ON BOUND STATES FOR SYSTEMS
OF WEAKLY COUPLED SCHRODINGER
EQUATIONS IN ONE SPACE DIMENSION

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Abstract. We establish the Birman-Schwinger relation for a class
of Schrödinger operators \(-d^2/dx^2 \otimes 1_\mathcal{H} + V\) on \(L^2(\mathbb{R}, \mathcal{H})\), where
\(\mathcal{H}\) is an auxiliary Hilbert space and \(V\) is an operator-valued po-
tential. As an application we give an asymptotic formula for the
bound states which may arise for a weakly coupled Schrödinger op-
erator with a matrix potential (having one or more thresholds). In
addition, for a two-channel system with eigenvalues embedded in
the continuous spectrum we show that, under a small perturbation,
such eigenvalues turn into resonances.

1. Introduction

In a recent paper [22] (see also [21]) we studied spectral and scat-
tering theory for the two-channel Schrödinger operator
\[
H = \tilde{H}_0 + V = \begin{pmatrix}
-\frac{d^2}{dx^2} & 0 \\
0 & -\frac{d^2}{dx^2} + 1
\end{pmatrix} + \begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}
\] (1.1)
on the Hilbert space \(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})\). In the low-energy limit, where
the spectral parameter tends to the boundary point of the continuous
spectrum of \(H\), viz. the point zero, we deduced asymptotic expan-
sions for the resolvent of \(H\) and, as an application, we obtained asymptotic ex-
expansions for the scattering matrix associated with the pair \((H, \tilde{H}_0)\)
as the energy parameter tends to zero. Besides being interesting from
the mathematical point of view, the study of spectral and scattering
theory for \(H\), having thresholds at 0 and 1, also works as a useful ex-
cercise towards analogous investigations for various multichannel quan-
tum system with more than one threshold (see, e.g., [23]) because it
describes many actual physical phenomena to a good approximation.

If we replace \(\tilde{H}_0\) in (1.1) by \(H_0 = -d^2/dx^2 \otimes 1_{\mathbb{C}^N}\) and \(V\) by an \(N \times N\)
matrix potential, we obtain the (usual) matrix Schrödinger operator
on \(L^2(\mathbb{R}, \mathbb{C}^N)\) having a single threshold at 0. The latter, of course, has
attracted a lot of attention during the years. Among recent results
we mention low-energy asymptotics for the corresponding scattering

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matrix [2, 3], Levinson’s theorem [14], Lieb-Thirring inequalities [20, 4] and quantum design [7].

A natural question, which seems not to have been addressed in the literature, concerns how negative energy levels may arise in a system of weakly coupled Schrödinger equations. In the scalar-valued setting, weakly coupled bound states for Schrödinger operators have been investigated in various dimensions (see [19, Chapter VI] and [31, 15, 16]). In this work we generalize the scalar-valued result obtained by Simon in dimension one [31] to the analogous matrix-valued setting.

We begin in the more abstract framework of Schrödinger operators with operator-valued potentials given formally by $H = -d^2/dx^2 \otimes 1_\mathcal{H} + V$ on $L^2(\mathbb{R}, \mathcal{H})$, where $\mathcal{H}$ is an auxiliary Hilbert space and the potential $V$ is a $\mathcal{B}(\mathcal{H})$-valued, measurable function on $\mathbb{R}$ such that $V(x)$ is symmetric for almost all $x$. In Section 3 we define the Hamiltonian $H$ by means of quadratic forms (Proposition 3.1) and in Section 4 we establish the celebrated Birman-Schwinger relation (Proposition 4.2), which transforms the eigenvalue problem for $H$ into an eigenvalue problem for a compact operator; the so-called Birman-Schwinger operator.

Equipped with the Birman-Schwinger relation we study weakly coupled bound states in Section 5. We restrict our attention to Schrödinger operators with matrix-valued potentials. In Section 5.1 we consider two-channel Hamiltonians with one and two thresholds, resp. First we consider $\mathbf{H}(g) = -d^2/dx^2 \otimes 1_{\mathbb{C}^2} + gV(x)$, where $V$ is a $2 \times 2$ matrix potential. Theorem 5.2 reveals how non-positive eigenvalues of an auxiliary matrix $\mathbf{S}$, defined in (5.2), give rise to negative eigenvalues $E_{ij}$ of $\mathbf{H}(g)$ provided $g$ is small enough. The eigenvalues $E_{ij}$ satisfy an asymptotic perturbation formula in which we derive the first few coefficients explicitly (see (5.3)). Second, we consider the above-mentioned Hamiltonian (1.1), henceforth denoted $\tilde{\mathbf{H}}(g)$, having thresholds at 0 and 1. In Theorem 5.6 we show how a negative eigenvalue of an auxiliary matrix $\tilde{\mathbf{S}}$, defined in (5.9), generates a negative eigenvalue of $\tilde{\mathbf{H}}(g)$. However, if one compares the proofs of Theorems 5.2 and 5.6 (in particular, the expressions for the matrices $\mathbf{T}_0$ and $\tilde{\mathbf{T}}_0$), it seems that the argument used in the proof of Theorem 5.2(ii), cannot be modified in order to treat the situation where zero is an eigenvalue of $\tilde{\mathbf{S}}$. Thus, it remains an attractive open problem to show that the zero eigenvalue of $\tilde{\mathbf{S}}$ (may) gives rise to a negative eigenvalue of $\tilde{\mathbf{H}}(g)$. In Section 5.2 we state the generalization of Theorem 5.2 to the $N$-channel Hamiltonian $-d^2/dx^2 \otimes 1_{\mathbb{C}^N} + \mathbf{V}(x)$, where $\mathbf{V}$ is an $N \times N$ matrix potential.

Having studied how negative eigenvalues arise for multichannel Hamiltonians under weak coupling, it is natural to address the problem of perturbation of embedded eigenvalues for a multichannel Schrödinger operator with a matrix-valued potential. In Section 6 we consider a two-channel Hamiltonian having eigenvalues embedded in its continuous spectrum. When perturbed by a “short range” potential, we show
that such eigenvalues move into the complex plane and become resonances. In particular, we verify Fermi’s golden rule (see, e.g., [27, 32]).

There is a vast literature on $2 \times 2$ operator-valued matrices, e.g. in system theory (see e.g. [8]) and in semigroup theory (see e.g. [11]). Most notably in this context is the substantial number of questions of a general nature which have been answered on spectral theory recently, see e.g. the survey [33]. However, the methods therein are not related to ours although some of the questions addressed clearly are, e.g. the appearance of resonances discussed by Mennicken and Motovilov [24].

2. Preliminaries

1) Vector-valued functions. Let $\mathcal{H}$ be a separable Hilbert space with scalar product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$. Then a function $\psi(x)$ from $\mathbb{R}$ to $\mathcal{H}$ is measurable if the scalar-valued functions $\langle \psi(x), \phi \rangle_{\mathcal{H}}$ are measurable, where $\phi$ denotes an arbitrary vector of $\mathcal{H}$. If $\psi(x)$ is such a measurable function, then $\| \psi(x) \|_{\mathcal{H}}$ is also measurable (as a function with non-negative values). Thus $L^p(\mathbb{R}, \mathcal{H})$ is defined as the set of equivalence classes of measurable functions $\psi(x)$ from $\mathbb{R}$ to $\mathcal{H}$, which satisfy that $\int_{\mathbb{R}} \| \psi(x) \|_{\mathcal{H}}^p dx$ is finite if $p < \infty$ and $\| \psi \|_{L^\infty} = \text{ess sup} \| \psi(x) \|_{\mathcal{H}} < \infty$ if $p = \infty$. The measure $dx$ is the Lebesgue measure. For any $p$ the $L^p(\mathbb{R}, \mathcal{H})$ space is a Banach space with norm $\| \cdot \|_p = \left( \int_{\mathbb{R}} \| \cdot \|_{\mathcal{H}}^p dx \right)^{1/p}$. In the case $p = 2$, $L^2(\mathbb{R}, \mathcal{H})$ is a complex and separable Hilbert space with scalar product $\langle \phi, \psi \rangle_2 = \int_{\mathbb{R}} \langle \phi, \psi \rangle_{\mathcal{H}} dx$ and corresponding norm $\| \psi \|_2 = \langle \psi, \psi \rangle_2^{1/2}$. For $n \in \mathbb{N}$, $1 \leq p < \infty$, the Sobolev space $W^{n,p}(\mathbb{R}, \mathcal{H})$ is defined as the space of those $\psi \in L^p(\mathbb{R}, \mathcal{H})$, for which all derivatives (weak sense) up to order $n$ are in $L^p(\mathbb{R}, \mathcal{H})$. If $p = 2$, $W^{n,2}(\mathbb{R}, \mathcal{H})$ is a separable Hilbert space denoted by $H^n(\mathbb{R}, \mathcal{H})$ with scalar product $\langle \phi, \psi \rangle_{H^n(\mathbb{R}, \mathcal{H})} = \int_{\mathbb{R}} \sum_{\alpha = 0}^n \langle (d/dx)^\alpha \phi, (d/dx)^\alpha \psi \rangle_{\mathcal{H}}$ and norm denoted by $\| \psi \|_{H^n(\mathbb{R}, \mathcal{H})}$.

2) Operators. Below $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. For a linear operator $T$, the notations $\mathcal{D}(T)$, $\text{Ran}(T)$, $\text{Ker}(T)$, $T^*$, $\overline{T}$, $\sigma(T)$, $\rho(T)$ are standard, see for example [25]. By $I$ we denote the identity operator. The resolvent of a self-adjoint operator $T$ is denoted by $R(T, z) = (T - zI)^{-1}$. By $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we denote respectively the sets of bounded and compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$. With the usual operator norm $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space. We set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathcal{S}_\infty(\mathcal{H}) := \mathcal{S}_\infty(\mathcal{H}, \mathcal{H})$.

3) Trace classes of compact operators. If $T \in \mathcal{S}_\infty(\mathcal{H})$ then the non-zero eigenvalues of $|T| = \sqrt{T^*T}$ are called the singular numbers or s-numbers of $T$. Let $\{s_j(T)\}$ denote the (possibly finite) non-increasing sequence of the singular numbers of $T$; every number counted according to its multiplicity as an eigenvalue of $|T|$. For $0 < p < \infty$ the von Neumann-Schatten class $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ is the set of $T \in \mathcal{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ for
which the functional
\[ \| T \|_{S_p(\mathcal{H}_1, \mathcal{H}_2)}^p := \sum_j |s_j(T)|^p \]
is finite. The functional \( \| \cdot \|_{S_p(\mathcal{H}_1, \mathcal{H}_2)} \) is a norm for \( p \geq 1 \) and the normed space \( S_p(\mathcal{H}_1, \mathcal{H}_2) \) is a Banach space. For \( p < 1 \) the functional is a quasinorm. For additional properties of the spaces \( S_p \) of compact operators we refer [5, Chapter 11]. The sets \( S_1(\mathcal{H}_1, \mathcal{H}_2) \) and \( S_2(\mathcal{H}_1, \mathcal{H}_2) \) are called the trace class and Hilbert-Schmidt class, respectively.

