On the Error in the Monte Carlo Pricing of Some Familiar European Path-Dependent Options

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Abstract

This paper studies the relative error in the crude Monte Carlo pricing of some familiar European path-dependent multi asset options. For the crude Monte Carlo method it is well-known that the convergence rate $O(n^{-1/2})$, where n is the number of simulations, is independent of the dimension of the integral. This paper shows that for a large class of pricing problems in the (multi-asset) Black-Scholes market also the *constant* in $O(n^{-1/2})$ is independent of the dimension. To be more specific, the constant is only dependent on the highest volatility amongst the underlying assets, time to maturity and degree of confidence interval. The main tool to prove this result is the isoperimetric inequality for Wiener measure.

Key words: option pricing, path-dependent options, Monte Carlo method, error estimates, the isoperimetric inequality, Wiener space.

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1 Introduction

The application of the Monte Carlo method to option pricing was first presented in Boyle [5] and it has proved to be an extremely useful tool for the valuation of contingent claims. The popularity of the Monte Carlo method in finance depends mainly on the fact that it provides a robust and simple method for performing integration. For example, Monte Carlo integration converges at a rate $O(n^{-1/2})$, where n is the number of simulations, that is independent of the dimension of the integral. For this reason, the Monte Carlo method is sometimes the only viable method for a large number of high-dimensional problems in finance.

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Previous work on the Monte Carlo pricing of European derivatives primarily focuses on different so called variance reduction techniques. Kemna and Vorst [11] consider the technique of control variates in the pricing of Asian options. Barraquand [1] exploits the idea of quadratic sampling. Glasserman, Heidelberger, and Shahabuddin [8] study importance sampling and stratification for the pricing of path-dependent options. These articles are just a small part of the research about the Monte Carlo technique in option pricing. For a more complete discussion, see Boyle, Broadie, and Glasserman [6].

The main purpose of this paper is to derive error estimates for the crude Monte Carlo pricing of European options, with particular emphasis on path-dependent options. We must underline that we will only consider the *crude* Monte Carlo technique. A discussion about the error in the so called quasi Monte Carlo method can be found in Caflish [7].

The present paper presents extensions of some previous results by Borell [4], where, among other things, he investigates the relative error in the Monte Carlo pricing of simple European options in a multi-dimensional Black-Scholes market. A simple option is an option that only depends on the underlying asset prices at the maturity date of the option. In particular, it is shown in [4] that for some simple options not only the convergence rate but also the *constant* in $O(n^{-1/2})$ is independent of the dimension. To be more specific, the constant is only dependent on the highest volatility amongst the underlying assets, time to maturity, and degree of confidence interval. This paper will show a similar result as in [4] for a large number of European styled *path-dependent* contracts.

The main results and the structure in this paper can be described as follows. In Section 2 we will, to begin with, consider an arbitrary market with a martingale measure Q. Assume that X is the payoff of a contingent-claim in the market, which we assume has a p-th moment, $p \geq 2$, and non-zero expectation $\alpha = E^Q[X]$. Section 2 shows that the size of the relative error for the Monte Carlo estimation of the expectation α is controlled by the value of the functional $D_p(X)$, $2 \leq p \leq \infty$, defined by

$$D_p(X) = \frac{\|X - \alpha\|_{L^p(Q)}}{|\alpha|}, \quad \text{for } 2 \leq p < \infty,$$

and

$$D_\infty(X) = \frac{\text{ess sup}\, X - \text{ess inf}\, X}{|\alpha|}$$

The main tools to prove this result are the Rosenthal and the Hoeffding inequalities. These classical inequalities will also be discussed in Section 2.

The goal for the remaining part of this paper, Section 3 and 4, is to establish upper bounds for $D_p(X)$ in the case that X is a contract on the Black-Scholes multi-asset market.

To be more specific, Section 3 compares the moments between path-dependent call (put) options and plain vanilla call (put) options in the Black-Scholes multi-asset market. The main result about call options in Section 3 can be stated as follows. Suppose that X is a payoff function of an option included in a certain class $\mathcal{C}_K$. This class will be properly defined in Section 3.1 and it turns out that options like lookbacks and Asian or Asian styled basket calls with strike price K, $K \geq 0$, will be members of the class $\mathcal{C}_K$. It will be proved that if the real number $\theta > 0$ is chosen such that

$$\|X\|_{L^1(Q)} = \| \max(\theta S_T^{(m)} - K, 0)\|_{L^1(Q)},$$

where $\{S_t^{(m)}\}$ denotes the price process of the most volatile asset of the underlyings, then

$$\|X\|_{L^p(Q)} \leq \| \max(\theta S_T^{(m)} - K, 0)\|_{L^p(Q)}$$

for each $1 \leq p < \infty$. A similar result for path-dependent *put* options will be derived as well.

To prove this and other results in Section 3 various geometric inequalities in Wiener space will be used. In particular, the isoperimetric inequality for Wiener measure.

The final section, Section 4, combines the results in the Sections 2 and 3. Section 4 shows how to obtain explicit upper bounds for the relative error in the Monte Carlo pricing of several different path-dependent contingent-claims. It will be proved that for a large class of pricing problems, the upper endpoint of a confidence interval of degree $100(1-\epsilon)\%$ for (the absolute value of) the relative error is bounded by

$$\frac{\kappa_\epsilon}{\sqrt{n}} \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}},$$

where σ_m denotes the highest volatility among the underlyings, T is time to maturity, and κ_ϵ is a constant only dependent on ϵ. The constant κ_ϵ is closely related to the best constant in Rosenthal's inequality and will be specified in Section 4.

