Proceedings of OMAE'02
21st International Conference on Offshore Mechanics and Arctic Engineering
June 23-28, 2002, Oslo, , Norway

OMAE2002-28444

STATISTICAL PROPERTIES OF ENVELOPE FIELD FOR GAUSSIAN SEA SURFACE

Krzysztof Podgórski
Department of Mathematical Sciences
Indiana University – Purdue University Indianapolis
Indianapolis, Indiana 46202
Email: kpodgorski@math.iupui.edu

Igor Rychlik
Department of Mathematical Statistics
Lund University
Box 118
S-221 00 Lund, Sweden
Email: igor@maths.lth.se

ABSTRACT

The envelope process is a useful analytical tool which is often used to study wave groups. Most research on statistical properties of the envelope, and thus of wave groups, was focused on one dimensional records. However for the marine application, an appropriate concept should be two dimensional in space and variable in time. Although a generalization to higher dimensions was introduced by Adler (1978), little work was done to investigate its features. Since the envelope is not defined uniquely and its properties depend on a chosen version, we discuss the definition of the envelope field for a two dimensional random field evolving in time which serves as a model of irregular sea surface. Assuming Gaussian distribution of this field we derive sampling properties of the height of the envelope field as well as of its velocity. The latter is important as the velocity of the envelope is related to the rate at which energy is transported by propagating waves. We also study how statistical distributions of group waves differ from the corresponding ones for individual waves and how a choice of a version of the envelope affects its sampling distributions. Analyzing the latter problem helps in determination of the version which is appropriate in an application in hand.

NOMENCLATURE

$E(\mathbf{p}, t)$ Envelope field evaluated at a position $\mathbf{p}$ and a time instant t.
g Gravity acceleration.
R_i Rayleigh distributed random amplitudes.
$S(\omega, \theta)$ Unitary spectrum for the sea surface, $\omega \in \mathbb{R}$ is the time angular frequency and its sign indicates the direction of propagation, $\theta \in (-\pi, \pi]$ is the azimuth.
$\tilde{S}(\omega, \theta)$ Physical spectrum for the sea surface, $\omega > 0$, and $\theta \in (-\pi, \pi]$.
$T(\omega, \theta)$ Transformation of arguments of the directional spectrum of the sea surface to arguments of the three dimensional spectrum.
V Envelope speed along the principal wave direction.
$\mathbf{V}_\alpha$ Envelope velocity in the direction given by an azimuth α.
$W(\mathbf{p}, t)$ Sea surface elevation above the mean level at a position $\mathbf{p}$ and a time instant t.
$\hat{W}(\mathbf{p}, t)$ Hilbert transform of $W(\mathbf{p}, t)$.
W_u, W_{uv} Partial derivatives of $W(\mathbf{p}, t)$ with respect to variables $u, v = x, y, t$ which are put as subindices, the same convention for $\hat{W}$ and E.
ϵ_i Uniform random phases.
Γ^+ Symmetry set determining a version of the envelope in the directional spectrum domain.
λ_{ijk} Spectral moments of the sea surface W.
$\boldsymbol{\lambda}^T$ Transpose of a vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, the vector is treated here as a one-column matrix and thus its transpose is a one-row matrix.
Λ^+ Symmetry set determining a version of the envelope in the general three dimensional spectral domain.
μ^u Envelope u-level crossing contour intensity.
μ_0^u Envelope u-level crossing intensity along the principal wave direction.
$\sigma(\boldsymbol{\lambda})$ General three dimensional spectrum of a Gaussian field $W(\boldsymbol{\tau})$.

INTRODUCTION

In his pioneering work Longuet-Higgins (1957) has introduced the decomposition of traveling random waves into the envelope (low frequency varying amplitude) and the carrier (high frequency oscillations). Since then the envelope process was studied on various occasions, for example, in the problem of finding the distribution of the global maximum of the underlying process, in finding the distribution of the height of wave crest, and in analysis of statistical properties of wave groups. Despite that in the original work of Longuet-Higgins the envelope was proposed for moving random surface, most of the future work was done for the envelope of univariate random processes with the notable exception of Adler (1978,1981), where the definition and some properties of the bivariate envelope field were discussed. We extend this original work and focus on applications to studies of Gaussian sea surfaces.

There are several reasons for which studying the envelope $E(\mathbf{p},t)$ for the bivariate field $W(\mathbf{p},t)$ evolving in time is particularly important for marine applications. First, $E(\mathbf{p},t)$ if appropriate chosen, is smoother than the underlying $W(\mathbf{p},t)$ while at high levels it generally follows its shape. Thus the high levels are exceeded by one process if they exceeded by the other and they can be used equivalently to derive statistics of the high waves. More important however, is application of the envelope field to studies of wave groups. The later are defined as, roughly speaking, collections of waves with large ones in the center accompanied with small vanishing waves at the ends. The wave groups are observed in empirical data where often a high wave is preceded or succeeded by another wave which is higher than average. Properties of such groups are important for ocean engineers. For example, a group of waves can be responsible for a capsize of the ship if she will not regain stability between oncoming high waves in the group. It is often reported that groups of waves do more damage than waves of the same size but separated by smaller waves [see, for example, Burcharth (1980)]. This is partially explained by the fact that energy propagate with the rate corresponding to the speed of groups of waves. For deep water waves this rate is slower than the speed of individual waves and it can be demonstrated by physical arguments that for waves having narrow band spectra it is the envelope that is responsible for the transport of energy.