4) Operator-valued functions. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be two separable Hilbert spaces. From above, a function \( \mathbb{R} \ni x \mapsto \psi(x) \in \mathcal{H} \) is measurable if and only if all the functions \( \mathbb{R} \ni x \mapsto \langle \psi(x), \phi \rangle_{\mathcal{H}} \in \mathbb{C} \) are measurable. As a result of Pettis Measurability Theorem (see, e.g., [10, Theorem II.1.2]) the following properties are equivalent for a \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \)-valued function \( \mathbb{R} \ni x \mapsto T(x) \):
(i) \( \forall \phi \in \mathcal{H}_2, \forall \psi \in \mathcal{H}_1, \mathbb{R} \ni x \mapsto \langle \phi, T(x) \psi \rangle_{\mathcal{H}_2} \in \mathbb{C} \) is measurable,
(ii) \( \forall \psi \in \mathcal{H}_1, \mathbb{R} \ni x \mapsto T(x) \psi \in \mathcal{H}_2 \) is measurable.

We say that a function \( \mathbb{R} \ni x \mapsto T(x) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) is measurable if it satisfies any one of the above properties (i)-(ii). In the affirmative case, \( \| T(x) \|_{S(\mathcal{H}_1, \mathcal{H}_2)} \) is also measurable because
\[
\| T(x) \|_{S(\mathcal{H}_1, \mathcal{H}_2)} = \sup_{\psi \in D_1} \left( \| T(x) \psi \|_{\mathcal{H}_2}/\| \psi \|_{\mathcal{H}_1} \right),
\]
where \( D_1 \) is a countable dense subset of \( \mathcal{H}_1 \). Moreover, we can define \( L^p(\mathbb{R}, \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)) \) as the linear space of (equivalence classes of) measurable functions \( T : \mathbb{R} \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) such that \( \| T(\cdot) \|_{S(\mathcal{H}_1, \mathcal{H}_2)} \in L^p(\mathbb{R}) \).

For the functional calculus for self-adjoint operators we recall the following result which can be found in, e.g., [6, Proposition V.1.2].

Proposition 2.1. If for each \( x \in \mathbb{R} \), \( T(x) \) is a self-adjoint operator on \( \mathcal{H} \) and \( \{ E_{T(x)}(A) \mid A \text{ a Borel set of } \mathbb{R} \} \) denotes its resolution of the identity, the following three properties are equivalent:
(i) \( \mathbb{R} \ni x \mapsto E_{T(x)}(A) \in \mathcal{B}(\mathcal{H}) \) is measurable for all Borel sets \( A \),
(ii) \( \mathbb{R} \ni x \mapsto e^{-itT(x)} \in \mathcal{B}(\mathcal{H}) \) is measurable for all \( t \in \mathbb{R} \),
(iii) \( \mathbb{R} \ni x \mapsto T(x) - \zeta)^{-1} \in \mathcal{B}(\mathcal{H}) \) is measurable for all \( \zeta \in \mathbb{C} \setminus \mathbb{R} \).

5) Fourier transform. Suppose \( \psi \in L^1(\mathbb{R}^d, \mathcal{H}) \). Then we define its Fourier transform \( (\mathcal{F}\psi)(\xi) = \hat{\psi}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\xi} \psi(x) \, dx \) which is an element of \( L^\infty(\mathbb{R}, \mathcal{H}) \). If \( \psi \in L^1(\mathbb{R}, \mathcal{H}) \cap L^2(\mathbb{R}, \mathcal{H}) \), then \( \hat{\psi} \in L^2(\mathbb{R}, \mathcal{H}) \) with \( \| \hat{\psi} \|_{L^2} = \| \psi \|_{L^2} \). The Fourier transform can then be extended by continuity to a unitary mapping of the Hilbert space \( L^2(\mathbb{R}, \mathcal{H}) \) into itself.

We have the following criterion.

Lemma 2.2. Let \( T \) be an operator on \( L^2(\mathbb{R}, \mathcal{H}) \) defined by
\[
(T\phi)(x) = \int_{\mathbb{R}} t(x, \xi) \phi(\xi) \, d\xi,
\]
(2.1)
where \( t(x, \xi) \in \mathcal{B}(\mathcal{H}) \) for each \( (x, \xi) \). Then \( T \) is a Hilbert-Schmidt operator on \( L^2(\mathbb{R}, \mathcal{H}) \) if and only if
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \text{tr}_{\mathcal{H}}[t(x, \xi)^* t(x, \xi)] \, d\xi \, dx < \infty.
\]
In this case,
\[
\|T\|_{\mathcal{S}_2(L^2(\mathbb{R}, \mathcal{H}))}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \text{tr}_{\mathcal{H}}[t(x, \xi)^* t(x, \xi)] \, d\xi \, dx.
\]

Proof: The Hilbert space \( \mathcal{H} \) is isomorphic to some \( L^2(Y) \) space and therefore it suffices to establish the statement for an operator \( T \) on \( L^2(\mathbb{R}, L^2(Y)) \) defined by (2.1) for some \( t(x, \xi) \in \mathcal{B}(L^2(Y)) \). Since \( L^2(\mathbb{R}, L^2(Y)) \) is isomorphic to \( L^2(\mathbb{R} \times Y) \), the rephrased assertion follows immediately from [25, Theorem VI.23]. \( \square \)

3. THE HAMILTONIAN \( H = H_0 + V \)

As in the scalar-valued case the quadratic form
\[
h_0[\psi, \psi] := \int_{\mathbb{R}} \| (d/dx) \psi(x) \|^2_{\mathcal{H}} \, dx
\]
(3.1)
is closed in \( L^2(\mathbb{R}, \mathcal{H}) \) on the domain \( H^1(\mathbb{R}, \mathcal{H}) \). Thus, this form generates a self-adjoint operator \( H_0 \) on \( L^2(\mathbb{R}, \mathcal{H}) \). The free Hamiltonian \( H_0 \) corresponds to the “Laplacian” \( -d^2/dx^2 \otimes 1_{\mathcal{H}} \) on \( L^2(\mathbb{R}, \mathcal{H}) \).

A potential \( V \) is a \( \mathcal{B}(\mathcal{H}) \)-valued, measurable function on \( \mathbb{R} \). Assume that \( V(x) \) is symmetric for almost all \( x \), i.e., \( V(x)^* = V(x) \) for almost all \( x \). The operator \( V(x) \in \mathcal{B}(\mathcal{H}) \) has a unique representation\(^1\) in the form \( V(x) = U(x)|V(x)| \), where \( |V(x)| \) is the modulus of \( V(x) \) defined by \( |V(x)| = (V(x)^* V(x))^{1/2} = (V(x) V(x))^1/2 \). We have that \( |V(x)| \) is a non-negative, self-adjoint operator belonging to \( \mathcal{B}(\mathcal{H}) \) and, moreover, \( \| |V(x)| \|_{\mathcal{B}(\mathcal{H})} = \| V(x) \|_{\mathcal{B}(\mathcal{H})} \). The operator \( U(x) \) is a partial isometry with initial domain \( \text{Ran} |V(x)| \), final domain \( \text{Ran} V(x) \) and \( \text{Ker} U(x) = \text{Ker} V(x) \). Observe that \( U(x)^* U(x) = P_{\text{Ran} V(x)} \) and \( U(x) U(x)^* = P_{\text{Ran} |V(x)|} \), where \( P_M \) denotes the orthogonal projection onto a closed subspace \( M \). The modulus \( |V(x)| \) possesses exactly one non-negative, self-adjoint square-root \( |V(x)|^{1/2} \in \mathcal{B}(\mathcal{H}) \). The square-root \( |V(x)|^{1/2} \) commutes with every bounded operator which commutes with \( |V(x)| \). We may define \( V(x)^{1/2} = U(x) |V(x)|^{1/2} \) such that \( V(x) = V(x)^{1/2} V(x)^{1/2} \). Moreover, \( V(x)^{1/2} \in \mathcal{B}(\mathcal{H}) \) with \( \| V(x)^{1/2} \|_{\mathcal{B}(\mathcal{H})} = \| |V(x)|^{1/2} \|_{\mathcal{B}(\mathcal{H})}^{1/2} \) and adjoint \( (V(x)^{1/2})^* = |V(x)|^{1/2} U(x)^* \). From Proposition 2.1 it follows that \( |V|, |V|^{1/2} \) and \( V^{1/2} \) are \( \mathcal{B}(\mathcal{H}) \)-valued measurable functions on \( \mathbb{R} \).

\(^1\)The representation is not unique if the potential vanishes on a set of positive measure.
**Proposition 3.1.**

(i) If \( V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H})) \) then the real-valued quadratic form

\[
v[\psi, \psi] := \int_{\mathbb{R}} \langle V(x)^{1/2} \psi(x), |V(x)|^{1/2} \psi(x) \rangle_{\mathcal{H}} \, dx
\]

is \( H_0 \) form-bounded with relative bound zero.

(ii) If \( V \in L^1(\mathbb{R}, \mathcal{S}_2(\mathcal{H})) \) then \( v \) is \( H_0 \) form-compact.

It follows from Proposition 3.1(i) and the KLMN theorem [26, Theorem X.17] that the form sum

\[
h[\psi, \psi] := h_0[\psi, \psi] + v[\psi, \psi]
\]

is closed and semi-bounded from below on \( H^1(\mathbb{R}^d, \mathcal{H}) \) and thus generates a self-adjoint operator \( H = H_0 + V \) on \( L^2(\mathbb{R}, \mathcal{H}) \). From Proposition 3.1(ii) and Weyl’s essential spectrum theorem it follows that \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0, \infty) \).