2 Error Estimates for the Monte Carlo Method

From now in this section assume that $(\Omega, \mathcal{F}, Q)$ is a given probability space and X is a random variable in $L^p(\Omega, \mathcal{F}, Q)$, $2 \leq p \leq \infty$, with $E^Q[X] = \alpha \neq 0$. In addition, let $X_1, \dots, X_n$ be stochastically independent observations on X and set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Here $\bar{X}_n$ is called the Monte Carlo estimator of X. In what follows we abbreviate

$$\|\cdot\|_p = \|\cdot\|_{L^p(Q)},$$

if $2 \le p < \infty$, and

$$l(X) = \operatorname{ess\,sup} X - \operatorname{ess\,inf} X.$$

Finally, put

$$D_p(X) = \frac{\|X - \alpha\|_p}{|\alpha|}, \quad 2 \le p < \infty,$$

and

$$D_\infty(X) = \frac{l(X)}{|\alpha|}.$$

If R_n denote the relative error after n simulations, i.e.

$$R_n = \left| \frac{\bar{X}_n - \alpha}{\alpha} \right|,$$

the Chebychev inequality gives that for any $0 < \epsilon < 1$

$$Q\left(R_n \le \frac{C_\epsilon}{\sqrt{n}} \right) \ge 1 - \epsilon, \tag{1}$$

where

$$C_\epsilon = \frac{D_2(X)}{\sqrt{\epsilon}}. \tag{2}$$

Thus, equation (1) yields that the convergence rate of the relative error in the Monte Carlo estimation is $O(n^{-1/2})$ with a constant that is bounded by C_ϵ.

By applying the central limit theorem it is easily seen that the convergence rate $O(n^{-1/2})$ is the best possible in the sense that if $\delta_n \to 0^+$ when $n \to \infty$ then

$$Q\left(R_n \le \frac{\delta_n}{n^{1/2}} \right) \to 0, \quad \text{as } n \to \infty.$$

However, in certain cases it may be possible to improve the constant C_ϵ in equation (2). The purpose of the remaining part of this section is to find better estimates of the constant C_ϵ.

First we recall a heuristic and well known argument which shows that it is plausible to improve the constant C_ϵ, provided n is sufficiently large as will be the case in most Monte Carlo simulations. In fact, by letting

$$\xi_i = \frac{X_i - \alpha}{\|X_i - \alpha\|_2}, \quad i = 1, 2, \dots , \tag{3}$$

the central limit theorem gives that the random variable

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i$$

is approximately distributed as a normal random variable with mean 0 and variance 1. Thus, for any $\lambda > 0$,

$$\begin{aligned} Q\Big(R_n \le \lambda\Big) &= Q\left(\Big|\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i\Big| \le \frac{\lambda\sqrt{n}}{D_2(X)}\right) \\ &\approx 2\Phi\left(\frac{\lambda\sqrt{n}}{D_2(X)}\right) - 1, \end{aligned} \tag{4}$$

where Φ denote the standard normal distribution function, that is

$$\Phi(y) = \int_{-\infty}^{y} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

Suppose Φ^{-1} is the inverse of Φ. By letting

$$\lambda = \frac{D_2(X)}{\sqrt{n}} \Phi^{-1}\Big(1 - \frac{\epsilon}{2}\Big), \quad 0 < \epsilon < 1,$$

in equation (4) we obtain

$$Q\left(R_n \le \frac{\tilde{C}_\epsilon}{\sqrt{n}}\right) \approx 1 - \epsilon,$$

where

$$\tilde{C}_\epsilon = \sqrt{\epsilon}\, \Phi^{-1}\Big(1 - \frac{\epsilon}{2}\Big) C_\epsilon$$

with C_ϵ defined as in equation (2). If ϵ is close to zero then it is evident that the constant $\tilde{C}_\epsilon$ will be considerably much smaller than C_ϵ.

In order to make the above argument more precise we will recall some classical inequalities for random walks. The next theorem is known as the Rosenthal inequality.

Theorem 1. *Let ζ and ζ' denote independent Poisson random variables with parameter $1/2$ and let Γ be the gamma function. Suppose $X \in L^p(Q)$, $2 \le p < \infty$, is a symmetric random variable. Then*

$$\|\sum_{i=1}^{n} X_i\|_p \le r_p \max\Big(\sqrt{n}\|X\|_2,\, n^{1/p}\|X\|_p\Big),$$

where

$$r_p = \begin{cases} \left(1 + \frac{2^{p/2}}{\pi^{1/2}}\Gamma\left(\frac{p+1}{2}\right)\right)^{1/p}, & 2 < p < 4, \\ \|\zeta - \zeta'\|_p, & p \ge 4. \end{cases}$$

A proof of Theorem 1 with $r_p = 2^p$ can be found in Rosenthal [17]. The value of the constant r_p given in Theorem 1 can be found in Ibragimov and Sharakhmetov [10]. Table 1 shows an upper bound for the value of the constant r_p for various values on p. The table will be useful in the sequel.

p	3	4	5	6	7	8	9	10
r_p ($\le$)	1.37	1.41	1.60	1.77	1.94	2.10	2.26	2.41

Table 1: An upper bound for the value of the constant r_p in the Rosenthal inequality.

By a standard result in probability theory, if X and X' are i.i.d. random variables in $L^p(Q)$ with $E^Q[X] = 0$ then $\|X\|_p \le \|X - X'\|_p$ (cf. p. 263 in Loeve [14]). Since the random variable $X - X'$ is symmetric the Rosenthal inequality implies

Corollary 1. *If $X \in L^p(Q)$, $2 \le p < \infty$, and $\alpha = E^Q[X]$ then*

$$\|\sum_{i=1}^{n} X_i - n\alpha\|_p \le 2\, r_p \max\Big(\sqrt{n}\|X - \alpha\|_2,\, n^{1/p}\|X - \alpha\|_p\Big),$$

where r_p is defined as in Theorem 1.

The next theorem is often referred to as the Hoeffding inequality.

Theorem 2. *If $X \in L^\infty(Q)$ and $\alpha = E^Q[X]$ then*

$$Q\left(\Big|\sum_{i=1}^{n} X_i - n\alpha\Big| \ge n\lambda\right) \le 2\exp\left(-\frac{2n\lambda^2}{l^2(X)}\right)$$

for every $\lambda > 0$.