In truly two dimensional set-up where even individual waves are hard to describe a formal manner, it is difficult to introduce the notion of wave groups. On the other hand the envelope field is defined in an arbitrary dimension and its properties naturally extend from the one-dimensional case. Take for example the sea surface given by the swell spectrum shown in Figure 1, [details on the model of this spectrum can be found in Torsethaugen (1996)]. The difference between dynamics of surface and envelopes can be illustrated by recording contour movements in two time instants within $5[s]$. For each of the field, let us consider the contours crossing the significant wave height level (the signif-

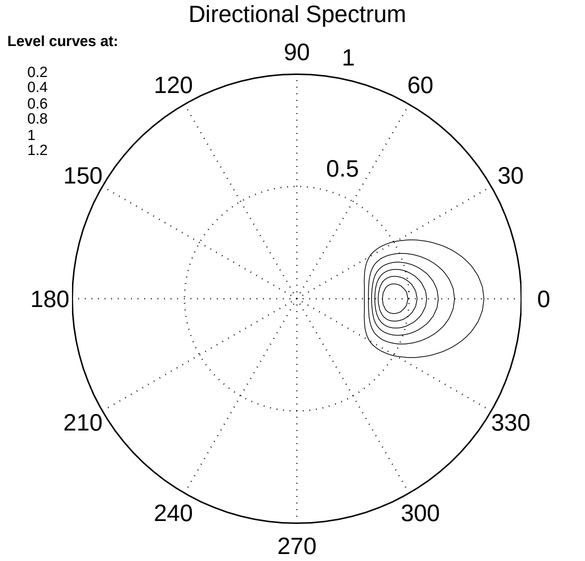

Figure 1. EXAMPLE OF SWELL SPECTRUM

icant wave height, in this case, is $2.2[m]$, thus the crossing level or the significant crest height is $1.1[m]$ above the mean sea level). For the sea surface, in order to obtain a more transparent picture of the contours we have also added crossing contours at the level equal to 90% of the significant wave height. Several important features can be noticed from the graphs presented in Figures 2, 5. First, the envelope field is indeed grouping the waves as its contours cover areas in which we observe clearly separated contours of the sea surface. Next, the displacements for the envelope are evidently smaller than for the sea surface – the envelope appears to move slower than the sea surface. Note also that waves entering the envelope contours are growing while these which are leaving are diminishing – expected behavior when the waves group are moving slower then the individual waves.

In this paper, we approach the observed and related features of the envelope in a systematic way using statistical properties of this multivariate field. We devote some space to discuss the definition of the multivariate envelope which does not lead to a unique concept. In fact there is certain freedom associated with a choice of the envelope. Thus we start with a discussion of the "proper" choice of the envelope for evolving sea surface. Later we derive some statistical distribution of "truly" spatial characteristics of the envelope and its velocities. Finally, these distributions are compared with the analogous distributions obtained for the underlying sea surface in both analytical and numerical manner. Numerical computations are performed for a Gaussian sea having a JONSWAP directional spectrum. Our numerical studies are supported by the MATLAB toolbox WAFO – Wave Analysis in Fatigue and Oceanography – containing a comprehensive package of numerical subroutines and programs for statistical analysis of random waves. This toolbox is available free of charge at `http//www.maths.lth.se/matstat/wafo`.