**Proof of Proposition 3.1.** The “kernel” of the resolvent of \( H_0 \) is given by (see, e.g., [28, Theorem 9.5.2])

\[
Q(x - y; \sqrt{E}) = \frac{e^{-\sqrt{|E|}|x-y|}}{2\sqrt{|E|}}, \quad E < 0. \tag{3.2}
\]

(i) To show that the form \( v \) is infinitesimally \( H_0 \) form-bounded, it suffices to show that the form

\[
w[\phi] = \langle |V|^{1/2}(H_0 - E)^{-1/2} \phi, |V|^{1/2}(H_0 - E)^{-1/2} \phi \rangle_{L^2(\mathbb{R}, \mathcal{H})}
\]

is bounded on \( L^2(\mathbb{R}, \mathcal{H}) \) and that its norm

\[
\|w\| := \inf_{\phi \in L^2(\mathbb{R}, \mathcal{H})} \frac{|\langle |V|^{1/2}(H_0 - E)^{-1/2} \phi, |V|^{1/2}(H_0 - E)^{-1/2} \phi \rangle|}{\|\phi\|^2}
\]

tends to zero as \( E \to -\infty \). By the definition of \( \|w\| \), and since \( U \) in \( V^{1/2} = U|V|^{1/2} \) is a partial isometry, we have that

\[
\|w\| \leq \|V|^{1/2}(H_0 - E)^{-1/2}\|^2_{\mathcal{B}(L^2(\mathbb{R}, \mathcal{H}))}. \tag{3.3}
\]

Therefore, it is enough to show that the right-hand side of the latter tends to zero as \( E \to -\infty \).

We consider first \( V \in L^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H})) \). For such \( V \) we have that

\[
\|V|^{1/2}(H_0 - E)^{-1/2}\|^2_{\mathcal{B}(L^2)} = \|V|^{1/2}(H_0 - E)^{-1}|V|^{1/2}\|^2_{\mathcal{B}(L^2)} \tag{3.4}
\]

Let \( \alpha = \sqrt{|E|} \) and \( \phi \in L^2(\mathbb{R}, \mathcal{H}) \). Then Hölder’s inequality yields that

\[
\|\|V|^{1/2}(H_0 + \alpha^2)^{-1}|V|^{1/2}\phi\|_{\mathcal{H}} \leq \frac{1}{2\alpha} \|V(x)\|^{1/2}_{\mathcal{B}(\mathcal{H})} \|V\|_{L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))} \|\phi\|_{L^2(\mathbb{R}, \mathcal{H})}.
\]
The latter implies that
\[
\left\| |V|^{1/2}(H_0 + \alpha^2)^{-1} |V|^{1/2} \phi \right\|^2_{L^2(\mathbb{R}, \mathcal{H})} 
\leq \int_{\mathbb{R}} \frac{1}{4\alpha^2} \left\| V(x) \right\|_{L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))} \left\| |V|^{1/2} \right\|^2_{L^2(\mathbb{R}, \mathcal{H})} \left\| \phi \right\|^2_{L^2(\mathbb{R}, \mathcal{H})} \, dx 
\leq \frac{1}{4\alpha^2} \left\| V \right\|^3_{L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))} \left\| \phi \right\|^2_{L^2(\mathbb{R}, \mathcal{H})}.
\]

In conjunction with (3.4), the latter shows that the right-hand side of (3.3) tends to zero as \( E \to -\infty \), which establishes assertion (i) for \( V \in L^\infty(\mathbb{R}, \mathcal{H}) \). A standard approximation argument yields the assertion for general \( V \).

(ii). It suffices to show that the form
\[
w[\phi] = \left\langle |V|^{1/2}(H_0 - E)^{-1/2} \phi, |V|^{1/2}(H_0 - E)^{-1/2} \phi \right\rangle
\]
defines a compact operator in \( L^2(\mathbb{R}, \mathcal{H}) \). Under the assumption in (i) we already know that \( w \) generates a bounded, self-adjoint operator \( W \) in \( L^2(\mathbb{R}, \mathcal{H}) \). Let us show that \( W \) is a Hilbert-Schmidt operator. From
\[
\text{tr}_{L^2}(W^*W) = \text{tr} \left( |V|^{1/2}(H_0 - E)^{-1} (|V|^{1/2})^* |V|^{1/2}(H_0 - E)^{-1} (|V|^{1/2})^* \right),
\]
we see that it is enough to show that \( W_1 = |V|^{1/2}(H_0 - E)^{-1} (|V|^{1/2})^* \) and \( W_2 = |V|^{1/2}(H_0 - E)^{-1} (|V|^{1/2})^* \) are Hilbert-Schmidt operators on \( L^2(\mathbb{R}, \mathcal{H}) \). It is enough to show it for \( W_2 \); the proof for \( W_1 \) is similar. The operator \( W_2 \) has integral "kernel"
\[
K_{W_2}(x - y; \alpha) = V(x)^{1/2}(2\alpha)^{-1}e^{-\alpha|x-y|}V(y)^{1/2}^*, \quad \alpha = \sqrt{-E} > 0.
\]
Using the criterion in Lemma 2.2 and the assumption in (ii), we estimate as follows:
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \text{tr}_\mathcal{H}[K_{W_2}(x - y; \alpha)^*K_{W_2}(x - y; \alpha)] \, dx \, dy = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \frac{e^{-\alpha|x-y|}}{2\alpha} \right)^2 \, dx \, dy
\times \text{tr}_\mathcal{H}[|V(x)|^{1/2})^*|V(x)|^{1/2})^* |V(x)|^{1/2} |V(y)|^{1/2})^* \, dx \, dy
\leq \frac{1}{4\alpha^2} \int_{\mathbb{R}} ||V(x)||_{s_2(\mathcal{H})} \, dx \int_{\mathbb{R}} ||V(y)||_{s_2(\mathcal{H})} \, dy.
\]
This shows that \( W_2 \) is a Hilbert-Schmidt operator in \( L^2(\mathbb{R}, \mathcal{H}) \). □

We note that \( V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H})) \) implies that \( |V| \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H})) \) and, in view of Proposition 3.1(i), \( |V| \) is infinitesimally \( H_0 \) form-bounded.
Consequently, the following mappings are bounded:

\[
V, |V| : H^1(\mathbb{R}, \mathcal{H}) \rightarrow H^{-1}(\mathbb{R}, \mathcal{H})
\]

(3.5)

\[
|V|^{1/2}, V^{1/2} : H^1(\mathbb{R}, \mathcal{H}) \rightarrow L^2(\mathbb{R}, \mathcal{H})
\]

(3.6)

\[
|V|^{1/2}, V^{1/2} : L^2(\mathbb{R}, \mathcal{H}) \rightarrow H^{-1}(\mathbb{R}, \mathcal{H}).
\]

(3.7)

The qualitative behaviour of any possible negative eigenvalues of \(H_0 + gV\) as \(g \to 0\) is described by the following simple result.

**Proposition 3.2.** If \(V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))\) then any negative eigenvalues of \(H_0 + gV\) approach zero as \(g\) tends to zero.

**Proof.** Following [31] it suffices to show that there are positive constants \(g_0\) and \(C\) such that \(H_0 + gV \geq -Cg\) for all \(g_0 > g > 0\).

Let \(\mathcal{F}\) denote the Fourier transform of vector-valued functions in \(L^2(\mathbb{R}, \mathcal{H})\) (see Part 5 in Section 2). We observe that, as for scalar-valued functions, a function \(\phi\) whose Fourier transform is integrable is bounded and continuous with the usual estimate

\[
\|\phi(x)\|_{\mathcal{H}} \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \|\mathcal{F}\phi(\xi)\|_{\mathcal{H}} d\xi.
\]

(3.8)

For an arbitrary \(\gamma > 0\), Hölder’s inequality yields that

\[
\left( \int_{\mathbb{R}} \|\mathcal{F}\phi(\xi)\|_{\mathcal{H}} d\xi \right)^2 \\
\leq \left( \int_{\mathbb{R}} (\xi^2 + \gamma^2)^{-1} d\xi \right) \left( \int_{\mathbb{R}} (\xi^2 + \gamma^2) \|\mathcal{F}\phi(\xi)\|_{\mathcal{H}}^2 d\xi \right) \\
= \frac{\pi}{\gamma} \|(-i\xi + \gamma)\mathcal{F}\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} = \frac{\pi}{\gamma} \|(H_0^{1/2} + \gamma)\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} \\
\leq \frac{2\pi}{\gamma} \left\{ \|H_0^{1/2}\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} + \|\gamma\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} \right\}.
\]

(3.9)

Let \(d = \max\{1/\gamma, 1\}\). Then (3.8) and (3.9) imply that

\[
\|\phi(x)\|_{\mathcal{H}} \leq d \left\{ \|H_0^{1/2}\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} + \|\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} \right\}
\]

(3.10)

for any \(\phi \in \mathcal{D}(H_0^{1/2}) = H^1(\mathbb{R}, \mathcal{H})\). The Sobolev type inequality (3.10) implies that

\[
h_g[\phi] \geq \|H_0^{1/2}\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} - g \int_{\mathbb{R}} \|V(x)\|_{\mathcal{H}} \|\phi(x)\|_{\mathcal{H}}^2 dx \\
\geq (1 - gd_1) \|H_0^{1/2}\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})} - gd_1 \|\phi\|^2_{L^2(\mathbb{R}, \mathcal{H})}
\]

where \(d_1 = d\|V\|^2_{L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))}\). When we take \(C = d_1\), \(g_0 = d_1^{-1}\) and \(0 < g < g_0\), we arrive at \(h_g[\phi] \geq -Cg\) as desired. \(\Box\)
4. The Birman-Schwinger relation

The Birman-Schwinger relation has been established rigorously for various classes of operators in the scalar-valued setting (see, e.g., [30, 31, 17]. It asserts that $E$ is a negative eigenvalue of $H = -d^2/dx^2 + V$ if and only if $-1$ is an eigenvalue of the operator $V^{1/2}(-d^2/dx^2 - E)^{-1}V^{1/2}$. Formally this is obvious since $\phi = V^{1/2}\psi$ is a solution to $V^{1/2}(-d^2/dx^2 - E)^{-1}V^{1/2}\phi = -\phi$.

Here we provide a simple proof of the Birman-Schwinger relation in our concrete operator-valued setting. For this purpose we introduce the Birman-Schwinger operator $K_E(V) = V^{1/2}(H_0 - E)^{-1}V^{1/2}$, $E < 0$, where $H_0$ is the nonnegative, self-adjoint operator associated with the quadratic form $h_0$ in (3.1). Setting $\alpha^2 = -E$, its integral "kernel" is given by

$$K_\alpha(x, y) = V(x)^{1/2}(2\alpha)^{-1}e^{-\alpha^2|x-y|}V(y)|^{1/2}, \quad \alpha > 0.$$ 

We have the following result.

**Lemma 4.1.** If $V \in L^1(\mathbb{R}, S_2(\mathcal{H}))$ then the Birman-Schwinger operator $K_E(V)$, $E < 0$, is a Hilbert-Schmidt operator on $L^2(\mathbb{R}, \mathcal{H})$; in particular $K_E(V)$ is a compact operator. Moreover, $\|K_E(V)\|_{\mathcal{B}(L^2(\mathbb{R}, \mathcal{H}))} \to 0$ as $E \to -\infty$.

