RESIDUE CURRENTS OF CAUCHY-FANTAPPIE-LERAY TYPE AND IDEALS OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We define a residue current of a holomorphic mapping, or more generally a holomorphic section to a holomorphic vector bundle, by means of Cauchy-Fantappie-Leray type formulas, and prove that a holomorphic function that annihilates this current belongs to the corresponding ideal locally. We also prove that the residue current coincides with the Coleff-Herrera current in the case of a complete intersection. The construction is of global nature and we also provide some geometric applications.

1. INTRODUCTION

Let $f = (f_1, \dots, f_m)$ be a holomorphic mapping at $0 \in \mathbb{C}^n$. If f is a complete intersection it is wellknown that a holomorphic function ϕ belongs to the ideal (f) if an only if $\phi R^f = 0$, where R^f is the Coleff-Herrera residue current associated to f. For more general f no such characterization of the ideal (f) is known. In this paper we introduce a current R^f for an arbitrary f which coincides with the Coleff-Herrera current when f is a complete intersection, and such that $\phi R^f = 0$ implies that ϕ belongs to the ideal.

It turns out to be natural to adopt an invariant point of view so we assume that f is a holomorphic section to the dual bundle E^* of a holomorphic m-bundle $E \to X$ over a complex n-manifold X. We then have mappings on the exterior algebra over E,

$$\delta_f \colon \Lambda^{\ell+1}E \to \Lambda^{\ell}E,$$

where δ_f is contraction (interior multiplication) with $2\pi i f$. In particular, if $e_1, \dots, e_m$ is a local (holomorphic) frame for E, and $e_1^*, \dots, e_m^*$ is the dual frame, then $f = \sum f_j e_j^*$, and if $\psi = \sum \psi_j e_j$ is a section to $E = \Lambda^1 E$, then

$$\delta_f \psi = 2\pi i \sum f_j \psi_j.$$

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Thus the holomorphic function ϕ belongs the the ideal $(f_1,\dots,f_m)$ locally, if and only of there is a holomorphic ψ such that $\delta_f\psi=\phi$.

To motivate the definition of the new currents let us recall briefly how this kind of division problems can be solved (when possible) by the Koszul complex. The first step is to start looking for smooth solutions or current solutions. To this end we introduce the spaces $\mathcal{E}_{0,k}(X,\Lambda^\ell E)$ of smooth sections to the exterior algebra of $E^*\oplus T^*_{0,1}$ which are $(0,k)$-forms with values in $\Lambda^\ell E$, and the corresponding spaces $\mathcal{D}'_{0,k}(X,\Lambda^\ell E)$ of currents. Thus an element $\phi\in\mathcal{E}_{0,k}(X,\Lambda^\ell E)$ can be written

$$\phi={\sum_{|I|=\ell}}'\phi_I\wedge e_I,$$

where ϕ_I are $(0,k)$-forms, and the prime denotes summation over increasing multiindices. Notice that δ_f extends to mappings $\mathcal{E}_{0,k}(X,\Lambda^\ell E)\to\mathcal{E}_{0,k}(X,\Lambda^{\ell-1}E)$ and $\mathcal{D}'_{0,k}(X,\Lambda^\ell E)\to\mathcal{D}'_{0,k}(X,\Lambda^{\ell-1}E)$. Outside

$$Y=f^{-1}(0)$$

one can clearly find a smooth solution to $\delta_f u_{1,0}=\phi$. Moreover, since δ_f and $\bar\partial$ anticommute we then have that $\delta_f\bar\partial u_{1,0}=-\bar\partial\delta_f=-\bar\partial\phi=0$ so, again at least outside Y, we can solve $\delta u_{2,1}=\bar\partial u_{1,0}$. Proceeding in this way we can successively find $u_k\in\mathcal{E}_{0,k-1}(X,\Lambda^k E)$ such that

$$\delta_f u_{1,0}=\phi,\quad \delta_f u_{k+1,k}=\bar\partial u_{k,k-1},\quad k\geq 1. \tag{1.1}$$

Suppose now that all $u_{k,k-1}$ have current extensions across Y such that the equations (1.1) still hold. In particular this means that $\bar\partial u_{m,m-1}=0$ and if, in addition, we assume that X is Stein, then by successively solving $\bar\partial v_{k,k-1}=u_{k,k-1}+\delta_f v_{k+1,k}$, we finally arrive at the holomorphic solution $\psi=u_{1,0}+\delta_f v_{2,1}$ to $\delta_f\psi=\phi$. Thus, if ϕ is not in the ideal (f) then some of the equations in (1.1) cannot hold across Y; this means that there are residues.

It is convenient to introduce some further notation. Since $\bar\partial$ and δ_f anticommute, $\mathcal{L}^{\ell,k}(X,E)=\mathcal{D}'_{0,k}(X,\Lambda^{-\ell}E)$ is a double complex and the corresponding total complex is

$$\xrightarrow{\nabla_f}\mathcal{L}^{r-1}(X,E)\xrightarrow{\nabla_f}\mathcal{L}^{r}(X,E)\xrightarrow{\nabla_f}, \tag{1.2}$$

where

$$\mathcal{L}^r(X,E)=\bigoplus_{\ell+k=r}\mathcal{L}^{\ell,k}(X,E)=\bigoplus_{\ell}\mathcal{D}'_{0,\ell+r}(X,\Lambda^\ell E)$$

and $\nabla_f=\delta_f-\bar\partial$. It is readily verified that the exterior product $\wedge$ induces a mapping

$$\mathcal{L}^r(X,E)\times\mathcal{L}^s(X,E)\to\mathcal{L}^{r+s}(X,E)$$

and that ∇_f is an antiderivation, i.e.,

$$\nabla(g\wedge h)=\nabla_f g\wedge h+(-1)^r g\wedge\nabla_f h$$

if $g \in \mathcal{L}^r(X,E)$ and $h \in \mathcal{L}^s(X,E)$. In particular, a function ϕ defines an element in $\mathcal{L}^0(X,E)$, and the system of equations (1.1) means that $u \in \mathcal{L}^{-1}(X,E)$ and

$$\nabla_f u = \phi. \tag{1.3}$$

If u is an element in $\mathcal{L}^{-1}(X,E)$, let $u_{k,k-1}$ denote the component with values in $\Lambda^k E$. Recall that $\Lambda^m E$ is a line bundle over X, the so-called determinant bundle $\det E$. Thus $u_{m,m-1}$ is a $\det E$-valued $(0,m-1)$-form (current).

Notice that if $\nabla_f u = 1$ in $X\backslash Y$, then $\nabla_f(\phi u) = \phi$ if ϕ is holomorphic. In order to find a solution to $\nabla_f u = 1$ outside Y, let us assume that E is equipped with some hermitean metric $|\cdot|$, and let s be the section to E with pointwise minimal norm such that $\delta_f s = |f|^2$. If the metric is given by the hermitean matrix σ_{jk} in the local frame e_j^*, i.e.,

$$|\xi|^2 = \sum_{jk} \sigma_{j\bar{k}} \xi_j \bar{\xi}_k,$$

for sections $\xi = \sum \xi_j e_j^*$ to E^*, then

$$s = \sum s_j e_j = \frac{1}{2\pi i}\Big(\sum_k \sigma_{j\bar{k}} \bar{f}_k\Big) e_j. \tag{1.4}$$

Since $\delta_f s = |f|^2$ is nonvanishing outside Y,

$$u = \frac{s}{\nabla_f s} = \frac{s}{\delta_f s - \bar\partial s} = \sum_\ell \frac{s \wedge (\bar\partial s)^{\ell-1}}{(\delta_f s)^\ell} = \sum_\ell \frac{s \wedge (\bar\partial s)^{\ell-1}}{|f|^{2\ell}} \tag{1.5}$$

is welldefined (observe that $\bar\partial s$ has even degree) and $\nabla_f u = 1$ there.

If $E = \mathbb{C}^m \times X$ with the trivial metric, e_j^* is the trivial frame, and $f = \sum f_j e_j^*$, then $s = \sum_1^m \bar{f}_j e_j / 2\pi i$ and we get the Bochner-Martinelli form

$$u = \sum_\ell \frac{1}{(2\pi i)^\ell} \frac{\sum \bar{f}_j e_j \wedge (\sum \bar\partial \bar{f}_j \wedge e_j)^{\ell-1}}{|f|^{2\ell}}$$

in f. The term with $\ell = m$ corresponds to the classical Bochner-Martinelli form; the full Bochner-Martinelli (or more generally Cauchy-Fantappie-Leray) form was introduced in [1] in order to construct integral formulas with weight factors in a convenient way. In the noncomplete intersection case not only the top degree term $u_{m,m-1}$ but also lower order terms give rise to residues as we will see.

Observe that if $\operatorname{Re}\lambda > m$, then the forms $|f|^{2\lambda}u$ and $\bar\partial|f|^{2\lambda} \wedge u$ are (locally) bounded and hence define currents on X. Our basic result states that the smooth form u has a current extension across Y and provides the residue current R^f.