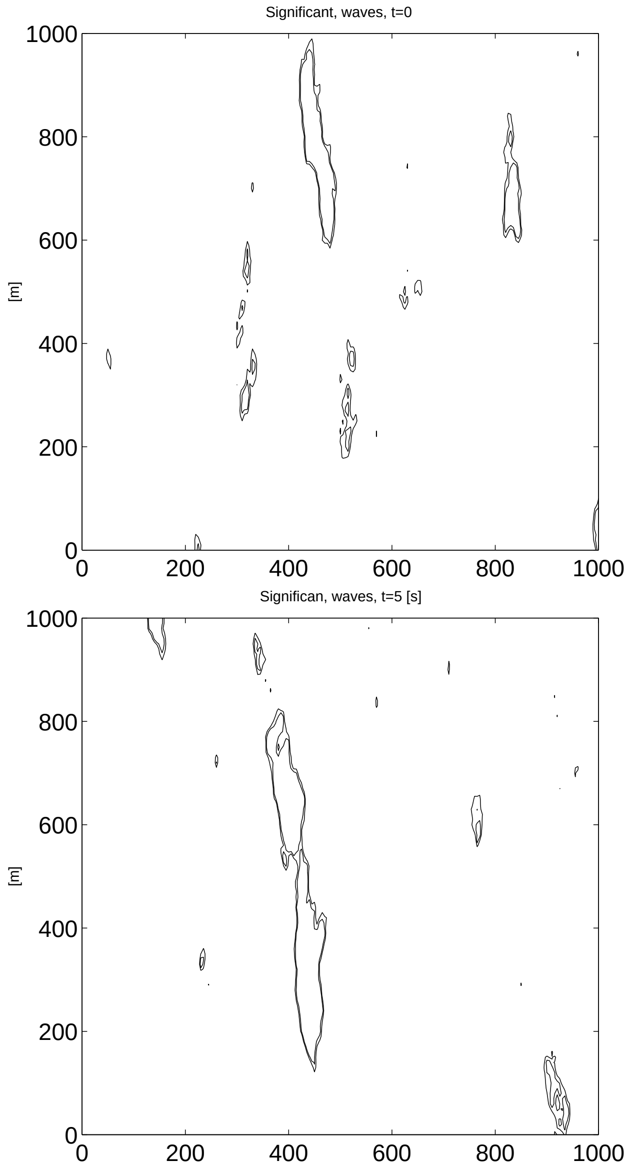

Figure 2. LEVEL CROSSING CONTOURS FOR SEA SURFACE

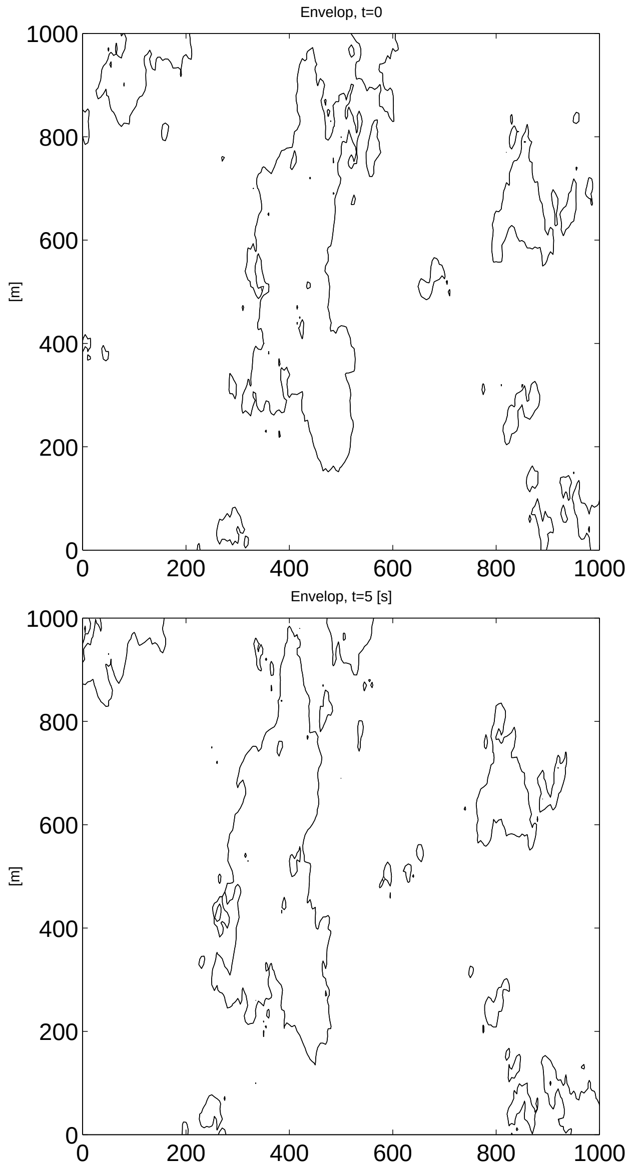

Figure 3. LEVEL CROSSING CONTOURS FOR ENVELOPE

DEFINITION OF ENVELOPE FIELD

Sea surface

Throughout the paper we assume that the sea surface is modeled by a homogeneous Gaussian field defined uniquely by its (continuous) spectrum for which one can take, for example, a JONSWAP spectrum. Although this model is intrinsically continuous, in order to give a more explicit definition of the envelope field let us consider its discretized version. The general case is discussed in Baxevani et al. (2002).

Consider a discrete set over the $\mathbb{R}^3$ given by $\{\boldsymbol{\lambda}_j\} = \{(\lambda_{1j}, \lambda_{2j}, \lambda_{3j})\}$, $j \in \mathbb{N}$, of which none is equal to zero. The spectrum of $W(\boldsymbol{\tau}) = W(\mathbf{p}, t)$ is given at $\boldsymbol{\lambda}_j$ by masses $\sigma(\boldsymbol{\lambda}_j)$. Since we deal with a real valued random field, the set has to be symmetric, i.e. if $\boldsymbol{\lambda}_j$ is in it, then also $-\boldsymbol{\lambda}_j$ must be there and both the frequencies have equal masses $\sigma(\boldsymbol{\lambda}_j) = \sigma(-\boldsymbol{\lambda}_j)$. The spectral representation of the sea surface is given by

$$W(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda^+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \cos(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j), \tag{1}$$

where Λ^+ is an arbitrary set in $\mathbb{R}^3$ such that $-\Lambda^+ \cap \Lambda^+ = \emptyset$ and $-\Lambda^+ \cup \Lambda^+ = \mathbb{R}^3/\{\mathbf{0}\}$. Moreover, $\{R_j\}$ and $\{\epsilon_j\}$ are two independent sequences of independent identically distributed random variables, the first one distributed according to the Rayleigh density $f(r) = re^{-r^2/2}$, $r > 0$, and the second one distributed uniformly on $[0, 2\pi]$.

Note that because of the symmetry of σ the statistical distributions of W do not depend on the choice of Λ^+. However, it is important to bear in mind that the choice of Λ^+ affects sampling properties of the envelope which is discussed later. Therefore the choice of Λ^+ becomes an important issue. Clearly, there are infinitely many possible such choices however, because of some additional symmetry of random sea surface, it is natural to take Λ^+ containing

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\right\}. \tag{2}$$

The spectral moments of W, if they are finite, are defined as

$$\lambda_{ijk} = 2\int_{\Lambda^+} \lambda_1^i \lambda_2^j \lambda_3^k d\sigma(\boldsymbol{\lambda}). \tag{3}$$

If $i+j+k$ is even, then λ_{ijk} does not depend on a choice of Λ^+ because of the symmetry of σ. However for odd $i+j+k$, different Λ^+ can, in general, lead to different λ_{ijk} which will be important when the distribution of the envelope field is discussed later in the paper.