**Proof.** We argue as for the operator $W_2$ in the proof of Proposition 3.1(ii). We omit the details. \hfill \Box

Having introduced the compact Birman-Schwinger operator we may formulate the Birman-Schwinger relation.

**Proposition 4.2.** Let $V \in L^1(\mathbb{R}, S_2(\mathcal{H}))$. Then $E < 0$ is an eigenvalue of $H = H_0 + V$ (defined by a quadratic form) having multiplicity $\neq$ if and only if $-1$ is an eigenvalue of $K_E(V)$ having geometric multiplicity $\neq$.

To establish Proposition 4.2 we need the following two results.

**Lemma 4.3.** If $V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ then for $E < 0$ the operators $|V|^{1/2} \times (H_0 - E)^{-1/2}$ and $V^{1/2}(H_0 - E)^{-1/2}$ are bounded on $L^2(\mathbb{R}, \mathcal{H})$.

**Proof.** Since $V$ is $H_0$ form-bounded, it follows immediately from (3.6) and (3.7) in conjunction with the fact that the operator $(H_0 - E)^{-1/2}$ is a bounded map from the domain $L^2(\mathbb{R}, \mathcal{H})$ to the range $H^1(\mathbb{R}, \mathcal{H})$. \hfill \Box

**Lemma 4.4.** Let $S$ and $T$ be bounded operators on the Hilbert space $\mathcal{K}$. Then $\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}$. Moreover, $\lambda \neq 0$ is an eigenvalue of $ST$ having geometric multiplicity $m$ if and only if $\lambda$ is an eigenvalue of $TS$ having geometric multiplicity $m$.

**Proof.** This is a simplified version of Theorem 2(i) in [9]. \hfill \Box

**Proof of Proposition 4.2.** Let $h_0$ be the form of $H_0$, let $v$ be the form of $V$ and let $h = h_0 + v$ be their form sum. According to Lemma 4.3
the operators $|V|^{1/2}(H_0 - E)^{-1/2}$ and $V^{1/2}(H_0 - E)^{-1/2}$ are bounded on $L^2(\mathbb{R}, \mathcal{H})$ and, consequently, the operator $I + [|V|^{1/2}(H_0 - E)^{-1/2}]^*V^{1/2}(H_0 - E)^{-1/2}$ is bounded on $L^2(\mathbb{R}, \mathcal{H})$. Moreover, the operator $A^{-1} = (H_0 - E)^{1/2}$ has domain $H^1(\mathbb{R}, \mathcal{H})$ and range $L^2(\mathbb{R}, \mathcal{H})$. Thus, we may introduce an auxiliary sesquilinear form $a$ defined on the form domain $H^1(\mathbb{R}, \mathcal{H}) \times H^1(\mathbb{R}, \mathcal{H})$ by

$$a[\phi, \psi] = \langle [I + (|V|^{1/2}A)^*V^{1/2}A]^{-1} \phi, A^{-1} \psi \rangle.$$  \hspace{1cm} (4.1)

We re-write $a$ and find that

$$a[\phi, \psi] = \frac{\langle A^{-1} \phi, A^{-1} \psi \rangle + \langle (|V|^{1/2}A)^*V^{1/2}AA^{-1} \phi, A^{-1} \psi \rangle}{a_1[\phi, \psi]} a_2[\phi, \psi].$$  \hspace{1cm} (4.2)

Clearly,

$$a_1[\phi, \psi] = h_0[\phi, \psi] - E\langle \phi, \psi \rangle$$ \hspace{1cm} (4.3)

and, since $|V|^{1/2}A$ is bounded on $L^2(\mathbb{R}, \mathcal{H})$,

$$a_2[\phi, \psi] = \langle V^{1/2}AA^{-1} \phi, [|V|^{1/2}A]^*A^{-1} \psi \rangle = \langle V^{1/2}A, V^{1/2}AA^{-1} \psi \rangle = v[\phi, \psi].$$  \hspace{1cm} (4.4)

Hence, (4.2)-(4.4) shows that the forms $a$ and $h - E$ are identical.

Suppose that $E < 0$ is an eigenvalue of $H = H_0 + V$, i.e., there exists an eigenfunction $\psi \in \mathcal{D}(H)$, $\psi \neq 0$, such that $(H - E)\psi = 0$. This is equivalent to $(h - E)[\psi, \phi] = 0$ for all $\phi \in H^1(\mathbb{R}, \mathcal{H})$. Since the forms $h$ and $a$ are identical, we may introduce $u = A^{-1} \phi$ and deduce that

$$0 = a[\psi, \phi] = \langle [I + (|V|^{1/2}A)^*V^{1/2}A]^{-1} \psi, A^{-1} \phi \rangle, \forall \phi \in H^1(\mathbb{R}, \mathcal{H}),$$

$$= \langle [I + (|V|^{1/2}A)^*V^{1/2}A]^{-1} \psi, u \rangle, \forall \phi \in L^2(\mathbb{R}, \mathcal{H}),$$

because $u$ runs through $L^2(\mathbb{R}, \mathcal{H})$ as $\phi$ runs through $H^1(\mathbb{R}, \mathcal{H})$. Consequently, $(I + (|V|^{1/2}A)^*V^{1/2}A) v = 0$, where $v = A^{-1} \psi = (H_0 - E)^{1/2} \psi$, so $-1 \in \sigma_p(|V|^{1/2}A)^*V^{1/2}A)$. By reversing the arguments leading to the latter conclusion, we infer that

$$E \in \sigma_p(H_0 + V) \text{ if and only if } -1 \in \sigma_p(|V|^{1/2}A)^*V^{1/2}A)$$  \hspace{1cm} (4.5)

Since $(H_0 - E)^{1/2}$ is injective from the domain $L^2(\mathbb{R}, \mathcal{H})$ to the range $H^1(\mathbb{R}, \mathcal{H})$, the arguments above also show that the multiplicities of the eigenvalues $E$ and $-1$ must be equal. In view of Lemma 4.4 and the definition of $A$, (4.5) implies that

$$E \in \sigma_p(H_0 + V) \text{ if and only if } -1 \in \sigma_p(V^{1/2}(H_0 - E)^{-1/2}|V|^{1/2}(H_0 - E)^{-1/2})^*$$  \hspace{1cm} (4.6)

and the multiplicities of $E$ and $-1$ are equal. But $|V|^{1/2}(H_0 - E)^{-1/2})^* = (H_0 - E)^{-1/2}|V|^{1/2}$ and therefore, in view of the definition of $K_E(V)$, (4.6) yields that $E \in \sigma_p(H_0 + V)$ if and only if $-1 \in \sigma_p(K_E(V)).$  \hspace{1cm} \(\square\)
If \( g \) is fixed and \( \lambda(\alpha) \) is an eigenvalue of \( K_\alpha(V) \) then the Birman-Schwinger relation asserts that any solution \( \alpha_g > 0 \) of
\[
g\lambda(\alpha_g) = -1
\]
is associated to the eigenvalue \( E(g) = -\alpha_g^2 \) of \( H(g) \). The latter equation plays a crucial role in Section 5.

Define the operators \( L_\alpha \) and \( M_\alpha \) by their "kernels":
\[
L_\alpha(x, y) = \frac{1}{2\alpha} V(x)^{1/2} |V(y)|^{1/2},
\]
\[
M_\alpha(x, y) = \frac{1}{2\alpha} V(x)^{1/2} [e^{-\alpha|x-y|} - 1]|V(y)|^{1/2}
\]
Moreover, we introduce the operator \( M_0 \) with "kernel"
\[
M_0(x, y) = -\frac{1}{2} V(x)^{1/2} |x - y||V(y)|^{1/2}.
\]

Imitating [31] we obtain the following result.

**Lemma 4.5.** If \( \int_{\mathbb{R}} (1 + |x|^2) \|V(x)\|_{S_2(\mathcal{H})} \, dx < \infty \) then the following assertions are valid:

(i) The operator \( M_0 \) is Hilbert-Schmidt on \( L^2(\mathbb{R}, \mathcal{H}) \).

(ii) As \( \alpha \downarrow 0 \), the operator \( M_\alpha \) converges to \( M_0 \) in the Hilbert-Schmidt norm on \( L^2(\mathbb{R}, \mathcal{H}) \).

(iii) The Birman-Schwinger operator \( gK_\alpha(V) \) has eigenvalue \(-1\) if and only if the same is true for \( g(1 + gM_\alpha)^{-1}L_\alpha \).

**Proof.**

(i) It follows from the estimate
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \text{tr}_\mathcal{H}[M_0(x, y)^*M_0(x, y)] \, dx \, dy \\
\leq \frac{1}{2} \int \int (|x|^2 + |y|^2) \|V(x)\|_{S_2(\mathcal{H})} \|V(y)\|_{S_2(\mathcal{H})} \, dx \, dy < \infty.
\]

(ii) We want to show that
\[
\int \int \text{tr}_\mathcal{H} [(M_\alpha - M_0)(x, y)^*(M_\alpha - M_0)(x, y)] \, dx \, dy \longrightarrow 0
\]
as \( \alpha \downarrow 0 \). Now,
\[
\text{tr}_\mathcal{H} [(M_\alpha - M_0)(x, y)^*(M_\alpha - M_0)(x, y)] \\
= \left| \frac{1}{2\alpha} (e^{-\alpha|x-y|} - 1) + \frac{1}{2}|x - y| \right|^2 \text{tr}_\mathcal{H} [V(x)\|V(y)\|],
\]
and since \( \frac{1}{2\alpha} (e^{-\alpha|x-y|} - 1) + \frac{1}{2}|x - y| \rightarrow 0 \) as \( \alpha \downarrow 0 \), we have the point-wise convergence
\[
\text{tr}_\mathcal{H} [(M_\alpha - M_0)(x, y)^*(M_\alpha - M_0)(x, y)] \longrightarrow 0 \text{ as } \alpha \downarrow 0.
\]
Moreover,
\[
\text{tr}_\mathcal{H}[M_\alpha(x, y)^* M_\alpha(x, y)] = \left| \frac{1}{2\alpha} \left( e^{-\alpha|x|} - 1 \right) \right|^2 \text{tr}_\mathcal{H}[|V(x)||V(y)|] \\
\leq \left| \frac{1}{2} |x - y| \right|^2 \text{tr}_\mathcal{H}[|V(x)||V(y)|] = \text{tr}_\mathcal{H}[M_0(x, y)^* M_0(x, y)]. \tag{4.13}
\]

It follows from (i) and (4.12)-(4.13) in conjunction with Lebesgue’s dominated convergence theorem that (4.11) holds.