Theorem 1.1. *Let f be a (locally nontrivial) holomorphic section to $E^* \to X$ and let $Y = \{f = 0\}$. The forms $|f|^{2\lambda}u$ and $\bar\partial|f|^{2\lambda} \wedge u$*

have analytic continuations as currents to $\mathrm{Re}\,\lambda > -\epsilon$*, and if* U *and* R^f *denote the values at* $\lambda = 0$*, then* U *is a current extension of* u *and*

$$\nabla_f U = 1 - R^f, \tag{1.6}$$

where R^f *has support on* Y*; moreover,*

$$R^f = R^f_{p,p} + \cdots + R^f_{q,q},$$

where $R^f_{j,j} \in \mathcal{D}'_{0,j}(X, \Lambda^j E)$*,* $p =$ *codim* Y *and* $q = \min(m, n)$*.*

If f is regular, then $R^f = R^f_{m,m}$ has measure coefficients, cf., Example 4, and hence $hR^f = 0$ if h is any (continuous) function that vanishes on Y. In the general case we have

Theorem 1.2. *If* h *is a holomorphic function that vanishes on* Y*, then* $\bar{h}R^f = 0$*. Moreover, if in addition* $|h| \le C|f|^k$*, then* $hR^f_{k,k} = 0$*.*

For degree reasons we thus have: if $h \in \mathcal{O}(X, \Lambda^\nu E)$ and $|h| \le C|f|^{\min(n,m-\nu)}$ locally, then $h \wedge R^f = 0$.

As intended, these theorems lead to results about ideals. To begin with we have

Corollary 1.3. *Assume that* X *is a Stein manifold,* $\phi \in \mathcal{O}(X)$ *and* $\phi R^f = 0$*. Then* ϕ *belongs to the ideal* I_f*, i.e.,* $\delta_f \psi = \phi$ *has a holomorphic solution.*

Proof. Since

$$\nabla_f(\phi U) = \nabla_f \phi \wedge U + \phi \nabla_f U = \phi(1 - R^f) = \phi,$$

thus $\nabla_f w = \phi$ has a current solution w, and therefore a holomorphic solution ψ, as in the introduction, cf., also (3.1). □

Combining Theorem 1.2 and Corollary 1.3 we get the following classical theorem due to Briancon and Skoda [6]: If f is an arbitrary holomorphic mapping, ϕ is holomorphic, and

$$|\phi| \le C|f|, \tag{1.7}$$

then $\phi^{\min(m,n)}$ belongs (locally) to the ideal I_f. If f is a complete intersection, then $R^f = R^f_{m,m}$ and hence $\nabla_f R^f = 0$ means that $\delta_f R^f = 0$. It follows that ϕ belongs to I_f if and only if $\phi R^f = 0$; this is the Passare-Dickenstein-Sessa theorem, [7] and [11]. In the non-complete intersection case, in general $\phi R^f = 0$ is not a necessary condition to belong to the ideal, see Example 2.

To formulate slightly more general statements we consider the complex

$$0 \leftarrow \mathcal{O}(X) \xleftarrow{\delta_f} \mathcal{O}(X, E) \xleftarrow{\delta_f} \mathcal{O}(X, \Lambda^2 E) \xleftarrow{\delta_f} \ldots \tag{1.8}$$

and its corresponding homology groups $H_\ell(\mathcal{O}(X, \Lambda^* E))$.

Theorem 1.4. *Let X be a Stein manifold and let f be a (nontrivial) holomorphic section to $E^* \to X$. If $\phi \in \mathcal{O}(X, \Lambda^\ell E)$ and $\delta_f \phi = 0$, then $\delta_f \psi = \phi$ has a holomorphic solution if and only if $\nabla_f w \wedge R^f = \phi \wedge R^f$ has a smooth solution w.*

Notice that $\nabla_f R^f = 0$. The last condition can be thought of as ϕ being ∇_f-exact on R^f. In particular this holds if $\phi \wedge R^f$ vanishes.

Proof. If the holomorphic solution ψ exists, then we can just take $w = \psi$ since $\nabla_f \psi = \delta_f \psi$. Conversely, if the solution w exists, then

$$\nabla_f(U \wedge \phi + R^f \wedge w) = (1 - R^f) \wedge \phi + R^f \wedge \nabla_f w = \phi,$$

and arguing as in the introduction, see also Section 3, it follows that there is a holomorphic solution to $\delta_f \psi = \phi$. □

For degree reasons we get

Corollary 1.5. *If $\ell > m - p$, then $H_\ell(\mathcal{O}(X, \Lambda^* E)) = 0$. If $\phi \in \mathcal{O}(X, \Lambda^{m-p} E)$ and $\delta_f \phi = 0$, then $\delta_f \psi = \phi$ has a holomorphic solution if and only if $\phi \wedge R^f = \phi \wedge R^f_{p,p} = 0$.*

Assuming that f is a complete intersection, i.e., $p = m$, we get back the Passare-Dickenstein-Sessa theorem. From Theorems 1.2 and 1.4 we get

Corollary 1.6. *Suppose that $\phi \in \mathcal{O}(X, \Lambda^\ell E)$ and $\delta_f \phi = 0$. If moreover $|\phi| \leq C|f|^{\min(m-\ell, n)}$, then ϕ is exact.*

When $\ell = 0$ this is (implies) the Briancon-Skoda theorem.

If f is a complete intersection, i.e., $p = m$, then it follows from Proposition 2.2 below, that $R^f = R^f_{m,m}$ is independent of the metric and hence intrinsically defined. In fact, it coincides with the Coleff-Herrera residue current.

Theorem 1.7. *Suppose that f is a complete intersection. Then in any local holomorphic frame $e_1, \ldots, e_m$ to E,*

$$R_f = \frac{1}{(2\pi i)^m} \bar\partial \frac{1}{f_m} \wedge \ldots \wedge \bar\partial \frac{1}{f_1} \wedge e_1 \wedge \ldots \wedge e_m. \tag{1.9}$$

Here the right hand side denotes the Coleff-Herrera current times $e_1 \wedge \ldots \wedge e_m$; it is an immediate consequence that this current defines an intrinsic $\det E$-valued current. More explicitly, if h is an invertible holomorphic mapping then

$$(\det h) \bar\partial \frac{1}{hf_m} \wedge \ldots \wedge \bar\partial \frac{1}{hf_1} = \bar\partial \frac{1}{f_m} \wedge \ldots \wedge \bar\partial \frac{1}{f_1}.$$

This was proved already in [8].

The local form of Theorem 1.7, and with R^f defined by the trivial metric (i.e., by the Bochner-Martinelli form), was proved in [13], but we can supply a more transparent argument. In short, we define currents V and $U \wedge V$ such that $\nabla_f V = 1 - R^f_{ch}$, where R^f_{ch} is the right hand side of (1.9), and $\nabla_f(U \wedge V) = V - U$. These two equalities together then yield the theorem.

In Section 2 we define the currents and prove Theorems 1.1 and 1.2, and in Section 4 we discuss Coleff-Herrera-Passare currents defined by analytic continuation and supply a proof of Theorem 1.7. In Section 7 we have collected some examples and applications.

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Contents

2. Residues of Cauchy-Fantappie-Leray type

The proof of Theorems 1.1 is based on the possibility to resolve singularities, i.e., Hironaka's theorem, and the following simple lemma, whose proof is obtained, e.g., just by a series of integrations by parts, see Section 6.

Lemma 2.1. *Let v be a strictly positive smooth function in $\mathbb{C}$, ψ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |s|^{2\lambda} \psi(s) \frac{ds \wedge d\bar{s}}{s^p}$$

and

$$\lambda \mapsto \int \bar\partial(v^\lambda |s|^{2\lambda}) \wedge \psi(s) \frac{ds}{s^p}$$

both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At $\lambda = 0$ they are both independent of v, and the second one is a distribution of ψ supported at the origin and they only depends on powers of $\partial/\partial s$ of the test function ψ. Moreover, if $\psi(s) = \bar{s}\phi(s)$ or $\psi = d\bar{s} \wedge \phi$, then the value of the second integral at $\lambda = 0$ is zero.

Proof of Theorem 1.1. Both the definition and the statement is clearly local and therefore we can assume that the bundle E is trivial. The proof relies on the possibility to resolve singularities locally with Hironaka's theorem. Given a small enough neighborhood U of a point in

X we have an n-dimensional complex manifold $\tilde{U}$ and a proper holomorphic mapping $\Pi\colon \tilde{U} \to U \subset X$ such that if $\tilde{Y} = \Pi^{-1}(Y)$, then Π restricted to $\tilde{U} \setminus \tilde{Y}$ is biholomorpic, and so that moreover $\tilde{Y}$ has normal crossings in $\tilde{U}$. This means that locally in $\tilde{U}$ we have that $\Pi^* f_j = u_j \mu_j$, where u_j are nonvanishing and μ_j are monomials in some local coordinates τ_j on $\tilde{U}$. To each normal crossing one can find a further resolution in a certain toric manifold, following [4] and [13], so that the pullback of one of the f_j divides all the other ones. To simplify notation, let us assume, somewhat abusively, that already μ_1 divides all the other μ_j.

