T_0(\omega) = T(\omega) = t$.

In Section 2 we present the generalized Rice's formula for expected number of marked crossings. The formula will be used to derive $F_\epsilon(t, h)$ - the distribution of T_ϵ, Z_ϵ. (This will be done in Section 3.) However the formula is valid only for almost all $\epsilon \geq 0$ values while we are interested in the distribution of T_ϵ, Z_ϵ for $\epsilon = 0$. Consequently conditions for continuity of $F_\epsilon(t, h)$ as a function of ϵ are needed. These issues and the computation of $F_0(t, h)$ will be discussed in Section 4, while in Section 5 some numerical examples will be given. For clarity of presentation, several proofs are moved to an appendix.

2 Generalized Rice's formula

Assume that $X(t)$, $t \in [0, 1]$, is strictly stationary process. For a fixed u, let $\mu^+(u)$ be the expected number of times the process X crosses the level u in the upward direction. The following classical result is called Rice's formula, see Leadbetter et al. (1983).

Theorem 3 *If the process $X(t)$ is Gaussian and the derivative $\dot{X}(t)$ exists in quadratic mean ($h^{-1}(X(t+h) - X(t)) \to \dot{X}(t)$ in quadratic mean as h goes to zero) then*

$$\mu^+(u) = \int_0^\infty z f_{\dot{X}(0), X(0)}(z, u)\, dz, \tag{2}$$

where $f_{\dot{X}(0), X(0)}(z, u)$ is the joint density of $\dot{X}(0), X(0)$.

Rice (1944, 1945) and Kac (1943), obtained (2) for a certain class of Gaussian processes and polynomials with random coefficients, respectively. The formula (2) has been generalized in many directions. In this section we shall present the generalization which is particularly suited for the problem discussed in this paper.

Under the assumptions of the theorem, if $f(u) = f_{X(0)}(u) > 0$ then $\mu^+(u)/f(u)$ is a version of the following conditional expectation

$$E[|\dot{X}(0)|1_{\{\dot{X}(0)>0\}}\Big|X(0) = u] = \frac{\mu^+(u)}{f(u)},$$

where $1_{\{\cdot\}}$ is equal to one if the statement "$\cdot$" is true and zero otherwise. In general, the left hand of the last equation is defined only for almost all u. Next the number of upcrossings can be computed using the following integral

$$\int_0^1 1_{\{\dot{X}(s)>0, X(s)=u\}}\, \mathrm{d}\mathcal{H}^0(s),$$

where $\mathcal{H}^0$ is the zero dimensional Hausdorff measure counting the number of points. Consequently, we may write that for stationary Gaussian processes

$$\begin{aligned} \mu^+(u) &= E[\int_0^1 1_{\{\dot{X}(s)>0, X(s)=u\}}\, \mathrm{d}\mathcal{H}^0(s)] \\ &\overset{\text{a.a.}u}{=} E[|\dot{X}(0)|1_{\{\dot{X}(0)>0\}}\Big|X(0) = u] f_{X(0)}(u), \end{aligned} \tag{3}$$

where $\overset{\text{a.a.}u}{=}$ means that equality is valid for almost all u. Actually this result is true for any stationary a.s. absolutely differentiable process X as long as $E[|\dot{X}(0)|] < \infty$. The formula (3) follows from Theorem 4, which in a more general version is given in Zähle (1984). In that paper the Hausdorf area and the coarea theorem, see Federer (1969), has been used to study properties of crossings for random fields. The one-dimensional version of the result, given here, follows also from Banach (1925), (see Rychlik (2000) for details).

Theorem 4 *Assume $X(t)$ is a.s. absolutely continuous and $Y(t)$ is vector valued a.s. measurable, e.g. cadlag. Let $q : [0, L] \times C \to [0, \infty)$ be a measurable function. If $E[|\dot{X}(t)|] < +\infty$*

for all $t \in [0, L]$ then for any Borel sets A, B

$$\int_R E\Big[\int_B 1_{\{Y(s)\in A, X(s)=x\}} q(s,X)\, \mathrm{d}\mathcal{H}^0(s)\Big]\, dx$$
$$= \int_R \int_B E[|\dot{X}(s)| q(s,X) 1_{\{Y(s)\in A\}} \Big| X(s) = x] f_{X(s)}(x)\, \mathrm{d}s\, dx. \tag{4}$$

There is some difficulty in interpreting (4). The conditional expectation

$$E[|\dot{X}(s)| q(s,X) 1_{\{Y(s)\in A\}} \Big| X(s) = x] f_{X(s)}(x) = H(x,s)$$

is only well defined with probability one with respect to (x, s) ($H(x, s)$ can be seen as a class of functions). In order to use the formula we need first to specify the version of conditional expectation we wish to consider. The problem of choosing a "proper" conditional density is a delicate issue and will be discussed later on.

In the following remark we demonstrate that Eq.(3) follows from Theorem 4.

Remark 5 *Consider formula (4) and choose $B = [0, 1]$, $Y(s) = \dot{X}(s)$, $A = (0, +\infty)$ and $q(s, X) = 1_{\{X(s)\le u\}}$ then*

$$\int_{-\infty}^{u} E\Big[\int_0^1 1_{\{\dot{X}(s)>0, X(s)=x\}}\, \mathrm{d}\mathcal{H}^0(s)\Big]\, dx$$
$$= \int_{-\infty}^{u} \int_0^1 E[|\dot{X}(s)| 1_{\{\dot{X}(s)>0\}} \Big| X(s) = x] f_{X(s)}(x)\, \mathrm{d}s\, dx.$$

Now by differentiating both sides of the equation on u and by assumed stationarity of X we obtain (3). Note that the differentiation on u implies that Eq. (3) is proved only for almost all u.

3 The distribution of T_ϵ, Z_ϵ.

We shall use Theorem 4 to derive formulae for the distribution of T_ϵ, Z_ϵ. Assume now that ω is a random process. Similarly as in Remark 5 we shall define different components in the left hand side of Eq. (4) so that

$$\int E\Big[\int_B 1_{\{Y(s)\in A, X(s)=x\}} q(s,X)\, \mathrm{d}\mathcal{H}^0(s)\Big]\, dx = \int P(0 < T_x(\omega) \le t, Z_x(\omega) \le h)\, dx.$$

Let $X(s) = \dot{\omega}(s)$, $Y(s) = \omega(s)$, $A = (-\infty, h]$, $B = (0, t]$, $0 < t < L$, and

$$q(s, \dot{\omega}) = \begin{cases} 1 & \text{if} \quad T_u = s \text{ where } u = \max(0, \dot{w}(s)), \\ 0 & \text{otherwise.} \end{cases}$$

(Note that in Definition 1, T_ϵ is a function of ω. However, T_ϵ can also be defined as a function of the derivative $\dot{\omega}$, since $T_\epsilon(\omega) = T_\epsilon(c + \omega)$ for any constant c.) Now it is easy to see that

$$\int_0^\epsilon \int_0^t q(s, \dot{\omega}) 1_{\{\dot{\omega}(s) = x, \omega(s) \le h\}}(s) \, \mathrm{d}\mathcal{H}^0(s) \, dx = \int_0^\epsilon 1_{\{0 < T_x(\omega) \le t, Z_x(\omega) \le h\}} \, dx.$$

and hence the next theorem follows from Theorem 4.