(iii) It follows from (4.13) that \( \|M_\alpha\|_{\mathcal{B}(L^2(\mathbb{R}, \mathcal{H}))} \leq ||M_\alpha||_{HS} \leq ||M_0||_{HS} \). Hence, \( \|M_\alpha\|_{\mathcal{B}(L^2(\mathbb{R}, \mathcal{H}))} \) is bounded independently of \( \alpha \in (0, \alpha_0] \) for some \( \alpha_0 > 0 \). Therefore, for \( g \) small enough, \( \|gM_\alpha\|_{\mathcal{B}(L^2(\mathbb{R}, \mathcal{H}))} < 1 \) and, consequently, \( (1 + gM_\alpha)^{-1} \) exists and is bounded for these \( g \) and \( \alpha \). In particular, we may write \( 1 + gK_\alpha(V) = (1 + gM_\alpha)[1 + g(1 + gM_\alpha)^{-1}L_\alpha] \) from which the assertion follows. \( \square \)

5. WEAKLY COUPLED BOUND STATES

Throughout this section operators (resp. vectors) are denoted by boldface capital (resp. small) letters to emphasize their matrix (resp. vector) structure.

5.1. TWO-CHANNEL HAMILTONIANS WITH MATRIX-VALUED POTENTIALS.

We consider the case where the potential is a \( 2 \times 2 \) matrix-valued potential \( V(x) \) with measurable functions \( V_{ij} \) on \( \mathbb{R} \) as entries. The Euclidean inner product and norm in \( \mathbb{C}^2 \) are denoted by \( \langle \cdot, \cdot \rangle_{\mathbb{C}^2} \) and \( \| \cdot \|_{\mathbb{C}^2} \), respectively.

**Assumption 5.1.**

(a) \( V(x) \) is symmetric, i.e. \( \overline{V_{ji}} = V_{ij} \).

(b) \[ \int_{\mathbb{R}} (1 + |x|^2) \|V(x)\|_{\mathcal{B}(\mathbb{C}^2)} \, dx < \infty. \]

(c) The functions \( V_{ij} \) are real-valued.

5.1.1. TWO-CHANNEL HAMILTONIAN WITH A SINGLE THRESHOLD.

First we consider the Hamiltonian \( \mathbf{H}(g) = \mathbf{H}_0 + gV(x) \) in \( L^2(\mathbb{R}, \mathbb{C}^2) \), defined in Proposition 3.1 by means of forms. Formally, we may write the Hamiltonian as

\[
\mathbf{H}(g) = \mathbf{H}_0 + g\mathbf{V} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix} + g \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \tag{5.1}
\]

in \( L^2(\mathbb{R}, \mathbb{C}^2) = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \). Under Assumption 5.1 we know that its essential spectrum equals the half-axis starting at the (threshold) point zero.

Define the matrices \( \mathbf{S} \) and \( \mathbf{T} \) by

\[
\mathbf{S} = \int_{\mathbb{R}} V(x) \, dx, \quad \mathbf{T}_0 = -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} V(x)|x-y|V(y) \, dy \, dx. \tag{5.2}
\]
We establish the following result.

**Theorem 5.2.** Let $V$ obey Assumption 5.1(a)-(b) and let $H(g) = H_0 + gV(x)$ be the self-adjoint Hamiltonian on $L^2(\mathbb{R}, \mathbb{C}^2)$ defined in Proposition 3.1 by means of forms.

(i) Assume that the matrix $S$, defined in (5.2), has $n \leq 2$ negative eigenvalues, denoted by $s_i$, with multiplicities $\kappa_i$. Then, for a small enough $g$, the two-channel Hamiltonian $H(g)$ has precisely $\sum_{i=1}^{n} \kappa_i$ negative eigenvalues (taking into account multiplicity) $E_{ij}$ satisfying the formulas

$$(-E_{ij}(g))^{1/2} = -\frac{g}{2} s_i + \frac{g^2}{2} \langle v_{ij}, T_0 v_{ij} \rangle_{\mathbb{C}^2} + O(g^3),$$

(iii) suppose that $V$ obey Assumption 5.1(c) and that the matrix $S$ has $n$ non-positive eigenvalues, denoted by $s_i$, with multiplicities $\kappa_i$. If the eigenvectors $v_{ij}$, $j = 1, \ldots, n$, $j = 1, \ldots, \kappa_i$, associated with the eigenvalue zero of $S$ satisfy $\langle v_{0j}, T_0 v_{0j} \rangle_{\mathbb{C}^2} \neq 0$ then the conclusion of part (i) remains valid.

**Proof.** According to the Birman-Schwinger relation formulated in Proposition 4.2, $E(g) < 0$ is an eigenvalue of $H(g)$ if and only if $-1$ is an eigenvalue of $gK_\alpha(V)$ with $\alpha^2 = -E(g)$. Furthermore, in view of Lemma 4.5(iii), the operator $gK_\alpha(V)$ has eigenvalue $-1$ if and only if the same is true for $g(1 + gM_\alpha)^{-1}L_\alpha$. Now let us denote the (unknown) eigenvalues and eigenfunctions of $(1 + gM_\alpha)^{-1}L_\alpha$ by $\mu_k(g, \alpha)$ and $\Psi_k(x; g, \alpha)$, respectively, viz.

$$(1 + gM_\alpha)^{-1}L_\alpha \Psi_k(x; g, \alpha) = \mu_k(g, \alpha) \Psi_k(x; g, \alpha).$$

Let $u_k \in \mathbb{C}^2$ be a constant vector. We insert

$$\Psi_k(x; g, \alpha) = \frac{1}{2\alpha} (1 + gM_\alpha)^{-1}|V(x)|^{1/2}u_k$$

into (5.4) and obtain

$$R_g u_k = \mu_k(g, \alpha) u_k,$$

where $R_g$ is the matrix

$$R_g = \frac{1}{2\alpha} \int_{\mathbb{R}} V^{1/2}(x)[(1 + gM_\alpha)^{-1}|V|^{1/2}](x) \, dx.$$

Define $S$ as in (5.2) and, moreover, define

$$T(\alpha) = \int_{\mathbb{R}} V^{1/2}(x)|M_\alpha| |V|^{1/2}(x) \, dx.$$

Then we have that

$$R_g = \frac{1}{2\alpha} S - \frac{g}{2\alpha} T(\alpha) + O(g^2).$$
for small $g$.

(i) By assumption the matrix $S$ has $n$ negative eigenvalues, denoted by $s_k$. For simplicity we assume that the eigenvalues $s_k$ are simple. The corresponding eigenvectors are denoted by $v_k$. We apply the regular perturbation theory to the eigenvalue problem (5.5) and we find that

$$
\mu_k(g, \alpha) = \frac{1}{2\alpha} s_k - \frac{g}{2\alpha} \langle v_k, T(\alpha) v_k \rangle_{C^2} + O(g^2).
$$

Define the matrix $T_0$ as in (5.2) and

$$
T_1 = \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} V(x) |x - y|^2 V(y) \, dx \, dy.
$$

Then we have that $T(\alpha) = T_0 + \alpha T_1 + O(\alpha^2)$. In this way we find that the eigenvalues associated with the eigenvalue problem (5.5) are

$$
\mu_k(g, \alpha) = \frac{1}{2\alpha} s_k - \frac{g}{2\alpha} \langle v_k, T_0 v_k \rangle_{C^2} + O(g^2).
$$

Together with the comments following the proof of Proposition 4.2, the latter implies that the solution to (4.7) is

$$
\alpha_g = -\frac{g}{2} s_k + \frac{g^2}{2} \langle v_k, T_0 v_k \rangle_{C^2} + O(g^3).
$$

Clearly, (5.6) implies that each negative eigenvalue $s_k$ of $S$ gives rise to precisely one negative eigenvalue $E_k(g)$ of $H(g)$ obeying the asymptotic formula

$$
(-E_k(g))^{1/2} = -\frac{g}{2} s_k + \frac{g^2}{2} \langle v_k, T_0 v_k \rangle_{C^2} + O(g^3).
$$

(ii) We investigate the situation where zero is an eigenvalue of $S$ (as above we restrict ourselves to the case where zero is simple). Let $S v_0 = 0$ for some $v_0 \neq 0$. Taylor’s formula yields

$$
\langle v_0, \int_{\mathbb{R}} \int_{\mathbb{R}} V(x) e^{-\alpha |x-y|} V(y) \, dx \, dy v_0 \rangle_{C^2}
$$

$$
= \frac{1}{2\alpha} \langle v_0, S^2 v_0 \rangle_{C^2} - \frac{1}{2} \langle v_0, \int_{\mathbb{R}} \int_{\mathbb{R}} V(x) |x - y| V(y) \, dx \, dy v_0 \rangle_{C^2}
$$

$$
+ \alpha \langle v_0, \int_{\mathbb{R}} \int_{\mathbb{R}} V(x) O(|x - y|^2) V(y) \, dx \, dy v_0 \rangle_{C^2}.
$$

Since $S v_0 = 0$ by assumption, the first term equals zero. As $\alpha \downarrow 0$, we obtain that

$$
\langle v_0, T_0 v_0 \rangle_{C^2} = \lim_{\alpha \downarrow 0} \langle v_0, \int_{\mathbb{R}} \int_{\mathbb{R}} V(x) e^{-\alpha |x-y|} V(y) \, dx \, dy v_0 \rangle_{C^2}
$$

$$
= \lim_{\alpha \downarrow 0} \sum_{i,j,k} \int e^{-\alpha |x-y|} V_{ik}(x) V_{kj}(y)(v_0)_i (\overline{v_0})_j \, dx \, dy. \quad (5.7)
$$
Let $\mathcal{F}$ denote the one-dimensional Fourier transform and let $(\mathcal{F}V)(\xi)$ denote the matrix with elements $(\mathcal{F}V)_{ij}(\xi) = (\mathcal{F}V_{ij})(\xi) = 1/(\xi^2 + \alpha^2)$, because $V$ is symmetric and $V_{ij}$ are real-valued. Using the latter in conjunction with the Fourier transform of $(1/2\alpha)e^{-\alpha|x|}$, which equals $1/(\xi^2 + \alpha^2)$, we find that

$$
\text{r.h.s. of (5.7)} = \lim_{\alpha \downarrow 0} \sum_{i,j,k} \int \frac{1}{\xi^2 + \alpha^2} (\mathcal{F}V_{ik})(\xi)(\mathcal{F}V_{kj})(\xi)(\psi_0)_i(\psi_0)_j \, d\xi
$$

$$
= \int \frac{1}{\xi^2} \langle\psi_0, (\mathcal{F}V)^* (\mathcal{F}V)(\xi)\psi_0\rangle_{C^2} \, d\xi
$$

$$
= \int \frac{1}{\xi^2} \| (\mathcal{F}V)(\xi)\psi_0 \|_{C^2}^2 \, d\xi \geq 0, \quad (5.8)
$$

By assumption, $\langle\psi_0, T_0\psi_0\rangle_{C^2} \neq 0$ and therefore (5.8) implies that there is also a negative eigenvalue of $H(g)$ associated with the eigenvalue zero of $\mathbf{S}$. \hfill \Box

Remark 5.3. The reasoning in the proof of Theorem 5.2(ii) requires that the entries $V_{ij}$ in the potential $\mathbf{V}$ are real-valued. A substantial improvement would be to establish the result for complex-valued entries.

