

# LIEB-THIRRING INEQUALITY FOR THE AHARONOV-BOHM HAMILTONIAN AND EIGENVALUE ESTIMATES

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ABSTRACT. The diamagnetic inequality is shown for the Aharonov-Bohm Hamiltonian  $H_{AB}$  in  $L^2(\mathbb{R}^2)$ . As an application the Lieb-Thirring inequality is established for the perturbed Schrödinger operator  $H_{AB} - V$ . In addition, CLR-type estimates and large coupling constant asymptotics for  $H_{AB} - qV$ ,  $q > 0$ , are discussed.

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## 1. INTRODUCTION AND MAIN RESULTS

The Aharonov-Bohm Schrödinger operator in  $L^2(\mathbb{R}^2)$  is given formally by the expression

$$H_{AB} = (\nabla + i\mathbf{A})^2, \tag{1.1}$$

where  $\nabla$  is the two-dimensional gradient on  $\mathbb{R}^2$  and the Aharonov-Bohm magnetic vector potential is given by

$$\mathbf{A}(x_1, x_2) = \alpha \left( \frac{x_2}{x_1^2 + x_2^2}, \frac{-x_1}{x_1^2 + x_2^2} \right), \tag{1.2}$$

where the magnetic flux parameter  $\alpha$  is a noninteger; as usual, it suffices to consider  $\alpha \in (0, 1)$  due to gauge symmetry. Clearly  $\text{curl} \mathbf{A} = 0$  for  $(x, y) \neq 0$  and  $\mathbf{A} \cdot \mathbf{x} = 0$ .

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*Date:* November 2, 2002.

The first author is a Marie Curie Post-Doc Fellow, supported by the European Union under grant no. HPMF-CT-2000-00973.

The Schrödinger operator in (1.1)-(1.2) arises when one considers the dynamics of a non-relativistic, spinless quantum particle interacting with a magnetic field confined in a thin, infinite solenoid. Ignoring the irrelevant coordinate along the solenoid, the problem reduces to two dimensions. If, moreover, the radius of the solenoid tends to zero, whilst the flux of the magnetic field remains constant then one obtains a particle moving in  $\mathbb{R}^2$  subject to a  $\delta$ -type magnetic field and the corresponding Schrödinger operator is given by (1.1)-(1.2). This operator has been studied intensively (see, e.g., [22, 1, 8]).

Within the theory of Schrödinger operators with magnetic fields  $L(\mathbf{A}) = (\nabla + i\mathbf{A})^2$  associated with a vector potential  $\mathbf{A} = (A_1, \dots, A_d)$  satisfying  $A_j \in L^2_{loc}(\mathbb{R}^d)$ , one of the fundamental facts is the diamagnetic inequality [3], viz.,  $|e^{-tL(\mathbf{A})}u| \leq e^{-tL_0}|u|$  for all  $t \geq 0$  and all  $u \in L^2(\mathbb{R}^d)$ ; here  $L_0$  denotes the negative Laplacian.

In Section 4 we show that this inequality is valid also for the Aharonov-Bohm (abbrev. A-B) Hamiltonian  $H_{AB}$  in  $L^2(\mathbb{R}^2)$ .

**Theorem 1.1.** *The inequality*

$$|e^{-tH_{AB}}u| \leq e^{-tL_0}|u|$$

holds for all  $t \geq 0$  and all  $u \in L^2(\mathbb{R}^2)$

This is a non-trivial result since the components of the A-B vector potential do not belong to  $L^2_{loc}(\mathbb{R}^2)$ . The proof uses a recent criterion (see Section 3) for the domination of semigroups due to Ouhabaz [16]. This criterion is a generalization (from operators to forms) of the celebrated Simon-Hess-Schrader-Uhlenbeck criterion for domination of semigroups [9].

As an application of the diamagnetic inequality we establish the Lieb-Thirring inequality for the perturbed A-B Schrödinger operator  $H_{AB} - V$  in Section 6. Here the electrostatic potential  $V$  is a nonnegative, measurable function on  $\mathbb{R}^2$  belonging to an appropriate class of functions which guarantees that the form sum  $H_{AB} - V$  generates a semi-bounded, self-adjoint operator in  $L^2(\mathbb{R}^2)$  with discrete spectrum below zero.