Now assume that ϕ is a test form with compact support in a set U as above. Since Π is proper, the support of $\tilde{\phi} = \Pi^*\phi$ can be covered by a finite number of local coordinate neighborhoods as above. In such a coordinate neighborhood, cf., (1.4), the pullback $\Pi^* s$ is $\bar{\mu}_1$ times a smooth form, thus $\Pi^*(s \wedge (\bar\partial s)^{\ell-1})$ is $\bar{\mu}_1^\ell$ times a smooth form, and $\Pi^*|f|^2 = |\mu_1|^2 v$ where v is a strictly positive smooth function. (In fact, if $\mu_j = \mu_1 \mu_j'$ and $\mu' = \sum \mu_j' e_j^*$ then $|\mu'| > 0$ since $\mu_1' = 1$.) If $\operatorname{Re}\lambda > m$ therefore

$$\int |f|^{2\lambda} u \wedge \phi = \sum_{\ell=1}^{m} \int |f|^{2\lambda} \frac{s \wedge (\bar\partial s)^{\ell-1}}{|f|^{2\ell}} \wedge \phi$$

is a finite sum of integrals like

$$\int_\tau v^\lambda |\mu_1|^{2\lambda} \frac{\alpha \wedge \tilde{\phi}}{\mu_1^\ell} \rho, \tag{2.1}$$

where α is a smooth form and ρ is a cutoff function from some partition of unity. Since

$$\mu_1 = \tau_{j_1}^{\alpha_{j_1}} \cdots \tau_{j_k}^{\alpha_{j_k}}$$

is a monomial, an application of Lemma 2.1 for each τ_j that divides μ_1 gives the desired analytic continuation. Since

$$\nabla u^\lambda = |f|^{2\lambda} - \bar\partial |f|^{2\lambda} \wedge u \tag{2.2}$$

and clearly $|f|^{2\lambda}$ has a continuation to $\operatorname{Re}\lambda > -\epsilon$ which is 1 for $\lambda = 0$, the desired continuation of the last term follows, and if we define the currents U and R^f as the values of the correpsonding terms at $\lambda = 0$, then (1.6) follows from (2.2). In particular, it follows that R^f has support on Y.

For the more precise form of R^f we have to consider the definition of more carefully. Notice that the term $R^f_{\ell,\ell}$ is the analytic continuation of $\bar\partial |f|^{2\lambda} \wedge s \wedge (\bar\partial s)^{\ell-1}/|f|^{2\ell}$ to $\lambda = 0$. The latter term, integrated against the test form ϕ (of bidegree $(n, n-\ell)$), gives rise to a finite sum of terms like

$$\int_\tau \bar\partial (v_1^\lambda |\mu_1|^{2\lambda}) \wedge \frac{\alpha \wedge \tilde{\phi}}{\mu_1^\ell} \rho. \tag{2.3}$$

Again Lemma 2.1 gives the proposed analytic continuation, and this time the value at $\lambda = 0$ only depends on the germ of $\tilde{\phi}$ at $\{\mu_1 = 0\} = \tilde{Y}$, which means that $R^f_{\ell,\ell}$ has support on Y, as we already know. However, we want to prove that it vanishes at $\lambda = 0$ if $\ell < p$. To this end we may assume that $\phi = \phi_I \wedge d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_{n-\ell}}$ where ϕ_I is an $(n,0)$-form. Now $d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_{n-\ell}}$ vanishes on Y for degree reasons if $\ell < p$ and hence $\Pi^*(d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_k})$ vanishes on $\tilde{Y} = \{\mu_1 = 0\}$. However, this is a form in $d\bar{\tau}_j$ with antiholomorphic coefficients and therefore each of its terms either contains a factor $d\bar{\tau}_j$ or a factor $\bar{\tau}_j$ for some τ_j that divides μ_1. However, in both cases (2.3) vanishes for $\lambda = 0$ according to the lemma. Thus Theorem 1.1 is proved. □

Proof of Theorem 1.2. Suppose that h is holomorphic and vanishes on Y. Then Π^*h vanishes on $\tilde{Y}$, i.e., where $\mu_1 = 0$, and hence Π^*h must be divisible with each τ_j that divides μ_1. If therefore $\phi = \bar{h}\psi$, then (2.3) vanishes at $\lambda = 0$ according to the lemma. This means that $\bar{h}R^f = 0$.

In the same way, if $|h| \leq C|f|^k$, then (each term of) Π^*h must be divisible by μ_1^k, and hence any denominator μ_1^ℓ is killed by Π^*h if $\ell \leq k$. This means that $h \wedge R^f_{\ell,\ell} = 0$ if $\ell \leq k$. □

We now consider what happens if we have two different metrics.

Proposition 2.2. *Suppose that U and U' are the currents with respect to two different metrics on E, then there is a current $U \wedge U' \in \mathcal{L}^{-2}(X, E)$ such that*

$$\nabla_f(U \wedge U') = U' - U + M, \tag{2.4}$$

where

$$M = M^{p+1,p} + \cdots + M^{m,m-1}.$$

In particular, if f is a complete intersection, then

$$\nabla_f(U \wedge U') = U' - U.$$

Thus; if f is a complete intersection and R^f and R'^f denote the two currents, it follows that $0 = \nabla_f(U' - U) = R^f - R'^f$, and so the current R^f is independent of the metric. In the general case

$$0 = \nabla_f^2(U \wedge U') = \nabla_f(U' - U - M) = R'_f - R_f - \nabla_f M.$$

Proof. Let u and u' be the forms corresponding to the two different metrics and let $|f|^2$ be the norm with respect to either of the two metrics. Then

$$\nabla(|f|^{2\lambda} u \wedge u') = |f|^{2\lambda} u' - |f|^{2\lambda} u - \bar{\partial}|f|^{2\lambda} \wedge u \wedge u'.$$

We claim that $|f|^{2\lambda} u \wedge u'$ as well as $\bar{\partial}|f|^2 \wedge u \wedge u'$ define currents $U \wedge U'$ and M when $\lambda = 0$. If so, then (2.4) follows. Since both Π^*u and Π^*u' are sums of terms like α_ℓ/μ_1^ℓ, as before the integral

$$\int \bar{\partial}(|f|^{2\lambda}) \wedge u \wedge u' \wedge \phi$$

is a finite sum of terms like

$$\int (v^\lambda \mu_1^{2\lambda}) \frac{\alpha \wedge \tilde{\phi}}{\mu_1^\ell} \rho.$$

Again this integral has an analytic continuation to $\lambda = 0$ that vanishes if ϕ has bidegree (n, k) and $k > n - p$. Thus the current M is defined and has components $M^{p+1,p} + \cdots + M^{m,m-1}$. In the same way the current $U \wedge U'$ is defined. □

Remark 1. Let $h \colon \tilde{X} \to X$ be a proper holomorphic mapping and let $\tilde{E} \to \tilde{X}$ and $\tilde{E}^*$ be the pullbacks of E and E^*, equipped with the induced hermitean metric. If f is a holomorphic section to E^*, then h^*f is a holomorphic section to $\tilde{E}^*$, which is naturally identified with the dual of $\tilde{E}$. If the section s to E corresponds to f as before, then h^*s corresponds to h^*f and thus h^*u is the form corresponding to h^*f in $\tilde{X} \setminus \tilde{Y}$. It follows directly that

$$R^f = h_* R^{h^*f}$$

where h_* denotes pushforward. □

3. Cohomology of the ∇_f-complex

Assume that $\mathcal{L}^{\ell,k}$ is a double complex with vanishing cohomology in the k-direction except at $k = 0$ where the cohomology is A_ℓ. Then A_ℓ is itself a

Let $H^m(\mathcal{L}(X, E))$ be the cohomology groups of the complex (1.2), and let $H^m(\mathcal{L}_{smooth}(X, E))$ be the cohomology groups of the corresponding complex of spaces of smooth forms. If we assume that X is a Stein manifold, then the double complexes $\mathcal{L}^{\ell,k} = \mathcal{D}'_{0,k}(X, \Lambda^{-\ell}E)$ and $\mathcal{L}^{\ell,k}_{smooth}(X, E) = \mathcal{E}_{0,k}(X, \Lambda^{-\ell}E)$ both have vanishing cohomology in the k-direction except at $k = 0$ where the cohomology is $\mathcal{O}(X, \Lambda^{-\ell}E)$.

Lemma 3.1. *Assume that $\mathcal{L}^{\ell,k}$ is a double complex with vanishing cohomology in the k-direction except at $k = 0$ where the cohomology is A_ℓ. Then A_ℓ is itself a complex and its cohomology at $\ell = r$ is isomorphic (via the natural the mapping) to the cohomology at r of the total complex $\mathcal{L}^m = \oplus_{\ell+k=m} \mathcal{L}^{\ell,k}$.*

From the lemma we get the isomorphisms

$$H_{-m}(\mathcal{O}(X, \Lambda^* E)) \simeq H^m(\mathcal{L}_{smooth}(X, E)) \simeq H^m(\mathcal{L}(X, E)) \tag{3.1}$$

induced by the natural inclusion mappings. Thus, if $\phi \in \mathcal{L}^r(X, E)$ and $\nabla_f \phi = 0$, then $\nabla_f u = \phi$ has a smooth solution if and only if it has a current solution. If in addition ϕ is holomorphic, i.e., in $\mathcal{O}(X, \Lambda^{-r}E)$, then $\nabla_f u = \phi$ has a holomorphic solution if and only if it has a smooth (or current) solution $\nabla_f u = \phi$.

Since $H_\ell(\mathcal{O}(X, \Lambda^* E)) = 0$ for $\ell < 0$ and $\ell > m - p$, see Corollary 1.5, we get that

Proposition 3.2. *The cohomology of* (1.2) *vanishes if* $r > 0$ *and* $r < -(m-p)$.

4. Comparison to Coleff-Herrera-Passare residues

We shall now discuss solutions to $\nabla_f U = 1 - R$ defined by principal value currents and Coleff-Herrera residue currents (or more generally Coleff-Herrera-Passare currents). For simplicity we essentially restrict to the case of a complete intersection. Since the questions are local we also assume we are in an open set in $\mathbb{C}^n$.