Example 5.4 (Square-well potentials). Let $\chi_{[0,1]}$ denote the characteristic function associated with the interval $[0,1]$. Choose the following entries of $\mathbf{V}$:

$$
V_{11}(x) = -5\chi_{[0,1]}(x), \quad V_{22}(x) = -3\chi_{[0,1]}(x),
$$

$$
V_{12}(x) = V_{21}(x) = -3a\chi_{[0,1]}(x), \quad a > 0.
$$

Then the matrix $\mathbf{S}$ equals

$$
\mathbf{S} = \begin{pmatrix}
-5 & -3a \\
-3a & -3
\end{pmatrix}
$$

and it has two real eigenvalues given by $-4 \pm \sqrt{1 + 9a^2}$. Thus the following cases are possible: 1) If $a > \sqrt{5/3}$ there is exactly one negative eigenvalue of $\mathbf{S}$, namely $-4 - \sqrt{1 + 9a^2}$. 2) If $a < \sqrt{5/3}$ there are two negative eigenvalues of $\mathbf{S}$, namely $-4 \pm \sqrt{1 + 9a^2}$. 3) If $a = \sqrt{5/3}$ there are two nonpositive eigenvalues of $\mathbf{S}$, namely $-4 - \sqrt{1 + 9a^2}$ and $0$.

5.1.2. Two-channel Hamiltonian with two thresholds. As an example of a Hamiltonian with more than one threshold, we consider the one in (1.1), having thresholds at 0 and 1. Henceforth its free Hamiltonian is denoted by $H_0$. The essential spectrum of $H_0$ is the union of the half-axes starting at the thresholds, i.e. $\sigma_{ess}(H_0) = [0, \infty)$. The resolvent of $H_0$ is given by

$$
(H_0 + \alpha^2)^{-1} = \begin{pmatrix}
(-d^2/dx^2 + \alpha^2)^{-1} & 0 \\
0 & (-d^2/dx^2 + \alpha^2 + 1)^{-1}
\end{pmatrix}, \quad \alpha > 0.
$$
where the entries have the integral kernels
\[ \frac{1}{2\alpha} e^{-\alpha|y-x|} \text{ and } \frac{1}{2\sqrt{\alpha^2 + 1}} e^{-\sqrt{\alpha^2 + 1}|y-x|}. \]

It is easy to show that the assertions of Proposition 3.1 are valid if one replaces \( H_0 \) by \( \tilde{H}_0 \). In this way we obtain a self-adjoint realization of the formal Hamiltonian \( \tilde{H}_0 + gV \) in \( L^2(\mathbb{R}, \mathbb{C}^2) \). Moreover, the Birman-Schwinger relation in Proposition 4.2 holds for \( \tilde{H}(g) = \tilde{H}_0 + gV \).

Define the operators \( \tilde{L}_\alpha \) and \( \tilde{M}_\alpha \) by their “kernels”
\[
\tilde{L}_\alpha(x, y) = \frac{1}{2\alpha} V(x)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V(y)|^{1/2},
\]
\[
\tilde{M}_\alpha(x, y) = V(x)^{1/2} \begin{pmatrix} \frac{1}{2}[e^{-\alpha|y-x|} - 1] & 0 \\ 0 & \frac{1}{2\sqrt{\alpha^2 + 1}} e^{-\sqrt{\alpha^2 + 1}|y-x|} \end{pmatrix} |V(y)|^{1/2}.
\]

Moreover, we introduce the operator \( \tilde{M}_0 \) by its kernel
\[
\tilde{M}_0(x, y) = V(x)^{1/2} \begin{pmatrix} \frac{1}{2}|y-x| & 0 \\ 0 & \frac{1}{2}e^{-|y-x|} \end{pmatrix} |V(y)|^{1/2}.
\]

By making a few obvious changes to the proof of Lemma 4.5 we obtain the following result.

**Lemma 5.5.** Assume that \( \int_{\mathbb{R}} (1 + |x|^2) \|V(x)\|_{L(\mathbb{C}^2)} \, dx < \infty \). If \( K_\alpha, L_\alpha, M_\alpha \) and \( M_0 \) in Lemma 4.5 are replaced by \( \tilde{K}_\alpha, \tilde{L}_\alpha, \tilde{M}_\alpha \) and \( \tilde{M}_0 \), then the assertions (i)-(iii) of Lemma 4.5 are still valid.

Define the matrices
\[
\tilde{S} = \begin{pmatrix} \int_{\mathbb{R}} V_{11}(x) \, dx & \int_{\mathbb{R}} V_{12}(x) \, dx \\ 0 & 0 \end{pmatrix},
\]
(5.9)

\[
\tilde{T}_0 = \int_{\mathbb{R}} \int_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V(x) \begin{pmatrix} \frac{1}{2}|y-x| & 0 \\ 0 & \frac{1}{2}e^{-|y-x|} \end{pmatrix} V(y) \, dx \, dy. \quad (5.10)
\]

For the Hamiltonian \( \tilde{H}(g) \) we are able to derive an analogue of part (i) in Theorem 5.2.

**Theorem 5.6.** Let \( V \) obey Assumption 5.1(a)-(c) and let \( \tilde{H}(g) = \tilde{H}_0 + gV(x) \) be the self-adjoint Hamiltonian on \( L^2(\mathbb{R}, \mathbb{C}^2) \) defined in Proposition 3.1 by means of forms.

Assume that the matrix \( \tilde{S} \), defined in (5.9), has a negative eigenvalue \( \tilde{s} \) (such an eigenvalue is simple if it exists). Then, for a small enough coupling constant \( g \), the eigenvalue \( \tilde{s} \) of \( \tilde{S} \) gives rise to exactly one negative eigenvalue \( \tilde{E} \) of the two-channel Hamiltonian \( \tilde{H}(g) \). The negative eigenvalue \( \tilde{E} \) satisfies the formula
\[
(-\tilde{E}(g))^{1/2} = -\frac{g}{2} \tilde{s} + \frac{g^2}{2} \langle \tilde{v}, \tilde{T}_0 \tilde{v} \rangle_{\mathbb{C}^2} + O(g^3),
\]
(5.11)

where \( \tilde{T}_0 \) is defined in (5.10) and \( \tilde{v} \) is the eigenvector corresponding to the eigenvalue \( \tilde{s} \) of \( \tilde{S} \).
Proof. Imitating the proof of Theorem 5.2 we arrive at the eigenvalue
problem
\[
\tilde{R}_g u_k = \mu_k(g, \alpha) u_k,
\]
where \( \tilde{R}_g \) is the matrix
\[
\tilde{R}_g = \frac{1}{2\alpha} \int_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(1 + g \tilde{M}_\alpha\right)^{-1} |V|^{1/2}(x) \, dx.
\]
Define the matrix \( \tilde{S} \) as in (5.9) and, moreover, define
\[
\tilde{T}(\alpha) = \int_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(1 + g \tilde{M}_\alpha\right) |V|^{1/2}(x) \, dx.
\]
Then we may write
\[
\tilde{R}_g = \frac{1}{2\alpha} \tilde{S} - \frac{g}{2\alpha} \tilde{T}(\alpha) + O(g^2)
\]
for small \( g \). From here on everything depends on the possible eigenvalues of \( S \). Let
\[
a = \int_{\mathbb{R}} V_{11}(x) \, dx.
\]
The following cases may occur: I. If \( a \neq 0 \) then there are two subcases.
1.1. If \( a > 0 \) then \( \tilde{S} \) has the eigenvalue zero and the positive eigenvalue
\( a \), each of multiplicity one. 1.2. If \( a < 0 \) then \( \tilde{S} \) has the eigenvalue
zero and the negative eigenvalue \( a \), each having multiplicity one. II. If
\( a = 0 \) then \( \tilde{S} \) has the eigenvalue zero with multiplicity one.

Repeating the reasoning in the first part of the proof of Theorem 5.2 we show that a negative eigenvalue of \( \tilde{S} \) (from 1.2 it has multiplicity
one) generates exactly one negative eigenvalue of \( \tilde{H}(g) \) provided \( g \) is
small enough. \( \Box \)

Remark 5.7. The matrix \( \tilde{S} \) always has the eigenvalue zero. It remains
an open problem to settle whether or not the latter gives rise to a
negative eigenvalue of \( \tilde{H}(g) \) for a sufficiently small \( g \).

5.2. \textbf{\textit{N-channel Hamiltonian with matrix-valued potentials.}} In
this section we consider the case where the potential is a \( N \times N \)
matrix-valued potential \( V(x) \) with measurable functions \( V_{ij} \) on \( \mathbb{R} \) as entries.

\textbf{Assumption 5.8.}
(a) \( \overline{V(x)} \) is symmetric, i.e., \( \overline{V_{ij}} = V_{ij} \).
(b)
\[
\int_{\mathbb{R}} (1 + |x|^2) \|V(x)\|_{B(C^1)} \, dx < \infty.
\]
(c) The functions \( V_{ij} \) are real-valued.

Define the matrices \( S \) and \( T_0 \) by
\[
S = \int_{\mathbb{R}} V(x) \, dx, \quad T_0 = -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} V(x)|x - y|V(y) \, dy \, dx.
\]
We have the following result.

**Theorem 5.9.** Let $\mathbf{V}$ obey Assumption 5.8(a)-(b) and let $\mathbf{H}(g) = \mathbf{H}_0 + g\mathbf{V}(x)$ be the self-adjoint Hamiltonian on $L^2(\mathbb{R}, \mathbb{C}^2)$ defined in Proposition 3.1 by means of forms.

(i) Assume that the matrix $\mathbf{S}$, defined in (5.13), has $n$ negative eigenvalues, denoted by $s_i$, with multiplicities $\nu_i$. Then, for a small enough $g$, the N-channel Hamiltonian $\mathbf{H}(g)$ has precisely $\sum_{i=1}^n \nu_i$ negative eigenvalues (taking into account multiplicity) $E_{ij}$ satisfying the formulas

\[
(-E_{ij}(g))^{1/2} = -\frac{g}{2} s_i + \frac{g^2}{2} \langle \mathbf{v}_{ij}, \mathbf{T}_0 \mathbf{v}_{ij} \rangle_{\mathbb{C}^n} + O(g^3), \quad (5.14)
\]

where $\mathbf{T}_0$ is defined in (5.13) and $\mathbf{v}_{ij}$ are the eigenvectors corresponding to the eigenvalue $s_i$ of $\mathbf{S}$.

