Rational points on weighted plane curves

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Let $w = (w_1, \ldots, w_n)$ be an n–tuple of positive integers. The w–degree of a monomial $x^m = x_1^{m_1} \cdots x_n^{m_n}$ is

$$\deg_w(x^m) = (w_1 m_1 + \cdots + w_n m_n)/\operatorname{lcm}(w_1, \ldots, w_n).$$

For integers $r \geq 0$, let S_r be the free $\mathbb{Z}$–module spanned by monomials $x^m = x_1^{m_1} \cdots x_n^{m_n}$ of w–degree r. The direct sum $\bigoplus_{r \geq 0} S_r$ is then a commutative graded ring which we denote by S. The weighted projective space $\mathbb{P}(w)$ with weights $w = (w_1, \ldots, w_n)$ is defined to be $\operatorname{Proj}(S)$. There are other equivalent definitions (see e.g. [3, 7]). If K is a field, then K–points of $\mathbb{P}(w)$ correspond to equivalence classes of non-zero homomorphisms $\Gamma \to K$, where Γ is the multiplicative semigroup of monomials in S. Two such homomorphisms $\psi_1 : \Gamma \to K$ and $\psi_2 : \Gamma \to K$ are equivalent if

$$\psi_1(x^m) = \alpha^{\deg_w(x^m)} \psi_2(x^m)$$

for some $\alpha \in K^*$. If K is a number field, then there is a natural toric height function $H : \mathbb{P}(w)(K) \to \mathbb{R}_{>0}$ defined in the following way (see [1, 9]). Let

$$l = \operatorname{lcm}(w_1, \ldots, w_n),$$
$$l_i = l/w_i, \quad i = 1, 2, \ldots, n.$$

Let $\psi : \Gamma \to K$ be a homomorphism representing $P \in \mathbb{P}(w)(K)$. Then

$$H(P) = \prod_v \sup_{1 \leq i \leq n} \left| \psi(x_i^{l_i}) \right|_v = \prod_v \sup_{x^m \in S_r} |\psi(x^m)|_v^{1/r}$$

for any positive integer r. Here v runs over all places of K and $|\ |_v$ are normalised such that $\prod_v |\alpha|_v = 1$ for all $\alpha \in K^*$. It is clear from the product formula that $H(P)$ is independent of the choice of $\psi : \Gamma \to K$. In this paper we are only concerned with $\mathbb{Q}$–points. In that case we may write

$$H(P) = \sup_{0 \leq i \leq n} \left| \psi(x_i^{l_i}) \right|,$$

where $\psi : \Gamma \to \mathbb{Z}$ is a representative of $P \in \mathbb{P}(w)(\mathbb{Q})$ such that

$$\gcd(\psi(x_1^{l_1}), \ldots, \psi(x_n^{l_n})) = 1.$$

We shall assume throughout the paper that $\mathbb{Q}$–points are always represented by such "primitive" $\psi : \Gamma \to \mathbb{Z}$.

For any subscheme $X \subset \mathbb{P}(w)$ we may define a counting function

$$N(X, B) = \#\{P \in X(\mathbb{Q}) \ : \ H(P) \leq B\}.$$

We may also consider the cardinality

$$N(X; \mathbf{B}) = \#\left\{P \in X(\mathbb{Q}) \ : \ \left|\psi(x_i^{l_i})\right| \leq B_i \ (0 \leq i \leq n)\right\},$$

where $\mathbf{B} = (B_1, \ldots, B_n)$. The following result is a generalisation of a theorem due to Heath-Brown [6]. He considers the case $w = (1, 1, 1)$.