For a proof of Theorem 2, see Hoeffding [9]. The bound in Theorem 2 is not the best possible, see Talagrand [19] for a further discussion.

More results on tail probabilities and moment estimations for sums of independent random variables can be found in Petrov [15].

We are now in the position to establish the main result in Section 2.

Theorem 3. *Suppose $X \in L^p(Q)$, $2 \leq p \leq \infty$, and $\alpha = E^Q[X] \neq 0$. Moreover, assume that the constant r_p is defined as in Theorem 1. If R_n denote the relative error after n simulations, i.e.*

$$R_n = \left| \frac{\bar{X}_n - \alpha}{\alpha} \right|,$$

then

$$Q\left(R_n < \frac{C_\epsilon^*}{\sqrt{n}} \right) > 1 - \epsilon, \quad \epsilon > 0,$$

if C_ϵ^* is any of the numbers

$$C_\epsilon = \frac{1}{\sqrt{\epsilon}} D_2(X),$$

$$C_\epsilon^{(r)} = \frac{2\, r_p}{\epsilon^{1/p}} \max\left(D_2(X),\, n^{\frac{1}{p}-\frac{1}{2}}\, D_p(X) \right), \quad 2 < p < \infty,$$

or

$$C_\epsilon^{(h)} = \sqrt{\frac{\ln(2/\epsilon)}{2}}\, D_\infty(X).$$

Proof. The case $C_\epsilon^* = C_\epsilon$ has already been shown. To prove the other cases, let ξ_i be defined as in equation (3) and observe that

$$R_n = D_2(X) \left| \frac{1}{n} \sum_{i=1}^n \xi_i \right|.$$

Firstly, let

$$C_\epsilon^* = C_\epsilon^{(r)}$$

and suppose that $\lambda > 0$. The Chebychev inequality yields

$$Q\left(\left| \frac{1}{n} \sum_{i=1}^n \xi_i \right| \geq \lambda \right) \leq \frac{\| \sum_{i=1}^n \xi_i \|_p^p}{n^p \lambda^p}$$

which in combination with Corollary 1 implies

$$Q\left(\left| \frac{1}{n} \sum_{i=1}^n \xi_i \right| \geq \lambda \right) \leq \frac{(2r_p)^p}{\lambda^p n^{p/2}} \max\left(1,\, n^{\frac{1}{p}-\frac{1}{2}} \|\xi_i\|_p\right)^p.$$

Note that $\|\xi_i\|_p = D_p(X)/D_2(X)$. Set

$$\lambda = \frac{2r_p}{\sqrt{n}\epsilon^{1/p}} \max\left(1,\, n^{\frac{1}{p}-\frac{1}{2}} \frac{D_p(X)}{D_2(X)}\right)$$

and we are done.

Now suppose

$$C_\epsilon^* = C_\epsilon^{(h)}.$$

Theorem 2 gives

$$Q\left(\left|\frac{1}{n}\sum_{i=1}^{n} \xi_i\right| \geq \lambda\right) \leq 2\exp\left(-\frac{2n\lambda^2}{l^2(\xi_i)}\right)$$

for any $\lambda > 0$. In particular, if we chose

$$\lambda = \sqrt{\frac{\ln(2/\epsilon)}{2n}}\, l(\xi_i),$$

the proof is complete. □

We conclude this section by making some comments on the error constant $C_\epsilon^{(r)}$ in Theorem 3. Observe that $C_\epsilon^{(r)}$ is dependent on the number of simulations n. However, if n is sufficiently large or $D_p(X)$ sufficiently small then

$$\max\left(D_2(X),\, n^{\frac{1}{p}-\frac{1}{2}} D_p(X)\right) = D_2(X)$$

and thus, under some additional assumptions, the constant $C_\epsilon^{(r)}$ is independent of n.

In Section 4 we will compare the value of the constants C_ϵ and $C_\epsilon^{(r)}$ in some special cases.

3 Comparison of Moments

In this section we will compare the moments between some familiar path-dependent contracts and plain vanilla call or put options.

3.1 Preliminaries and Main Result

From now on the sample space $\Omega = C_0([0,T];\mathbb{R}^m)$ consists of all functions $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ such that, for each $i = 1, \ldots, m$, the function $\omega_i : [0,T] \mapsto \mathbb{R}$ is continuous and $\omega_i(0) = 0$. The space Ω is equipped with the norm $\|\cdot\|_{C_0}$, defined by

$$\|\omega\|_{C_0} = \max_{i=1,\ldots,m} \max_{0\leq t\leq T} |\omega_i(t)|.$$

The measure Q will henceforth denote Wiener measure on Ω. Defining

$$W_t(\omega) = \omega(t), \quad 0 \le t \le T, \ \omega \in \Omega,$$

the process $\{W_t\}_{t=0}^T$ is a standard m-dimensional Brownian motion with respect to Q. A vector in $\mathbb{R}^m$ is interpreted as an m by 1 matrix.

We use the convention that

$$e^\zeta = (e^{\zeta_1}, e^{\zeta_2}, \dots, e^{\zeta_m}), \quad \zeta \in \mathbb{R}^m$$

$$\zeta\eta = (\zeta_1\eta_1, \zeta_2\eta_2, \dots, \zeta_m\eta_m), \quad \zeta, \eta \in \mathbb{R}^m,$$

$$(\zeta\omega)(t) = \zeta\omega(t), \quad \zeta \in \mathbb{R}^m, \ \omega \in \Omega, \ 0 \le t \le T,$$

and

$$(e^\omega)(t) = e^{\omega(t)}, \quad \omega \in \Omega, \ 0 \le t \le T.$$

Assume in the remaining part of this paper that the dynamics of the underlying asset price vector S under the measure Q is given by

$$S_t = S_0 e^{\nu(t) + \sigma C W_t}, \quad 0 \le t \le T,$$

where C is a non-singular m by m matrix such that each row c_i in C satisfies $|c_i| = 1$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$,