Theorem 6 *Consider ω with a.s. absolute continuous derivative $\dot{\omega}$ such that $E[|\ddot{\omega}(t)|] < \infty$ for all $t \in [0, L]$. If for all $t \in [0, L]$ the distribution of the derivative process has a density $f_{\dot{\omega}(t)}(u)$ then*

$$\begin{aligned} P(T_\epsilon = 0, Z_\epsilon \le h) &= \mathrm{E}\big[1_{\{\omega(0) \le h\}} 1_{\{T_\epsilon(\omega) = 0\}}\big], & (5) \\ P(T_\epsilon = L, Z_\epsilon \le h) &= \mathrm{E}\big[1_{\{\omega(L) \le h\}} 1_{\{T_\epsilon(\omega) = L\}}\big], & (6) \end{aligned}$$

and for any $t \in (0, L)$,

$$P(0 < T_\epsilon \le t, Z_\epsilon \le h) \overset{a.a.\epsilon}{=} \int_0^t \mathrm{E}\Big[|\ddot{\omega}(s)| \, q(s, \dot{\omega}) 1_{\{\omega(s) \le h\}} \,\Big|\, \dot{\omega}(s) = \epsilon \Big] f_{\dot{\omega}(s)}(\epsilon) \, \mathrm{d}s, \qquad (7)$$

where $\overset{a.a.\epsilon}{=}$ means that equality holds for almost all $\epsilon \ge 0$.

Proof: The theorem follows from Theorem 4 if we demonstrate measurability of $q(s, \dot{\omega})$. This is done in Lemma 8. □

Assume that a version of conditional expectation is chosen and then the distribution function of T_ϵ, Z_ϵ is for almost all ϵ equals the following function

$$F_\epsilon(t, h) = \begin{cases} \mathrm{E}\big[1_{\{\omega(0) \le h\}} 1_{\{T_\epsilon(\omega) = 0\}}\big] & \text{if } t = 0, \\ \mathrm{E}\big[1_{\{\omega(0) \le h\}} 1_{\{T_\epsilon(\omega) = 0\}}\big] + F(h, t, \epsilon) & \text{if } t \in (0, L), \\ \mathrm{E}\big[1_{\{\omega(0) \le h\}} 1_{\{T_\epsilon(\omega) = 0\}}\big] + \mathrm{E}\big[1_{\{\omega(L) \le h\}} 1_{\{T_\epsilon(\omega) = L\}}\big] + F(h, L, \epsilon) & \text{if } t = L, \end{cases} \qquad (8)$$

where

$$F(t,h,\epsilon) = \int_0^t \mathrm{E}\Big[\,|\ddot{\omega}(s)|\, q(s,\dot{\omega}) 1_{\{\omega(s)\le h\}} \,\Big|\, \dot{\omega}(s) = \epsilon\Big]\, f_{\dot{\omega}(s)}(\epsilon)\,\mathrm{d}s. \tag{9}$$

Corollary 7 *Under the assumptions of Theorem 6, if $F_\epsilon(L,h)$ is a continuous function of ϵ and h, then $P(Z \le h) = F_0(L,h)$.*

Proof: By Lemma 2, $P(Z_\epsilon \le h) \le P(Z \le h) \le P(Z_\epsilon \le h - L\epsilon)$ and the corollary follows. □

Since T_ϵ may not converge to T_0, the analysis of the joint distribution of T_ϵ, Z_ϵ is more difficult. The key is to demonstrate that with probability one there is only one t such that $\omega(t) = Z(\omega)$. The sufficient conditions for the convergence of T_ϵ to T are given in the following lemma (the proof is given in the appendix).

Lemma 8 *Assume ω is a.s. continuously differentiable and the distribution of the derivative $\dot{\omega}(t)$ has a density $f_{\dot{\omega}(t)}(x)$ for all $t \in [0,L]$; then each of the following hold:*

(0) The functions $q(t,\dot{\omega})$, $1_{\{T_\epsilon(\omega)=0\}}(\omega)$ and $1_{\{T_\epsilon(\omega)=L\}}(\omega)$ are measurable.

(I) If the density $f_{\omega(t)}(x)$ exists and is bounded in x and in $t \in [0,L]$ then, for any $\epsilon \ge 0$, $P(Z_\epsilon = h) = 0$ and $P(T_\epsilon = t) = 0$ for all h and $t \in (0,L)$.

(II) If, in addition to the assumptions in (I), for any $\delta > 0$ the joint density $f_{\omega(s)-\omega(L),\dot{\omega}(s)}(x,y)$ exists and is bounded in x,y and in $s \in [0, L-\delta]$, then

$$\begin{aligned}\lim_{\epsilon\to 0} P(T_\epsilon = 0, Z_\epsilon \le h) &= P(T = 0, Z \le h),\\ \lim_{\epsilon\to 0} P(T_\epsilon = L, Z_\epsilon \le h) &= P(T = L, Z \le h).\end{aligned}$$

(III) If, in addition to the assumptions in (I) and (II), for any $\delta > 0$ the joint density $f_{\omega(t)-\omega(s),\dot{\omega}(s),\dot{\omega}(t)}(x,y,z)$ exists and is bounded in x,y,x and in $(s,t) \in A_\delta = \{(s,t) \in [0,L]\times[0,L] : |t-s| \ge \delta\}$ then $(T_\epsilon, Z_\epsilon) \to (T,Z)$ a.s. as ϵ goes to zero, and

$$\lim_{\epsilon\to 0} P(T_\epsilon \le t, Z_\epsilon \le h) = P(T \le t, Z \le h). \tag{10}$$

We finish this section with a formula for $f_\epsilon(t,x)$, the joint density of T_ϵ, Z_ϵ.

Corollary 9 *Assume that the joint density $f_{\omega(t),\dot\omega(t)}(x,u)$ exits for all $t \in [0,L]$ then, for almost all ϵ, the density of T_ϵ, Z_ϵ is given by*

$$f_\epsilon(t,x) = f_\epsilon^E(t,x) + f_\epsilon^C(t,x), \tag{11}$$

where

$$f_\epsilon^E(t,x) = \mathrm{E}\big[1_{\{T_\epsilon=0\}} \,\big|\, \omega(0)=x\big]\, f_{\omega(0)}(x)\, \delta_0(t) + \mathrm{E}\big[1_{\{T_\epsilon=L\}} \,\big|\, \omega(L)=x\big]\, f_{\omega(L)}(x)\, \delta_L(t),$$

and

$$f_\epsilon^C(t,x) = \mathrm{E}\big[\,|\ddot\omega(t)|\, q(t,\dot\omega) \,\big|\, \dot\omega(t)=\epsilon, \omega(t)=x\big]\, f_{\omega(t),\dot\omega(t)}(x,\epsilon),$$

where $\delta_s(t) = \delta(t-s)$ is the Dirac function. The conditional expectations are understood to be those used in (9).