For the sea surface, due to the dispersion relation, we have additionally

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(\omega, \theta) = \left(\frac{\omega^2}{g}\cos\theta, \frac{\omega^2}{g}\sin\theta, \omega\right),$$

where $\omega \in \mathbb{R}$ and $\theta \in (-\pi, \pi]$, while g is the gravity acceleration. Moreover,

$$\sigma(\boldsymbol{\lambda}) = S(\omega, \theta)\Delta\omega\Delta\theta,$$

where S is the unitary spectral density and $\Delta\omega$, $\Delta\theta$ are increments over a cell of the grid of ω's and θ' corresponding to $\{\boldsymbol{\lambda}_j\}$.

The problem with directional spectra for the sea surface is that they are degenerated in the full three dimensional space. This is due to the dispersion relation which reduce dimension of the spectral domain by one. In order to manipulate between the mathematically convenient three dimensional domain $\mathbb{R}^3$ of σ and the physically justified two dimensional domain $\mathbb{R}\times(-\pi,\pi]$ we define a natural transformation between these two domains

$$T(\omega, \theta) = \begin{cases} (-\omega, \theta - \pi) & \text{if } \theta \in [0, \pi], \\ (-\omega, \theta + \pi) & \text{if } \theta \in [-\pi, 0). \end{cases}$$

This a little bit technical definition can be interpreted more intuitively if we consider $\mathbb{R}\times(-\pi,\pi]$ as the union of two polar coordinate systems one corresponding to positive ω, $[0,\infty)\times(-\pi,\pi]$, and other, the anti-system $(-\infty, 0]\times(-\pi,\pi]$, corresponding to negative ω. The transformation T takes a point from one system to the anti-system and then rotates it there by π. The original symmetry condition for σ translates now to

$$S(T(\omega, \theta)) = S(\omega, \theta). \tag{4}$$

Further, the set Λ^+ corresponds in the reduced domain to $\Gamma^+ = \boldsymbol{\lambda}^{-1}(\Lambda^+)$ and a choice of $\Lambda^+ \subseteq \mathbb{R}^3$ is equivalent to a choice of $\Gamma^+ \subset \mathbb{R}\times(-\pi,\pi]$ such that

$$\begin{aligned} &T(\Gamma^+) \cap \Gamma^+ = \emptyset \\ &T(\Gamma^+) \cup \Gamma^+ = \mathbb{R}/\{0\} \times (-\pi, \pi]. \end{aligned}$$

Note, that the unitary spectrum is uniquely related to the physical spectrum $\tilde{S}(\omega, \theta)$, $\omega > 0$, which is more frequently seen in engineering applications, by

$$\tilde{S}(\omega, \theta) = 2S(\omega, \theta), \omega > 0.$$

Because of mathematical convenience throughout the paper we often use the unitary spectrum instead of the physical spectrum.

Envelope field

The Hilbert transform of the process W given by (1) has the form

$$\hat{W}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\lambda}_j \in \Lambda^+} \sqrt{2\sigma(\boldsymbol{\lambda}_j)} R_j \sin(\boldsymbol{\lambda}_j^T \boldsymbol{\tau} + \epsilon_j).$$

The Hilbert transform has the same unitary spectrum and thus the same distribution as the original field. Also at each fixed point $(\mathbf{p}, t)$ the two are independent. However treated as stochastic fields they are dependent. For example, the covariances between derivatives of W and $\hat{W}$ are given through the spectral moments of W as, for example,

$$Cov(\hat{W}_x, W) = \lambda_{100} = -Cov(\hat{W}, W_x).$$

We have already remarked that these covariances are affected by a choice of Λ^+ and thus so is the dependence structure of W and $\hat{W}$.

The real envelope process $E(\mathbf{p}, t)$ is defined as

$$E(\boldsymbol{\tau}) = \sqrt{W(\boldsymbol{\tau})^2 + \hat{W}(\boldsymbol{\tau})^2}. \tag{5}$$

Note, that the envelope field is positive and always stays above the sea surface and it is also depending on a choice of Λ^+ because of the dependence between W and $\hat{W}$. The following two subsections illustrates importance of the choice of Λ^+.

"Narrow-banded" example The following example is often used to illustrate the concept of the envelope for narrow banded processes. Assume that Λ^+ consists of the two atoms $\boldsymbol{\lambda}-\boldsymbol{\delta}$, $\boldsymbol{\lambda}+\boldsymbol{\delta}$ and $\sigma(\boldsymbol{\lambda}-\boldsymbol{\delta})=\sigma(\boldsymbol{\lambda}+\boldsymbol{\delta})=1/2$. We have