$$S_0 = (S_0^{(1)}, \dots, S_0^{(m)}) \in (0, \infty)^m$$

denotes the initial asset prices and

$$\sigma = (\sigma_1, \dots, \sigma_m) \in (0, \infty)^m$$

stands for the volatilities of the underlying assets and $T > 0$ can be thought of as the maturity date of some given option. Moreover, $\nu \in \Omega$ is defined by

$$\nu_i(t) = (r - \frac{\sigma_i^2}{2})t, \quad 0 \le t \le T, \ i = 1, \dots, m,$$

where $r \in \mathbb{R}$ is the interest rate. Finally, suppose that

$$\max_{i=1,\dots,m} \sigma_i = \sigma_m,$$

that is, asset price number m has the greatest volatility. Henceforth we will denote the price process of the i:th asset, $i = 1, \dots, m$, by $\{S_t^{(i)}\}_{t=0}^T$.

Next we will define a certain class $\mathcal{L}$ of functionals F on the space $C([0,T]; \mathbb{R}_+^m)$. We will say that $F \in \mathcal{L}$ if $F > 0$ and for any fixed $s \in \mathbb{R}_+^m$ the map

$$\omega \mapsto \ln F(se^\omega)$$

is Ω-Lipschitz with constant 1, i.e.

$$|\ln F(se^{\omega+\tilde{\omega}}) - \ln F(se^{\omega})| \leq \|\tilde{\omega}\|_{C_0}$$

for all $\omega, \tilde{\omega} \in \Omega$. Clearly, this definition equivalently means that $F > 0$ and

$$F(se^{\omega+\tilde{\omega}}) \leq e^{\|\tilde{\omega}\|_{C_0}} F(se^{\omega})$$

for all $\omega, \tilde{\omega} \in \Omega$.

In addition, let $K \geq 0$ and set $\mathcal{C}_K = (\mathcal{L} - K)^+$, that is, $F_K \in \mathcal{C}_K$ if, and only if, there exists an $F \in \mathcal{L}$ such that

$$F_K = \max(F - K, 0).$$

In particular, $\mathcal{C}_0 = \mathcal{L}$ and $\mathcal{C}_K \subseteq \mathcal{C}_L$ if $K \leq L$.

We next give some examples of functionals in the class $\mathcal{C}_K$.

1. Assume that μ_j, $j = 1, \dots, m$, are positive and bounded Borel measures on $[0, T]$. It is evident that the functional

   $$F^{(\mu)}(se^{\omega}) = \sum_{j=1}^{m} \int_0^T s_j e^{\omega_j(t)} \mu_j(dt)$$

   belongs to the class $\mathcal{L}$. Thus, for any fixed $K \geq 0$, if

   $$F_K^{(\mu)}(se^{\omega}) = \max(\, F^{(\mu)}(se^{\omega}) - K,\, 0)$$

   then $F_K \in \mathcal{C}_K$. Moreover, set

   $$X = F_K^{(\mu)}(S).$$

   Depending on the measures μ_j, $j = 1, \dots, m$, X equals the payoff function of an Asian or an Asian styled basket call option with fixed strike. It should be emphasised that the measures μ_j, $j = 1, \dots, m$, must be positive, otherwise $F^{(\mu)}$ will not be a member of the class $\mathcal{L}$. The case of signed measures μ_j will be further discussed in Section 4.3.

2. If $M \subseteq [0, T]$ and $\beta_j \in \mathbb{R}_+$, $j = 1, \dots, m$, then the functional

   $$F^{(M,\beta)}(se^{\omega}) = \max_{j=1,\dots,m} \sup_{t \in M} \beta_j \, s_j e^{\omega_j(t)} \tag{5}$$

   is included in $\mathcal{L}$. Hence, for fixed $K \geq 0$,

   $$F_K^{(M,\beta)}(se^{\omega}) = \max(\, F^{(M,\beta)}(S) - K,\, 0)$$

   is a member of $\mathcal{C}_K$. By setting

   $$X = F_K^{(M,\beta)}(S),$$

   X is the payoff function of a lookback option with fixed strike if $m = 1$ and a so called maximum option if $M = \{T\}$.

3. Other examples of functionals in the class $\mathcal{C}_K$ can be constructed by taking the maximum or minimum of members of $\mathcal{C}_K$. To see this, if $F, G \in \mathcal{L}$ then it is evident that

$$\max(F, G) \in \mathcal{L} \quad \text{and} \quad \min(F, G) \in \mathcal{L}.$$

Thus, if $K \leq L$ and $F_K \in \mathcal{C}_K$, $G_L \in \mathcal{C}_L$ then

$$\max(F_K, G_L) \in \mathcal{C}_L \quad \text{and} \quad \min(F_K, G_L) \in \mathcal{C}_L,$$

since $\mathcal{C}_K \subseteq \mathcal{C}_L$.

Our main result about path-dependent *call* options is

Theorem 4. *If $F_K \in \mathcal{C}_K$, $K \geq 0$, and θ is chosen such that*

$$\|F_K(S)\|_1 = \|\max(\theta S_T^{(m)} - K, 0)\|_1,$$

then

$$\|F_K(S)\|_p \leq \|\max(\theta S_T^{(m)} - K, 0)\|_p$$

for each $1 \leq p < \infty$.

Theorem 4 will be proved in Section 3.2.

Next we will define a class of path dependent *put* options. Let $\mathcal{L}$ be defined as previous and set

$$\mathcal{P}_K = \max(K - \mathcal{L}, 0), \quad K > 0.$$

Theorem 5 below is the counterpart to Theorem 4 for put options. The proof of Theorem 5 is indicated in the next subsection.

Theorem 5. *If $F_K \in \mathcal{P}_K$, $K > 0$, and θ is chosen such that*

$$\|F_K(S)\|_1 = \|\max(K - \theta S_T^{(m)}, 0)\|_1,$$

then

$$\|F_K(S)\|_p \leq \|\max(K - \theta S_T^{(m)}, 0)\|_p$$

for each $1 \leq p < \infty$.