4 Evaluation of the density of T, Z.

In this section we shall discuss the choice of the version of the conditional expectation in the definition of $F(h,t,\epsilon)$ in (9) and computation of the density $f_0(s,x)$ given by (11).

If for vector valued variables X, Y the joint density $f_{X,Y}(x,y)$ exists then the most natural definition of the conditional expectation is

$$E[g(Y)|X=x]f_X(x) = \int g(y) f_{X,Y}(x,y)\, dy. \tag{12}$$

(Here g is a measurable function such that $E[|g(Y)|] < \infty$.) Obviously, if the joint density of $(\omega(t), \dot\omega(t), \ddot\omega(t), q(t,\dot\omega))$ is known then (12) can be used to compute the function $F(t,h,\epsilon)$ or the density $f_\epsilon(s,x)$. However $q(t,\dot\omega)$ is a function of an infinite sequence $\{\omega(s_i), \dot\omega(s_i)\}_{i=1}^\infty$, see below, and hence the density is in general unknown. Consequently we shall define the expectation as a limit of a sequence of suitable approximations.

Let $\{s_i\}_{i=1}^{\infty}$, $s_1 = 0, s_2 = L$, be a dense subset of $[0, L]$ and let

$$\begin{aligned} q^N(t,\dot\omega) &= 1_{\{(\omega(t)-\omega(s_i))\geq 0 \text{ or } \dot\omega(s_i)\leq \dot\omega(t)^+) \text{ for all } i\leq N\}}, \\ q^N_\epsilon(0,\omega) &= 1_{\{(\omega(0)-\omega(s_i))\geq 0 \text{ or } \dot\omega(s_i)\leq \epsilon) \text{ for all } i\leq N\}}, \\ q^N_\epsilon(L,\omega) &= 1_{\{(\omega(L)-\omega(s_i))\geq 0 \text{ or } \dot\omega(s_i)\leq \epsilon) \text{ for all } i\leq N\}}, \end{aligned}$$

where $x^+ = \max(0, x)$. Assume that for any N the non-degenerated density of

$$\omega(s_1), \dot\omega(s_1), \ldots, \omega(s_N), \dot\omega(s_N), \omega(t), \dot\omega(t), \ddot\omega(t) \tag{13}$$

exists. We use (12) to compute the following functions

$$f_\epsilon(t, x; N) = f^E_\epsilon(t, x; N) + f^C_\epsilon(t, x; N), \tag{14}$$

where

$$\begin{aligned} f^E_\epsilon(t,x;N) =& \mathrm{E}\big[q^N_\epsilon(0,\omega) \,\big|\, \omega(0) = x\big]\, f_{\omega(0)}(x)\, \delta_0(t) \\ &\qquad\qquad + \mathrm{E}\big[q^N_\epsilon(L,\omega) \,\big|\, \omega(L) = x\big]\, f_{\omega(L)}(x)\, \delta_L(t), \\ f^C_\epsilon(t,x;N) =& \mathrm{E}\big[\, |\ddot\omega(t)|\, q^N(t,\dot\omega) \,\big|\, \dot\omega(t) = \epsilon, \omega(t) = x\big]\, f_{\omega(t),\dot\omega(t)}(x, \epsilon). \end{aligned}$$

Since $q^N(t, \dot\omega) \geq q(t, \dot\omega)$ while $q^N_\epsilon(0, \omega) \geq 1_{\{T_\epsilon=0\}}$, $q^N_\epsilon(L, \omega) \geq 1_{\{T_\epsilon=L\}}$ then for any version of conditional expectation used to define $f_\epsilon(t, x)$ in (11), the functions $f_\epsilon(t, x; N)$ are, for almost all ϵ, upper bounds for $f_\epsilon(t, x)$, i.e.

$$f_\epsilon(t, x) \overset{\text{a.a.}\epsilon}{\leq} f_\epsilon(t, x; N)$$

for all (t, x).

Next we shall derive two sufficient conditions for the convergence of $f_\epsilon(t, x; N)$ to a version of $f_\epsilon(t, x)$ for almost all ϵ. Under either, the limit of $f_\epsilon(t, x; N)$, which always exists since the functions are positive and non-increasing, can be used to define the conditional expectation in (11). Thus we can say that $f_\epsilon(t, x)$ is the limit of $f_\epsilon(t, x; N)$ for all $\epsilon \geq 0$. The first condition, (A), will be used in construction of numerical algorithm to compute the density of T, Z, while the second, (B), is easy to check for ω with known joint densities of values and the derivatives.

Theorem 10 *Assume that, for any N and $t \in [0, L]$, (13) has a non-degenerated density. Further, suppose*

$$\int_0^L \mathrm{E}\big[\,|\ddot{\omega}(s)|\,\big|\,\dot{\omega}(s) = \epsilon\big]\, f_{\dot{\omega}(s)}(\epsilon)\, \mathrm{d}s < \infty. \tag{15}$$

(A) If

$$\lim_{N\to\infty} \int_{-\infty}^{+\infty} \int_0^L f_\epsilon(s, x; N)\, \mathrm{d}s\, \mathrm{d}x = 1 \tag{16}$$

then, for all $t \in (0, L)$ and almost all $\epsilon \geq 0$, $f_\epsilon(t, x) = \lim_{N\to\infty} f_\epsilon(t, x; N)$.

(B) If $\ddot{\omega}$ is a.s. continuous and, for a fixed $t \in [0, L]$ and any $\delta > 0$, the densities $f_{\dot{\omega}(s)-\dot{\omega}(t)}(y)$, $f_{\omega(s)-\omega(t),\dot{\omega}(s)-\dot{\omega}(t)}(x, y)$ are bounded in x, y and in s such that $|t-s| \geq \delta$ then, for almost all $\epsilon \geq 0$, $f_\epsilon(t, x) = \lim_{N\to\infty} f_\epsilon(t, x; N)$.

Proof: Proof is given in the appendix.