For a general holomorphic mapping $f = (f_1, \ldots, f_m)$ we define the current

$$\Big[\frac{1}{f_1\cdots f_k}\bar\partial\frac{1}{f_{k+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_m}\Big] \tag{4.1}$$

as the value at $\lambda = 0$ of

$$\frac{|f_1\cdots f_k|^{2\lambda}\bar\partial|f_{k+1}|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_m|^{2\lambda}}{f_1\cdots f_m}.$$

The existence of the necessary analytic continuation is proved as before by a resolution of singularities; see also the proof of Theorem 4.1 below where the case with a complete intersection is implicitly proved. For the equivalence of this definition and definitions as limits of integrals over appropriate cycles, see [12] and the references given there, and [5]. From our definition of these currents it is quite easy to prove the fundamental computational rules from [11].

To begin with, the current (4.1) is clearly commuting in the indices $j \le k$ and alternating in the indices $j \ge k+1$. Moreover, it is easily seen that the formal Leibniz' formula for $\bar\partial$ holds. However, the current is to be considered as a whole unit, this is the reason for the brackets, so in general it is not true that one can multiply formally with f_j and cancel out the denominator.

Example 1. Let $f_2(z) = z_1^2$ and $f_1(z) = z_1z_2$. Then, for instance, we have that

$$z_1z_2\Big[\frac{1}{z_1z_2}\bar\partial\frac{1}{z_1^2}\Big] = \frac{z_1z_2|z_1z_2|^{2\lambda}\bar\partial|z_1|^{4\lambda}}{z_1z_2z_1^2}\Big|_{\lambda=0} =$$
$$= \frac{2}{3}\frac{|z_2|^{2\lambda}\bar\partial|z_1|^{6\lambda}}{z_1^2}\Big|_{\lambda=0} = \frac{2}{3}\bar\partial\frac{1}{z_1^2}.$$

□

However, in the complete intersection case this phenomenon never occurs; it is indeed possible to cancel any denominator by multiplication.

Theorem 4.1. *Assume that f is a complete intersection. Then*

$$f_1\left[\frac{1}{f_1\cdots f_k}\bar\partial\frac{1}{f_{k+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_m}\right] = \left[\frac{1}{f_2\cdots f_k}\bar\partial\frac{1}{f_{k+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_m}\right]$$

and

$$f_m\left[\frac{1}{f_1\cdots f_k}\bar\partial\frac{1}{f_{k+1}}\wedge\cdots\wedge\bar\partial\frac{1}{f_m}\right]=0.$$

The theorem follows from [11] once the equivalence with the Coleff-Herrera-Passare currents is established. For future reference it is anyway convenient for us to supply a direct proof.

Proof. We will use a resolution Π as in the proof of Theorem 1.1. We first notice that if $\mu_j=\tau_{j_1}^{\alpha_{j_1}}\cdots\tau_{j_\ell}^{\alpha_{j_\ell}}$, then

$$\Pi^*\bar\partial|f_j|^{2\lambda}=\bar\partial|u_j\mu_j|^{2\lambda}=\lambda|u_j\mu_j|^{2\lambda}\big(\alpha_{j_1}\frac{d\bar\tau_{j_1}}{\bar\tau_{j_1}}+\ldots+\alpha_{j_\ell}\frac{d\bar\tau_{j_\ell}}{\bar\tau_{j_\ell}}+\frac{d\bar u_j}{\bar u_j}\big).$$

The right hand side of the first equality acting on the test form $\phi=dz_1\wedge\ldots\wedge dz_n\wedge\phi^{n-m+k}$ of bidegree $(n,n-m+k)$ is as before a finite sum of terms like

$$\int\frac{v^{2\lambda}|\mu_2\cdots\mu_k|^{2\lambda}\bar\partial|u_{k+1}\mu_{k+1}|^{2\lambda}\wedge\ldots\wedge\bar\partial|u_m\mu_m|^{2\lambda}\wedge\alpha\wedge\tilde\phi^{n-m+k}}{\mu_2\ldots\mu_m}\rho,$$

where $\alpha=\Pi^*(dz_1\wedge\ldots\wedge dz_n)$. Expanding each factor $\bar\partial|u_j\mu_j|^{2\lambda}$ we get a number of terms to compute by means of Lemma 2.1. Let s^ℓ be a factor in say μ_{k+1}. We claim that the integral with the factor $\lambda|u_{k+1}\mu_{k+1}|^{2\lambda}d\bar s/\bar s$ can give a contribution when $\lambda=0$ only if s is not a factor in any other μ_j (and not $=u_{k+1}$). Indeed, assume that also s occurs in μ_j. The form $d\bar f_{k+2}\wedge\ldots d\bar f_m\wedge\phi^{n-m+k}$ has degree $n-1$ (in $d\bar z$) and hence it vanishes on on $\{f_{k+1}=f_j=0\}$ since this set has codimension 2 (since f_j,f_{k+1} is a complete intersection). Therefore, $\Pi^*(d\bar f_{k+2}\wedge\ldots d\bar f_m\wedge\phi^{n-m+k})$ vanishes where $s=0$, so as before, cf., the proof of Theorem 1.1 it follows that each of its terms contains either $d\bar s$ or $\bar s$. Both cases lead to something that vanishes at $\lambda=0$. If s^ℓ is a factor in μ_1 we can therefore insert $|s^\ell|^{2\lambda}$ in the nominator of the expression above without affecting the value at $\lambda=0$, according to Lemma 2.1. (The value at $\lambda=0$ is independent of how many factors $|s|^\lambda$ we put in the nominator as long as there is no factor $\bar s$ in the denominator.) Thus we can actually multiply the nominator with $|u_1\mu_1|^{2\lambda}$, and this proves the first statement of the theorem.

For the second statement we have to show that

$$\int\frac{v^{2\lambda}|\mu_1\cdots\mu_k|^{2\lambda}\bar\partial|u_{k+1}\mu_{k+1}|^{2\lambda}\wedge\ldots\wedge\bar\partial|u_m\mu_m|^{2\lambda}\wedge\alpha\wedge\tilde\phi^{n-m+k}}{\mu_1\ldots\mu_{m-1}}\rho$$

vanishes at $\lambda=0$, due to the lack of a factor μ_m in the denominator. Again, suppose that s is a factor in μ_m and consider the possible contribution from a term with the factor $\lambda|u_m\mu_m|^{2\lambda}d\bar s/\bar s$. If s is a factor also in some other μ_j, then the integral vanishes for $\lambda=0$ for the same reason as before. If no other μ_j contains s, then the integral of the term vanishes for $\lambda=0$ since then there is no s in the denominator. □

We can now define a new current satisfying a similar ∇_f-equation as the Cauchy-Fantappie-Leray type current U from Theorem 1.1. Indeed, consider the current

$$V = \Big[\frac{1}{f_1}\Big]\frac{e_1}{2\pi i} + \Big[\frac{1}{f_2}\bar\partial\frac{1}{f_1}\Big]\wedge\frac{e_1\wedge e_2}{(2\pi i)^2} + \\ + \Big[\frac{1}{f_3}\bar\partial\frac{1}{f_2}\wedge\bar\partial\frac{1}{f_1}\Big]\wedge\frac{e_1\wedge e_2\wedge e_3}{(2\pi i)^3} + \cdots \tag{4.2}$$

If f is a complete intersection, then a simple computation, using Theorem 4.1, yields that

$$\nabla_f V = 1 - R'_f,$$

where

$$R'_f = \Big[\bar\partial\frac{1}{f_m}\wedge\ldots\wedge\bar\partial\frac{1}{f_1}\Big]\wedge\frac{e_1\wedge\ldots\wedge e_m}{(2\pi i)^m}.$$

We are now ready to prove Theorem 1.7; as already announced we actually prove

Proposition 4.2. *Suppose that f is a complete intersection, let V be the current defined by (4.2), and let U be the extension of the Bochner-Martinelli form provided by Theorem 1.6. Then there is a current $U\wedge V$ such that $\nabla_f(U\wedge V) = V - U$.*

Proof. Let us define

$$v^\lambda = \frac{|f_m\cdots f_1|^{2\lambda}}{f_1}\frac{e_1}{2\pi i} + \frac{|f_m\cdots f_2|^{2\lambda}\bar\partial|f_1|^{2\lambda}}{f_2 f_1}\wedge\frac{e_1\wedge e_2}{(2\pi i)^2} + \ldots \\ \cdots + \frac{|f_m\cdots f_{k+1}|^{2\lambda}\bar\partial|f_k|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}}{f_{k+1}\cdots f_1}\wedge\frac{e_1\wedge\ldots\wedge e_{k+1}}{(2\pi i)^{k+1}} + \ldots \\ \cdots + \frac{|f_m|^{2\lambda}\bar\partial|f_{m-1}|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}}{f_m\cdots f_1}\wedge\frac{e_1\wedge\ldots\wedge e_m}{(2\pi i)^m}.$$

A simple computation yields that

$$\nabla_f v^\lambda = |f_1\cdots f_m|^{2\lambda} - \frac{\bar\partial|f_m|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}}{f_m\cdots f_1}\wedge e_1\wedge\ldots\wedge e_m + \mathcal{R}$$

where $\mathcal{R}$ denote terms like

$$\frac{|f_m\cdots\widehat{f_j}\cdots f_k|^{2\lambda}\bar\partial|f_j|^{2\lambda}\wedge\bar\partial|f_{k-1}|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}}{f_k\cdots f_1},$$

i.e., terms in which we have $\bar\partial|f_j|^{2\lambda}$ although f_j does not occur in the denominator. Let u be the Bochner-Martinelli form. By a desingularization as before one shows that $|f|^{2\lambda}u\wedge v^\lambda\wedge\phi$ has an analytic continuation as a current to $\lambda > -\epsilon$ and we define the current $U\wedge V$ as the value at $\lambda = 0$. Notice that

$$\nabla_f(|f|^{2\lambda}u\wedge v^\lambda) = |f|^{2\lambda}v^\lambda - |f|^{2\lambda}|f_m\cdots f_1|^{2\lambda}u \\ - \bar\partial|f|^{2\lambda}\wedge u\wedge v^\lambda + |f|^{2\lambda}u\wedge\mathcal{R}. \tag{4.3}$$

We thus have to prove that the first two terms at the right hand side become V and U when $\lambda = 0$ and that the remaining terms vanish.