Theorem. *Let $X \subset \mathbb{P}(w_1, w_2, w_3)$ be defined by $F = 0$, where $F(x) = \sum_m a_m x^m \in S_d$ is irreducible over $\mathbb{Q}$. Let $\varepsilon > 0$, $B_1, B_2, B_3 \geq 1$ be given and put*

$$V = B_1 B_2 B_3 \quad \text{and} \quad T = \sup_{\substack{x^m \in S_d \\ a_m \neq 0}} (B_1^{w_1 m_1} B_2^{w_2 m_2} B_3^{w_3 m_3})^{1/l}.$$

Then

$$N(X; \mathbf{B}) \ll_{w,d,\varepsilon} \left(T^{-1/d^2} V^{1/d+\varepsilon}\right)^{\frac{w_1 w_2 w_3}{g l^2}},$$

where $g = \gcd(w_1, w_2, w_3)$. In particular, if $B \geq 1$, then

$$N(X, B) \ll_{w,d,\varepsilon} \left(B^{2/d+\varepsilon}\right)^{\frac{w_1 w_2 w_3}{g l^2}}.$$

The rest of this paper is devoted to the proof of this result. We shall use the conventions from [6] that the implied constants may depend on w and d, and that $\|G\|$ denotes the the maximum modulus of the coefficients of a polynomial $G \in \mathbb{Z}[x_1, \ldots, x_n]$. The proof is similar to Heath-Brown's proof in the case $w = (1, 1, 1)$.

For any prime p we have a map $X(\mathbb{Q}) \to X_p(\mathbb{F}_p)$ where $X_p = X \otimes \mathbb{F}_p$. It is given by $\psi \mapsto \overline{\psi}$, where $\psi : \Gamma \to \mathbb{Z}$ represents $P \in X(\mathbb{Q})$ and $\overline{\psi} : \Gamma \to \mathbb{F}_p$ is the composition of $\psi : \Gamma \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{F}_p$. Remember that we assume that

$$\gcd(\psi(x_1^{l_1}), \psi(x_2^{l_2}), \psi(x_3^{l_3})) = 1$$

so $\overline{\psi} : \Gamma \to \mathbb{F}_p$ is non-zero. Let

$$S(X; \mathbf{B}, p) = \left\{P \in X(\mathbb{Q}) \ : \ \left|\psi(x_i^{l_i})\right| \leq B_i \ (1 \leq i \leq 3), \overline{P} \in (X_p)_{\mathrm{sm}}(\mathbb{F}_p)\right\},$$

and

$$S(X;\mathbf{B}) = \left\{ P \in X(\mathbb{Q}) \ : \ \left|\psi(x_i^{l_i})\right| \leq B_i \ (1 \leq i \leq 3), P \in X_{\mathrm{sm}}(\mathbb{Q}) \right\},$$

where X_{sm} is the smooth locus of X. Note that

$$N(X;\mathbf{B}) = \#S(X;\mathbf{B}) + O(1) \tag{1}$$

since $\#X_{\mathrm{sing}}(\mathbb{Q}) = O(1)$ for the singular locus X_{sing} of X.

Lemma 1. *Let r be the largest integer less than $\log(\|F\|\, V)$, where $V = B_1B_2B_3$. If $A \geq \log^2(\|F\|\, V)$, then there are distinct primes $p_1, \ldots, p_r$ such that and $A \ll p_i \ll A$ for $i = 1, 2, \ldots, r$ and*

$$S(X;\mathbf{B}) = \bigcup_{i=1}^{r} S(X;\mathbf{B}, p_i).$$

Proof. Let $y_0, \ldots, y_n$ be all the monomials of S_D for some D. Assume that

$$\mathbb{P}(w) \to \mathbb{P}^n, \quad (x_0, x_1, x_2) \mapsto (y_0, \ldots, y_n)$$

is an embedding and let Y be the image of X. Then, by the Jacobian criterion, Y_{sing} is defined by

$$G_1 = \cdots = G_k = 0$$

for some forms $G_i \in \mathbb{Z}[y_0, \ldots, y_n]$ which satisfy $\log\|G_i\| \ll \log\|F\|$ and $\deg(G_i) \ll \deg_w(F)$. Hence,

$$\log|G_i(\psi(y_0), \ldots, \psi(y_n))| \ll \log(\|F\|\, V),$$

if $\psi : \Gamma \to \mathbb{Z}$ represents $P \in S(X;\mathbf{B})$.