Lemma 8 and Theorem 10 gives conditions for the validity of the following approximation scheme. First, define $f_\epsilon(s, z)$ to be $\lim_{N\to\infty} f_\epsilon(t, z; N)$ then, with

$$F_\epsilon(t, h; N) = \int_{-\infty}^{h} \int_0^t f_\epsilon(s, z; N)\, ds\, dz$$

we have

$$\lim_{N\to\infty} F_\epsilon(t, h; N) = F_\epsilon(t, h) \overset{\text{a.a.}\epsilon}{=} P(T_\epsilon \leq t, Z_\epsilon \leq h) \to P(T \leq t, Z \leq h),$$

as $\epsilon \to 0$. Consequently, if $f_\epsilon(t, x; N)$ are continuous and the sequence $\{f_\epsilon(t, x; N)\}$ converge uniformly then

$$f_{T,Z}(t, x) = \lim_{N\to\infty} f_0(t, x; N).$$

Often continuity of $f_\epsilon(t, x; N)$ can be proved. Checking uniform convergence of $f_\epsilon(t, x; N)$ is a much more difficult problem. However in practice, the hardest problem is the computation of $f_\epsilon(t, x; N)$, what shall be discussed next.

When (numerical) computation is possible, then often one can also justify the assumption of the continuity of $f_\epsilon(t, x; N)$ as a function of ϵ. The uniform convergence assumption can then be replaced by the following condition: Assume that for sufficiently small δ and all $\epsilon \in [0, \epsilon_0]$

$$\iint f_\epsilon(s, x; N)\, \mathrm{d}s\, \mathrm{d}x < 1 + \delta,$$

then $f_{T,Z}(t,x) \approx f_0(t,x;N)$ in the sense that the distribution of T, Z can be approximately computed using $f_0(t,x;N)$ with an error of

$$|P(T \leq t, Z \leq h) - F_0(t,h;N)| \leq \delta$$

for all values of (t,h). Finally, in order to reduce the amount of numerical integration, the definition of the density $f_0(t,x;N)$ needs to be modified as follows

$$f_N(t,x) = f_\epsilon^E(t,x;N) + \mathrm{E}\big[\,|\ddot{\omega}(t)|\,\tilde{q}^N(t,\omega)\,\big|\,\dot{\omega}(t)=\epsilon, \omega(t)=x\big]\, f_{\omega(t),\dot{\omega}(t)}(x,\epsilon),$$

where

$$\tilde{q}^N(t,\omega) = 1_{\{\omega(s_i)\leq\omega(t) \text{ for all } i\leq N\}}(\omega).$$

Obviously $\tilde{q}^N(t,\omega) \geq q^N(t,\omega)$ and hence, if

$$\iint f_N(s,x)\,\mathrm{d}s\,\mathrm{d}x \leq 1+\delta,$$

then $f_N(t,x)$ can be used as an approximation to the density of T, Z. This is how the density of T, Z has been computed in the examples presented in the next section. Since the density $f_N(t,h)$ is an $N+1$ dimensional integral, which has to be computed numerically, it is important to choose the grid s_i in an optimal way so that the derived approximation is accurate even for small values of N, see Rychlik and Lindgren (1991) for some strategies to choose s_i for Gaussian ω.

4.1 Approximation of T, Z-density for Gaussian ω.

In this subsection, we shall give sufficient conditions involving only the covariance structure of the Gaussian process ω under which the assumptions of Lemma 8 and Theorem 10 are satisfied.

Let ω be a stationary Gaussian process such that the fourth spectral moment λ_4 exists and the fourth order derivative of the covariance function satisfies

$$r^{(4)}(\tau) = \lambda_4 - o(|\log|\tau||^{-\alpha}),$$

as $\tau \to 0$, for some $\alpha > 1$. Then $\ddot{\omega}(s)$ is a.s. continuous, see Cramér and Leadbetter (1967), and hence the assumptions of Theorem 6 and Corollary 9 are satisfied. If in addition, we assume that spectral measure of ω contains a continuous component, then, for any $s \neq t$, $(\omega(s), \omega(t), \dot{\omega}(s), \dot{\omega}(t))$ has a nonsingular joint density, with covariance matrix that depends in a continuous way on $t - s$. Hence the assumptions (I-III) of Lemma 8 are satisfied and we can conclude that

$$\lim_{\epsilon \to 0} P(T_\epsilon \leq t, Z_\epsilon \leq h) = P(T \leq t, Z \leq h).$$

Since the assumptions of Theorem 10 (B) are also satisfied for any value of t, we can then define

$$f_\epsilon(t, x) = \lim_{N \to \infty} f_\epsilon(t, x; N).$$

Finally, since the spectral measure contains continuous components, it is clear that, for any N, the joint density

$$\omega(s_1), \dot{\omega}(s_1), \ldots, \omega(s_N), \dot{\omega}(s_N), \omega(t), \dot{\omega}(t), \ddot{\omega}(t).$$

exists and that the functions $f_\epsilon(t, x; N)$ are continuous.

The non-stationary Gaussian process used in the next section is derived from the stationary one by means of conditioning on values at a finite number of points t_i, $0 < t_1 < \ldots < t_n < L$. Studies of this type of processes are motivated by practical applications for which the random function is observed at some fixed time points, see Sjö (2001) for more detailed discussion. If the measurements consists of the random function plus some random noise (such as zero mean iid. Gaussian variables), then one can modify the arguments presented for stationary Gaussian ω to this non-stationary case. However if there is no measurement error the densities of $\omega(t_i)$ are degenerated and hence the assumptions of Lemma 8 and Theorem 10 are not satisfied.

In order to resolve this problem one has to choose points $t, s_1, \ldots, s_N$ which are disjoint with the conditioning times t_i and then $f_\epsilon(t, x; N)$ exists and is still continuous. Next it is easy to reformulate the assumptions in the lemma (and the theorem) so that this special case is covered. For clarity of presentation we have chosen not to do it. In order to give some indication of what type of modification is required we give an example: in Lemma 8, (III) the definition of A_δ has to be changed to the following one

$$A_\delta\{(s,t) \in [0,L] \times [0,L] : |t-s| \geq \delta\} \cap \bigcap_{i=1}^{n} \{(s,t) : |s-t_i| \geq \delta, |t-t_i| \geq \delta\}.$$

5 Examples

We will demonstrate the result with some examples. We start with a stationary Gaussian process with a rather periodical behavior, it has a typical oceanographic spectrum, and evaluate the density for the global maximum for two different L. As a final example we study a non-stationary Gaussian process that is created from a stationary ditto by conditioning on the value at some points. We use the notation $f^C(t,x)$, $f^E(t,x)$, instead of $f_0^C(t,x;N)$, $f_0^E(t,x;N)$, respectively. The computer programs uses toolbox WAFO—Wave Analysis in Fatigue and Oceanography, which is available without charge at `http://www.maths.lth.se/matstat/wafo/`.