For the first term we have to verify that each integral like

$$\int |f|^{2\lambda} \frac{|f_m \cdots f_{k+1}|^{2\lambda} \bar{\partial}|f_k|^{2\lambda} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda}}{f_{k+1} \cdots f_1} \wedge \phi$$

is unaffected at $\lambda = 0$ if we delete the factor $|f|^{2\lambda}|f_m \cdots f_{k+2}|^{2\lambda}$ from the nominator. In a local chart in the desingularization, cf., the proof of Theorem 1.6, let μ_j be the factor that divides all the other μ_k, so that $\Pi^*|f|^2 = v|\mu_j|^2$. If $j \leq k$, then the integral (of the pullback) vanishes as we saw in the first part of the proof of Theorem 4.1 (with or without the factor $|v_j \mu_j|^{2\lambda}$). Therefore we may assume that $j > k$. On the other hand, for each $\nu > k$ we saw in the same proof that we can insert or delete the factor $|u_\nu \mu_\nu|^{2\lambda}$ (in particular $v|mu_j|^{2\lambda}$) in the nominator without affecting the value at $\lambda = 0$.

It is quite easy to see that the second term (at the right hand side of) (4.3) is U when $\lambda = 0$; in fact, any integral like (2.1) is unaffected if a factor $|u_k \mu_k|^{2\lambda}$ is inserted. The third term vanishes for the same reason as M in Proposition 2.2 vanishes in the complete intersection case.

The remaining terms give rise, after desingularization, to integrals of the kind that was discussed in the second part of the proof of Theorem 4.1, and so they vanish at $\lambda = 0$. Thus the proof is complete. □

5. Definition as principal value integrals

We prefer the definition above with meromorphic continuation. However in this section we point out that one can just as well define the currents as one-parameter principal value integrals. The approach does not differ essentially from. e.g., [13] so we just sketch the arguments. However, we derive the asymptotic behaviour of $|f|^{2\lambda}u$ and $\bar{\partial}|f|^{2\lambda} \wedge u$ directly from the desingularization instead of utilize a Bernstein-Sato functional relation. The result is

Theorem 5.1. *With the notation from Theorem 1.1 we have that*

$$\lim_{\epsilon \to 0} \int_{|f|^2 > \epsilon} u \wedge \psi = \int U \wedge \psi \tag{5.1}$$

and

$$\lim_{\epsilon \to 0} \int_{|f|^2 = \epsilon} u \wedge \psi = \int R^f \wedge \psi, \tag{5.2}$$

where the second limit is taken over all regular values for $|f|^2$.

Proof. We claim that the second statement follows from the first one. In fact, by Stokes' theorem,

$$\int_{|f|^2=\epsilon} u\wedge\phi = -\int_{|f|^2>\epsilon}(\bar\partial u\wedge\phi - u\wedge\bar\partial\phi) = -\int_{|f|^2>\epsilon}\big((\delta_f u-1)\wedge\phi - u\wedge\bar\partial\phi\big), \tag{5.3}$$

since $(\delta_f-\bar\partial)u=1$ in $X\setminus Y$. Assuming that (5.1) holds, the right hand side of (5.3) tends to

$$-\int\big((\delta_f U-1)\wedge\phi - U\wedge\bar\partial\phi\big) = -\int(\nabla_f U-1)\wedge\phi = \int R^f\wedge\phi,$$

by (1.6), and thus (5.2) holds.

To prove (5.1), let

$$I(\epsilon)=\int_{|f|^2>\epsilon} u\wedge\phi.$$

Since $|u|\lesssim|f|^{-2m+1}$ it follows that $|I(\epsilon)|\lesssim\epsilon^{-2m+1}$. For $\mathrm{Re}\,\lambda>2m$ therefore its Mellin transform

$$M(\lambda)=\int_0^\infty I(\eta)d\eta^\lambda=\int|f|^{2\lambda}u\wedge\phi$$

is defined. From the proof of Theorem 1.1 and Lemma 6.1 it follows that $M(\alpha+i\beta)$ is rapidly decreasing in β, locally uniformly in α. It is now a standard result that $I(\mu)$ admits an asymptotic expansion for small μ. In fact, by the inversion formula,

$$I(\mu)=\frac{1}{2\pi i}\int_{\alpha+i\mathbb{R}}\frac{M(\lambda)d\lambda}{\lambda\mu^\lambda},$$

for large $\alpha>0$, and since the integrand is rapidly decreasing and has its leftmost pole at the origin with residue $M(0)$, we can apply Cauchy's theorem and obtain

$$I(\mu)=M(0)+\frac{1}{2\pi i}\int_{-\epsilon/2+i\mathbb{R}}\frac{M(\lambda)d\lambda}{\lambda\mu^\lambda}=M(0)+\mathcal{O}(\mu^{\epsilon/2}).$$

Iterating one gets the asymtotic expansion

$$I(\mu)=M(0)+\sum_{k\le C}\sum_{a_j<A}\mu^{a_j}(\log\mu)^k+\mathcal{O}(\mu^A).$$

□

6. Proof of Lemma 2.1

In this section we provide a proof of Lemma 2.1 and we also have the following addition that was used in the previous section.

Lemma 6.1. *In addition to the statements in Lemma 2.1 we also have that both the functions of λ are rapidly decreasing when $\lambda=\alpha+i\beta$ and $|\beta|\to\infty$, locally uniformly in α.*

Proof of Lemmas 2.1 and 6.1. If $\operatorname{Re}\lambda > p$ we have that

$$\int v^\lambda |s|^{2\lambda}\psi \frac{ds\wedge d\bar{s}}{s^p} = \int |s|^{(2(\lambda-p)}(v^\lambda\psi)ds\wedge \bar{s}^p d\bar{s} =$$

$$-\frac{1}{\lambda-p+1}\int d|s|^{2(\lambda-p+1)}\wedge (v^\lambda\psi)\wedge \bar{s}^{p-1}d\bar{s} =$$

$$\frac{1}{\lambda-p+1}\int |s|^{2(\lambda-p+1)}\wedge \frac{\partial}{\partial s}(v^\lambda\psi)ds\wedge \bar{s}^{p-1}d\bar{s} = \dots$$

$$\frac{1}{(\lambda-p+1)(\lambda-p+2)\cdots(\lambda+1)}\int |s|^{2(\lambda-1)}\frac{\partial^{p-1}}{\partial s^{p-1}}(v^\lambda\psi)ds\wedge \bar{s}d\bar{s} =$$

$$\frac{1}{(\lambda-p+1)(\lambda-p+2)\cdots(\lambda+1)\lambda}\int |s|^{2\lambda}\frac{\partial^{p}}{\partial s^{p}}(v^\lambda\psi)ds\wedge d\bar{s}.$$

Since the integral in the last term exists for $\operatorname{Re}\lambda > 1$ we have found the analytic continuation to $\operatorname{Re}\lambda > -1$ since there are no poles there. To find the value at $\lambda = 0$ we let $\lambda \to 0$ in the next to last integral, and we then get

$$\pm\frac{1}{(p-1)!}\int \frac{\partial^{p-1}}{\partial s^{p-1}}(\psi)\frac{ds\wedge d\bar{s}}{s}$$

which does not depend on v.

For the second statement, first notice that it is, by an integration by parts, equal to the first integral, but with ψ replaced by $\partial\psi/\partial\bar{s}$. Hence it is analytic for $\operatorname{Re}\lambda > -1$ as well, and the value at $\lambda = 0$ is then

$$\pm\frac{1}{(p-1)!}\int \frac{\partial}{\partial\bar{s}}\frac{\partial^{p-1}}{\partial s^{p-1}}(\psi)\frac{ds\wedge d\bar{s}}{s} = c_p\frac{\partial^{p-1}}{\partial s^{p-1}}(\psi)\Big|_0,$$

which vanishes if $\psi = \bar{s}\phi$.

To reveal the general meromorphic continuation and the decay in the Im direction it is enough to consider

$$\int |s|^{2\lambda}v^\lambda\phi\frac{ds\wedge d\bar{s}}{s},$$

where ϕ is smooth with compact support. By the non-holomorphic change of variables, $\sigma = \sqrt{v}s$, we get

$$\int |\sigma|^{2\lambda}\tilde{\phi}\frac{d\sigma\wedge d\bar{\sigma}}{\sigma}$$

and with polar coordinates (and $2\lambda = \alpha$)

$$\int_0^\infty r^\alpha\chi(r)dr,$$

where $\chi(r)$ is smooth. By repeated integrations by parts we then get that

$$c_N(\lambda)\int_0^\infty r^{\alpha+N}D^N\chi(r)dr,$$

where $c_N(\lambda) = \mathcal{O}(|\lambda|^{-N})$ away from the poles. This proves the lemma. $\square$

7. Some remarks and examples

We begin with some further remarks on the case with a complete intersection.

Remark 2. We have thus seen that R^f is intrinsically defined when f is a complete intersection and locally coincides with the Coleff-Herrera current. It is well-known that the Coleff-Herrera current is equal to its own standard extension, see [5], and it thus follows that the the same holds for our current R^f, in particular for the Bochner-Martinelli current.