$$\begin{aligned}
W(\boldsymbol{\tau}) &= R_1\cos\left[(\boldsymbol{\lambda}+\boldsymbol{\delta})^T\boldsymbol{\tau}+\epsilon_1\right]+R_2\cos\left[(\boldsymbol{\lambda}-\boldsymbol{\delta})^T\boldsymbol{\tau}+\epsilon_2\right]\\
&= 2R_1\cos(\boldsymbol{\lambda}^T\boldsymbol{\tau}+\bar\epsilon)\cos(\boldsymbol{\delta}^T\boldsymbol{\tau}+\bar\epsilon)+\\
&\quad +(R_2-R_1)\cos\left[(\boldsymbol{\lambda}-\boldsymbol{\delta})^T\boldsymbol{\tau}+\epsilon_2\right]\\
\hat W(\boldsymbol{\tau}) &= R_1\sin\left[(\boldsymbol{\lambda}+\boldsymbol{\delta})^T\boldsymbol{\tau}+\epsilon_1\right]+R_2\sin\left[(\boldsymbol{\lambda}-\boldsymbol{\delta})^T\boldsymbol{\tau}+\epsilon_2\right]\\
E(\boldsymbol{\tau}) &= \sqrt{(R_1-R_2)^2+4R_1R_2\cos^2(\boldsymbol{\delta}^T\boldsymbol{\tau}+\bar\epsilon)},
\end{aligned}$$

where $\bar e=(e_1+e_2)/2$.

Since (R_2-R_1) is small relatively to R_1 with large probability, we often observe

$$W(\boldsymbol{\tau}) \approx 2R_1\cos(\boldsymbol{\lambda}^T\boldsymbol{\tau}+\bar\epsilon)\cos(\boldsymbol{\delta}^T\boldsymbol{\tau}+\bar\epsilon) \quad (6)$$

$$E(\boldsymbol{\tau}) \approx 2R_1|\cos(\boldsymbol{\delta}^T\boldsymbol{\tau}+\bar\epsilon)|. \quad (7)$$

If we assume that $|\boldsymbol{\delta}|$ is essentially smaller than $|\boldsymbol{\lambda}|$, then the signal $X(\boldsymbol{\tau})$ is a cosine function corresponding to the high frequency $|\boldsymbol{\lambda}|$ and modulated by a cosine amplitude having the low frequency $|\boldsymbol{\delta}|$. We see that the envelope coincides with this low frequency varying amplitude. This simple example illustrates the usual interpretation of the envelope process as a process which is governing slow frequency modulation of amplitudes of high frequency components in the signal.

We conclude this example with analysis what can happen if we choose a wrong version of the envelope. Clearly, for the two atoms $\boldsymbol{\lambda}-\boldsymbol{\delta}$, $\boldsymbol{\lambda}+\boldsymbol{\delta}$, in the full spectral domain there exist the two anti-atoms $\boldsymbol{\delta}-\boldsymbol{\lambda}$, $-\boldsymbol{\lambda}-\boldsymbol{\delta}$, and we can choose Λ^+ in such a way that now $\boldsymbol{\delta}-\boldsymbol{\lambda}$, $\boldsymbol{\lambda}+\boldsymbol{\delta}$ belong to it. The whole above analysis remains valid except that now we switch $\boldsymbol{\delta}$ with $\boldsymbol{\lambda}$. Consequently, now the envelope will by modulated by high frequency $|\lambda|$ and thus its usual interpretation as the low frequency component fails completely.

Crossings intensity in the principal wave direction

Let us consider the intensity of crossings of a u-level by the envelope in the direction $y=0$. It follows from the Rice formula that this intensity is given by

$$\mu_0^u = E(|E_x(\mathbf{0})||E(\mathbf{0})=u)\cdot\frac{u}{\lambda_{000}}e^{-u^2/(2\lambda_{000})}, \quad u>0.$$

Straightforward computations of the above conditional expectation lead to

$$\mu_0^u=\sqrt{\frac{2\lambda_{200}}{\pi}}\sqrt{1-\frac{\lambda_{100}^2}{\lambda_{000}\lambda_{200}}}\cdot\frac{u}{\lambda_{000}}e^{-u^2/(2\lambda_{000})}, \quad u>0.$$

[See also Lindgren (1989).] The highest intensity is reached for the level $u=\sqrt{\lambda_{000}}$ often called the reference level for the envelope, and is equal

$$\mu_0^u=\sqrt{\frac{2}{\pi\cdot e}}\frac{1}{\lambda_{000}}\sqrt{\lambda_{200}\lambda_{000}-\lambda_{100}^2}.$$

We observe that the intensity of envelope crossing in the direction $y=0$ depends on the choice of Λ^+ only through λ_{100} and in such a way that larger $|\lambda_{100}|$ corresponds to lower crossing intensity. We expect that the envelope to smooth the sea surface so the goal is to obtain low crossing intensity. We reduce this problem to minimizing the spectral functional

$$\lambda_{100}=2\int_{\Lambda^+}\lambda_1\,d\sigma(\boldsymbol{\lambda})=2\int_{\Gamma^+}\frac{\omega^2}{g}\cos\theta S(\omega,\theta)\,d\omega d\theta.$$

In general, the optimal choice will depend on the form of a spectrum in hand. For example, consider a spectrum having symmetry properties similar to these exhibited in Figure 4 by the JONSWAP spectrum used in the examples. Considering the symmetry given by (4), it is rather obvious that the choice of $\Gamma^+=\{(\omega,\theta):\theta\in(-\frac{\pi}{2},\frac{\pi}{2}],\omega\in\mathbb{R}/\{0\}\}$ is the optimal in such a situation.