5.1 Stationary process

The intuition about the stationary process may say that the position of the global maximum should be uniformly distributed, except at the endpoints. Our examples will show that it is not always true. The density is always perfectly symmetric around the midpoint of the interval, but its shape depends on the width of the interval relative to the "periodicity" and the irregularity of the process. A very narrow-banded process, gives a rather different result than a broad-banded process. For the sake of clarity we present f^E and f^C in separate plots, since f^E actually is two one-dimensional functions, $f^E(0,x)$ and $f^E(L,x)$, while $f^C(s,x)$ is truly two-dimensional.

The process in our stationary example has zero mean, approximately 0.12 zero-upcrossings per unit interval, and 0.17 local maxima per unit interval. First we have evaluated the density of the position and height of the global maximum in the interval $[0,3]$ where the expected number of local maxima is roughly 0.5 (Figure 1), and then in the interval $[0,9]$, which is

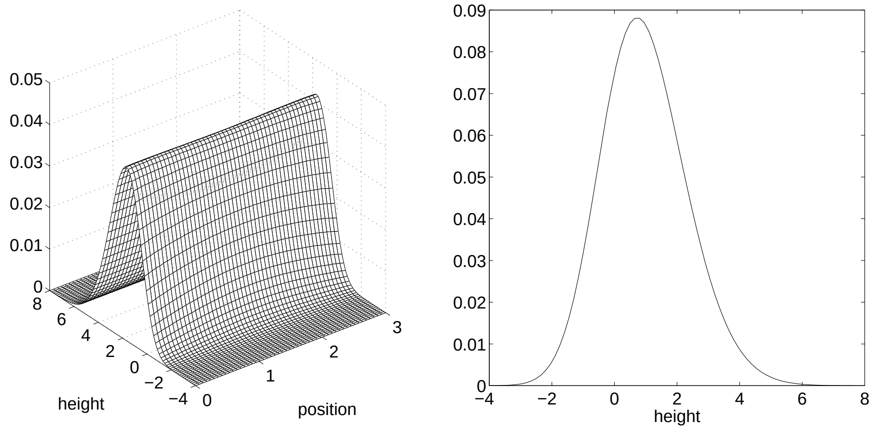

Figure 1: Density of position and height of the global maximum in the short interval $[0, 3]$. *Left:* Interior density $f^C(s, x)$. *Right:* Left endpoint density $f^E(0, x)$, the right endpoint density $f^E(3, x)$ is identical.

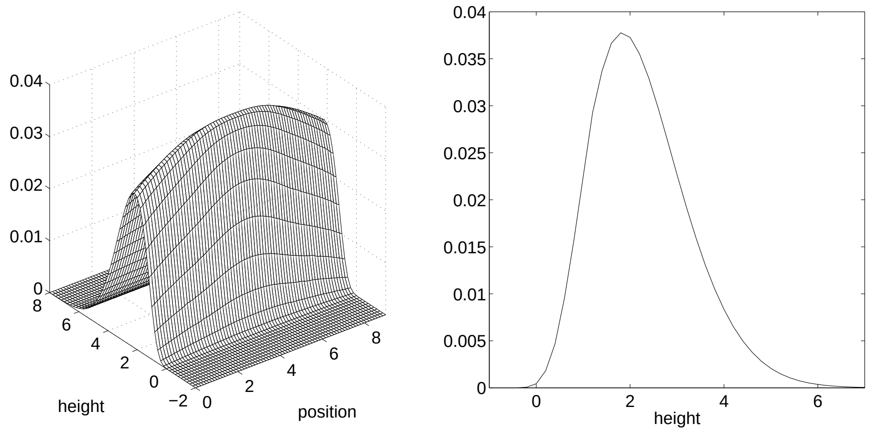

Figure 2: Density of position and height of the global maximum in the longer interval $[0, 9]$. *Left:* Interior density $f^C(s, x)$. *Right:* Left endpoint density $f^E(0, x)$, the right endpoint density $f^E(9, x)$ is identical.

three times wider and where we expect 1.5 local maxima (Figure 2).

The short interval gives an almost uniform density for the position, but also larger probability to have the global maximum at one of the endpoints. The proportion is 41% in the interior of the interval to 59% at the endpoints. In the wider interval it is more likely that the global maximum is in the central part of the interval, especially the smaller maxima are likely to be there, while the high local maxima are almost uniformly distributed. The proportion between interior and endpoints in this case is 82% to 18%. With the shorter interval it is also more probable to have a low global maximum with height below 0: $\mathrm{P}(Z \leq 0) = 0.18$ for the short interval, while $\mathrm{P}(Z \leq 0) = 0.004$ for the longer.

We also compared the evaluated densities to empirical results based on simulation; here we present the results for the short interval. We simulated 500 replications of a stationary process with the given spectral density. In the interval $[0, 3]$ the global maximum were observed. Figure 3 shows the simulated results compared to the evaluated density. The cumulative functions integrated from the sub-densities are normalised to have maximum value 1.

5.2 Process conditioned on observations – a non-stationary process

This section exemplifies the situation where we would like to find the maximum of a process, given that we have a number of observations of it. We started by simulating a stationary Gaussian process, the simulation was 'observed' at four randomly located points, and a zero mean Gaussian measurement error was added with variance 1/10 of the variance of the stationary process. Conditional on the observations (with errors), we obtain a new process that is a non-stationary Gaussian process with mean function equal to the conditional mean, and covariance function equal to the conditional covariance.

In a real situation the covariance function of the stationary process normally has to be estimated, but in this example it is taken as known, i.e., the covariance function we simulated from. The stationary covariance function used is of the type $a \exp(-b\tau^2)$, with parameters such that the covariance function is almost zero after $\tau = 1.2$, so at the distance

