Zero-temperature dynamics for the ferromagnetic Ising model on random graphs

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Abstract

We consider Glauber dynamics at zero temperature for the ferromagnetic Ising model on the usual random graph model on N vertices, with on average γ edges incident to each vertex, in the limit as $N \to \infty$. Based on numerical simulations, Svenson [1] reported that the dynamics fails to reach a global energy minimum for a range of values of γ. The present paper provides a mathematically rigorous proof that this failure to find the global minimum in fact happens for *all* $\gamma > 0$. A lower bound on the residual energy is also given.

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1 Introduction

To find the global minimum in a complicated energy landscape using a local search algorithm may be a difficult task. One example where this is well-known is spin glasses, whose ground states are hard to find or even to characterize. Perhaps surprisingly, local search algorithms based on single spin flips (i.e., Glauber dynamics) may also fail to find the global minumum in a model as simple the purely ferromagnetic Ising model. In particular, this may happen when the search is carried out at zero temperature, meaning that moves that increase the energy are not merely discouraged, but even disallowed. Svenson [1] studied this phenomenon for the Ising model on a random graph, and the purpose of this note is to shed some further light on his findings. For other interesting aspects of zero-temperature dynamics in the ferromagnetic Ising model, see, e.g., Camia, De Santis and Newman [2].

To define our context, we recall the usual models of random graphs. We start with a complete graph on N vertices, with one edge linking each of the $\binom{N}{2}$ pairs of vertices; let us denote this graph by $G_N = (V_N, E_N)$. Following the terminology of Janson, Łuczak and Ruciński [3], we define binomial and uniform random graphs as follows.

- The **binomial random graph**, denoted $G_{\text{bin}}(N,p)$, with $p \in [0,1]$, is the random graph obtained from G_N by removing each edge with probability $(1-p)$, thus keeping it with probability p, and doing this independently for each edge.

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- The **uniform random graph**, denoted $G_{\text{unif}}(N, M)$, with $M \in \{0, \ldots, \binom{n}{2}\}$, is the graph obtained by considering all

$$\binom{\binom{N}{2}}{M}$$

possible subgraphs of G_N with the property of containing all N vertices and exactly M edges, and picking one of these subgraphs at random, i.e., according to uniform distribution.

These two models are closely related – a fact that we will exploit later. The most interesting way to scale the parameters p and M as $N \to \infty$ is to fix a $\gamma > 0$ and to let $p = \frac{\gamma}{N-1}$ and $M = \frac{\gamma N}{2}$, respectively in the two models. This keeps the expected number of retained edges incident to a given vertex constant equal to γ in both models. One reason to use this sort of scaling is that a phase transition takes place at $\gamma = 1$, in that the probability of having a "giant" connected component, containing a nonvanishing fraction of the vertices as $N \to \infty$, is 0 if $\gamma \leq 1$ and 1 if $\gamma > 1$. See [3] for this and much more precise information on this phase transition.

For the uniform random graph with such scaling, we need to take care of the annoying detail that $\frac{\gamma N}{2}$ is not always an integer. To keep things well-defined in this case too, we set $G_{\text{unif}}(N, \frac{\gamma N}{2})$ to be simply $G_{\text{unif}}(N, \lfloor \frac{\gamma N}{2} \rfloor)$, i.e., we drop the noninteger part of $\frac{\gamma N}{2}$.

The **ferromegnetic Ising model** is a certain correlated assignment of the values $+1$ (spin up) and -1 (spin down) to the vertices of a graph $G = (V, E)$, in a way that favors configurations whith a lot of agreement between neighboring vertices. **Zero-temperature dynamics** for the ferromagnetic Ising model on G is a discrete-time $\{+1, -1\}^V$-valued Markov chain with transition mechanism as follows: At each integer time, a vertex $v \in V$ is chosen at random (uniformly), and then the spin at this vertex is flipped (changed from $+1$ to -1 or vice versa) if and only if such a flip does not cause the spin at v to be aligned with strictly fewer vertices than before. (This sort of dynamics may also be defined in continuous time; it will be easy to see that our results below go through irrespectively of whether time is discrete and continuous.)