The following proposition is quite easy to verify and we omit the proof.

Proposition 7.1. *Suppose that f is a complete intersection and either f is regular, or $m = n$ (so that Y is a discrete set). If there is a (*$\det E$*-valued) $(0, m)$-current R such that $\nabla_f V = 1 - R$ is solvable with some $V \in \mathcal{L}^{-1}(X, E)$, and furthermore $\bar{h}R = 0$ for all holomorphic h that vanishes on Y. Then R is (locally) equal to the Coleff-Herrera current.*

It seems to be unknown whether this holds for an arbitrary complete intersection. $\square$

Example 2 (A simple example of a non-complete intersection). Let $f_1 = z_1$ and $f_2 = z_1 z_2$. Then $Y = \{z_1 = 0\}$ so f is not a complete intersection. If we adopt the trivial metric then

$$s(z) = \frac{1}{2\pi i}\bar{z}_1 e_1 + \bar{z}_1\bar{z}_2 e_2 = \frac{1}{2\pi i}\bar{z}_1(e_1 + \bar{z}_2 e_2),$$

so

$$u = \frac{s}{|f|^2} + \frac{s\wedge\bar\partial s}{|f|^4} = \frac{1}{2\pi i}\frac{e_1 + \bar{z}_2 e_2}{z_1(1+|z_2|^2)} + \frac{1}{(2\pi i)^2}\frac{d\bar{z}_2\wedge e_2\wedge e_1}{z_1^2(1+|z_2|^2)^2}.$$

Since $|f|^2 = |z_1|^2(1+|z_2|^2)$ it follows, cf., Section 4, that the extension U is obtained by extending $1/z_1$ and $1/z_1^2$ to the currents $[1/z_1]$ and $[1/z_1^2]$, respectively. It follows that

$$R^f = \frac{1}{2\pi i}\bar\partial\Big[\frac{1}{z_1}\Big]\wedge\frac{e_1+\bar{z}_2 e_2}{1+|z_2|^2} + \frac{1}{(2\pi i)^2}\bar\partial\Big[\frac{1}{z_1^2}\Big]\wedge\frac{d\bar{z}_2\wedge e_2\wedge e_1}{(1+|z_2|^2)^2}.$$

Thus a holomorpic function ϕ annihilates R^f if and only if $\phi\bar\partial[1/z_1^2] = 0$, i.e., $\phi(0, z_2) = \partial\phi/\partial z_1(0, z_2) = 0$, whereas ϕ belongs to the ideal if and only if just $\phi(0, z_2) = 0$. $\square$

Example 3 (A less trivial example). Let now $f_1 = z_1^2$ and $f_2 = z_1 z_2$ in a neighborhood $\mathcal{U}$ of 0, again with the trivial metric. Then $Y = \{z;\ z_1 = 0\}$ and

$$u = \frac{1}{2\pi i}\frac{1}{z_1}\frac{\bar{z}_1 e_1 + \bar{z}_2 e_2}{|z|^2} + \frac{1}{(2\pi i)^2}\frac{1}{z_2^2}\frac{\bar{z}_2 d\bar{z}_1 - \bar{z}_1 d\bar{z}_2}{|z|^4}\wedge e_2\wedge e_1,$$

where $|z|^2 = |z_1|^2 + |z_2|^2$. To find the extension U given by Theorem 1.6 and R^f, we consider the proper mapping $\Pi\colon \widetilde{\mathcal{U}} \to \mathcal{U}$, where $\widetilde{\mathcal{U}}$ is the blow up at 0 of U. We can cover $\widetilde{\mathcal{U}}$ by the two coordinate neighborhoods

$$\Omega_1 = \{\tau;\ (\tau_1\tau_2, \tau_1) = z \in \mathcal{U}\} \text{ and } \Omega_2 = \{\sigma;\ (\sigma_1, \sigma_1\sigma_2) = z \in \mathcal{U}\}.$$

To compute $\tilde{u} = \Pi^* u$ in Ω_1, notice that

$$\tilde{s} = \Pi^* s = \bar{\tau}_1^2 \bar{\tau}_2 (\bar{\tau}_2 e_1 + e_2)/2\pi i,$$

so that

$$\tilde{u} = \frac{\tilde{s}}{\Pi^*|f|^2} + \frac{\tilde{s} \wedge \bar{\partial}\tilde{s}}{\Pi^*|f|^4} = \frac{1}{2\pi i}\frac{1}{\tau_2\tau_1^2}\frac{\bar{\tau}_2 e_1 + e_2}{1+|\tau_2|^2} - \frac{1}{(2\pi i)^2}\frac{1}{\tau_2^2\tau_1^4}\frac{d\bar{\tau}_2 \wedge e_2 \wedge e_1}{(1+|\tau_2|^2)^2}.$$

The extension $\tilde{U}$ across $\tilde{Y}$ is simply

$$\tilde{U} = \frac{1}{2\pi i}\Big[\frac{1}{\tau_2\tau_1^2}\Big]\frac{\bar{\tau}_2 e_1 + e_2}{1+|\tau_2|^2} - \frac{1}{(2\pi i)^2}\Big[\frac{1}{\tau_2^2\tau_1^4}\Big]\frac{d\bar{\tau}_2 \wedge e_2 \wedge e_1}{(1+|\tau_2|^2)^2}$$

and therefore

$$R^{\Pi^* f} = \frac{1}{2\pi i}\bar{\partial}\Big[\frac{1}{\tau_2\tau_1^2}\Big] \wedge \frac{\bar{\tau}_2 e_1 + e_2}{1+|\tau_2|^2} - \frac{1}{(2\pi i)^2}\bar{\partial}\Big[\frac{1}{\tau_2^2\tau_1^4}\Big] \wedge \frac{d\bar{\tau}_2 \wedge e_2 \wedge e_1}{(1+|\tau_2|^2)^2}$$

in Ω_1. By Leibniz' formula

$$\bar{\partial}\Big[\frac{1}{\tau_2^2\tau_1^4}\Big] = \Big[\frac{1}{\tau_2^2}\bar{\partial}\frac{1}{\tau_1^4}\Big] + C d\bar{\tau}_2$$

so that actually

$$R^{\Pi^* f} = \frac{1}{2\pi i}\bar{\partial}\Big[\frac{1}{\tau_2\tau_1^2}\Big] \wedge \frac{\bar{\tau}_2 e_1 + e_2}{1+|\tau_2|^2} - \frac{1}{(2\pi i)^2}\Big[\frac{1}{\tau_2^2}\bar{\partial}\frac{1}{\tau_1^4}\Big] \wedge \frac{d\bar{\tau}_2 \wedge e_2 \wedge e_1}{(1+|\tau_2|^2)^2}.$$

Since

$$\bar{\partial}\Big[\frac{1}{\tau_2\tau_1^2}\Big] = \Big[\frac{1}{\tau_2}\bar{\partial}\frac{1}{\tau_1^2}\Big] + \Big[\frac{1}{\tau_1^2}\bar{\partial}\frac{1}{\tau_2}\Big]$$

it follows that $R_{1,1}^{\Pi^* f}$ has support on $\{\tau;\ \tau_1\tau_2 = 0\}$, whereas $R_{2,2}^{\Pi^* f}$ has support only on $\{\tau;\ \tau_1 = 0\}$. In the same way one can check that $R^{\Pi^* f}$ in Ω_2 is supported on $\{\sigma;\ \sigma_1 = 0\}$. Since $R^f = \Pi_* R^{\Pi^* f}$, cf., Remark 1, it follows that $R_{1,1}^f$ has support on $\{z_1 = 0\}$ as expected, whereas $R_{2,2}^f$ has support at the origin.

To compute $R_{2,2}$, let $\phi = \Phi dz_1 \wedge dz_2$ be a test form in $\mathcal{U}$. Then

$$\int_{\mathcal{U}} R_{2,2}^f \wedge \phi = \int_{\tilde{\mathcal{U}}} R_{2,2}^{\Pi^* f} \wedge \Pi^*\phi.$$

Over Ω_1 the last integral is

$$-\frac{1}{(2\pi i)^2}\int_\tau \Big[\frac{1}{\tau_2^2}\bar{\partial}\frac{1}{\tau_1^4}\Big] \wedge \frac{d\bar{\tau}_2 \wedge e_2 \wedge e_1}{(1+|\tau_2|^2)^2} \wedge \Phi(\tau_1\tau_2, \tau_1) d(\tau_1\tau_2) \wedge d\tau_1 =$$

$$= \frac{1}{(2\pi i)^2}\int_\tau \Big[\frac{1}{\tau_2^2}\bar{\partial}\frac{1}{\tau_1^3}\Big] \wedge \frac{d\bar{\tau}_2 \wedge e_2 \wedge e_1}{(1+|\tau_2|^2)^2} \wedge \Phi(\tau_1\tau_2, \tau_1) d\tau_1 \wedge d\tau_2.$$