For comparison, the natural choice corresponding to (2) will result in λ_{100} smaller by

$$-4\int_{\omega>0}\int_{\theta\in(-\pi,-\pi/2]\cup(\pi/2,\pi]}\frac{\omega^2}{g}\cos\theta S(\omega,\theta)\,d\omega d\theta.$$

This is essentially negligible if we deal with directional spectra obtained by vanishing spreading functions. Such spectra are, for all practical purposes, equal to zero for $\theta\in(-\pi,-\pi/2]\cup(\pi/2,\pi]$. However this condition will no longer be true if, for example, there will be an additional swell portion of the spectrum corresponding to these values of azimuth θ.

Finally, consider $\Lambda^+=\{(x_1,x_2,x_3),x_2>0\}$. If spectrum S is symmetric with respect to θ, then it is clear that in this case λ_{100} is equal to zero. Thus the intensity of envelope crossing is equal to $\sqrt{\frac{2}{\pi e}}\sqrt{\lambda_{200}/\lambda_{000}}$. For comparison, the intensity crossing of the sea surface at the reference level zero is given by $\frac{1}{\pi}\sqrt{\lambda_{200}/\lambda_{000}}$, thus the ratio of this intensities is equal to $\sqrt{2\pi/e}\approx 1.52$. This "wrong" envelope has by 50% more crossings than the sea surface.

STATISTICAL DISTRIBUTIONS FOR ENVELOPE

Intensity of envelope contours

In the previous sections we have computed intensity of the crossings by the envelope along the principle wave direction.

Table 1. VERSIONS OF ENVELOPE IN TERMS OF Λ^+ and Γ^+.

#	Λ^+	Γ^+	Comment
1	$x_1 > 0$	$\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$	Maximizes λ_{100},
2	$x_2 > 0$	$\theta \in (0, \pi]$	$\lambda_{100} = 0$,
3	$x_3 > 0$	$\omega > 0$	Natural choice for the sea surface.

Since we are studying the envelope field it is more natural to consider the intensity of the contour lines rather than individual crossings along a straight line. It follows from generalized Rice's formula that this intensity is given by

$$\mu^u = E\left(\sqrt{E_x^2 + E_y^2}|E(\mathbf{0}) = u\right)\frac{u}{\lambda_{000}}e^{-u^2/(2\lambda_{000})}, \quad u > 0.$$

After some rather standard calculations involving the covariances between W, $\hat{W}$, W_x, W_y, $\hat{W}_x$, and $\hat{W}_y$, one can obtain that this intensity is given by

$$\mu^u = E\left(\sqrt{X_1^2 + X_2^2}\right)\frac{u}{\lambda_{000}}e^{-u^2/(2\lambda_{000})}, \quad u > 0,$$

where (X_1, X_2) is a Gaussian vector with variances $\lambda_{200} - \lambda_{100}^2/\lambda_{000}$, $\lambda_{020} - \lambda_{010}^2/\lambda_{000}$, respectively, and the covariance equal to $\lambda_{110} - \lambda_{100}\lambda_{010}/\lambda_{000}$. Thus functional form of this intensity is identical to the intensity of crossing along a line although constants are different. In the special case of the sea surface we obtain some simplifications. First, it is usually assumed that the coordinates system for $W(\boldsymbol{\tau})$ is taken in such a way that $\lambda_{110} = 0$. Moreover, for spectra $S(\omega, \theta)$ exhibiting symmetry with respect to θ (for example, for the spectra shown in Figures 1 and 4) we have also $\lambda_{010} = 0$. Consequently, X_1 and X_2 are independent Gaussian with variances $\lambda_{200} - \lambda_{100}^2/\lambda_{000}$ and λ_{020}, respectively.

Velocity

There are variety of ways to introduce a concept of velocity for moving surfaces [see Baxevani et al. (2002)]. We focus here on the velocity describing the motion of a contour level in the specified direction given by an azimuth α. We define this velocity by the equations

$$\begin{bmatrix} E_x & E_y \\ -\sin\alpha & \cos\alpha \end{bmatrix}\mathbf{V}_\alpha = -\begin{bmatrix} E_t \\ 0 \end{bmatrix}, \qquad (8)$$

where the first equation in the system guarantees that the motion following $\mathbf{V}_\alpha$ stays on the same envelope level and in this sense describes motion of the constant level contours, while the second equation implies that the velocity points always in the direction α, so the motion is along a straight line if α is constant.

Let us assume that $\alpha = 0$, i.e. that we are interested in the constant direction coinciding with the principle direction of waves. It follows from (8) and (5) that the speed $V = |\mathbf{V}_\alpha|$ is given by

$$V = -\frac{W_t \cdot W + \hat{W}_t \cdot \hat{W}}{W_x \cdot W + \hat{W}_x \cdot \hat{W}}. \qquad (9)$$

"Narrow banded" example continued Let us assume that in this example we have $\boldsymbol{\lambda} - \boldsymbol{\delta} = (\omega_1^2/g, 0, \omega_1)$ and $\boldsymbol{\lambda} + \boldsymbol{\delta} = (\omega_2^2/g, 0, \omega_2)$, where $\omega_1 = \omega - \delta$ and $\omega_2 = \omega + \delta$ for some $\delta > 0$. Using the approximation (6) we obtain that the high frequency modulation speed, i.e. the speed of individual waves, is given by

$$V_W = g\frac{2\omega}{2\omega^2 + 2\delta^2},$$

while by (7) the envelope is propagating with the speed

$$V = g\frac{1}{2\omega}.$$

This illustrative example demonstrates that the speed ratio $V_W/V = \frac{1}{1+\delta^2/\omega^2} < 2$ and is approximately equal to 2 if $\delta^2 \ll \omega^2$. As we will see this result extends also to the envelope generated by JONSWAP spectra.