The **energy** of a spin configuration $\xi \in \{+1, -1\}^V$ simply counts the number of edges in the graph whose endvertices take opposite spin values. Following [1], we shall normalize the energy by the number $|V|$ of vertices, and therefore define the energy D of a spin configuration $\xi \in \{+1, -1\}^V$ as

$$D(\xi) = |V|^{-1} \sum_{\langle x,y \rangle \in E} \frac{1 - \xi(x)\xi(y)}{2} . \tag{1}$$

Clearly, the energy is decreasing as a function of time, under zero-temperature dynamics. What Svenson [1] did was to study in more detail how the energy behaves starting from a random spin configuration on a random graph chosen according to $G_{\text{unif}}(N, \frac{\gamma N}{2})$. By a random spin configuration, we here mean that the initial spin at each vertex is $+1$ or -1 with probability $\frac{1}{2}$ each, independently for each vertex. In particular, does the energy reach its global minimum $D = 0$? Based on numerical simulations, Svenson concluded that for large N, the answer to this question seems to depend on the value of γ. More precisely, he observed that for small or large γ, the global minimum is reached, while for a range of intermediate values, the dynamics fails to find this minimum. It turns out that this apparent qualitative dependence on γ is just a consequence of N not being sufficiently large in the simulations. In one of our main results (Theorem 1.2 below), we shall show, with mathematical rigor, that, in the limit as $N \to \infty$, the energy D fails to approach 0 for *any* nonzero value of γ.

Write D_t for the energy at time t of the zero-temperature dynamics. Since D_t is decreasing and bounded, it has a well-defined limit as $t \to \infty$; we denote this limit by D_∞. Furthermore, when considering a sequence of graphs indexed by their number N of vertices, we write $D_\infty^{(N)}$ for the limiting energy as $t \to \infty$ in the graph with N vertices. Our main results are as follows.

Theorem 1.1 *Fix $\gamma > 0$, and pick, for each positive integer N, a random graph according to $G_{\text{bin}}(N, \frac{\gamma}{N-1})$, and run zero-temperature dynamics starting from a random spin configuration, chosen by giving each vertex independently spin $+1$ or -1 with probability $\frac{1}{2}$ each. Then, for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mathbf{P}\left(D_\infty^{(N)} > \frac{\gamma^5 e^{-6\gamma}}{256} - \varepsilon\right) = 1\,. \tag{2}$$

Theorem 1.2 *Fix $\gamma > 0$, and pick, for each positive integer N, a random graph according to $G_{\text{unif}}(N, \frac{\gamma N}{2})$, and run zero-temperature dynamics starting from a random spin configuration, chosen by giving each vertex independently spin $+1$ or -1 with probability $\frac{1}{2}$ each. Then, for any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mathbf{P}\left(D_\infty^{(N)} > \frac{\gamma^5 e^{-6\gamma}}{256} - \varepsilon\right) = 1\,.$$

Neither of these results are sharp: the lower bound $\frac{\gamma^5 e^{-6\gamma}}{256}$ on the limiting energy can be improved in both theorems, by more careful considerations along the lines of the proofs presented here. We conjecture that the limiting energy is asymptotically nonrandom as $N \to \infty$, in the sense that there exists a deterministic function $f(\gamma)$ such that for any $\varepsilon > 0$ we have

$$\lim_{N\to\infty} \mathbf{P}(f(\gamma) - \varepsilon < D_\infty^{(N)} < f(\gamma) + \varepsilon) = 1$$

in both random graph models G_{bin} and G_{unif}. To actually compute $f(\gamma)$ seems like a complicated combinatorial problem.