3.2 The Isoperimetric Inequality for Wiener measure; proof of Theorem 4

In order to present the isoperimetric inequality for Wiener measure we first introduce the so called Cameron-Martin space H. The space H consists of all functions $h = (h_1, h_2, \ldots, h_m)$ such that, for each $i = 1, \ldots, m$, the function $h_i : [0,T] \mapsto \mathbb{R}$ is absolutely continuous with a square integrable derivative and $h_i(0) = 0$. The space H is equipped with the norm $\|\cdot\|_H$, defined by

$$\|h\|_H = \Big(\sum_{i=1}^{m} \int_0^T \big(h_i'(t)\big)^2 \, dt \Big)^{\frac{1}{2}}, \quad h \in H.$$

Next we state the so called isoperimetric inequality for Wiener measure.

Theorem 6. *Let O be the set of all $h \in H$ such that $\|h\|_H \leq 1$. If A is a Borel set in Ω and*

$$Q(A) = \Phi(a),$$

then

$$Q(A + \lambda O) \geq \Phi(a + \lambda)$$

for each real number $\lambda \geq 0$.

Theorem 6 is a special case of the isoperimetric inequality for Gaussian measures, which was discovered independently by Borell [3] and Sudakov and Tsirelson [18]. In both papers the proof was based on the isoperimetric inequality on the sphere. Recently, Ledoux [13] has developed a short and self contained proof based on the Ornstein-Uhlenbeck semigroup. A stochastic calculus version of the proof for the Gaussian isoperimetric inequality can be found in Barthe and Maurey [2].

A consequence of Theorem 6 is

Corollary 2. *Assume that $F \in \mathcal{L}$. If, for $a, b > 0$,*

$$Q\big(\, F(S) \geq b \,\big) = Q\big(\, S_T^{(m)} \geq a \,\big),$$

then

$$Q\big(\, F(S) \geq \vartheta b \,\big) \leq Q\big(\, S_T^{(m)} \geq \vartheta a \,\big)$$

for each real number $\vartheta \geq 1$.

Proof. Fix $\vartheta \geq 1$ and let $\lambda \geq 0$ be given by the equation $\vartheta = \exp(\sigma_m \sqrt{T} \lambda)$. We first prove that

$$Q\big(\, F(S) \leq \vartheta b \,\big) \geq Q\big(\inf_{\|h\|_H \leq 1} F\big(S(\cdot + \lambda h)\big) \leq b \,\big). \tag{6}$$

Firstly, note that the random variable $\inf_{||h||_H \leq 1} F\big(S(\cdot + \lambda h)\big)$ is Borel measurable since $F(S(\cdot))$ is continuous and O is a compact subset of Ω, and thus, the supremun can be taken over a dense denumerable subset. Now, suppose that $h = (h_1, \dots , h_m) \in H$ and let $\langle \, , \, \rangle$ denote the inner product in $\mathbb{R}^m$. Since

$$h(t) = \int_0^t h'(x)dx, \quad 0 \leq t \leq T,$$

the Cauchy-Schwarz inequality gives for $0 \leq t \leq T$ and $j = 1, \dots , m$

$$\begin{aligned}\langle c_j, h(t)\rangle &\leq \Big(\sum_{i=1}^m \Big(\int_0^t |h_i'(u)| du \Big)^2 \Big)^{\frac{1}{2}} \\ &\leq \sqrt{t} \Big(\sum_{i=1}^m \int_0^t \big(h_i'(u)\big)^2 du \Big)^{\frac{1}{2}} \\ &= \sqrt{t} \|h\|_H\end{aligned}$$

and thus $\|\sigma Ch\|_{C_0} \leq \sigma_m \sqrt{T} \|h\|_H$. Next, observe that if $\omega \in \Omega$ then

$$\begin{aligned}F\big(S(\omega)\big) &= F\big(S(\omega + h)e^{-\sigma Ch}\big) \\ &\leq e^{\|\sigma Ch\|_{C_0}} F\big(S(\omega + h)\big) \\ &\leq e^{\sigma_m \sqrt{T} \|h\|_H} F\big(S(\omega + h)\big)\end{aligned}$$

and consequently

$$\inf_{||h||_H \leq 1} F\big(S(\omega + \lambda h)\big) \geq \frac{1}{\vartheta} F\big(S(\omega)\big),$$

which proves equation (6).

Next, note that Theorem 6 implies

$$Q\big(\inf_{||h||_H \leq 1} F\big(S(\cdot + \lambda h)\big) \leq b \big) \geq Q\big(e^{-\sigma_m \sqrt{T} \lambda} S_T^{(m)} \leq a\big).$$

Moreover,

$$Q\big(e^{-\sigma_m \sqrt{T} \lambda} S_T^{(m)} \leq a \big) = Q\big(S_T^{(m)} \leq \vartheta a \big)$$

and therefore, according to equation (6),

$$Q\big(F(S) \leq \vartheta b \big) \geq Q\big(S_T^{(m)} \leq \vartheta a \big),$$

which implies the statement in Corollary 2. $\square$

Proof of Theorem 4. Suppose $F_K \in \mathcal{C}_K$ and

$$\|F_K(S)\|_1 = \| \max(\theta S_T^{(m)} - K, 0)\|_1. \tag{7}$$

We want to prove that

$$E^Q\big[F_K(S)^p\big] \leq E^Q\big[\max(\theta S_T^{(m)} - K, 0)^p\big]$$

for each $1 < p < \infty$.