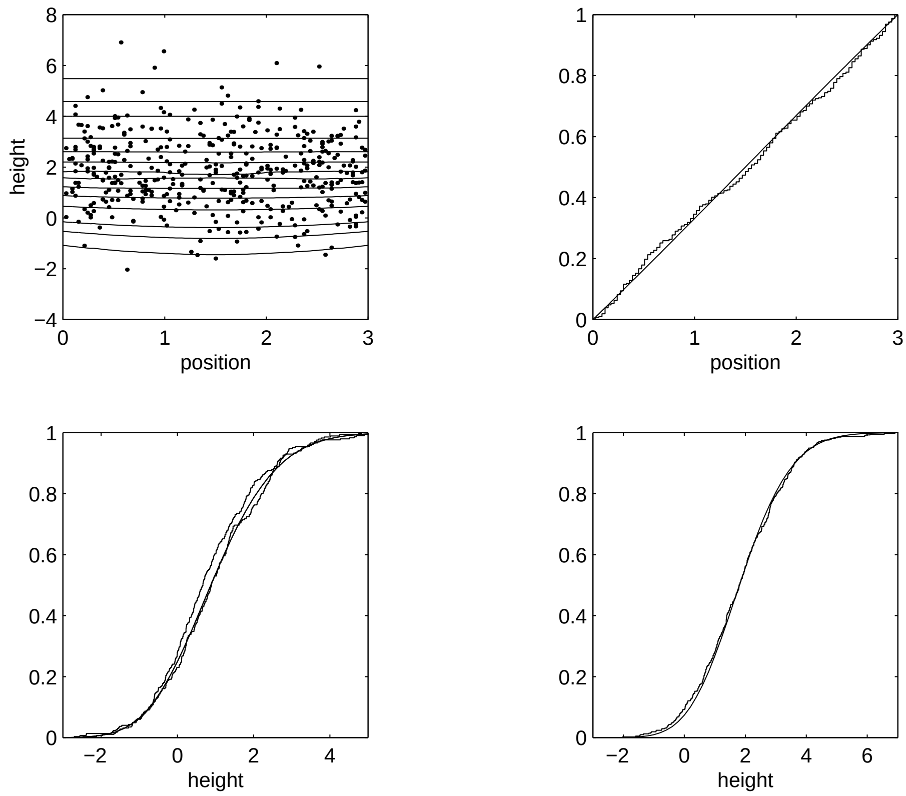

Figure 3: Simulation of position T and height Z of the global maximum in $[0, 3]$, 500 replications. The outcome was such that 202 maxima were located in the interior, 143 at the left endpoint, and 155 at the right, which is close to the analytical proportion. *Top Left* : Contours of the density in Figure 1 (*Left*) with interior pairs of (T, Z) marked by dots. *Top Right*: Empirical distribution of T, given that the maximum is in the interior of the interval (irregular line), together with the cumulative distribution function, integrated from f^C and normalised. *Bottom Left*: Empirical distribution of Z at the left endpoint, and at the right endpoint (two irregular lines), together with the cumulative distribution function integrated from $f^E(0, x)$ and normalised. *Bottom Right*: Empirical distribution of Z, given that the maximum is in the interior (irregular line), together with the cumulative distribution function integrated from f^C and normalised.

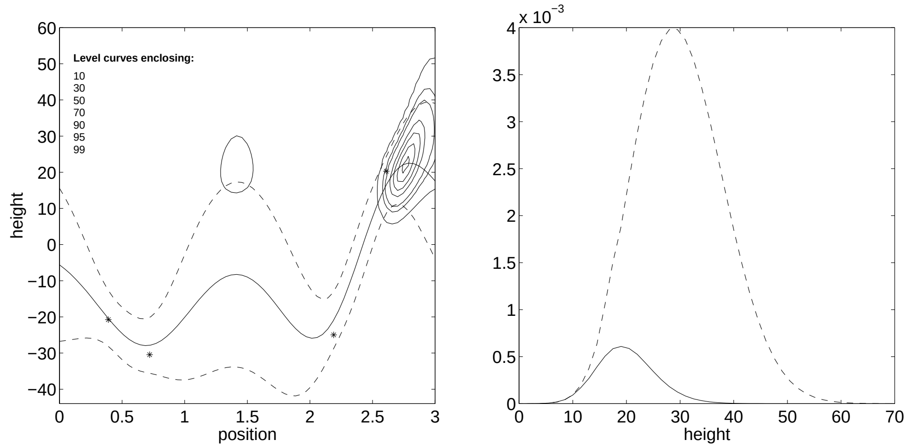

Figure 4: *Left*: Contours of f^C for a process conditioned on observations. The respective contours encloses 10%, 30%, etc. of the total integral of the density. The observations are marked by '$*$'. The conditional mean, i.e., the reconstructed mean, is given by a solid curve, flanked by approximative confidence bands (dashed curves). *Right*: Left endpoint density $f^E(0,x)$ (solid), and right endpoint density $f^E(3,x)$ (dashed).

1.2 the influence of an observation is negligible. The measurement error variance is $a/10$. The reason for choosing only four observations is clarity, with many observations the density gets very concentrated.

To the left in Figure 4 there is a contour plot of the interior density f^C. The location and value of the observations are marked by '$*$'. The plot also shows the conditional mean function, point-wise flanked by approximative confidence bands evaluated pointwise as the mean function $\pm$ two times the standard deviation. To the right in Figure 4 are the endpoint densities. The proportions are 91% in the interior, 0.8% to the left, and 8.2% to the right.

To illustrate the influence of the observation error we have repeated the same evaluation, but this time with the observation error variance equal to 0. In this situation the non-stationary process has variance 0 at the observations, i.e., the situation commented in Subsection 4.1. The result is shown in Figure 5. The dashed curves coincide with the solid at the observations since the conditional variance is 0 there. The contours of this density are more erratic depending on numerical problems due to the fact that the distribution is nearly singular close to the observations. Exactly at the observation points, the density is 0 for all heights. The proportions this time are 95.7% in the interior, 0.1% to the left, and 4.2% to the right.

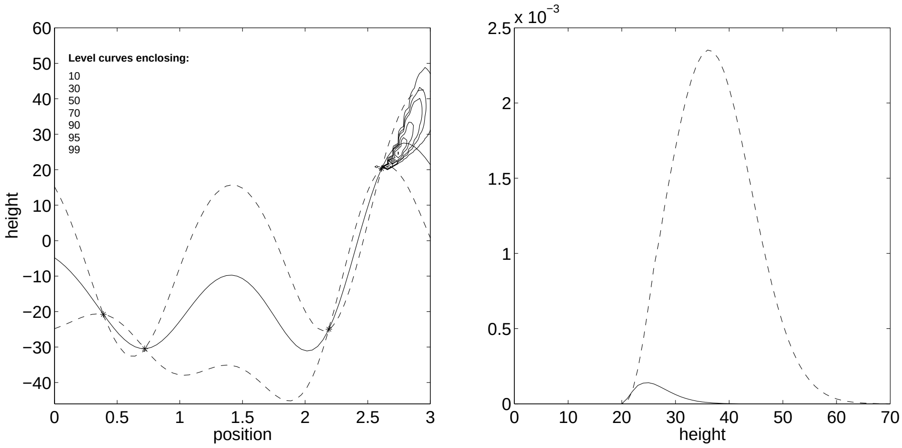

Figure 5: *Left*: Contours of f^C for a process conditioned on observations without observation error. The observations are marked by '$*$'. The conditional mean, i.e., the reconstructed mean, is given by a solid curve, flanked by approximative confidence bands (dashed curves). *Right*: Left endpoint density $f^E(0,x)$ (solid), and right endpoint density $f^E(3,x)$ (dashed).