Let $p_1, \ldots, p_r$ be the first primes larger than cA for some constant c. Since $A \gg r^2$ this yields $p_i \ll cA$ for $i = 1, 2, \ldots, r$. Let $Q \in Y_{\mathrm{sm}}(\mathbb{Q})$ be the image of $P \in S(X;\mathbf{B})$. Then Q is represented by

$$\mathbf{y} = (\psi(y_0), \ldots, \psi(y_n)) \in \mathbb{Z}^{n+1}.$$

Moreover, $\overline{Q} \in Y_p(\mathbb{F}_p)$ is represented by $\overline{\mathbf{y}} \in \mathbb{F}_p^{n+1}$ since

$$\gcd(\psi(y_0), \ldots, \psi(y_n)) = 1.$$

Without loss of generality we may assume that $G_1(\mathbf{y}) \neq 0$. If $p_i \mid G_1(\mathbf{y})$ for $i = 1, 2, \ldots, r$, then

$$r\log(cA) \leq \log|G_1(\mathbf{y})| \ll \log(\|F\|\, V) \leq r.$$

This is contradictory for some suitable choice of $c = c(w,d)$. Hence, $\overline{Q} \in (Y_p)_{\mathrm{sm}}(\mathbb{F}_p)$ for some $p \in \{p_1, \ldots, p_r\}$. That is, $\overline{P} \in S(X;\mathbf{B}, p_i)$ for some $i = 1, 2, \ldots, r$. □

Lemma 2. *Let $Y \subset \mathbb{P}(w)$ be the intersection $F = G = 0$ for some $F, G \in S$. If F and G do not have a common factor in $\mathbb{Q}[x_1, x_2, x_3]$, then*

$$\#Y(\mathbb{Q}) \leq l^2(\deg_w(F))(\deg_w(G)).$$

Proof. Let $\pi : \mathbb{P}^2 \to \mathbb{P}(w)$ be given by $(x_1, x_2, x_3) \mapsto (x_1^{w_1}, x_2^{w_2}, x_3^{w_3})$. Then the forms $\pi^*F, \pi^*G \in \mathbb{Z}[x_1, x_2, x_3]$ have degrees $l \deg_w(F)$ and $l \deg_w(G)$, respectively. The lemma thus follows from Bezout's theorem provided that π^*F and π^*G do not have a common factor. One can check that a common factor of π^*F and π^*G is equal to $H(x_1^{w_1}, x_2^{w_2}, x_3^{w_3})$ for some common factor H of F and G. □

Lemma 3. *Let p be a prime such that*

$$p > 4 \left(T^{-1/d^2} V^{1/d+\varepsilon}\right)^{\frac{w_1 w_2 w_3}{g l^2}},$$

where $g = \gcd(w_1, w_2, w_3)$. Then $\#S(X; \mathbf{B}, p) = O_\varepsilon(p)$.

Proof. Since $\#X_p(\mathbb{F}_p) = O(p)$ (see [8]), it suffice to show that there are $O_\varepsilon(1)$ points $P \in S(X; \mathbf{B}, p)$ with $\overline{P} = \overline{P_1}$ for a fixed $P_1 \in S(X; \mathbf{B}, p)$. Let $P_1, \ldots, P_n$ be all such points. The idea is to find $G \in S_D$ such that $F \nmid G$ but $G(P_i) = 0$ for $i = 1, 2, \ldots, n$. Then $n \leq dDl^2$ according to lemma 2. That is, $\#S(X; \mathbf{B}, p) = O_\varepsilon(p)$ provided that $D = O_\varepsilon(1)$.