The remaining sections are organized as follows. In Section 2 we treat the case G_{bin} of binomial random graphs – which is slighly easier to handle compared to G_{unif} due to the independence between different edges – and arrive at a proof of Theorem 1.1. Then, in Section 3, we show how this result can be carried over to the case of uniform random graphs, thus proving Theorem 1.2. Finally, in Section 4, we indicate some generalizations of these results that follow from easy modifications of our proofs.

2 The binomial case

To see that it is indeed possible for the zero-temperature dynamics to get stuck in a configuration whose energy is nonzero, consider the graph on 6 vertices in Figure 1. If we run zero-temperature dynamics on this graph, and the spin configuration at some time T is as in the figure, then that spin configurations will remain for all $t \geq T$, so that (using the defining formula (1)) we get $D_\infty = D_t = \frac{1}{6}$.

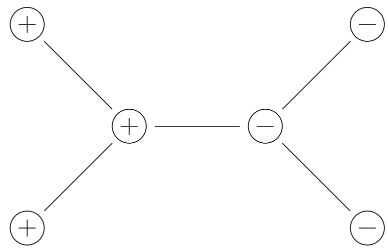

Figure 1: An example of graph $G = (V, E)$ with a spin configuration $\xi \in \{-1, 1\}^V$ that is stuck despite the energy being nonzero. Any change of a single spin variable will result in a configuration with strictly larger energy. The same example appeared in [1, Fig. 2].

The idea of our proof is to show that for large N, the random graph $G_{\text{bin}}(N, \frac{\gamma}{N-1})$ contains (with probability close to 1) an abundance of connected components shaped as the graph in Figure 1, and that a substantial fraction of these start off in the spin configuration indicated in the figure. Each of those connected components will contribute a small share to the final energy $D_\infty^{(N)}$, and these contributions will (asymptotically) add up to the quantity $\frac{\gamma^5 e^{-6\gamma}}{256}$ in (2).

The key lemma is the following Law of Large Numbers-type result for certain connected components in the random graph $G_{\text{bin}}(N, \frac{\gamma}{N-1})$. For an edge $e \in E_N$, define the random variable H_e to be 1 if, in the random graph, e is the "center bar" (i.e., the horizontal bar in the letter "H") of a connected component that contains exactly 6 vertices and has the H-shaped topology of the graph in Figure 1. Also let $H = \sum_{e \in E_N} H_e$. In other words, H is the number of connected components in the random graph that have 6 vertices and the topology in Figure 1. As with the energy, we write $H_e^{(N)}$ and $H^{(N)}$ when the graph is the N:th one in a sequence indexed by N.

Lemma 2.1 *Fix $\gamma > 0$, and pick a random graph according to $G_{\text{bin}}(N, \frac{\gamma}{N-1})$ for each N. For any $\varepsilon > 0$, we have*

$$\lim_{N\to\infty} \mathbf{P}\left(\tfrac{1}{8}\gamma^5 e^{-6\gamma} - \varepsilon < \frac{H^{(N)}}{N} < \tfrac{1}{8}\gamma^5 e^{-6\gamma} + \varepsilon \right) = 1\,.$$

Proof: Let us begin by checking that the expected value $\mathbf{E}\left[\frac{H^{(N)}}{N}\right]$ tends to $\frac{1}{8}\gamma^5 e^{-6\gamma}$ as $N \to \infty$. For an edge $e \in E_N$ to form the horizontal bar in an H-shaped connected component on 5 vertices, there are $\frac{(N-2)(N-3)}{2} \times \frac{(N-4)(N-5)}{2}$ different ways to choose the other four edges, and for each such choice we have probability $p^5(1-p)^{6(N-6)+10}$ that all five edges but none of their neighboring edges are present. Hence

$$\mathbf{E}\left[H_e^{(N)}\right] = \frac{(N-2)(N-3)(N-4)(N-5)}{4} p^5 (1-p)^{6N-26} \tag{3}$$

and by summing over all $\binom{n}{2}$ edges in G_N, we get

$$\begin{aligned} \mathbf{E}\left[H^{(N)}\right] &= \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)}{8} p^5 (1-p)^{6N-26} \\ &= \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)}{8(N-1)^5} \gamma^5 \left(1 - \frac{\gamma}{N-1}\right)^{6N-26} . \end{aligned}$$