The classic Lieb-Thirring inequality for a  $d$ -dimensional Schrödinger operator  $L_0 - V$  in  $L^2(\mathbb{R}^d)$ , with  $L_0 = -\Delta$  as above and  $d \geq 3$ , says that

$$\sum_j |e_j(L_0 - V)|^\gamma \leq b_d(\gamma) \int_{\mathbb{R}^d} V(x)^{\gamma + \frac{d}{2}} dx, \quad (1.3)$$

where  $e_j(L_0 - V)$  denotes the negative eigenvalues of  $L_0 - V$ ,  $\gamma > 0$  and  $V \in L^{\gamma + (d/2)}$ . The constant  $b_d(\gamma)$  is expressible in terms of  $\Gamma$ -functions (see, e.g., [12]). The Lieb-Thirring inequality plays a crucial role in the problem of stability of matter (see, e.g., [12]). One way of establishing (1.3) is to use the Cwikel-Lieb-Rozenblum (abbrev. CLR)

estimate (see, e.g., [18]) which, in its original form, reads

$$N_-(L_0 - V) \leq C(d) \int_{\mathbb{R}^d} V(x)^{d/2} dx, \quad d \geq 3. \quad (1.4)$$

Here  $N_-$  denotes the number of negative eigenvalues of a self-adjoint operator, provided its negative spectrum is discrete. The single assumption, under which (1.4) is valid, is the finiteness of the integral on its right-hand side. In [24, p 99-100] it is shown how one can obtain the Lieb-Thirring inequality provided (1.4) holds.

The Lieb-Thirring inequality for  $d$ -dimensional Schrödinger operators with magnetic fields  $L(\mathbf{A}) - V$ , with  $d \geq 3$  and  $A_j \in L^2_{loc}(\mathbb{R}^d)$ , takes the same form and can be obtained from the CLR-estimate for  $L(\mathbf{A}) - V$  which is shown by means of the diamagnetic inequality (see, e.g., [24, p 168]).

In two dimensions there exist CLR-type estimates both for  $L_0 - V$  (see [25, 5]) and  $L(\mathbf{A}) - V$  (see [20]), provided  $A_j \in L^2_{loc}(\mathbb{R}^2)$  for the latter operator. However, unlike in higher dimensions, these estimates, having a different form, do not produce Lieb-Thirring inequalities. Moreover, in our case, the A-B magnetic potential does not belong to  $L^2_{loc}(\mathbb{R}^2)$ . Therefore the question on Lieb-Thirring inequalities for the perturbed A-B Schrödinger operator  $H_{AB} - V$  was up to now open.

In the present paper we establish the following Lieb-Thirring inequality for the perturbed A-B Schrödinger operator.

**Theorem 1.2.** *Let  $\lambda_j$  denote the negative eigenvalues of  $H_{AB} - V$ . If, moreover,  $\gamma > 0$  and  $V \in L^{\gamma+1}(\mathbb{R}^2)$  then*

$$\sum_j |\lambda_j|^\gamma \leq C(\gamma) \int_{\mathbb{R}^2} V(x)^{\gamma+1} dx,$$

where the constant  $C(\gamma)$  fulfills the following upper bounds for the most interesting values of  $\gamma$ :

$$C(\gamma) \leq \begin{cases} 0.5981 & \text{for } \gamma = 1/2, \\ 0.6175 & \text{for } \gamma = 1, \\ 0.7527 & \text{for } \gamma = 3/2. \end{cases}$$

We note that the expression we obtain for the best constant in Theorem 1.2 is implicit; see (6.7).

The diamagnetic inequality is one out of two crucial ingredients in the proof of Theorem 1.2. The other is an abstract CLR-estimate for generators of semigroups dominated by positive semigroups. To make the paper self-explanatory we formulate this rather recent result, obtained by Rozenblum and Solomyak, in Theorem 5.1 (see Section 5).

One of the important applications of the eigenvalue estimates for Schrödinger operators is to deduce the asymptotic formulas for the

eigenvalues when the coupling constant is present and it tends to infinity. The technology of getting the asymptotic formulas from the estimates is well-established now (see, e.g., [18] and [6, 7]), and what is required from the estimates is that they have correct order in the coupling constant. For a weakly singular magnetic field such estimates were obtained by Lieb (see [24]) and Melgaard-Rozenblum [14] in dimensions  $d \geq 3$ , and by Rozenblum-Solomyak [20] in dimension  $d = 2$  (see also [21]). The only existing estimate for the A-B Schrödinger operator, by Balinsky, Evans and Lewis [4], does not have correct order in the coupling constant, and therefore does not imply an asymptotic formula. As a consequence of our diamagnetic inequality, we find that the same sort of eigenvalue estimates hold for the A-B Schrödinger operator, as for the less singular magnetic field, which automatically produces asymptotic formulas.