Assume that the positive integer D is fixed and bounded in terms of w,d, and ε. It will become clear later how to choose D. We may assume that $S_{(k+1)D} = S_D S_{kD}$ for all $k \geq 0$ since this is true for all sufficiently large D. Let $x^{m'} \in S_d$ be a monomial such that

$$(B_1^{w_1 m'_1} B_2^{w_2 m'_2} B_3^{w_3 m'_3})^{1/l} = T = \sup_{\substack{x^m \in S_d \\ a_m \neq 0}} (B_1^{w_1 m_1} B_2^{w_2 m_2} B_3^{w_3 m_3})^{1/l},$$

where $F(x) = \sum_m a_m x^m$. Let $x^{m_1}, \ldots, x^{m_e} \in S_D$ be the monomials which are not divisible by $x^{m'}$. It is proved in [6] that it is impossible for F to divide a non-trivial polynomials such as

$$G(x) = \sum_{i=1}^{e} c_i x^{m_i} \in S_D$$

(the proof is for $w = (1, 1, 1)$ but the argument carries over to the general case). Let $\psi_i : \Gamma \to \mathbb{Z}$ represent $P_i \in \mathbb{P}(w)(\mathbb{Q})$ for $i = 1, 2, \ldots, n$. Then there exists a non-trivial $c = (c_1, \ldots, c_e) \in \mathbb{Z}^e$ such that $G(P_i) = 0$ for $i = 1, 2, \ldots, n$ if and only if the matrix

$$M = \begin{pmatrix} \psi_1(x^{m_1}) & \cdots & \psi_1(x^{m_e}) \\ \vdots & & \vdots \\ \psi_n(x^{m_1}) & \cdots & \psi_n(x^{m_e}) \end{pmatrix}$$

has rank at most $e-1$. We may assume that $e<n$ so $\operatorname{rank}(M)<e$ if and only if the $e\times e$–minors of M vanish. Without loss of generality we may consider the minor

$$\Delta=\begin{vmatrix}\psi_1(x^{m_1}) & \cdots & \psi_1(x^{m_e})\\ \vdots & & \vdots\\ \psi_e(x^{m_1}) & \cdots & \psi_e(x^{m_e})\end{vmatrix}.$$

If we write $m_j=(m_{j1},m_{j2},m_{j3})$ for $j=1,2,\ldots,e$, then

$$|\psi_i(x^{m_j})|^l=\prod_{k=1}^{3}\left|\psi_i(x_k^{l_k})\right|^{w_k m_{jk}}\leq\prod_{k=1}^{3}B_k^{w_k m_{jk}}\quad\text{for}\quad 1\leq i,j\leq e.$$

Hence,

$$|\Delta|\leq e^e(B_1^{w_1e_1}B_2^{w_2e_2}B_3^{w_3e_3})^{1/l},$$

where

$$e_i=\sum_{j=1}^{e}m_{ji}\quad\text{for}\quad i=1,2,3.$$

We shall see below that $\operatorname{ord}_p(\Delta)\geq e(e-1)/2$. Consequently, if

$$p>e^{\frac{2}{e-1}}B_1^{\frac{2w_1e_1}{le(e-1)}}B_2^{\frac{2w_2e_2}{le(e-1)}}B_3^{\frac{2w_3e_3}{le(e-1)}},$$

then $\Delta=0$.

Now $m_1,\ldots,m_e$ are the different solutions $a=(a_1,a_2,a_3)\in\mathbb{Z}^3_{\geq0}$ of

$$\begin{cases}w_1a_1+w_2a_2+w_3a_3=Dl\\ a_1<m'_1\text{ or }a_2<m'_2\text{ or }a_3<m'_3.\end{cases}$$

Hence, if

$$\begin{aligned}E(w,k)&=\left\{a\in\mathbb{Z}^3_{\geq0}\,:\,w_1a_1+w_2a_2+w_3a_3=k\right\},\\ s_i(w,k)&=\sum_{a\in E(w,k)}a_i\quad\text{for}\quad i=1,2,3,\end{aligned}$$

then

$$\begin{aligned}e&=\#E(w,Dl)-\#E(w,Dl-dl),\\ e_i&=s_i(w,Dl)-(s_i(w,Dl-dl)+m'_i\#E(w,Dl-dl)).\end{aligned}$$