Using also the observation that

$$\lim_{N\to\infty}\left(1-\frac{\gamma}{N-1}\right)^{6N-26} = e^{-6\gamma}\,, \tag{4}$$

we obtain

$$\lim_{N\to\infty}\mathbf{E}\left[\frac{H^{(N)}}{N}\right] = \tfrac{1}{8}\gamma^5 e^{-6\gamma}\,. \tag{5}$$

The idea is now to establish that the variance $\mathbf{Var}\left[\frac{H^{(N)}}{N}\right] = \mathbf{E}\left[\left(\frac{H^{(N)}}{N} - \mathbf{E}\left[\frac{H^{(N)}}{N}\right]\right)^2\right]$ tends to 0 as $N\to\infty$. Once that is done, Chebyshev's inequality yields

$$\mathbf{P}\left(\left|\frac{H^{(N)}}{N} - \mathbf{E}\left[\frac{H^{(N)}}{N}\right]\right| \ge \frac{\varepsilon}{2}\right) \le \frac{4\mathbf{Var}\left[\frac{H^{(N)}}{N}\right]}{\varepsilon^2} \to 0 \text{ as } N\to\infty\,. \tag{6}$$

For any $\delta>0$, we can then use (5) and (6) to deduce that for all sufficiently large N we have

$$\left|\mathbf{E}\left[\frac{H^{(N)}}{N}\right] - \tfrac{1}{8}\gamma^5 e^{-6\gamma}\right| < \frac{\varepsilon}{2}$$

and

$$\mathbf{P}\left(\left|\frac{H^{(N)}}{N} - \mathbf{E}\left[\frac{H^{(N)}}{N}\right]\right| < \frac{\varepsilon}{2}\right) > 1-\delta$$

so that

$$\mathbf{P}\left(\left|\frac{H^{(N)}}{N} - \tfrac{1}{8}\gamma^5 e^{-6\gamma}\right| < \varepsilon\right) > 1-\delta\,.$$

Sending $\delta\to 0$ then proves the lemma.

It remains to show that

$$\lim_{N\to\infty}\mathbf{Var}\left[\frac{H^{(N)}}{N}\right] = 0\,. \tag{7}$$

To this end, we first calculate $\mathbf{E}\left[H_e^{(N)}H_f^{(N)}\right]$ for $e,f\in E_G$. Similar considerations as those leading to (3) yield

$$\mathbf{E}\left[H_e^{(N)}H_f^{(N)}\right] = \begin{cases} \frac{(N-2)(N-3)(N-4)(N-5)}{4}p^5(1-p)^{6N-26} & \text{if } e=f \\ 0 & \text{if } e\sim f \\ \frac{(N-4)(N-5)\cdots(N-11)}{16}p^{10}(1-p)^{12(N-12)+56} & \text{otherwise} \end{cases}$$

where $e\sim f$ means that e and f share one vertex. Hence

$$\begin{aligned}\mathbf{E}\left[\left(H^{(N)}\right)^2\right] &= \sum_{e\in E_G}\sum_{f\in E_G}\mathbf{E}\left[H_e^{(N)}H_f^{(N)}\right] = \\ &= \frac{N(N-1)\cdots(N-5)}{8}p^5(1-p)^{6N-26} + \frac{N(N-1)\cdots(N-11)}{64}p^{10}(1-p)^{12N-88}\end{aligned}$$

so that

$$\mathbf{Var}\left[H^{(N)}\right] = \mathbf{E}\left[\left(H^{(N)}\right)^2\right] - \left(\mathbf{E}\left[H^{(N)}\right]\right)^2 =$$