Let us emphasize that the magnetic flux parameter  $\alpha$  is a noninteger throughout the paper. If  $\alpha$  is an integer, the resulting operator is gauge equivalent to the negative Laplacian in  $L^2(\mathbb{R}^2)$ . This, however, does not reflect itself in the Lieb-Thirring inequality but the eigenvalue estimates in Section 7 are no longer valid, as one can see from the factor  $A^{-2}$  in formula (7.3).

## 2. THE UNPERTURBED HAMILTONIAN $H_{AB}$

For  $\mathbf{A} = (A_1, A_2)$  in (1.2) we observe that

$$A_1, A_2 \in L_{loc}^\infty(\mathbb{R}^2 \setminus \{0\}).$$

Put

$$\Omega_n = \mathbb{R}^2 \setminus B(0, 1/n), \quad n \geq 2,$$

where  $B(0, r)$  denotes the ball with centre 0 and radius  $r$ .

We define on  $L^2(\Omega_n)$  (for each  $n \geq 2$ ) the form

$$h_n[u, v] = \sum_{j=1}^2 \int_{\Omega_n} \left( \frac{\partial u}{\partial x_j} + iA_j u \right) \overline{\left( \frac{\partial v}{\partial x_j} + iA_j v \right)} \quad (2.1)$$

on the domain  $\mathcal{D}(h_n) = H_0^1(\Omega_n)$ . The form is closed since  $A_1, A_2 \in L^\infty(\Omega_n)$ . The associated self-adjoint, nonnegative operators are denoted by  $H_n$ .

Define, in addition, the (closed) form  $l_n$  with the same form expression and domain as  $h_n$  but with  $A_1 = A_2 = 0$ . The associated self-adjoint, nonnegative operators are denoted by  $L_n$ .

Define now the form  $h$  by

$$\begin{aligned} h[u, v] &= h_n[u, v] \text{ if } u, v \in \mathcal{D}(h_n), \\ \mathcal{D}(h) &= \cup_n \mathcal{D}(h_n) = \cup_{n \geq 2} H_0^1(\Omega_n). \end{aligned}$$

**Lemma 2.1.** *The form  $h$  is closable.*

*Proof.* The form  $h$  is closable if and only if any sequence  $\{u_n\}$ ,  $u_n \in \mathcal{D}(h)$ , for which

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^2} = 0 \text{ and } \lim_{n, m \rightarrow \infty} h[u_n - u_m] = 0, \quad (2.2)$$

satisfies  $\lim_{n \rightarrow \infty} h[u_n] = 0$ .

First observe that (2.2) implies

$$C := \sup_n h[u_n]^{1/2} < \infty. \quad (2.3)$$

Take  $\epsilon > 0$  and choose  $n_0$  such that

$$h[u_n - u_m] \leq \epsilon \text{ when } n, m \geq n_0. \quad (2.4)$$

Let  $n_1 \geq n_0$ . Choose  $K \subset \mathbb{R}^2 \setminus \{0\}$  such that  $K$  is compact and  $\text{supp } u_{n_1} \subset K$ .

In view of (2.2) and  $K \subset \mathbb{R}^2 \setminus \{0\}$  it follows that

$$\int_K |(\nabla + i\mathbf{A})(u_n - u_m)|^2 dx \leq h[u_n - u_m] \longrightarrow 0 \text{ as } n, m \rightarrow \infty, \quad (2.5)$$

$$\int_K |u_n|^2 dx \longrightarrow 0 \text{ as } n \rightarrow \infty, \quad (2.6)$$

and, since  $\mathbf{A}$  is bounded on  $K$ ,

$$\int_K |\mathbf{A}u_n|^2 dx \longrightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.7)$$

Now,

$$\begin{aligned} & \left| \left( \int_K |\mathbf{A}(u_n - u_m)|^2 \right)^{1/2} - \left( \int_K |\nabla(u_n - u_m)|^2 dx \right)^{1/2} \right| \\ & \leq \left( \int_K |(\nabla + i\mathbf{A})(u_n - u_m)|^2 dx \right)^{1/2}. \end{aligned} \quad (2.8)$$