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Appendix

Proof of Lemma 2:

We demonstrate first that $\omega(t) > Z_\epsilon$ implies $\dot{\omega}(t) < \epsilon$. Suppose $\omega(t) > Z_\epsilon(\omega)$, and $\dot{\omega}(t) > \epsilon \geq 0$ ($\dot{\omega}(t) = \epsilon$ and $\omega(t) > Z_\epsilon$ is impossible). Since $\omega(t) > \omega(L)$ then there is $t_0 = \inf\{s \in (t, L) : \dot{\omega}(s) < 0\}$. Obviously $\dot{\omega}(s) \geq 0$ for all $s \in (t, t_0)$. Now, for continuous $\dot{\omega}$ there is $s \in (t, t_0)$ such that $\dot{\omega}(s) = \epsilon$ and $\omega(s) > Z_\epsilon(\omega)$, which is a contradiction. Consequently, if $\omega(t) > Z_\epsilon$ then $\dot{\omega}(t) < \epsilon$, hence the distance from $Z_\epsilon(\omega)$ to $Z(\omega)$ has to be less then ϵL. The second statement of the lemma is obvious. $\square$

Proof of Lemma 8:

In the proof we shall employ Bulinskaya lemma, given next for completness, see Cramér, Leadbetter (1967), p. 76 for the proof. Adler (1981) Theorem 3.2.1 presents similar type of result for fields which in slightly modified form will be used in following proofs.

Lemma 11 *(Bulinskaya (1961)) Let u be fixed. If one dimensional density $f_t(x)$ of the process $\zeta(t)$ is bounded in x and in $0 \le t \le 1$, and if $\zeta(t)$ has, with probability one, a continuous sample derivative $\dot{\zeta}(t)$, then the probability is zero that $\dot{\zeta}(t) = 0$, $\zeta(t) = u$ simultaneously, for any t in $0 \le t \le 1$.*

We beginn with the proof of (0). First we give an alternative characterizations of functions $q(t, \dot{\omega})$ and the indicators $1_{\{T_\epsilon(\omega)=0\}}(\omega)$, $1_{\{T_\epsilon(\omega)=L\}}(\omega)$ and then demonstrate their measurability. Only measurability of $q(t, \dot{\omega})$ will be given. (Measurability of the indicators can be shown in a similar way.)

Let us introduce $\Delta\omega(s,t) = \omega(t) - \omega(s)$ (obviously $\Delta\omega(s,t) = \int_t^s \dot{\omega}(z)\, dz$ is a function of $\dot{\omega}$). The partial derivative $\Delta\omega_{12}(s,t) = \dot{\omega}(t) - \dot{\omega}(s)$ then, for a fixed t

$$\begin{aligned} q(t,\dot{\omega}) = 1 \quad \Leftrightarrow \quad & \Delta\omega(0,t) > 0 \text{ and } \Delta\omega(t,L) \le 0 \text{ and } \bigwedge_{0<s<t} \Delta\omega(s,t) > 0 \text{ or } \Delta\omega_{12}(s,t) > 0 \\ & \text{and } \bigwedge_{t \le s < L} \Delta\omega(t,s) \le 0 \text{ or } \Delta\omega_{12}(t,s) < 0. \end{aligned} \tag{17}$$

(Simply, $\bigwedge_{0<s<t} \Delta\omega(s,t) > 0$ or $\Delta\omega_{12}(s,t) > 0$ means that ω is higher or growing faster at t than at any other $s \in (0,t)$.) Next for any $\epsilon \ge 0$

$$T_\epsilon(\omega) = 0 \Leftrightarrow \Delta\omega(0,L) \le 0 \text{ and } \bigwedge_{0<s<L} \Delta\omega(0,s) \le 0 \text{ or } \dot{\omega}(s) < \epsilon, \tag{18}$$

$$T_\epsilon(\omega) = L \Leftrightarrow \Delta\omega(0,L) > 0 \text{ and } \bigwedge_{0<s<L} \Delta\omega(s,L) > 0 \text{ or } \dot{\omega}(s) < \epsilon. \tag{19}$$

Let $\{s_i\}_{i=1}^{\infty}$ be a countable, dense subset of $[0,L]$, such that $s_1 = 0$, $s_2 = L$. Define

$$\begin{aligned} A^-(t,\omega) &= \bigwedge_n \bigvee_k \bigwedge_i \{(s_i \le t - n^{-1}) \Rightarrow (\Delta\omega(s_i,t) \ge k^{-1} \text{ or } \Delta\omega_{12}(s_i,t) \ge k^{-1})\} \\ A^+(t,\omega) &= \bigwedge_n \bigvee_k \bigwedge_i \{(s_i \ge t + n^{-1}) \Rightarrow (\Delta\omega(t,s_i) \le 0 \text{ or } \Delta\omega_{12}(t,s_i) \le -k^{-1})\} \end{aligned}$$

where n, k, i are integers. Now, it is easy to see that for continuously differentiable ω

$$q(t, \dot{\omega}) = 1 \Leftrightarrow \Delta\omega(0,t) > 0 \text{ and } \Delta\omega(t,L) \leq 0 \text{ and } A^-(t,\omega) \text{ and } A^+(t,\omega) \text{ are true.} \quad (20)$$

Now, since $\Delta\omega(s,t), \Delta\omega_{12}(s,t)$ are measurable functions $(s, t, \dot{\omega})$ and hence, for fixed s_i, x, y,

$$1_{\{\Delta\omega_{12}(s_i,t)\leq x\}\cup\{\Delta\omega(s_i,t)\leq y\}}(t,\omega)1_{\{\Delta\omega(0,t)>0\}\cap\{\Delta\omega(t,L)\leq 0\}}(t,\omega)$$

is a measurable function of (t, ω). Consequently $q(s, \dot{\omega})$ is measurable too and (0) is prooved.

We turn now to the proof of (I). First, note that the assumed existence of the density of $\dot{\omega}(t)$ implies that $P(T_\epsilon = t) = 0$ for any fixed $t \in (0, L)$. Next we turn to the condition that $P(Z_\epsilon = h) = 0$ for any fixed h and $\epsilon \geq 0$. Obviously, since the density of $\omega(0)$ and $\omega(L)$ exists then $P(T_\epsilon = 0, Z_\epsilon = h) = 0$ and $P(T_\epsilon = L, Z_\epsilon = h) = 0$. Now, since the density of $\omega(t)$ is bounded then employing Bulinskaya lemma for $\zeta(t) = \omega(t) + \epsilon t$ implies that for a fixed value h and ϵ

$$P(\omega(t) = h, \dot{\omega}(t) = \epsilon, \text{ for any } t \in [0, L]) = 0, \quad (21)$$

which implies $P(T_\epsilon \in (0, L), Z_\epsilon = h) = 0$, and hence (I) is proved.