$$\begin{aligned}
&= \frac{N(N-1)\cdots(N-5)}{8} p^{10}(1-p)^{12N-52} \\
&\quad \times \left(\frac{1}{p^5(1-p)^{6N-26}} + \frac{(N-6)(N-7)\cdots(N-11)}{8}(1-p)^{-36} - \frac{N(N-1)\cdots(N-5)}{8}\right) \\
&= \frac{N(N-1)\cdots(N-5)}{8(N-1)^{10}} \gamma^{10}(1-\tfrac{\gamma}{N-1})^{12N-52} \\
&\quad \times \left(\frac{(N-1)^5}{\gamma^5(1-\frac{\gamma}{N-1})^{6N-26}} + \frac{(N-6)(N-7)\cdots(N-11)}{8}(1-\tfrac{\gamma}{N-1})^{-36} - \frac{N(N-1)\cdots(N-5)}{8}\right).
\end{aligned}$$

By expanding the last two polynomials in N, and using (4) again, we get that

$$\begin{aligned}
\mathbf{Var}\left[H^{(N)}\right] &= \frac{\gamma^{10}e^{-12\gamma}}{8N^4}\left(\frac{(N-1)^5}{\gamma^5 e^{-6\gamma}} + \frac{\left(1-\frac{\gamma}{N-1}\right)^{-36} - 1}{8} N^6 + O(N^5)\right) \\
&= \left(\left(1-\tfrac{\gamma}{N-1}\right)^{-36} - 1\right) O(N^2) + O(N)
\end{aligned}$$

as $N \to \infty$. Note also that $\lim_{N\to\infty}\left(\left(1-\frac{\gamma}{N-1}\right)^{-36} - 1\right) = 0$, so that

$$\lim_{N\to\infty} \frac{\mathbf{Var}\left[H^{(N)}\right]}{N^2} = 0\,. \tag{8}$$

But

$$\mathbf{Var}\left[\frac{H^{(N)}}{N}\right] = \frac{\mathbf{Var}\left[H^{(N)}\right]}{N^2},$$

so (8) immediately implies (7), and the proof is complete. □

Now that we are equipped with Lemma 2.1, the proof of Theorem 1.1 is quite simple.

Proof of Theorem 1.1: Consider the initial random assignment of spins to the random graph. Each connected component of the type studied in Lemma 2.1 has probability 2^{-6} of getting precisely the spin configuration in Figure 1, and probability 2^{-6} of getting of getting the "opposite" spin configuration, obtained by flipping all six spins. Each of these two configurations has a nonzero energy and is a fixed-point of the zero-temperature dynamics. The probabilites of the two configurations add up to 2^{-5}. The usual Law of Large Numbers therefore ensures that, given $\varepsilon > 0$ and sufficiently many such H-shaped components, the probability that at least a fraction $2^{-5} - \varepsilon$ of them start off in such a spin configuration can be made arbitrarily close to 1. If we write $\tilde{H}^{(N)}$ for the number of such connected components receiving the desired kind of initial spin configuration, then we can combine Lemma 2.1 with the usual Law of Large Numbers to conclude that

$$\lim_{N\to\infty} \mathbf{P}\left(\left|\frac{H^{(N)}}{N} - \tfrac{1}{8}\gamma^5 e^{-6\gamma}\right| < \varepsilon \text{ and } \left|\frac{\tilde{H}^{(N)}}{N} - \frac{H^{(N)}}{32N}\right| < \varepsilon\right) = 1\,,$$

for given $\varepsilon > 0$. Hence,

$$\lim_{N\to\infty} \mathbf{P}\left(\left|\frac{\tilde{H}^{(N)}}{N} - \frac{\gamma^5 e^{-6\gamma}}{256}\right| < \tfrac{33\varepsilon}{32}\right) = 1\,.$$

Sice ε was arbitrary we may replace $\frac{33\varepsilon}{32}$ by ε in the last expression, so that in particular we have

$$\lim_{N\to\infty} \mathbf{P}\left(\frac{\tilde{H}^{(N)}}{N} > \frac{\gamma^5 e^{-6\gamma}}{256} - \varepsilon\right) = 1\,.$$

But $\frac{\tilde{H}^{(N)}}{N} \geq D_\infty^{(N)}$, so the theorem follows. □

3 The uniform case

In this section we carry over our analysis from the case of binomial random graphs to that of uniform random graphs. The key result that we need to obtain is the following uniform random graph analogue of Lemma 2.1.