According to (2.7), the first term on the left-hand side of the latter tends to zero as  $n, m \rightarrow \infty$  and, due to (2.5), the same holds for the right-hand side. Thus,

$$\int_K |u_n - u_m|^2 + |\nabla(u_n - u_m)|^2 dx \longrightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Since the Dirichlet term, in the latter expression, is closable it follows from the latter in conjunction with (2.6) that

$$\int_K |\nabla u_n|^2 dx \rightarrow 0, \quad \int_K |u_n|^2 dx \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (2.9)$$

Now,

$$\begin{aligned} h[u_n] &= h[u_n, u_n - u_{n_1}] + h[u_n, u_{n_1}] \\ &\leq h[u_n]^{1/2} h[u_n - u_{n_1}]^{1/2} + h[u_n, u_{n_1}]. \end{aligned} \quad (2.10)$$

It follows from (2.3) and (2.4) that

$$h[u_n]^{1/2}h[u_n - u_{n_1}]^{1/2} \leq C\epsilon^{1/2} \text{ when } n \geq n_0. \quad (2.11)$$

Since  $\mathbf{A}$  is bounded on  $K$  we infer from (2.9) and (2.7) that

$$h[u_n, u_{n_1}] = \int_K \left\{ u_n \bar{u}_{n_1} + (\nabla + i\mathbf{A})u_n \overline{(\nabla + i\mathbf{A})u_{n_1}} \right\} dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.12)$$

Using (2.11)-(2.12) in (2.10) shows that  $\lim_{n \rightarrow \infty} h[u_n] = 0$  as desired.  $\square$

The closure of  $h$  is denoted  $\bar{h}$  and the associated semi-bounded (from below), self-adjoint operator is denoted by  $H_{AB}$ . We define  $l$  in a similar way, viz.

$$\begin{aligned} l[u, v] &= l_n[u, v] \text{ if } u, v \in \mathcal{D}(l_n), \\ \mathcal{D}(l) &= \cup_n \mathcal{D}(l_n) = \cup_{n \geq 2} H_0^1(\Omega_n). \end{aligned}$$

Then  $l$  is closable. The closure  $\bar{l}$  has domain  $\mathcal{D}(\bar{l}) = H^1(\mathbb{R}^2)$ . The associated nonnegative, self-adjoint operator is just the negative Laplacian in  $L^2(\mathbb{R}^2)$ ; we denote it by  $L_0$ .

### 3. SEMIGROUP CRITERION

Throughout this section  $\mathcal{H}$  denotes our Hilbert space  $L^2(\mathbb{R}^2)$ . For a given  $u \in \mathcal{H}$  we denote by  $\bar{u} := \operatorname{Re} u - i\operatorname{Im} u$  the conjugate function of  $u$ . By  $|u|$  we denote the absolute value of  $u$  (i.e. the function  $x \mapsto |u(x)| := \sqrt{u(x) \cdot \overline{u(x)}}$ ) and by  $\operatorname{sign} u$  the function defined by

$$\operatorname{sign} u(x) = \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

Let  $c$  be a sesquilinear form which satisfies

$$\mathcal{D}(c) \text{ is dense in } \mathcal{H}, \quad (3.1)$$

$$\operatorname{Re} c[u, u] \geq 0, \quad \forall u \in \mathcal{D}(c), \quad (3.2)$$

$$|c[u, v]| \leq M \|u\|_c \|v\|_c, \quad \forall u, v \in \mathcal{D}(c), \quad (3.3)$$

where  $M$  is a constant and  $\|u\|_c = \sqrt{\operatorname{Re} c[u, u] + \|u\|^2}$  and, moreover,

$$\langle \mathcal{D}(c), \|\cdot\|_c \rangle \text{ is a complete space.} \quad (3.4)$$

**Definition 3.1.** Let  $\mathcal{K}$  and  $\mathcal{L}$  be two subspaces of  $\mathcal{H}$ . We shall say that  $\mathcal{K}$  is an ideal of  $\mathcal{L}$  if the following two assertions are fulfilled:

- 1)  $u \in \mathcal{K}$  implies  $|u| \in \mathcal{L}$ .
- 2) If  $u \in \mathcal{K}$  and  $v \in \mathcal{L}$  such that  $|v| \leq |u|$  then  $v \cdot \operatorname{sign} u \in \mathcal{K}$ .