(ii) Suppose that $\mathbf{V}$ obey Assumption 5.8(c) and that the matrix $\mathbf{S}$ has $n$ non-positive eigenvalues, denoted by $s_i$, with multiplicities $\nu_i$. If the eigenvectors $\mathbf{v}_{0j}$, $j = 1, \ldots, \nu_i$, associated with the eigenvalue zero of $\mathbf{S}$ satisfy $\langle \mathbf{v}_{0j}, \mathbf{T}_0 \mathbf{v}_{0j} \rangle_{\mathbb{C}^n} \neq 0$ then the conclusion of part (i) remains valid.

**Proof.** The proof is a straightforward generalization of the proof of Theorem 5.2. \[\square\]

**Remark 5.10.** One of the referees pointed out that Theorem 5.9 was proven in Šeba [29]. Therein, however, Theorem 3 is incorrect because the quantity $\int_{\mathbb{R}} (1/p^2)(a_0, \mathcal{F}(V)^2(p)a_0) \, dp$ (in Šeba’s notation) is not necessarily different from zero. Moreover, the Birman-Schwinger relation (in the matrix-valued setting) is stated without proof.

6. **Perturbation of Embedded Eigenvalues**

For the sake of completeness we consider perturbation of two-channel diagonal Hamiltonians with one-dimensional Schrödinger operators as component Hamiltonians, having eigenvalues embedded in its continuous spectrum.

6.1. **Two-channel Hamiltonians.** Consider the formal expression

\[
\mathbf{H}(g) = \mathbf{H}(0) + g\mathbf{V} = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} + g \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix} \quad (6.1)
\]

in $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where

\[
H_{11} = -\frac{d^2}{dx^2} + W_{11}(x) \quad \text{and} \quad H_{22} = -\frac{d^2}{dx^2} + 1 + W_{22}(x). \quad (6.2)
\]

We impose the following assumptions on the potentials $W_{jj}$, $j = 1, 2$. 
**Assumption 6.1.** Suppose that the real-valued, measurable functions $W_{jj}$, $j = 1, 2$, satisfy:

(a) $W_{jj} \neq 0$.
(b) The bound

$$|W_{jj}(x)| \leq C \left( 1 + |x|^2 \right)^{-1-\delta}$$  \hspace{1cm} (6.3)

holds for some $C, \delta > 0$ and all $x$.
(c) $\int_{\mathbb{R}} W_{jj}(x) \, dx \leq 0$.
(d) $W_{jj}$ extends to a function analytic in the sector

$$\mathcal{A}_{\alpha_0} = \{ \, z \in \mathbb{C} : |\arg z| \leq \alpha_0 \}$$

for some $\alpha_0 > 0$. Moreover, the bound (6.3) holds in this sector.

Under Assumption 6.1(a)-(c) the operator $H_{11} = -\frac{d^2}{dx^2} + W_{11}(x)$ is self-adjoint in $L^2(\mathbb{R})$ and $\sigma(H_{11}) = \sigma_{d}(H_{11}) \cup \sigma_{ess}(H_{11}) = \sigma_{d}(H_{11}) \cup [0, \infty)$ with a non-empty discrete spectrum $\mu_1 < \mu_2 < \cdots < \mu_N < 0$, which is simple and finite [31]. The corresponding normalized eigenfunctions $\phi_n$, $n = 1, 2, \ldots, N$, are exponentially decaying. The analyticity requirement in Assumption 6.1(d) is convenient to adopt for analyzing the resonance behaviour. Similarly, the operator $H_{22} = -\frac{d^2}{dx^2} + 1 + W_{22}(x)$ is self-adjoint in $L^2(\mathbb{R})$ and $\sigma(H_{22}) = \sigma_{d}(H_{22}) \cup \sigma_{ess}(H_{22}) = \sigma_{d}(H_{22}) \cup [1, \infty)$ with a non-empty discrete spectrum $\nu_1 < \nu_2 < \cdots < \nu_M < 1$ which is simple and finite. The corresponding normalized eigenfunctions $\chi_m$, $m = 1, 2, \ldots, M$, are exponentially decaying.

Consider the unperturbed Hamiltonian $\mathbf{H}(0) = \text{diag}(H_{11}, H_{22})$. Assumption 6.1 ensures that

$$\sigma_{c}(\mathbf{H}(0)) = \sigma_{ess}(\mathbf{H}(0)) = \sigma_{ess}(H_{11}) \cup \sigma_{ess}(H_{22}) = [0, \infty) \cup [1, \infty) = [0, \infty).$$

Thus, the continuous spectrum of $\mathbf{H}(0)$ is the union of the two half-lines starting at 0 and 1. This motivates the definition of the threshold set $T = \{ 0, 1 \}$. Furthermore, $\sigma_{p}(\mathbf{H}(0)) = \sigma_{p}(H_{11}) \cup \sigma_{p}(H_{22})$. Among this (finite) set of eigenvalues, a (finite) subset is isolated or situated at the threshold 0, while the rest satisfying the condition $0 < \nu_m < 1$ is embedded in the continuous spectrum of $\mathbf{H}(0)$. For the sake of simplicity we make the following assumption.

**Assumption 6.2.** Suppose that none of the embedded eigenvalues $\nu_m$ of $\mathbf{H}(0)$ coincide with the threshold 0.

We impose the following conditions on the components of the perturbation $\mathbf{V}$.

**Assumption 6.3.** Suppose that the real-valued, measurable functions $V_{ij}$, $i, j = 1, 2$, satisfy:

(a) The bound

$$|V_{ij}(x)| \leq C \left( 1 + |x|^2 \right)^{-1-\delta}$$  \hspace{1cm} (6.4)
holds for some $C, \delta > 0$ and all $x$.
(b) $V_{ij}$ extends to a function analytic in the sector $A_{\alpha_0}$ (see Assumption 6.1(d)) for some $\alpha_0 > 0$. Moreover, the bound (6.4) holds in this sector.

6.2. Complex dilation. We use a complex deformation. For $\theta$ real define $S_\theta$ on $L^2(\mathbb{R})$ by the unitary operator

$$\quad (S_\theta \psi) = e^{\theta/2} \psi(e^{\theta} x), \quad \psi \in L^2(\mathbb{R}). \quad (6.5)$$

$S_\theta$ is a one-parameter unitary group on $L^2(\mathbb{R})$. It is easy to see that $S_\theta$ leave $\mathcal{D}(-d^2/dx^2) = H^2(\mathbb{R})$ invariant and that

$$H_{11,\theta} := S_\theta H_{11} S_\theta^{-1} = -e^{-2\theta} \frac{d^2}{dx^2} + W_{11,\theta}(x) = -e^{-2\theta} \frac{d^2}{dx^2} + W_1(e^{\theta} x),$$

Let $\mathcal{A}_0 = \{ \theta : |\text{Im} \theta| \leq \min\{\alpha_0, \pi/4\} \}$ (cf. Assumption 6.1(d)). Under Assumption 6.1, $H_{11,\theta}$ obviously has a continuation to a type (A) family of $m$-sectorial operators analytic in the sense of Kato [13] for $\theta \in \mathcal{A}_0$. Likewise,

$$H_{22,\theta} := S_\theta H_{22} S_\theta^{-1} = -e^{-2\theta} \frac{d^2}{dx^2} + 1 + W_{22,\theta}(x) = -e^{-2\theta} \frac{d^2}{dx^2} + 1 + W_2(e^{\theta} x)$$

has a continuation to a type (A) analytic family of operators on $A_0$. From standard Aguilar-Combes theory [1] we determine the spectra of $H_{11,\theta}$ and $H_{22,\theta}$:

$$\sigma(H_{11,\theta}) = \{ \mu_1, \mu_2, \ldots, \mu_N \} \cup \{ e^{-2\theta} \lambda : \lambda \in [0, \infty) \},$$

$$\sigma(H_{22,\theta}) = \{ \nu_1, \nu_2, \ldots, \nu_M \} \cup \{ e^{-2\theta} \lambda + 1 : \lambda \in [0, \infty) \}.$$

Having extended $S_\theta$ in (6.5) analytically to $A_0$ we may define

$$S_\theta \Psi = \begin{pmatrix} S_\theta & 0 \\ 0 & S_\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \Psi \in \mathcal{H}.$$ 

Due to its diagonal structure, the Hamiltonian

$$H_\theta(0) := S_\theta H(0) S_\theta^{-1} = \begin{pmatrix} H_{11,\theta} & 0 \\ 0 & H_{22,\theta} \end{pmatrix}$$

has a continuation to a type (A) analytic family of operators in the sector $A_0$. Furthermore,

$$\sigma(H_\theta(0)) = \sigma(H_{11,\theta}) \cup \sigma(H_{22,\theta})$$

$$= \{ \mu_1, \mu_2, \ldots, \mu_N \} \cup \{ \nu_1, \nu_2, \ldots, \nu_M \}$$

$$\cup \{ e^{-2\theta} \lambda : \lambda \in [0, \infty) \} \cup \{ e^{-2\theta} \lambda + 1 : \lambda \in [0, \infty) \}.$$ 

In particular, the eigenvalues embedded in $\sigma(H(0))$ are discrete eigenvalues of $H_\theta(0)$ for $\theta$ nonreal.

Henceforth $E_0$ denotes any of the embedded eigenvalues $\nu_m$ of $H(0)$. Let $R_0(\theta; \zeta)$ denote the resolvent of $H_\theta(0)$. Since $E_0$ is an isolated eigenvalue of $H_\theta(0)$, we may choose a contour $\Gamma$ around $E_0$ such that $\Gamma$ belongs to the resolvent set of $H_\theta(0)$ and $E_0$ is the only eigenvalue
of $H_\theta(0)$ contained inside of $\Gamma$. Moreover, let $P_\theta$ denote the eigenprojection associated with the eigenvalue $E_0$ and put

$$S_\theta^{(p)} := \frac{1}{2\pi i} \int_\Gamma \frac{R_0(\theta; \zeta)}{(E_0 - \zeta)^p} d\zeta, \quad p \geq 1. \quad (6.6)$$

Then $P_\theta = -S_\theta^{(0)}$ and $\tilde{R}_0(\theta; \zeta) := S_\theta^{(1)}$ is the reduced resolvent of $H_\theta(0)$ at the point $\zeta$. Define

$$V_\theta = S_\theta V S_\theta^{-1} = \begin{pmatrix} V_{11, \theta} & V_{12, \theta} \\ V_{21, \theta} & V_{22, \theta} \end{pmatrix} \quad \text{with} \quad V_{i,j, \theta}(x) = V_{i,j}(e^\theta x).$$

Then we have the following result.