Lemma 3.1 *Fix $\gamma > 0$, and pick a random graph according to $G_{\text{unif}}(N, \frac{\gamma N}{2})$ for each N. For any $\varepsilon > 0$, we have*

$$\lim_{N\to\infty} \mathbf{P}\left(\tfrac{1}{8}\gamma^5 e^{-6\gamma} - \varepsilon < \frac{H^{(N)}}{N} < \tfrac{1}{8}\gamma^5 e^{-6\gamma} + \varepsilon\right) = 1\,.$$

Once this lemma has been established, we get a proof of Theorem 1.2 by copying the proof of Theorem 1.1 word by word, with the sole exception of referring to Lemma 3.1 rather than to Lemma 2.1.

The precise connection between binomial and uniform random graphs is as follows. The distribution of a binomial random graph $G_{\text{bin}}(N,p)$ conditional on having exactly M edges present is (regardless of p), exactly that of the uniform random graph $G_{\text{unif}}(N,M)$. This property is immediate from the invariance of both distributions under permutation of the edge set E_N.

Another obvious property of the uniform random graph model, that will be useful in the proof of Lemma 3.1, is the following. If we pick a random graph according to $G_{\text{unif}}(N,M)$, and then add one edge at random, picked uniformly from the set of absent edges, then the resulting graph is distributed according to $G_{\text{unif}}(N,M+1)$. Likewise, we can go from $G_{\text{unif}}(N,M+1)$ to $G_{\text{unif}}(N,M)$ by deleting one of the present edges, chosen uniformly at random. These insertion and deletion procedures can of course be iterated, allowing us to go from $G_{\text{unif}}(N,M_1)$ to $G_{\text{unif}}(N,M_2)$ for any $M_1, M_2 \in \{0,1,\ldots,\binom{n}{2}\}$.

Proof of Lemma 3.1: This proof will involve two random graphs $G_{\text{bin}}(N, \frac{\gamma}{N-1})$ and $G_{\text{unif}}(N, \frac{\gamma N}{2})$. We will count the number of H-shaped connected components on six vertices in each of them, so that the notation $H^{(N)}$ risks getting overloaded. We therefore write $H_{\text{bin}}^{(N)}$ and $H_{\text{unif}}^{(N)}$ for these counts in the two different graphs.

Start from the binomial random graph $G_{\text{bin}}(N, \frac{\gamma}{N-1})$, and write $X_{\text{bin}}^{(N)}$ for the (random) number of edges in that graph. By the discussion preceding this proof, we can then obtian the uniform random graph $G_{\text{unif}}(N, \frac{\gamma N}{2})$ as follows:

- If $X_{\text{bin}}^{(N)} > \lfloor \frac{\gamma N}{2} \rfloor$ then delete one edge chosen at random (uniformly), and repeat until we are left with exactly $\lfloor \frac{\gamma N}{2} \rfloor$ edges.

- If $X_{\text{bin}}^{(N)} < \lfloor \frac{\gamma N}{2} \rfloor$ then add one edge chosen at random (uniformly), and repeat until we have exactly $\lfloor \frac{\gamma N}{2} \rfloor$ edges.