The formulas from lemma 5 in the appendix gives

$$\begin{aligned} e &= \frac{Ddgl^2}{w_1 w_2 w_3} + O(1), \\ e_i &= \frac{D^2 g l^2}{2w_i w_1 w_2 w_3}(dl - w_i m_i') + O(D), \\ \frac{2w_i e_i}{le(e-1)} &= \frac{w_1 w_2 w_3}{gl^2}\left(\frac{1}{d} - \frac{w_i m_i'}{d^2 l}\right) + O(D^{-1}). \end{aligned}$$

We can thus choose D, bounded in terms of w, d, and ε, such that

$$e^{\frac{2}{e-1}} B_1^{\frac{2w_1e_1}{le(e-1)}} B_2^{\frac{2w_2e_2}{le(e-1)}} B_3^{\frac{2w_3e_3}{le(e-1)}} \leq \left(T^{-1/d^2} V^{1/d+\varepsilon}\right)^{\frac{w_1w_2w_3}{gl^2}} < p.$$

It remains to show that $\operatorname{ord}_p(\Delta) \geq p(p-1)/2$. Let $y_0, \ldots, y_q$ be all the monomials of S_D and assume that $\overline{\psi}_1(y_0) \neq 0$. Let $z_i = y_i/y_0$ for $i = 1, 2, \ldots, q$ and put

$$a_i = \left(\frac{\psi_i(y_1)}{\psi_i(y_0)}, \ldots, \frac{\psi_i(y_q)}{\psi_i(y_0)}\right) \in \mathbb{Z}_p^q \quad \text{for} \quad i = 1, 2, \ldots, e.$$

Here $\mathbb{Z}_p$ denotes the p–adic numbers. Since $S_{(k+1)D} = S_D S_{kD}$ for all $k \geq 0$, we have

$$(S/(F))_{(y_0)} = \mathbb{Z}[z_1, \ldots, z_q]/(f_1, \ldots, f_k)$$

for some $f_1, \ldots, f_k \in \mathbb{Z}[z_1, \ldots, z_q]$. The assumption $\overline{P_1} \in (X_p)_{\text{sm}}(\mathbb{F}_p)$, implies that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_q} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial z_1} & \cdots & \frac{\partial f_k}{\partial z_q} \end{pmatrix}$$

has rank $q-1$ modulo p when evaluated at a_1. We can thus find a parametric solution of $f_1 = \cdots = f_k = 0$ in a neighbourhood of a_1 according to the implicit function theorem (see chapter III, §4 in [2]). Let $g_1, \ldots, g_{q-1} \in \mathbb{Z}_p[[z]]$ be such that if $b = (b_1, \ldots, b_q) \in \mathbb{Z}_p^n$ satisfies $f_i(b) = 0$ for $i = 1, 2, \ldots, k$ and $b \equiv a_1 \pmod{p}$, then

$$b_i = g_i(b_q) \quad \text{for} \quad i = 1, 2, \ldots, q-1.$$

Let

$$h_j(z) = \begin{cases} 1, & j = 0 \\ g_j(z + a_{iq}), & j = 1, 2, \ldots, q-1 \\ z + a_{iq}, & j = q, \end{cases}$$

and put $z_i = (a_{iq} - a_{1q})/p \in \mathbb{Z}_p$ for $i = 1, 2, \ldots, e$, where $a_i = (a_{i1}, \ldots, a_{iq})$. Then

$$\Delta = \left(\prod_{i=1}^{e} \psi_i(y_0) \right) \begin{vmatrix} h_{i_1}(pz_1) & \cdots & h_{i_e}(pz_1) \\ \vdots & & \vdots \\ h_{i_1}(pz_e) & \cdots & h_{i_e}(pz_e) \end{vmatrix}$$

for some $i_1, \ldots, i_e \in \{0, 1, \ldots, q\}$. By using elementary column operations over $\mathbb{Z}_p$ on this determinant we see that $\mathrm{ord}_p(\Delta) \geq p(p-1)/2$ (see [6]). $\square$