Write $F_K = (F - K)^+$ where $F \in \mathcal{L}$ and set

$$\psi(x) = Q\big(F(S) \geq x\big) - Q\big(\theta S_T^{(m)} \geq x\big).$$

Partial integration now gives

$$\begin{aligned} E^Q\big[\, F_K(S)^p \,\big] &- E^Q\big[\, \max(\theta S_T^{(m)} - K, 0)^p \,\big] \\ &= p(p-1) \int_K^\infty (x - K)^{p-2} \int_x^\infty \psi(y) dy \; dx. \end{aligned}$$

Thus, it is enough to prove that

$$\int_x^\infty \psi(y) dy \leq 0 \tag{8}$$

for all $x \geq K$.

Equation (7) gives

$$\int_K^\infty \psi(y) dy = 0.$$

Let $y^* = \inf\{y \geq K : \psi(y) \leq 0\}$. According to Corollary 2 we have that $\psi(y) \leq 0$ for all $y > y^*$, which implies equation (8) and the proof of Theorem 4 is complete. □

Theorem 5 can be proven in a similar way. The details are omitted.

3.3 A Remark on Barrier Options

This section is concluded with an example of a payoff function which satisfies the converse inequality compared to Theorem 4. The example we have chosen is a down-and-out call option.

From now on assume that the market only consists of one asset, i.e. $m = 1$. The price process will for simplicity be denoted $\{S_t\}_{t=0}^T$, where

$$S_t = S_0 e^{\nu(t) + \sigma W_t}, \quad 0 \leq t \leq T,$$

with $\nu(t) = (r - \sigma^2/2)t$ and $\sigma > 0$. In addition, set

$$\tau = \inf\{t \in M \,|\, S_t < H(t)\},$$

where M is a closed subset of $[0,T]$ and $H : M \to \mathbb{R}_+$ is continuous. In particular, the random variable τ is Borel measurable. The payoff function of a down-and-out call is defined as

$$\max(S_T - K, 0)1_{\{\tau > T\}}$$

with $K \geq 0$.

The following so called shift inequality will be useful in the sequel.

Theorem 7. *Assume that A is a Borel set in Ω. If $\|h\|_H = 1$ and*

$$Q(A) = \Phi(a),$$

then

$$\Phi(a - \lambda) \leq Q(A + \lambda h) \leq \Phi(a + \lambda)$$

for each $\lambda \geq 0$.

For a proof of Theorem 7, see Kuelbs and Li [12].

Corollary 3. *If, for $a, b > 0$,*

$$Q\big(S_T \geq a, \tau > T\big) = Q\big(S_T \geq b\big),$$

then

$$Q(S_T \geq \vartheta a, \tau > T) \geq Q(S_T \geq \vartheta b) \tag{9}$$

for each real number $\vartheta \geq 1$.

Proof. Set

$$h(t) = \frac{t}{\sqrt{T}}, \quad 0 \leq t \leq T.$$

If $\lambda \geq 0$ satisfies $\exp(\lambda\sigma h(T)) = \vartheta$ then

$$Q\big(\, S_T \geq \vartheta a,\, \tau > T\big) = Q\big(\, e^{-\lambda\sigma h(T)} S_T \geq a,\, \tau > T\big).$$

If $A = \{S_T \geq a,\, \tau > T\}$ we have

$$Q\big(\, e^{-\lambda\sigma h(T)} S_T \geq a,\, \tau > T\big) \geq Q(A + \lambda h).$$

Theorem 7 implies

$$\begin{aligned} Q(A + \lambda h) &\geq Q(e^{-\lambda\sigma h(T)} S_T \geq b) \\ &= Q(S_T \geq \vartheta b) \end{aligned}$$

and the proof is complete.

□

Proposition 1. *Suppose $K \geq 0$. If θ is chosen such that*

$$\| \max(S_T - K, 0) 1_{\{\tau > T\}} \|_1 = \| \max(\theta S_T - K, 0) \|_1,$$

then

$$\| \max(S_T - K, 0) 1_{\{\tau > T\}} \|_p \geq \| \max(\theta S_T - K, 0) \|_p$$

for each $1 \leq p < \infty$.

Proof. The result follows from Corollary 3 in the same way as Theorem 4 follows from Corollary 2. □

There is a similar result as in Proposition 1 for certain other barrier options such as up-and-out put options. The details are omitted here.

4 The Error in the Monte Carlo Pricing of Some Familiar European Path-Dependent Options

This section shows, using Theorems 4 and 5, how to obtain an explicit upper bound of $D_p(X)$ for different choices of payoff functions X and thereby establish error bounds for the Monte Carlo pricing of X. To begin with we will consider call options.

4.1 Call Options

First we state a lemma, taken from Borell [4].

Lemma 1. *Assume that $K \geq 0$ and $p \geq 2$. The function*

$$\theta \mapsto \frac{\| \max(\theta S_T^{(m)} - K, 0) \|_p}{\| \max(\theta S_T^{(m)} - K, 0) \|_1}, \quad \theta > 0,$$

is non-increasing.

Now, consider a lookback option written on the m:th asset with payoff function

$$F_K(S) = \max(\max_{t \in M} S_t^{(m)} - K, 0),$$

where $M \subseteq [0, T]$ and $K \geq 0$. Clearly, if $\theta \leq 1$ then

$$\| F_K(S) \|_1 \geq \| \max(\theta S_T^{(m)} - K, 0) \|_1.$$

Thus, according to Theorem 4 and Lemma 1,

$$\frac{\| F_K(S) \|_2}{\| F_K(S) \|_1} \leq \frac{\| \max(S_T^{(m)} - K, 0) \|_2}{\| \max(S_T^{(m)} - K, 0) \|_1}$$

and therefore

$$D_2\big(F_K(S)\big) \leq D_2\big(\max(S_T^{(m)} - K, 0)\big) \tag{10}$$

because

$$D_2(X) = \left(\frac{\|X\|_2^2}{\|X\|_1^2} - 1\right)^{\frac{1}{2}}$$

provided $E^Q[X] > 0$.

Since the right hand side in equation (10) can be evaluated analytically we may easily obtain an upper bound for $D_2(F_K(S))$ in the case that $F_K(S)$ is the payoff of a lookback option with fixed strike.