We are interested in the statistical distribution of V when measured at an arbitrarily selected point on the sea as well as the so-called biased distribution obtained by measuring this velocity on the fixed level contour. It is well known that these two distributions are essentially different, the first one is simply the distribution of random variable V while the second one has to be computed with a use of generalized Rice's formula. Relatively straightforward although tedious calculations [see Baxevani et al. (2002)] lead to the following form of the distributions of velocity V in the direction of the line $y = 0$:

$$-a \cdot \left(b + \sqrt{c \cdot d - e^2}\frac{X}{Y}\right),$$

$$\begin{aligned}
a &= \frac{1}{\lambda_{200} - \lambda_{100}^2/\lambda_{000}}, \\
b &= \lambda_{101} - \lambda_{100}\lambda_{001}/\lambda_{000}, \\
c &= \lambda_{200} - \lambda_{100}^2/\lambda_{000}, \\
d &= \lambda_{002} - \lambda_{001}^2/\lambda_{000}, \\
e &= \lambda_{101} - \lambda_{100}\lambda_{001}/\lambda_{000},
\end{aligned}$$

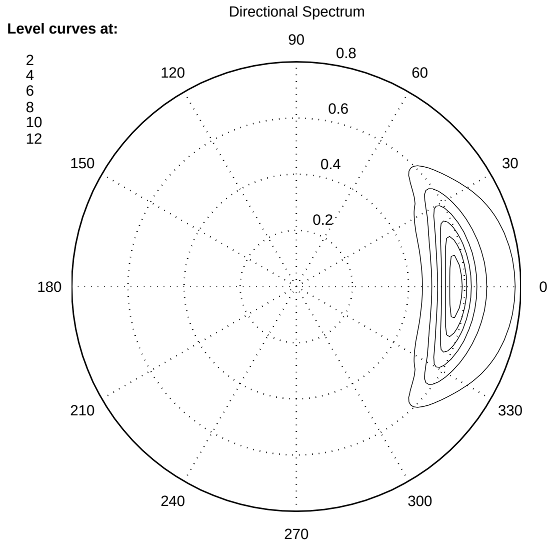

Figure 4. DIRECTIONAL JONSWAP SPECTRUM USED IN EXAMPLES.

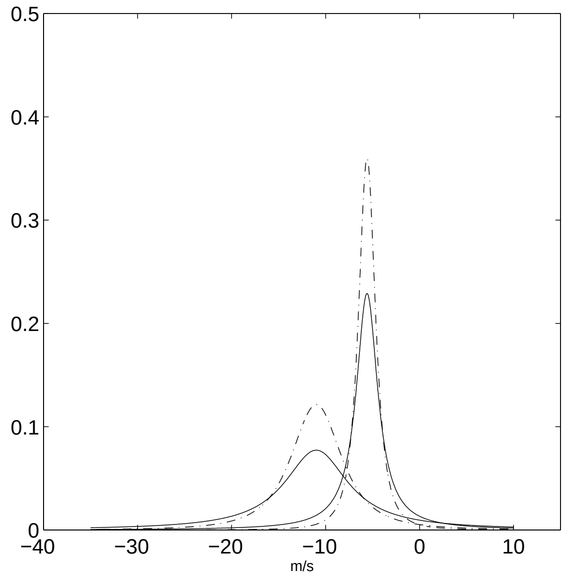

Figure 5. DISTRIBUTIONS OF VELOCITIES FOR ENVELOPE AND SEA SURFACE.

variables X and Y are independent, X having the standard normal distribution while the distribution of Y is

a) the standard normal if we deal with the unbiased sampling,

b) the Rayleigh distribution if we deal with the biased sampling distribution of velocity V sampled at points $(\mathbf{p},t)$ such that $E(\mathbf{p},t)=u$.

In the above, λ_{ijk} are spectral moments of $W(\mathbf{p},t)$ as defined by (3).

For comparison, the analogous velocity of the sea surface has the same form but with the constants $a=1/\lambda_{200}$, $b=\lambda_{101}$, $c=\lambda_{200}$, $d=\lambda_{002}$, and $e=\lambda_{101}$.

Notice for the second choice in Table 1 on for JONSWAP type spectra, i.e. $\lambda_{100}=0$, these coefficients coincide with the ones for the envelope. Thus statistically velocities of the envelope and of individual waves are identical which again demonstrates how bad things can go with a wrong choice of the envelope.

It is interesting that both the velocities have the biased sampling distributions on u-level contours which do not depend on the level u, i.e. they are the same independently of the elevation at which the velocity is measured.