Let  $a$  and  $b$  be two sesquilinear forms both of which satisfy (3.1)-(3.4). The semigroups associated to  $a$  and  $b$  are denoted by  $e^{-tB}$  and  $e^{-tB}$ , respectively.

The following result was established by Ouhabaz [16, Theorem 3.3 and its Corollary].

**Theorem 3.2** (Ouhabaz'96). *Assume that the semigroup  $e^{-tB}$  is positive. The following assertions are equivalent:*

- 1)  $|e^{-tA}f| \leq e^{-tB}|f|$  for all  $t \geq 0$  and all  $f \in \mathcal{H}$ .
- 2)  $\mathcal{D}(a)$  is an ideal of  $\mathcal{D}(b)$  and

$$\operatorname{Re} a[u, |v| \operatorname{sign} u] \geq b[|u|, |v|] \quad (3.5)$$

for all  $(u, v) \in \mathcal{D}(a) \times \mathcal{D}(b)$  such that  $|v| \leq |u|$ .

- 3)  $\mathcal{D}(a)$  is an ideal of  $\mathcal{D}(b)$  and

$$\operatorname{Re} a[u, v] \geq b[|u|, |v|] \quad (3.6)$$

for all  $u, v \in \mathcal{D}(a)$  such that  $u \cdot \bar{v} \geq 0$ .

The following lemma is useful in applications when one wishes to establish the criteria in Theorem 3.2.

**Lemma 3.3.** *Suppose that  $\Omega \subset \mathbb{R}^2$  is an open set. Let  $u, v \in H^1(\Omega)$  be such that  $u(x) \cdot \overline{v(x)} \geq 0$  for a.e.  $x \in \Omega$ . Then*

1.  $\operatorname{Im} \left( \frac{\partial u}{\partial x_j} \cdot \bar{v} \right) = |v| \operatorname{Im} \left( \frac{\partial v}{\partial x_j} \cdot \operatorname{sign} \bar{u} \right)$ .
2.  $|v| \operatorname{Im} \left( \frac{\partial x}{\partial x_j} \cdot \operatorname{sign} \bar{u} \right) = |u| \operatorname{Im} |u| \operatorname{Im} \left( \frac{\partial v}{\partial x_j} \cdot \operatorname{sign} \bar{u} \right)$ .

*Proof.* Let  $\chi_{\{u=0\}}$  denote the characteristic function on the set  $\{\mathbf{x} \mid u(\mathbf{x}) = 0\}$ . Since  $(\partial u / \partial x_j) \cdot \chi_{\{u=0\}} = 0$ , we have that

$$\frac{\partial u}{\partial x_j} \cdot \bar{v} = \frac{\partial u}{\partial x_j} \cdot \bar{v} \cdot \frac{v \cdot \bar{u}}{|u| \cdot |v|} \chi_{\{u \neq 0\}} \chi_{\{v \neq 0\}} = |v| \cdot \frac{\partial u}{\partial x_j} \cdot \frac{\bar{u}}{|u|} \cdot \chi_{\{u \neq 0\}}.$$

By taking the imaginary part on both sides of the latter equality, we obtain that

$$\operatorname{Im} \left( \frac{\partial u}{\partial x_j} \cdot \bar{v} \right) = |v| \operatorname{Im} \left( \frac{\partial u}{\partial x_j} \cdot \operatorname{sign} \bar{u} \right),$$

which verifies the first assertion. To prove the second assertion we start from

$$|v| \cdot u = |v| \cdot u \cdot \frac{v \cdot \bar{u}}{|u| \cdot |v|} \chi_{\{u \neq 0\}} \chi_{\{v \neq 0\}} = |u| \cdot v.$$

Hence,

$$\frac{\partial |u|}{\partial x_j} \cdot u + |v| \cdot \frac{\partial u}{\partial x_j} = \frac{\partial |u|}{\partial x_j} \cdot v + |u| \cdot \frac{\partial v}{\partial x_j}.$$

We multiply both sides by  $\operatorname{sign} \bar{u} = (\bar{u}/|u|) \chi_{\{u \neq 0\}}$  and take the imaginary parts on both sides to obtain

$$|v| \operatorname{Im} \left( \frac{\partial u}{\partial x_j} \cdot \operatorname{sign} \bar{u} \right) = \operatorname{Im} \left( \frac{\partial v}{\partial x_j} \cdot \bar{u} \chi_{\{u \neq 0\}} \right) = \operatorname{Im} \left( \frac{\partial v}{\partial x_j} \cdot \bar{u} \right).$$

The latter in combination with the first assertion (with  $u$  substituted by  $v$  and vice-versa) shows the second assertion.  $\square$

4. DIAMAGNETIC INEQUALITY FOR  $H_{AB}$ 

The usual diamagnetic inequality holds for vector potentials which belong to  $L^2_{loc}$  (see, e.g., [3]). In this section we establish the diamagnetic inequality for the Schrödinger operator  $H_{AB}$ , i.e. when  $A_j \notin L^2_{loc}$ ,  $j = 1, 2$ .