If we combine (1), lemma 1 and lemma 3 we get

$$N(F; \mathbf{B}) \ll_\varepsilon \left(T^{-1/d^2} V^{1/d+\varepsilon} \right)^{\frac{w_1 w_2 w_3}{gl^2}} \|F\|^\varepsilon .$$

The following observation completes the proof of the theorem, just put $B = V$.

Lemma 4. *Suppose that the coefficients of $F \in S_d$ are relatively prime. Then either $N(X, B) \leq (dl)^2$ or $\|F\| \ll B^{3d^3}$.*

Proof. Let $P_1, \ldots, P_n \in \mathbb{P}(w)(\mathbb{Q})$ be the points counted by $N(X, B)$ and $\psi_i : \Gamma \to \mathbb{Z}$ for $i = 1, 2, \ldots, n$ the corresponding primitive representatives. Let $y_1, \ldots, y_q$ be all the monomials of S_d. The rank of the matrix

$$M = \begin{pmatrix} \psi_1(y_1) & \cdots & \psi_1(y_q) \\ \vdots & & \vdots \\ \psi_n(y_1) & \cdots & \psi_n(y_q) \end{pmatrix}$$

is then at most $q - 1$ since the vector $a \in \mathbb{Z}^q$ consisting of the coefficients of F is a non-trivial solution of $Ma = 0$. If the rank is less than $q - 1$, then there exists another solution $c \in \mathbb{Z}^q$ which is linearly independent of a. The corresponding polynomial $G \in S_d$ is not divisible by F because $\deg_w(G) = d$ and F is irreducible. By lemma 2, F and G have at most $(dl)^2$ common zeros in $\mathbb{P}(w)(\mathbb{Q})$. On the other hand, if the rank of M is equal to $q - 1$, then one non-trivial solution $c = (c_1, \ldots, c_e) \in \mathbb{Z}^e$ of $Mc = 0$ is given by the $(q-1) \times (q-1)$–minors of

$$\begin{vmatrix} \psi_{i_1}(y_1) & \cdots & \psi_{i_1}(y_q) \\ \vdots & & \vdots \\ \psi_{i_{q-1}}(y_1) & \cdots & \psi_{i_{q-1}}(y_q) \end{vmatrix},$$

for some $1 \leq i_1 < \cdots < i_{q-1} \leq n$. By expanding the minors and using the trivial bound $q \leq \binom{d+2}{2} \leq 3d^2$, we get

$$|c_i| \leq ((q-1)B^d)^{q-1} \ll B^{3d^3} \quad \text{for} \quad i = 1, 2, \ldots, q.$$

The elements of a are relatively prime so c has to be an integer multiple of a. Hence, $\|F\| \ll B^{3d^3}$ as promised. $\square$

Appendix

Lemma 5. *Assume that the elements of* $a = (a_1, a_2, a_3)$ *are relatively prime positive integers and let*

$$E(a,n) = \left\{x \in \mathbb{Z}^3_{\geq 0} \; : \; a_1x_1 + a_2x_2 + a_3x_3 = n\right\},$$
$$s_i(a,n) = \sum_{x \in E(a,n)} x_i \quad \textit{for} \quad i = 1,2,3.$$