We shall now proof (II). Denote by $a(\epsilon) = 1_{\{T_\epsilon(\omega)=0\}}(\omega)$ and $b(\epsilon) = 1_{\{T_\epsilon(\omega)=L\}}(\omega)$. Obviously the functions a, b are positive, bounded and non-decreasing, see Eqs. (18-19). What we need to show is that

$$\lim_{\epsilon\to 0} a(\epsilon) = 1_{\{T=0\}}, \quad \lim_{\epsilon\to 0} b(\epsilon) = 1_{\{T=L\}}.$$

In order to prove the convergence we split C into three subsets. First, consider ω such that $T(\omega) = 0$, then for all $\epsilon \geq 0$ we have $T_\epsilon(\omega) = 0$ and hence $a(\epsilon) = 1$ (similarly $b(\epsilon) = 0$). The second case is when $0 < T(\omega) < L$. Then there exist $t \in (0, L)$ such that $\dot{\omega}(t) = 0$ and $\omega(t) > \omega(0)$. Since $\omega(0) < \omega(t)$ then $a(\epsilon)$ tends to zero as ϵ goes to zero. We turn now to check the limit of $b(\epsilon)$. Clearly if $\omega(L) < \omega(t)$ then also $b(\epsilon)$ tends to zero. Consequently the only interesting case is when $\omega(t) = \omega(L)$. We shall show next that this can happen with probability zero. Again using Bulinskaya lemma we have that for any $\delta > 0$

$$P(\omega(s) - \omega(L) = 0, \dot{\omega}(s) = 0 \text{ for any } s \in [0, L - \delta]) = 0.$$

By taking a sequence of δ converging to zero we obtain that

$$P(\omega(s) - \omega(L) = 0, \dot{\omega}(s) = 0 \text{ for any } s \in [0, L)) = 0. \tag{22}$$

and hence with probability one we have $\omega(L) < \omega(t)$.

Finally let us consider the case $T(\omega) = L$, then $\omega(L) > \omega(s)$ for all $s \in [0, L)$ and hence $a(\epsilon) = 0$ and $b(\epsilon) = 1$. Consequently we proved that a.s.

$$\begin{aligned}
\lim_{\epsilon \to 0} 1_{\{\omega(0) \leq h\}} 1_{\{T_\epsilon = 0\}}(\omega) &= 1_{\{\omega(0) \leq h\}} 1_{\{T = 0\}}(\omega), \\
\lim_{\epsilon \to 0} 1_{\{\omega(0) \leq h\}} 1_{\{T_\epsilon = L\}}(\omega) &= 1_{\{\omega(L) \leq h\}} 1_{\{T = L\}}(\omega),
\end{aligned}$$

and (II) follows.

We turn to (III). For ω such that $\omega(0) = Z(\omega)$, then we have $T_\epsilon(\omega) = T(\omega)$ and $Z_\epsilon(\omega) = Z(\omega)$. Next consider ω such that $Z(\omega) = \omega(L) > \omega(0)$. Then from (22) it follows that $\omega(L) > \omega(s)$ for all $s \in [0, L)$ and hence $T_\epsilon(\omega) = T(\omega)$ while $Z_\epsilon(\omega) = Z(\omega)$. Finally, we consider ω such that $\omega(t) = Z(\omega) > \omega(0)$ for some $t \in (0, L)$. (From the previous case we have that a.s. $\omega(t) > \omega(L)$.) Now for a fixed $\delta > 0$, by a minor modification of the proof of Theorem 3.2.1 in Adler (1981) (with field $X(s,t) = \omega(t) - \omega(s)$), one can show that

$$P(\omega(t) - \omega(s) = 0, \dot{\omega}(s) = 0, \dot{\omega}(t) = 0 \text{ for any } (s,t) \in A_\delta) = 0.$$

Now by letting δ go to zero we obtain that

$$P(\omega(t) - \omega(s) = 0, \dot{\omega}(s) = 0, \dot{\omega}(t) = 0 \text{ for any } (s,t) \in [0, L] \times [0, L], t \neq s) = 0,$$

and hence, with probability one, there is at most one $t \in (0, L)$ such that $\omega(t) = Z(\omega)$. Consequently t is an unique global maximum on $[0, L]$ and by Lemma 2 $(T_\epsilon, Z_\epsilon) \to (T, Z)$. This completes proof of (III). $\square$

Proof of Theorem 10:

We begin with statement (A); Functions $f_\epsilon(t, x; N)$ are bounded by $f_\epsilon(t, x; 0)$. By means of Fubini's theorem, the integrability of $f_\epsilon(t, x; 0)$ is implied by (15). Since $q^N(t, \dot{\omega}) \geq q(t, \dot{\omega})$,

$q_\epsilon^N(0,\omega) \geq 1_{\{T_\epsilon=0\}}$ and $q_\epsilon^N(L,\omega) \geq 1_{\{T_\epsilon=L\}}$, are decreasing sequences of random variables, then by (16)

$$f_\epsilon(t,x;N) - f_\epsilon(t,x) \geq 0$$

converges to zero as N tends to infinity.

Proof of (B); Since the densities $f_{\dot\omega(s)-\dot\omega(t)}$, $f_{\omega(s)-\omega(t),\dot\omega(s)-\dot\omega(t)}$ are bounded then for a fixed t the probability of the statements (I) and (II) is one.

(I) The number of $s \in [0,L]$ such that $\dot\omega(t) = \dot\omega(s)$ and $\ddot\omega(s) = 0$ is at most one.

(II) The number of $s \in [0,L]$ such that $\omega(t) = \omega(s)$ and $\dot\omega(t) = \dot\omega(s)$ is one.

Now if a sample ω satisfies conditions (I) and (II), then

$$q(t,\dot\omega) = 1 \Leftrightarrow \omega(t) > \omega(0) \text{ and } \omega(t) \geq \omega(L) \text{ and } \bigwedge_{s\in[0,L]} \omega(s) \leq \omega(t) \text{ or } \dot\omega(s) \leq \dot\omega(t). \tag{23}$$

Further, if, for $t = 0$, a sample ω satisfying conditions (I) and (II) then statement

$$q_\epsilon(0,\omega) = 1 \Leftrightarrow \bigwedge_{s\in[0,L]} \omega(s) \leq \omega(L) \text{ or } \dot\omega(s) \leq \epsilon, \tag{24}$$

and for $t = L$

$$q_\epsilon(L,\omega) = 1 \Leftrightarrow \bigwedge_{s\in[0,L]} \omega(s) \leq \omega(L) \text{ or } \dot\omega(s) \leq \epsilon. \tag{25}$$

(The proof of statements (23-25) involves elementary manipulations of (17-19) and hence is omitted.) Consequently with probability one

$$\lim_{N\to\infty} (q^N(t,\dot\omega), q_\epsilon^N(0,\omega) q_\epsilon^N(L,\omega)) = (q(t,\dot\omega), 1_{\{T_\epsilon(\omega)=0\}}, 1_{\{T_\epsilon(\omega)=L\}}). \tag{26}$$

Thus by dominated convergence theorem

$$\lim_{N\to\infty} f_\epsilon(t,x;N) = f_\epsilon(t,x),$$

for almost all ϵ and hence (B) is proved. $\square$