**Lemma 6.4.** Let Assumption 6.1 and Assumption 6.3 hold. Let $\Gamma$ be the contour described above and let $S_\theta^{(p)}$ be defined in (6.6).

(i) If $\text{Im} \theta \in (0, \alpha_0)$ then there exists a constant $C_\theta > 0$ such that

$$\max_{\zeta \in \Gamma} \|g V_\theta R_0(\theta; \zeta)\| \leq C_\theta |g|. \quad (6.7)$$

If $\zeta$ is replaced by $\zeta_0 = \min\{\mu_1, \nu_1\} - 1$ then the constant in (6.7) is independent of $\theta$ and the estimate holds for all $|\text{Im} \theta| < \alpha_0$.

(ii) For $p \geq 0$ there exists a constant $C_\theta > 0$ such that

$$\|g V_\theta S_\theta^{(p)}\| \leq C_\theta \frac{|\Gamma|}{2\pi [\text{dist} (E_0, \Gamma)]^p}.$$  

**Proof.** The contour $\Gamma$ is by assumption contained in the resolvent of $H_\theta(0)$. Since $R_0(\theta; \cdot)$ is bounded and continuous and $\Gamma$ is compact, there exists a constant $\tilde{C}_\theta$ such that $\max_{\zeta \in \Gamma} \|R_0(\theta; \zeta)\| \leq \tilde{C}_\theta$.

Thus, $\max_{\zeta \in \Gamma} \|V_\theta R_0(\theta; \zeta)\| \leq \tilde{C}_\theta C_V$, where $C_V$ denotes a bound on the norm of $V_\theta$, which is independent of $\theta$ by Assumption 6.3(b). This shows the first claim. Moreover, $\zeta_0$ is to the left of the numerical range $\Theta(H_\theta(0))$ of $H_\theta(0)$ at the unit distance. Hence $\|R_0(\theta; \zeta_0)\| = 1/|\text{dist} (\zeta_0, \Theta(H_\theta(0)))| = 1$. Therefore the constant $\tilde{C}_\theta$ in the above estimate may be replaced by 1. This verifies (i). The assertion (ii) follows immediately.

Hence, provided $g$ is small enough, it follows from Lemma 6.4(i) that $g V_\theta$ is $H_\theta(0)$-compact. The latter, in conjunction with [27, Lemma 1, page 16], implies that the perturbed operators $H_\theta(g) = H_\theta(0) + g V_\theta$ are a type (A) analytic family of operators for $\theta \in A_0$ and suitable small $g$. Since $E_0$ is an isolated, simple eigenvalue of $H_\theta(0)$, the analyticity of $H_\theta(g)$ allows us to apply regular perturbation theory. The next section is devoted to this task.
6.3. Perturbation series and Fermi’s golden rule. Following Kato [13, Sections II.2 and VII.1] and using Lemma 6.4 we infer that $H_\theta(g)$ has an eigenvalue near $E_0$ given by a convergent power series in $g$. The convergent series is given by

$$E(g) = E_0 + \sum_{j=1}^{\infty} E_j(g),$$

where

$$E_j(g) = \sum_{p_1, \ldots, p_j = j-1} (-1)^j \left[ \frac{\text{tr}}{j} \prod_{i=1}^{j} g V_\theta S_\theta^{(p_i)} \right]$$

(6.9)

In view of Lemma 6.4(ii) we see that $E_j(g) = O(g^j)$.

Let us compute the lowest-order terms of the series (6.8). Since Rank $P_\theta = 1$, $P_\theta$ can be represented as

$$P_\theta = \left\langle \cdot, \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} \right| \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} \right>$$

with $\chi_\theta := S_\theta \chi_m$, where $\chi_m$ is the eigenfunction associated with the eigenvalue $E_0$ of $H_{22}$. Indeed, $H_{22, \theta} = \chi_\theta^T = S_\theta H_{22} S_\theta^{-1} S_\theta \chi_m = E_0 S_\theta \chi_m = E_0 \chi_\theta$ and, consequently,

$$H_\theta(0) \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} = \begin{pmatrix} H_{11, \theta} & 0 \\ 0 & H_{22, \theta} \end{pmatrix} \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} = E_0 \begin{pmatrix} 0 \\ \chi_m \end{pmatrix}.$$  

We compute $E_1(g)$:

$$E_1(g) = \text{tr} (g V_\theta P_\theta) = g \left\langle \left( \begin{pmatrix} 0 \\ \chi_m \end{pmatrix}, V_\theta \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} \right) \right| \mathcal{H} = g \left\langle \chi_m, V_2 \chi_m \right|_{L^2(\mathbb{R})}.\tag{6.10}$$

We see that the first-order term is real and does not contribute to the resonance width. Next, we consider $E_2(g)$. According to (6.9),

$$E_2(g) = -g^2 \text{tr} \left( P_0 V_\theta \hat{R}_0(\theta; E_0 - i0) V_\theta P_0 \right).$$

Due to the standard constancy-in-$\theta$ argument (see e.g., [27, pages 55-56]), we may take the limit $\text{Im} \theta \to 0$ and in this way we arrive at

$$E_2(g) = -g^2 \text{tr} \left( P_0 V_\theta \hat{R}_0(0; E_0 - i0) V_\theta P_0 \right) = -g^2 \left\langle \chi_m, V_2 \chi_m \right|_{L^2(\mathbb{R})}.$$

(6.11)
where the notation \([(H_{jj} - E_0 + i0)^{-1}]\) refers to the reduced resolvent of \(H_{jj}, j = 1, 2\).

We restrict our focus to the imaginary part of \(E_2(g)\), which determines the resonance width to leading order. For this purpose we introduce

\[
R_k = \left( (-d^2/dx^2 + W_{kk}(x) - E_0 + t_k - i0)^{-1} \right), \quad k = 1, 2,
\]

where \(t_1 = 0\) and \(t_2 = 1\) are the thresholds. Clearly,

\[
\text{Im} E_2(g) = -g^2 \sum_{k=1}^{2} \langle V_k \chi_m | (\text{Im} R_k) V_k \chi_m \rangle_{L^2(\mathbb{R})}. \tag{6.12}
\]

Now, for \(E > 0\), the resolvent equation yields that

\[
\text{Im} \left( (-d^2/dx^2 + W_{kk}(x) - E - i0)^{-1} \right) =
\]

\[
t_k(E + i0)^* \text{Im} \left( (-d^2/dx^2 - E - i0)^{-1} \right) t_k(E + i0), \tag{6.13}
\]

where

\[
t_k(\zeta) = \left[ I + |W_{kk}|^{1/2} \left( -\partial_x^2 - \zeta \right)^{-1} \right] |W_{kk}|^{1/2} \text{Sgn}(W_{kk}) \right)^{-1}.
\]

The quantities \(t_k(E + i0)\) are well-defined in view of Assumption 6.1. Furthermore, again for \(E > 0\),

\[
\text{Im} \left( (-d^2/dx^2 - E - i0)^{-1} \right) = \frac{\pi}{2\sqrt{E}} \sum_{\sigma = \pm} (\gamma_{E})^{*} \gamma_{E}, \tag{6.14}
\]

where \(\gamma_{E} : H^1 \to \mathbb{C}\) is the trace operator which acts on the first Sobolev space \(H^1(\mathbb{R})\) as follows (see, e.g., [18, Section IV.1]),

\[
\gamma_{E} \phi = \tilde{\phi}(\sigma \sqrt{E}), \quad \sigma = \pm, \quad E > 0.
\]

Here, as usual, \(\tilde{\phi}\) denotes the Fourier transform of \(\phi\). Using (6.13) and (6.14) we can rewrite the expression (6.12) in the following way,

\[
\text{Im} E_2(g) = -g^2 \sum_{k=1}^{2} \langle V_k \chi_m, (\text{Im} R_k) V_k \chi_m \rangle_{L^2}
\]

\[
= -g^2 \sum_{k=1}^{2} \langle V_k \chi_m, t_k(E_0 - t_k + i0)^* \rangle \times \text{Im} \left( (-d^2/dx^2 - E_0 + t_k - i0)^{-1} t_k(E_0 - t_k + i0) V_k \chi_m \right)_{L^2}
\]
\[
\begin{align*}
&= -g^2 \sum_{k=1}^{2} (t_k(E_0 - t_k + i0) V_{k2} \chi_m, \text{Im} \left(-d^2/dx^2 - E_0 + t_k - i0\right)^{-1} \\
&\times t_k(E_0 - t_k + i0) V_{k2} \chi_m)_{L^2} \\
&= -g^2 \sum_{k=1}^{2} \sum_{\sigma = \pm} \frac{\pi}{2\sqrt{E_0 - t_k}} (\gamma_{E_0 - t_k} t_k(E_0 - t_k + i0) V_{k2} \chi_m, \gamma_{E_0 - t_k}^\sigma t_k(E_0 - t_k + i0) V_{k2} \chi_m)^C \\
&\times t_k(E_0 - t_k + i0) V_{k2} \chi_m) \\
&= -g^2 \sum_{k=1}^{2} \sum_{\sigma = \pm} \frac{\pi}{2\sqrt{E_0 - t_k}} |\gamma_{E_0 - t_k}^\sigma t_k(E_0 - t_k + i0) V_{k2} \chi_m|^2. \quad (6.15)
\end{align*}
\]

In this way we have established the following result.

**Theorem 6.5.** Let Assumption 6.1 and Assumption 6.3 hold. Let \(\nu_m\) be a simple eigenvalue of the operator \(H_{22}\) defined in (6.2) giving rise to the eigenvalue \(E_0 = \nu_m\) embedded in the continuous spectrum of \(H(0)\).
Let \(E_0\) satisfy Assumption 6.2. For a small enough coupling constant \(g\), the eigenvalue \(E_0\) of \(H(0)\) turns into a resonance, i.e. \(E_0 \not\in \sigma(H(g))\). The coordinates of its corresponding pole is given by (6.9)-(6.11). In particular, Fermi’s golden rule takes the explicit form (6.15).

**Remark 6.6.** If Assumption 6.2 is not fulfilled, i.e. we have an eigenvalue of \(H_{22}\) at the threshold point 0 of the continuous spectrum of \(H(0)\), complex dilation breaks down. An insight into this problem was established in [12]. For abstract Hamiltonians \(H(g)\) having the structure found in (6.1), it was shown that under small off-diagonal perturbations this eigenvalue never moves into the continuous spectrum.

7. ACKNOWLEDGMENTS

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