However, there are two disadvantages with this approach. Firstly, the right hand side in equation (10) may be very large since

$$D_p\big(\max(\theta S_T^{(m)} - K, 0)\big) \to \infty, \quad \text{as } K \to \infty,$$

for any $\theta > 0$ and any $p \geq 2$. Moreover, for an arbitrary payoff function $F_K(S)$, $F \in \mathcal{C}_K$, it may be very difficult to find an upper bound for θ other than zero such that

$$\|F_K(S)\|_1 \geq \|\max(\theta S_T^{(m)} - K, 0)\|_1.$$

However, there is a way to get around these problems. If $F_K \in \mathcal{C}_K$ then

$$F_K = \max(F - K, 0) = \max(F, K) - K$$

for some $F \in \mathcal{L}$. Thus, to price the option with payoff $X = F_K(S)$ we only need to estimate the expectation of the random variable

$$Y = \max(F(S), K).$$

It is easy to find an upper bound for $D_p(Y)$. Namely, since $\max(F, K) \in \mathcal{C}_0$ it follows

$$\frac{\|Y\|_p}{\|Y\|_1} \leq \frac{\|S_T^{(m)}\|_p}{\|S_T^{(m)}\|_1} = e^{\frac{1}{2}(p-1)\sigma_m^2 T}$$

which yields, in combination with the Minkowski inequality in the case $2 < p < \infty$,

$$D_p(Y) \leq \begin{cases} \big(e^{\sigma_m^2 T} - 1\big)^{\frac{1}{2}}, & \text{if } p = 2, \\ e^{\frac{1}{2}(p-1)\sigma_m^2 T} + 1, & \text{if } 2 < p < \infty. \end{cases} \tag{11}$$

Remarkably enough, the estimate is only dependent of p, the greatest volatility and time to maturity.

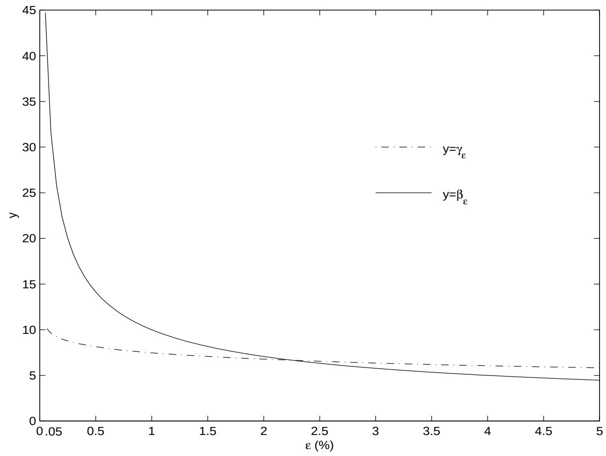

Figure 1: The graph of $\epsilon \mapsto \beta_\epsilon$ and $\epsilon \mapsto \gamma_\epsilon$ (cf. equation (12)) for $0.05\% \leq \epsilon \leq 5\%$.

Next we will consider the relative error for the Monte Carlo simulation of $E^Q[Y]$ and give two numerical examples, see the Figures 1 and 2. The first example will compare the value of the error constants C_ϵ and $C_\epsilon^{(r)}$, given in Theorem 3, for varying ϵ. Recall here that Theorem 3, among other things, stated that the upper endpoint of a $100(1-\epsilon)$ % confidence interval for the relative error in the Monte Carlo estimation of $E^Q[Y]$ is bounded by

$$\frac{\min(C_\epsilon, C_\epsilon^{(r)})}{\sqrt{n}}$$

where n is the number of simulations,

$$C_\epsilon = \frac{D_2(Y)}{\sqrt{\epsilon}} \quad \text{and} \quad C_\epsilon^{(r)} = \frac{2r_p}{\epsilon^{1/p}} \max\Big(D_2(Y),\, n^{\frac{1}{p}-\frac{1}{2}}\, D_p(Y)\Big).$$

If we assume that the option parameters are $\sigma_m = 0.3$ and $T = 1$ and that $n \geq 10^4$, then the estimate in equation (11) and some calculations give

$$\max\Big(D_2(Y),\, n^{\frac{1}{p}-\frac{1}{2}}\, D_p(Y)\Big) \leq \big(e^{\sigma_m^2 T} - 1\big)^{\frac{1}{2}}, \quad 4 \leq p \leq 70.$$

Thus, if $n \geq 10^4$ then

$$C_\epsilon^{(r)} \leq \frac{2r_p}{\epsilon^{1/p}} \big(e^{\sigma_m^2 T} - 1\big)^{\frac{1}{2}}, \quad 4 \leq p \leq 10.$$

Since the estimate in equation (11) also yields that

$$C_\epsilon \leq \frac{1}{\sqrt{\epsilon}} \big(e^{\sigma_m^2 T} - 1\big)^{\frac{1}{2}}$$

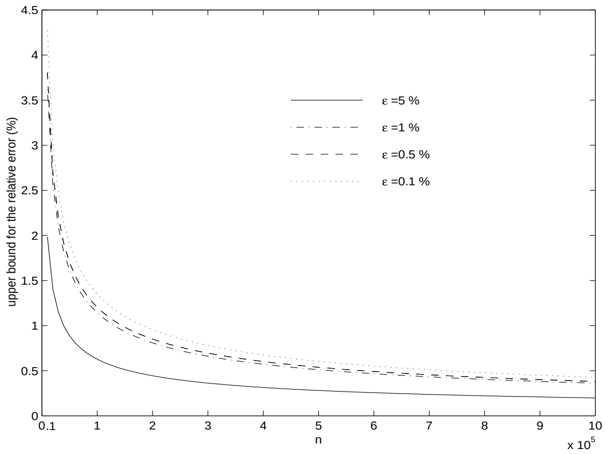

Figure 2: An upper bound for the relative error in the Monte Carlo estimation of the quantity $E^Q[Y]$ as a function of the number of simulations n. The option parameters are $\sigma_m = 0.3$ and $T = 1$. The value of n varies between 10^4 and 10^6.

it would be of interest to compare the numbers

$$\beta_\epsilon = \frac{1}{\sqrt{\epsilon}} \quad \text{and} \quad \gamma_\epsilon = \min_{p=4,5,\dots,10} \frac{2r_p}{\epsilon^{1/p}} \tag{12}$$

as functions of $\epsilon > 0$. The reason that the minimum is taken over the set $\{4, 5, \dots, 10\}$ is simply that we have computed the value of r_p for these numbers, see Table 1 in Section 2. Of course, one could include more numbers in the set $\{4, 5, \dots, 10\}$ and thereby obtain a smaller value of γ_ϵ. However, this will not radically change the value of γ_ϵ, at least not for interesting values of ϵ, say between 0.05% and 5%.