EXAMPLES

In this section we consider the directional Gaussian sea surface obtained from the JONSWAP spectrum $\tilde{S}(\omega,\theta) = S(\omega)D(\omega,\theta)$, where

$$S(\omega) = g^2\frac{\alpha}{\omega^5}e^{-1.25\omega_p^4/\omega^4}\rho^{\psi(\omega)},$$

with $\psi(\omega)=e^{-(\omega-\omega_p)^2/(2\sigma^2\omega_p^2)}$, where σ is a jump function of ω:

$$\sigma = \begin{cases} 0.07 & \text{if } \omega/\omega_p \le 1, \\ 0.09 & \text{if } \omega/\omega_p > 1. \end{cases}$$

and α is a scale, ρ controls the shape, and ω_p is the peak frequency. The spreading function is given by $D(\omega,\theta) = G_0\cos^{2c}(\theta/2)$. The alternative data driven parameters can be introduced by the relations [see Goda (1990)]:

$$\alpha = \beta_J H_{1/3}^2\omega_p^4,$$

and

$$\beta_J = \frac{0.06238(1.094-0.01915\log\rho)}{0.23+0.0336\rho-0.185(1.9+\rho)^{-1}},$$
$$\omega_p = 2\pi\frac{1-0.132(\rho+0.2)^{-0.559}}{T_{1/3}},$$

where $H_{1/3}$ is the significant waves height, $T_{1/3}$ their average period. The following values of the parameters where assumed

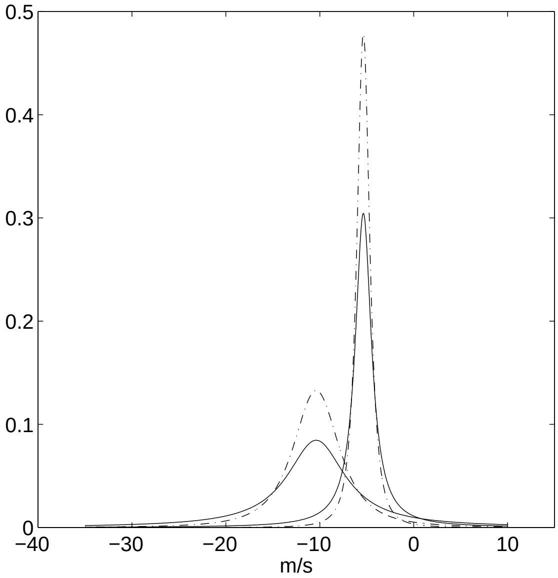

Figure 6. DISTRIBUTIONS OF VELOCITIES FOR CLASSICAL ENVELOPE AND ONE DIMENSIONAL SEA RECORD.

in the computed examples: $H_{1/3} = 7[m]$, the peak period $2\pi/\omega_p = 11[s]$, $\rho = 2.3853$, $c = 15$. The spectrum is shown in Figure 4. For the envelope we have chosen Λ^+ given in (2) which corresponds to $\Gamma^+ = (0,\infty) \times (-\pi, \pi]$.

In Figure 5, we present the unbiased and biased sampling distributions of velocities both for the envelope and for the sea surface. The solid lines represent the unbiased densities and the dashed-dotted ones corresponds to the biased sampling densities. We see that the biased sampling distribution which are more important for applications, are more concentrated around its center. The group velocity is smaller than that of individual waves as it is observed in the real life records. The peaks are at $-5.58[m/s]$ and $-10.98[m/s]$. Thus waves are roughly twice as fast as groups, the result in agreement with conclusions of the narrow banded example.

CONCLUSION

It is important to realize that even for studying the statistical properties of sea surface in the direction along $y = 0$, i.e. of $W(x,0,0)$, the envelope field is a different concept from the envelope process defined for one dimensional record $W(x,0,0)$, the latter often used in ocean engineering for analysis of wave movements. Indeed, we have computed also the distribution of velocities for the classical one dimensional envelope and in our case the resulting distributions are presented in Figure 6. As we can observe, the distributions are not identical, the one dimensional record distributions are slightly more peaky.

From the formal point of view the multidimensional envelope is not essentially harder to study than its one-dimensional version. Thus for the sea surface which is a three dimensional field, it is more appropriate but also manageable to study effectively such "fully" dimensional objects and concepts as wave contours, envelope contours, or vector velocities. In this work we demonstrate this for few simple examples. Through similar approach one can tackle many other important "multidimensional" problems. This is however left for future studies.

ACKNOWLEDGMENT

The second author would like to thank the Knut and Alice Wallenberg Foundation through the Gothenburg Stochastic Centre for providing partial support when he was working on this paper.

REFERENCES

Adler, R.J. (1978) On the envelope of a Gaussian random field, *J. Appl. Prob.* **15**, pp. 502-513.

Adler, R.J. (1981) *The geometry of random fields*, Wiley, New York.

Baxevani, A., Podgórski, K., Rychlik, I. (2002) Velocities for Random Surfaces, *Lund Univ. Technical Rep.*, pp. 1-49.

Burcharth, H.F. (1980) A comparison of natural waves and model waves with special reference to wave grouping, *Proc. 17th Conf. on Coastal Eng., Sydney, Australia*, **3**, *Lund Univ. Technical Rep.*, pp. 2992-3009.

Goda, Y. (1990) Random waves and spectra. *Handbook of Coastal and Ocean Engineering*, **1**, pp. 175-209.

Lindgren, G. (1989) Slepian models for χ^2-processes with dependent components with applications to envelope upcrossing. *J. Appl. Prob.*, **26**, pp. 36-49.

Longuet-Higgins, M. S. (1957) The statistical analysis of a random, moving surface. *Phil. Trans. Roy. Soc. A*, **249**, pp. 321-387.

Torsethaugen, K. (1996) Model for a doubly peaked wave spectrum Report No. STF22 A96204. *SINTEF Civil and Environm. Engineering*, Trondheim.