Denote by  $e^{-tH_n}$  (resp.  $e^{-tL_n}$ ) the semigroup associated with  $H_n$  (resp.  $L_n$ ) introduced in Section 2. For each  $n$  the diamagnetic inequality holds for these pairs of semigroups.

**Proposition 4.1.** *The inequality*

$$|e^{-tH_n} f| \leq e^{-tL_n} |f|$$

holds for all  $t \geq 0$  and all  $f \in L^2(\Omega_n)$  ( $n \geq 2$ ).

*Proof.* By the domination criterion in Theorem 3.2, assertion 3, it suffices to prove that

$$\operatorname{Re} h_n[u, v] \geq l_n[|u|, |v|] \quad (4.1)$$

for all  $u, v \in \mathcal{D}(h_n) = H_0^1(\Omega_n)$  obeying  $u \cdot \bar{v} \geq 0$ .

Let  $u, v \in H_0^1(\Omega_n)$  be such that  $u \cdot \bar{v} \geq 0$ . We have that

$$\begin{aligned} I_1 &:= \operatorname{Re} \int_{\Omega_n} \left\{ \frac{\partial u}{\partial x_1} \cdot \frac{\partial \bar{v}}{\partial x_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial \bar{v}}{\partial x_2} \right\} \\ &= \int_{\Omega_n} \left\{ \operatorname{Re} \left( \frac{\partial u}{\partial x_1} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Re} \left( \frac{\partial v}{\partial x_1} \cdot \operatorname{sign} \bar{v} \right) \right. \\ &\quad \left. + \operatorname{Re} \left( \frac{\partial u}{\partial x_2} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Re} \left( \frac{\partial v}{\partial x_2} \cdot \operatorname{sign} \bar{v} \right) \right\} \\ &\quad + \int_{\Omega_n} \left\{ \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Im} \left( \frac{\partial v}{\partial x_1} \cdot \operatorname{sign} \bar{v} \right) \right. \\ &\quad \left. + \operatorname{Im} \left( \frac{\partial u}{\partial x_2} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Im} \left( \frac{\partial v}{\partial x_2} \cdot \operatorname{sign} \bar{v} \right) \right\} \\ &= \int_{\Omega_n} \left\{ \operatorname{Re} \left( \frac{\partial u}{\partial x_1} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Re} \left( \frac{\partial v}{\partial x_1} \cdot \operatorname{sign} \bar{v} \right) \right. \\ &\quad \left. + \operatorname{Re} \left( \frac{\partial u}{\partial x_2} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Re} \left( \frac{\partial v}{\partial x_2} \cdot \operatorname{sign} \bar{v} \right) \right. \\ &\quad \left. + \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \cdot \operatorname{sign} \bar{u} \right) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right. \\ &\quad \left. + \operatorname{Im} \left( \frac{\partial u}{\partial x_2} \cdot \operatorname{sign} \bar{u} \right) \operatorname{Im} \left( \frac{\partial u}{\partial x_2} \cdot \operatorname{sign} \bar{u} \right) \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right\}, \end{aligned}$$

where we applied Lemma 3.3, part 2, in the last equality. From [15, Lemma 4.1] we have that

$$\frac{\partial |u|}{\partial x_1} = \operatorname{Re} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right), \quad \forall u \in H^1(\Omega_n) \supset H_0^1(\Omega_n).$$



Using the latter we find that

$$I_1 = \int_{\Omega_n} \left\{ \frac{\partial|u|}{\partial x_1} \cdot \frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2} \cdot \frac{\partial|v|}{\partial x_2} + \left[ \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right) \right]^2 \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right. \\ \left. + \left[ \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right) \right]^2 \frac{|v|}{|u|} \chi_{\{u \neq 0\}} \right\}. \quad (4.2)$$