Note that $\mathbf{E}\left[X_{\text{bin}}^{(N)}\right] = \binom{n}{2}\frac{\gamma}{N-1} = \frac{\gamma N}{2}$. Furthermore, by the Law of Large Numbers for the binomial distribution, we have, for any $\varepsilon > 0$, that

$$\lim_{N\to\infty} \mathbf{P}\left(\left|\frac{X_{\text{bin}}^{(N)}}{N} - \frac{\gamma}{2}\right| < \varepsilon\right) = 1\,,$$

or in other words

$$\lim_{N\to\infty} \mathbf{P}\left(\left|X_{\text{bin}}^{(N)} - \frac{\gamma N}{2}\right| < N\varepsilon\right) = 1\,.$$

This means that the probability of flipping at most $N\varepsilon$ edges when going from the binomial to the uniform random graph as described above, tends to 1 as $N \to \infty$. It is easy to see that each edge that is turned on or off can affect the number of H-shaped connected components of the desired kind by at most 2. Hence,

$$\lim_{N\to\infty} \mathbf{P}\left(\left|H_{\text{unif}}^{(N)} - H_{\text{bin}}^{(N)}\right| < 2N\varepsilon\right) = 1\,.$$

Dividing by N, and combining this information with Lemma 2.1, we get

$$\lim_{N\to\infty} \mathbf{P}\left(\left|\frac{H_{\text{unif}}^{(N)}}{N} - \frac{H_{\text{bin}}^{(N)}}{N}\right| < 2\varepsilon \text{ and } \left|\frac{H_{\text{bin}}^{(N)}}{N} - \tfrac{1}{8}\gamma^5 e^{-6\gamma}\right| < \varepsilon\right) = 1$$

so that

$$\lim_{N\to\infty} \mathbf{P}\left(\left|\frac{H_{\text{unif}}^{(N)}}{N} - \tfrac{1}{8}\gamma^5 e^{-6\gamma}\right| < 3\varepsilon\right) = 1\,.$$

Since ε was arbitrary, we can replace 3ε by ε, and the lemma is established. □

4 Some additional results

Let us finally point out a couple of easy generalizations of the results obtained in the previous sections.

1. Svenson [1] reports having observed similar phenomena for zero-temperature dynamics for the ferromagnetic Potts model, as for the Ising model. In the Potts model, there are $q \geq 3$ different spin values rather than the two spin values of the Ising model. If we define energy, analogously to the Ising case, as $\frac{1}{N}$ times the number of edges having different spins at its endvertices, then Theorems 1.1 and 1.2 carry over to the Potts case with

   $$\lim_{N\to\infty} \mathbf{P}\left(D_\infty^{(N)} > \frac{\gamma^5 e^{-6\gamma}(q-1)}{8q^5} - \varepsilon\right) = 1\,.$$

   This follows by applying Lemmas 2.1 and 3.1 similarly as in the proof of Theorem 1.1.

2. Write $I^{(N)}$ for the number of connected components in the random graph having 7 vertices and the topology indicated in Figure 2. By similar calculations as in the proofs of Lemmas 2.1 and 3.1, we get a Law of Large Numbers for $I^{(N)}$, stating that $\frac{I^{(N)}}{N}$ is close to $\frac{1}{8}\gamma^6 e^{-7\gamma}$ for large N, both in the binomial and in the uniform random graph. If the initial spin configuration on such a connected component is as in Figure 2, where "?" may be either a + or a −, then the three vertices to the left and the three to the right will stay put forever, while the middle vertex will keep changing its mind forever – a so-called blinker. For large N, close to a fraction $\frac{1}{32}$ of the connected components with the topology in question will start in such a spin configuration. Hence, the fraction of blinkers among the N vertices will, asymptotically for large N, be at least $\frac{\gamma^6 e^{-7\gamma}}{256}$. This bound, like the ones in Theorems 1.1 and 1.2, is open to improvement.

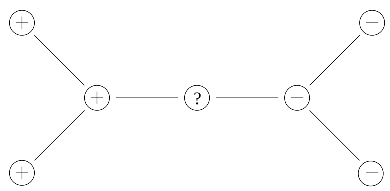

Figure 2: Suppose that we start zero-temperature dynamics on this graph with the indicated initial spin configuration, where the "?" may be either a + or a −. Then the three vertices to the left, as well as the three vertices to the right, will keep their spin values forever, whereas the middle vertex will keep flipping back and forth between a plus spin and a minus spin indefinitely.

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