Figure 1 shows the values of β_ϵ and γ_ϵ for $0.05\% \leq \epsilon \leq 5\%$. As the figure shows, if ϵ is approximately greater than 2% then $\beta_\epsilon < \gamma_\epsilon$, but if ϵ is smaller than 2% then $\beta_\epsilon > \gamma_\epsilon$. Moreover, if ϵ is close to 0 then γ_ϵ is considerably much smaller than β_ϵ.

The next example, which is presented in Figure 2, describes the number of simulations that is required to obtain a certain accuracy of the Monte Carlo estimation of the quantity $E^Q[Y]$. According to the previous discussion we may bound the upper endpoint of the confidence interval of degree $100(1-\epsilon)\%$ for the relative error by

$$\frac{\min(\beta_\epsilon, \gamma_\epsilon)}{\sqrt{n}} \left(e^{\sigma_m^2 T} - 1\right)^{\frac{1}{2}}.$$

In the example we have chosen the option parameters to be $\sigma = 0.3$ and $T = 1$, the same as in the previous example. Figure 2 shows that the relative error is smaller than 1% if $n > 3 * 10^5$ for any $\epsilon \geq 0.1\%$.

4.2 Put Options

Much of the discussion in Subsection 4.1 is also relevant for put options. It can be proved that the function

$$\theta \mapsto \frac{\| \max(K - \theta S^{(m)},\, 0)\|_p}{\| \max(K - \theta S^{(m)},\, 0)\|_1}, \quad \theta > 0,$$

is non-decreasing if $K > 0$. Moreover, every $F_K \in \mathcal{P}_K$ can be written

$$F_K = K - \min(F, K), \quad F \in \mathcal{L}.$$

Thus, to price a derivative with the payoff $F_K(S)$ it suffices to estimate the expectation of

$$Y = \min(F(S), K).$$

Since $\min(F, K) \in \mathcal{L}$ we now once again get the same estimate of $D_p\big(Y\big)$ as in equation (11). Thus, the example given in section 4.1 is relevant for put options as well.

4.3 Options with a Floating Strike Price

There is a large class of options that are not included in neither $\mathcal{C}_K$ nor $\mathcal{P}_K$, namely options with a floating strike price. That is, the fixed strike price K is replaced by $F^{(\mu)}(S)$, where $F^{(\mu)}$ is defined as in Subsection 3.1. This subsection discusses a method to estimate the relative error in the Monte Carlo pricing of these options, a method which has certain similarities with the use of control variates in the Monte Carlo method, see e.g. [7].

Firstly, consider the payoff functions

$$X = \max\big(F(S) - F^{(\mu)}(S), 0\big) \quad \text{or} \quad X = \max\big(F^{(\mu)}(S) - F(S), 0\big),$$

where $F \in \mathcal{L}$. For instance, if each measure μ_j, $j = 1, \ldots, m$, is a positive linear combination of Dirac measures on $[0, T]$, then $E^Q[F^{(\mu)}(S)]$ can easily be evaluated analytically and therefore it suffices to estimate the expectation of

$$Y = \max\big(F(S), F^{(\mu)}(S)\big) \quad \text{or} \quad Y = \min\big(F(S), F^{(\mu)}(S)\big).$$

As previously, the value of $D_p(Y)$ can be bounded as in equation (11).

4.4 A Remark on Barrier Options

Another large class of options which is not included in $\mathcal{C}_K$ or $\mathcal{P}_K$ are contracts which have a discontinuous payoff, that is, the payoff $X = F(S)$ where $F \notin C([0, T]; \mathbb{R}^m_+)$. Most barrier options are examples of such contracts. This subsection describes a method which gives an estimate of the relative error for some barrier options, namely those barrier options which have a bounded payoff.

Consider for instance an up-and-out call with payoff

$$X = \max(S_T - K, 0)1_{\{\tau > T\}}$$

where $K \geq 0$, S_t is defined as in subsection 3.3 and

$$\tau = \inf\{t \in M \mid S_t > H(t)\},$$

where $M \subseteq [0,T]$, $T \in M$, and $H : M \to (0, \infty)$. For simplicity, below M is finite. The range of the random variable X is obviously bounded by

$$l\big(X\big) = H(T) - K.$$

Moreover, one can easily find a lower bound α_{min} for $\alpha = E^Q[X]$, namely

$$\alpha \geq E^Q\big[\max(S_T - K, 0)1_{\{\max_{0 \leq t \leq T} S_t < H_{min}\}}\big] = \alpha_{min}$$

where $H_{min} = \min_{t \in M} H(t)$. Note that α_{min} can be evaluated analytically by using well-known formulas, see e.g. Rich [16]. Thus, Theorem 3 implies the error estimate

$$Q\left(R_n \leq \frac{C_\epsilon^{(h)}}{\sqrt{n}}\right) \geq 1 - \epsilon$$

where

$$C_\epsilon^{(h)} \leq \sqrt{\frac{\ln(2/\epsilon)}{2}}\,\frac{H(T) - K}{\alpha_{min}}.$$

Whether or not this is a good estimate depends mainly on the value of α_{min}.

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