If $b_1 = \gcd(a_2, a_3)$, $b_2 = \gcd(a_1, a_3)$, $b_3 = \gcd(a_1, a_2)$, *and*

$$\chi_i(n) = 1 + 2\sum_{j=1}^{b_i-1} \frac{\zeta_i^{jn}}{1-\zeta_i^{-ja_i}} \quad \textit{for} \quad i = 1,2,3,$$

where $\zeta_i = \exp(2\pi\sqrt{-1}/b_i)$, *then*

$$\#E(a,n) = \frac{1}{2a_1a_2a_3}\left(n^2 + n\sum_{i=1}^{3} a_i\chi_i(n)\right) + O_a(1),$$
$$s_i(a,n) = \frac{1}{a_1a_2a_3a_i}\left(\frac{n^3}{6} + \frac{n^2}{4}\left(\sum_{j=1}^{3} a_j\chi_j(n) - a_i\chi_i(n)\right)\right) + O_a(n).$$

Proof. It is straightforward to establish the formula for $\#E(a,n)$ by using its generating function. We only show the basic idea and leave the details to the reader. If we put $\xi_i = \exp(2\pi\sqrt{-1}/a_i)$ for $i = 1,2,3$, then

$$\sum_{n\geq 0} \#E(a,n)t^n = \prod_{i=1}^{3} \frac{1}{1-t^{a_i}} = \prod_{i=1}^{3}\left(\frac{1}{a_i}\sum_{j=0}^{a_i-1}\frac{1}{1-\xi_i^j t}\right) =$$
$$\frac{1}{a_1a_2a_3}\left[\frac{1}{(1-t)^3} + \sum_{i=1}^{3}\sum_{j=1}^{a_i-1}\frac{1}{(1-t)^2(1-\xi_i^j t)} + \right.$$
$$\left.\sum_{i=1}^{3}\sum_{j=0}^{a_i-1}\sum_{k=1}^{b_i-1}\frac{1}{(1-\xi_i^j t)(1-\zeta_i^k t)^2}\right] + \cdots \quad (2)$$

where the missing terms do not contain any quadratic factors and therefore only contribute $O_a(1)$ to $\#E(a,n)$. By using the formulas

$$\frac{1}{(1-t)^3} = \sum_{n\geq 0}\frac{1}{2}(n+1)(n+2)t^n,$$
$$\frac{1}{(1-At)(1-Bt)^2} = \sum_{n\geq 0}\left(\frac{A^{n+2}}{(A-B)^2} - \frac{AB^{n+1}}{(A-B)^2} + \frac{B^{n+1}(n+1)}{B-A}\right)t^n,$$

in (2) we obtain the asymptotic formula for $\#E(a,n)$. The generating function of $s_i(a,n)$ is

$$\sum_{n\geq 0} s_i(a,n)t^n = \frac{1}{(1-t^{a_i})}\prod_{j=1}^{3}\frac{1}{1-t^{a_j}} = \frac{1}{1-t^{a_i}}\sum_{n\geq 0}\#E(a,n)t^n,$$

so

$$s_i(a, qa_i + r) = \sum_{j=0}^{q-1}\#E(a, ja_i + r) \quad \text{for} \quad 0 \leq r < a_i.$$

One can check that

$$\sum_{j=0}^{q-1} j\chi_k(ja_i + r) = \begin{cases} O_a(q) & \text{if } i = k, \\ \frac{q^2}{2}\chi_k(r) + O_a(q) & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} 2a_1a_2a_3 \sum_{j=0}^{q-1} \#E(a, ja_i + r) &= \\ a_i^2 \sum_{j=0}^{q-1} j^2 + 2a_i r \sum_{j=0}^{q-1} j + a_i \sum_{k=1}^{3}\left(a_k \sum_{j=0}^{q-1} j\chi_k(ja_i + r)\right) + O_a(q) &= \\ a_i^2\left(\frac{q^3}{3} - \frac{q^2}{2}\right) + a_i r q^2 + a_i \frac{q^2}{2}\left(\sum_{k=1}^{3} a_k\chi_k(r) - a_i\chi_i(r)\right) + O_a(q). \end{aligned}$$

By putting $q = (n-r)/a_i$ in this last expression and collecting the powers of n we obtain the formula for $s_i(a,n)$. □

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