Since $(\sigma_1, \sigma_2) = (\tau_1\tau_2, 1/\tau_2)$ and $R^{\Pi^* f}_{2,2}$ has support only where $\tau_1 = 0$ (equivalently $\sigma_1 = 0$) we can compute the integral over the whole $\tilde{\mathcal{U}}$ by extending the integration to $\tau_2 \in \mathbb{P}^1$. For any test form $\psi(s)ds$ we have that

$$\frac{1}{2\pi i}\int_s \bar\partial\Big[\frac{1}{s^3}\Big] \wedge \psi(s)ds = \frac{1}{2}\frac{\partial^2\psi}{\partial s^2}(0),$$

and therefore (supressing $e_2 \wedge e_1$ and letting $\tau_2 = \sigma$) we get

$$\frac{1}{2}\frac{1}{2\pi i}\int_\sigma \Big[\frac{1}{\sigma^2}\Big]\frac{\sigma^2\Phi_{11}(0) + 2\sigma\Phi_{12}(0) + \Phi_{22}(0)}{(1+|\sigma|^2)^3} d\bar\sigma \wedge d\sigma = \\ = \frac{1}{2}\frac{1}{2\pi i}\int_\sigma \frac{\Phi_{11}(0)}{(1+|\sigma|^2)^3} d\bar\sigma \wedge d\sigma = \frac{1}{4}\Phi_{11}(0).$$

Thus we conclude that

$$R_{2,2}.\Phi(z)dz_1 \wedge dz_2 = \frac{1}{4}\Phi_{z_1 z_1}(0) e_2 \wedge e_1.$$

One can compute $R_{1,1}$ in a similar way. □

Example 4 (The regular case). If D is a holomorphic connection on E^* and $(Df)^m \neq 0$ on Y, then Y is a regular manifold and u is locally integrable so it defines a current U across Y, and

$$\nabla_f U = 1 - R^f, \tag{7.1}$$

where R^f is the ($\det E$-valued) current with support on Y such that

$$R^f \wedge (Df)^m = [Y] \wedge e_1^* \wedge e_1 \wedge \ldots \wedge e_m^* \wedge e_m. \tag{7.2}$$

Here $[Y]$ is the (m, m)-current of integration on Y. In fact, if $f = \sum f_j e_j^*$ in a local holomorphic frame, then the condition means that $\partial f_1 \wedge \ldots \wedge \partial f_m \neq 0$ on Y, and choosing f_j as part of a holomorphic local coordinate system, (7.2) is readily verified.

Suppose that v is any solution to $\nabla_f v = 1$ outside Y such that $v_{k,k-1} = \mathcal{O}(|f|^{-(2k-1)})$. If V is the natural extension of v as a locally integrable current in X, then V satisfies (7.1). In fact, $U \wedge V$ is locally integrable and it is readily checked that $\nabla_f(U \wedge V) = V - U$, so that $0 = \nabla_f V - \nabla_f U$. □

Example 5. In the case when X is a subset of $\mathbb{C}^n$ that contains the origin and $f_j = z_j$ it is natural to take $E = T^*_{1,0}$. Then δ_z is contraction on differential forms in $\mathbb{C}^n$ with the holomorphic vector field $2\pi i \sum z_j(\partial/\partial z_j)$ and if u is a reasonable solution to $\nabla_z u = 1$ in $X \setminus \{0\}$, it has a current extension U across 0, and

$$\nabla_z U = 1 - [0]$$

in the current sense. As mentioned above, this holds if u is smooth outside the origin and $u_\ell = \mathcal{O}(|z|^{-(2\ell-1)})$. Notice that now u_ℓ is a

$(\ell, \ell-1)$-form in the usual sense. For instance, the Bochner-Martinelli form (current)

$$U = \frac{\partial|z|^2}{\nabla_z \partial|z|^2} = \sum \frac{1}{(2\pi i)^\ell} \frac{\partial|z|^2 \wedge (\bar\partial\partial|z|^2)^{\ell-1}}{|z|^{2\ell}}$$

will do. Another choice, cf., Section 4, is V defined by

$$V_{1,0} = \frac{dz_1}{2\pi i z_1}, \quad V_{k+1,k} = \frac{dz_{k+1}}{2\pi i z_{k+1}} \wedge \bar\partial V_{k,k-1}.$$

□

Example 6 (Jacobi formulas). If α is a meromorphic $(1,0)$-form on a compact Riemann surface X then the sum of its residues is zero; this is just because the sum is equal to

$$\int_X \bar\partial\alpha,$$

which vanishes by Stokes' theorem. In $\mathbb{C}$ we have the following variant: If p and q are polynomials in z, then the sum of the residues of qdz/p in $\mathbb{C}$ is equal to minus the residue at infinity. In particular, the sum is zero if $\deg q \le \deg p - 2$, since then qdz/p is holomorphic at infinity. In this example we shall consider some generalizations of these two statements.

Assume that X is a compact complex manifold and let f be a holomorphic section to the n-bundle $E^* \to X$ and let U and R_f be defined as before with respect to some hermitean metric on E. If h is a $\det E^*$-valued holomorphic $(n,0)$-form, then $h \wedge R^f_{n,n}$ is a $\bar\partial$-exact scalar form, and hence

$$\int_X h \wedge R^f_{n,n} = 0.$$

Notice that we do not have to require that f be a complete intersection. This will be used in the proof of the following theorem due to Yger and Vidras, [14]; our contribution is that we can put it(s proof) into a geometric frame.

Theorem 7.2. *Let $P_1, \ldots, P_n$ be polynomials in $\mathbb{C}^n$ and assume that there are numbers δ_j, $0 < \delta_j \le deg(P_j)$, such that*

$$\sum \frac{|P_j(\zeta)|^2}{(1+|\zeta|^2)^{\delta_j}} \ge c \tag{7.3}$$

for large ζ in $\mathbb{C}^n$. Then if Q is a polynomial with $deg Q < \sum \delta_j - n$ it follows that

$$\int_{\mathbb{C}^n} Q d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge R^P = 0.$$

Since the common zero set in $\mathbb{C}^n$ is then compact it must be a finite set of points, and thus P is a complete intersection.

The simplest case of this statement is when all $\delta_j = \deg P_j$. Then the condition (7.3) is equivalent to that the principal terms of the polynomials have no common zeros outside 0 in $\mathbb{C}^n$, and this in turn is equivalent to that the homogenizations p_j of P_j have no common zeros at the hyperplane at infinity. Let $L = \mathcal{O}(-1)$ be the line bundle over $\mathbb{P}^n$ whose sections are represented by 1-homogeneous polynomials. Then $p = (p_1, \ldots, p_n)$ is a holomorphic section to $E^* = L^{\delta_1} \otimes \cdots \otimes L^{\delta_n}$, with no zeros on the hyperplane at infinity. Notice that $\det E^* = L^\delta$, where $\delta = \delta_1 + \cdots + \cdots \delta_n$. If q is the $\sum \delta_j$-homogenization of Q times $d\zeta_1 \wedge \ldots \wedge d\zeta_n$, then the condition $\deg Q \leq \delta - n - 1$ ensures that q is a global $\det E^*$-valued holomorphic $(n,0)$-form on $\mathbb{P}^n$, since the canonical bundle $\det T^*_{1,0}(\mathbb{P}^n)$ is isomorphic to $L^{-(n+1)}$. Thus

$$\int_{\mathbb{C}^n} q \wedge R^p = \int_{\mathbb{P}^n} q \wedge R^p = 0.$$

For the general case, one can make a preliminary reduction to the case when all δ_j are integers, see [14]. Then the assumption on Q is that $\deg Q \leq \delta - n - 1$. Choose a number $M > 0$ such that

$$\delta_j + M - \deg P_j > 0$$

for all j, and consider the section p as a section to

$$E^* = L^{\delta_1 + M} \otimes \cdots \otimes L^{\delta_n + M};$$

i.e., $p_j(z) = P_j(z'/z_0) z_0^{\delta_j + M}$ if $\zeta = z'$. If E^* is equipped with the natural metric, then

$$|p|^2 = \sum_j \frac{|P_j(z'/z_0) z_0^{\delta_j+M}|^2}{|z|^{2(\delta_j+M)}}.$$

The condition (7.3) then is equivalent to

$$|p(z)|^2 \geq c \frac{|z_0|^{2M}}{|z|^{2M}} = |h|^{2M} \tag{7.4}$$

for $[z]$ close to the hyperplane $z_0 = h(z) = 0$ in $\mathbb{P}^n$. Now $\det E^* = L^{\delta + nM}$ so if q is the corresponding $(0,n)$-form with values in $\det E^*$, then $q = h^{nM} g$ where g is a holomorphic $(n,0)$-form, and hence $q \wedge R^p = 0$ close to this hyperplane according to Theorem 1.2. Thus the theorem follows.

In the same way we can prove the following abstract version:

Theorem 7.3. *Let X be a compact manifold and let f_0 be a holomorphic section to the line bundle $L \to X$ and let $Y = \{f_0 = 0\}$. Moreover, suppose that $f_1, \ldots, f_n$ are holomorphic functions in $X \setminus Y$ (obs L trivial here) such that $f_j f_0^{d_j}$ are global sections to L^{d_j}, and assume that we have integers $\delta_j \leq d_j$ such that*

$$\sum |f_j f_0|^{\delta_j} \geq c$$

close to Y. If then q is a holomorphic $(n,0)$-form in $X\setminus Y$ such that qf_0^d is holomorphic in X, where $d=d_1+\cdots+d_n$, then

$$\int_{X\setminus Y} q\wedge R^f = 0,$$

where f is the section $f_1+\cdots+f_n$ to $L^{d_1}\oplus\cdots\oplus L^{d_n}$.