Next let  $u, v \in H_0^1(\Omega_n)$  with  $u \cdot \bar{v} \geq 0$ . Using  $\operatorname{Re} u \frac{\partial \bar{v}}{\partial x_1} = \operatorname{Re} \bar{u} \frac{\partial v}{\partial x_1}$  we have that

$$I_2 := \operatorname{Re} \int_{\Omega_n} \left\{ -iA_1 \frac{\partial u}{\partial x_1} \bar{v} - iA_2 \frac{\partial u}{\partial x_2} \bar{v} + iA_1 u \frac{\partial \bar{v}}{\partial x_1} + iA_2 u \frac{\partial \bar{v}}{\partial x_2} \right\} \\ = \int_{\Omega_n} \left\{ -\operatorname{Im}(-iA_1) \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \bar{v} \right) - \operatorname{Im}(-iA_2) \operatorname{Im} \left( \frac{\partial u}{\partial x_2} \bar{v} \right) \right. \\ \left. - \operatorname{Im}(iA_1) \operatorname{Im} \left( \bar{u} \frac{\partial v}{\partial x_1} \right) - \operatorname{Im}(iA_2) \operatorname{Im} \left( \bar{u} \frac{\partial v}{\partial x_2} \right) \right\}.$$

Using the first part of Lemma 3.3 we may rewrite  $I_2$  as

$$I_2 = \int_{\Omega_n} \left\{ -\operatorname{Im}(-iA_1) \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right) |v| - \operatorname{Im}(-iA_2) \right. \\ \times \operatorname{Im} \left( \frac{\partial u}{\partial x_2} \operatorname{sign} \bar{u} \right) |v| - \operatorname{Im}(iA_1) \operatorname{Im} \left( \frac{\partial v}{\partial x_1} \operatorname{sign} \bar{v} \right) |u| \\ \left. - \operatorname{Im}(iA_2) \operatorname{Im} \left( \frac{\partial v}{\partial x_2} \operatorname{sign} \bar{v} |u| \right) \right\}.$$

Next we apply the second part of Lemma 3.3 to the last two terms in  $I_2$ . It follows that

$$I_2 = \int_{\Omega_n} \left\{ (A_1 - A_1) \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right) |v| \right. \\ \left. + (A_2 - A_2) \operatorname{Im} \left( \frac{\partial u}{\partial x_2} \operatorname{sign} \bar{u} \right) |v| \right\} = 0. \quad (4.3)$$

For the last term in  $h_n$ , we have that

$$I_3 := \operatorname{Re} \int_{\Omega_n} (A_1^2 + A_2^2) u \cdot \bar{v} = \int_{\Omega_n} (A_1^2 + A_2^2) |u| |v| \quad (4.4)$$

for all  $u, v \in H_0^1(\Omega_n)$  such that  $u \cdot \bar{v} \geq 0$ .

Since  $\operatorname{Re} h_n[u, v] = \sum_{j=1}^2 I_j$ , we obtain from (4.2), (4.3) and (4.4) that

$$\operatorname{Re} h_n[u, v] = \int_{\Omega_n} \left\{ \frac{\partial|u|}{\partial x_1} \cdot \frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2} \cdot \frac{\partial|v|}{\partial x_2} + \left[ \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right) \right]^2 \right. \\ \left. \times \frac{|v|}{|u|} \chi_{\{u \neq 0\}} + \left[ \operatorname{Im} \left( \frac{\partial u}{\partial x_1} \operatorname{sign} \bar{u} \right) \right]^2 \frac{|v|}{|u|} \chi_{\{u \neq 0\}} + (A_1^2 + A_2^2) |u| |v| \right\}.$$

In the latter, the sum of last three terms is nonnegative, so we infer that

$$\begin{aligned} \operatorname{Re} h_n[u, v] &\geq \int_{\Omega_n} \left\{ \frac{\partial|u|}{\partial x_1} \cdot \frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2} \cdot \frac{\partial|v|}{\partial x_2} + (A_1^2 + A_2^2)|u||v| \right\} \\ &\geq \int_{\Omega_n} \left\{ \frac{\partial|u|}{\partial x_1} \cdot \frac{\partial|v|}{\partial x_1} + \frac{\partial|u|}{\partial x_2} \cdot \frac{\partial|v|}{\partial x_2} \right\} = l_n[|u|, |v|] \end{aligned}$$

for all  $u, v \in H_0^1(\Omega_n)$  obeying  $u \cdot \bar{v} \geq 0$ . This verifies (4.1).  $\square$

The semigroups associated with  $H_{AB}$  and  $L_0$ , introduced in Section 2, are denoted by  $e^{-tH_{AB}}$  and  $e^{-tL_0}$ , resp. By means of Proposition 4.1 we are ready to prove Theorem 1.1, i.e., the diamagnetic inequality for the operator  $H_{AB}$ .