□

Example 7 (Generalized logarithmic residues). Let $D=D'+\bar\partial$ be the holomorphic connection on E^* induced by our hermitean metric. Then $(Df)^m$ is a smooth $\det E^*$-valued $(n,0)$-form and if f is a complete intersection, then

$$(Df)^m/m!\wedge R^f=[Y^f],$$

if we identify $\tau=e_1^*\wedge e_1\wedge\ldots\wedge e_n^*\wedge e_n$ with 1. Here $[Y^f]$ denotes the current of integration on the regular part of Y such that each branch is counted with multiplicity. To see this just notice that in a local holomorphic frame $Df=D\sum f_je_j^*=\sum df_j\wedge e_j+\mathcal{O}(f)$, where $\mathcal{O}(f)$ denotes terms with some factor f_j. Since $\mathcal{O}(f)R^f=0$ we get

$$(Df)^m/m!\wedge R^f=\frac{1}{(2\pi i)^m}\bar\partial\frac{1}{f_1}\wedge df_1\wedge\ldots\bar\partial\frac{1}{f_m}\wedge df_m\wedge\tau,$$

and it is wellknown that the right hand side is equal to $[Y^f]\wedge\tau$, see for instance [9]. It is also known that there is a current A of bidegree $(m,m-1)$ such that $dA=\bar\partial A=[Y^f]-c_m(D)$, where $c_m(D)$ is the Chern form of top degree (a proof will be given below).

In general we have

Lemma 7.4. *If f is any holomorphic section, then for each k the current*

$$R^f_{\ell,\ell}\wedge(Df)^\ell \tag{7.5}$$

has measure coefficients.

Proof. Using the notation from the proof of Theorem 1.1 we have after the desingularization terms like

$$\int\bar\partial(|\mu_1|^{2\lambda}v^\lambda)\wedge\frac{\alpha\wedge\tilde\phi}{\mu_1^\ell}\rho\wedge(d\mu_1\wedge\mu'+\mu_1D\mu')^\ell.$$

Since the last factor is $\mu_1^\ell\gamma_1+d\mu_1\wedge\mu_1^{\ell-1}\gamma_2$ and μ_1 is a monomial, it is easy to see that the integrand is integrable when $\lambda=0$. □

We shall now show that an appropriate combination of the currents (7.5) gives a closed (m,m)-current $[Y^f]$, which coincides with the previous one in the complete intersection case, and an $(m,m-1)$-current A such that again

$$dA=\bar\partial A=[Y^f]-c_m(D). \tag{7.6}$$

To this end we need somewhat more complex geometry and give [9] as general reference, but see also [3]. The curvature tensor $\Theta = D^2$ is a Hom (E^*)-valued $(1,1)$-form, and it can be identified with the section $\widetilde{\Theta}$ to $\Lambda(E^* \oplus E \oplus T^*(X))$ which locally is defined by

$$\sum \Theta_{jk} \wedge e_j \wedge e_k^*,$$

if Θ_{jk} is the matrix (of $(1,1)$-forms) for Θ in the frame e_j^*. In particular $\delta_f \tilde{\Theta} = 2\pi i \Theta f$. Since f is holomorphic, $\bar{\partial} D f = D^2 f = \Theta f$. The connection D has a canonical extension to the bundle $\Lambda(E^* \oplus E \oplus T^*(X))$, and $\bar{\partial}\widetilde{\Theta} = D\widetilde{\Theta} = 0$. We also have to introduce the operator

$$\mathcal{D}_f = \delta_f - D.$$

If follows that

$$\nabla_f\big(Df - \frac{i}{2\pi}\widetilde{\Theta}\big) = 0 \text{ and } \mathcal{D}_f\big(Df - \frac{i}{2\pi}\widetilde{\Theta}\big) = 0. \tag{7.7}$$

Thus,

$$\nabla_f U \wedge (Df - \frac{i}{2\pi}\tilde{\Theta})^m/m! = (1 - R^f) \wedge (Df - \frac{i}{2\pi}\tilde{\Theta})^m/m!,$$

and if we extract the term with full degree in both e_j and e_j^*, we find that

$$\frac{i}{2\pi}\tilde{\Theta}^m/m! - \sum_{j=p}^{m} R^f_{j,j} \wedge (Df)^j/j! \wedge (\frac{i}{2\pi}\tilde{\Theta})^{m-j}/(m-j)!$$

is a $\bar{\partial}$-exact (m,m)-current (with values in det $E^* \otimes$det E). Let us define $[Y^f]$ such that the term on the right is $[Y^f] \wedge \tau$. Thus $[Y^f]$ is a (m,m)-current on Y with measure coefficients, according the the lemma above, and there is a $(m,m-1)$-current A such that $\bar{\partial} A = c_m(D) - [Y^f]$. When f is a complete intersection $R^f_{j,j} = 0$ for all j but $j = m$ so we get back the usual current.

To show that they actually are d-cohomologous we need an extra argument. In general, $\mathcal{D}_f^2$ is not zero; however it is a tensor, and in fact, if ξ is a smooth section to E, then

$$\mathcal{D}_f \xi = \delta_\xi\big(Df - \frac{i}{2\pi}\widetilde{\Theta}\big), \tag{7.8}$$

if δ_ξ denotes contraction with $2\pi i\xi$. (One can think of $\mathcal{D}_f^2$ as the curvature associated to the "connection" $\mathcal{D}_f$.) Let now

$$w = \frac{s}{\mathcal{D}_f s} = \sum_\ell \frac{s \wedge (\mathcal{D}_f s)^{\ell-1}}{|f|^{2\ell}}$$

outside the singularity Y.

Proposition 7.5. *The forms $|f|^{2\lambda}w$ and $d|f|^{2\lambda}\wedge w$ have analytic continuations as currents to $\lambda=0$. If W and $\mathcal{R}^f$ are the values at $\lambda=0$, then W is a current extension of w across Y, $\mathcal{R}^f$ is supported on Y, and*

$$\mathcal{D}_f W\wedge\big(Df-\frac{i}{2\pi}\widetilde{\Theta}\big)^m/m! = (1-\mathcal{R}^f)\wedge(Df-\frac{i}{2\pi}\tilde{\Theta})^m/m!. \tag{7.9}$$

Proof. Outside Y we have

$$\mathcal{D}_f w = 1-\frac{s}{(\mathcal{D}_f s)^2}\wedge\mathcal{D}_f^2 s = 1-\frac{s}{(\mathcal{D}_f s)^2}\wedge\delta_s\big(Df-\frac{i}{2\pi}\widetilde{\Theta}\big)$$

by (7.8). Since

$$\delta_s\big(Df-\frac{i}{2\pi}\widetilde{\Theta}\big)\big(Df-\frac{i}{2\pi}\widetilde{\Theta}\big)^m/m! = \delta_s\big(Df-\frac{i}{2\pi}\widetilde{\Theta}\big)^{m+1}/(m+1)! = 0$$

for degree reasons, (7.5) holds outside Y. The analytic continuations are obtained precisely as in the proof of Theorem 1.1, and then (7.9) follows. □

Remark 3. From the usual desingularization it is easily seen that $\partial|f|^{2\lambda}\wedge w$ vanishes at $\lambda=0$ so that actually $\mathcal{R}^f=\bar\partial|f|^{2\lambda}\wedge w\big|_{\lambda=0}$. □

If we again extract the terms of full degree in e_j and e_j^* from (7.9), and notice that the connection D acts as the usual exterior differential d on such a term (i.e., on a section to the trivial line bundle $\det E^*\otimes\det E$), we get currents B and C such that $dB=c_m(D)-C$. However, A is the component of B of bidegree $(m,m-1)$ and $[Y^f]$ is the component of C of bidegree (m,m), and a simple consideration of bidegrees yields (7.6).

In the case when f is regular, these formulas, with the same A and B, were found in [3] with a quite different proof.

□

References

[1] M. Andersson: *Integral representation with weights I*, Preprint Göteborg (2001).

[2] M. F. Atiyah: *Resolution of singularities and division of distributions*, Com. Pure. Appl. Math. **23** (1970), 145–150..

[3] B. Berndtsson: *Cauchy-Leray forms and vector bundles*, Ann. scient. Ec. Norm. Sup. **24** (1991), 319–337.

[4] C. A. Berenstein & R. Gay & A. Vidras & A. Yger: *Residue currents and Bezout identities*, Progress in Mathematics **114** Birkhäuser Yerlag (1993).

[5] J-E. Björk: *Residue currents and $\mathcal{D}$-modules on complex manifolds*, Preprint Stockholm (1996).

[6] J. Briancon & H. Skoda: *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbb{C}^n$.*, C. R. Acad. Sci. Paris Sér. A 278 (1974), 949–951.

[7] A. DICKENSTEIN & C. SESSA: *Canonical representatives in moderate cohomology*, Invent. Math. **80** (1985), 417–434..

[8] A. DICKENSTEIN & C. SESSA: *Résidus de formes méromorphes et cohomologie modérée*, Géométrie complexe (Paris, 1992), 35–59, Actualités Sci. Indust., 1438, Hermann, Paris, 1996.

[9] PH. GRIFFITHS & J. HARRIS: *Principles of Algbraic Geometry*, John Wiley and Sons, (1978).

[10] M. PASSARE: *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), no. 1, 75–152..

[11] M. PASSARE: *A calculus for meromorphic currents*, J. Reine Angew. Math. **392** (1988), 37-56.

[12] M. PASSARE & A. TSIKH: *Residue integrals and their Mellin transforms*, Canadian J. of Math. **47** (1995), 1037-1050.

[13] M. PASSARE & A. TSIKH & A. YGER: *Residue currents of the Bochner-Martinelli type*, Publ. Mat. **44** (2000), 85-117.

[14] A. YGER & A. VIDRAS: *On some generalizations of Jacobi's formula*, Ann. Scient. Éc. Norm. Sup.. **34 (2001), 131-157**

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