*Proof of Theorem 1.1.* Bear in mind that when  $k_1$  and  $k_2$  are closed forms bounded from below then  $k_1 \geq k_2$  means that  $\mathcal{D}(k_1) \subset \mathcal{D}(k_2)$  and  $k_1[u, u] \geq k_2[u, u]$  for  $u \in \mathcal{D}(k_1)$ . A sequence  $\{k_n\}$  of closed forms bounded from below is nonincreasing if  $k_n \geq k_{n+1}$  for all  $n$ .

The forms  $\{h_n\}$  defined in (2.1) on the domains  $\mathcal{D}(h_n) = H_0^1(\Omega_n)$  in  $L^2(\Omega_n)$ ,  $n \geq 2$ , is a nonincreasing sequence

$$\cdots \leq h_{n+1} \leq h_n \leq h_{n-1} \leq \cdots,$$

of closed, *non-densely defined* forms in  $L^2(\mathbb{R}^2)$ . The monotone convergence theorem for closed forms is also valid for non-densely defined forms [23, Theorem 4.1]. Hence

$$e^{-tH_{AB}} = \text{s-} \lim_{n \rightarrow \infty} e^{-tH_n}.$$

The latter equality in conjunction with Proposition 4.1 yields the assertion.  $\square$

## 5. ABSTRACT CLR-ESTIMATES

In this section we recall Rozenblum and Solomyak's abstract CLR-estimate for generators of positively dominated semigroup.

Let  $\Omega$  be a space of  $\sigma$ -finite measure  $\mu$  and let  $L^2 = L^2(\Omega, \mu)$ . Let  $B$  be a nonnegative, self-adjoint operator in  $L^2$  such that it generates a positivity preserving semigroup  $Q(t) = e^{-tB}$ . We assume also that  $Q(t)$  is an integral operator (see, e.g., [2]) with bounded kernel  $Q(t; x, y)$  subject to

$$M_B(t) := \operatorname{ess\,sup}_x Q(t; x, x), \quad M_B(t) = O(t^{-\beta}) \text{ at zero for some } \beta > 0. \quad (5.1)$$

We will write  $B \in \mathcal{P}$  if  $B$  satisfies the afore-mentioned assumptions.

If  $B \in \mathcal{P}$  and  $B \geq \alpha_0$  for some  $\alpha_0 \geq 0$  then for any  $\alpha \geq -\alpha_0$  the operator  $B_\alpha = B + \alpha$  also belongs to  $\mathcal{P}$ . The corresponding semigroup is  $Q_{B_\alpha}(t) = e^{-\alpha t} Q_B(t)$  and thus  $M_{B_\alpha}(t) = e^{-\alpha t} M_B(t)$ .

We say that the semigroup  $P(t) = e^{-tA}$  is dominated by  $Q(t)$  if the diamagnetic inequality holds, i.e., if any  $u \in L^2$  satisfies

$$|P(t)u| \leq Q(t)|u| \text{ a.e. on } \Omega. \quad (5.2)$$

In the latter case we write  $A \in \mathcal{PD}(B)$ .

Let now  $G$  be a nonnegative, continuous, convex function on  $[0, \infty)$ . To such a function we associate

$$g(\lambda) = \mathcal{L}(G)(\lambda) := \int_0^\infty z^{-1}G(z)e^{-z/\lambda} dz, \quad \lambda > 0, \quad (5.3)$$

provided the latter integral converges. In other words,  $g(1/\lambda)$  is the Laplace transform of  $z^{-1}G(z)$ .

For a nonnegative, measurable function  $V$  such that the latter is form-bounded with respect to  $B$  with a bound less than one, we associate the operators  $B - V$ ,  $A - V$  by means of quadratic forms (see [17, Theorem X.17]). The number of negative eigenvalues (counting multiplicity) of  $B - V$  is denoted by  $N_-(B - V)$ ; if there is some essential spectrum below zero, we set  $N_-(B - V) = \infty$ .

Rozenblum and Solomyak [19, Theorem 2.4] have established the following abstract CLR-estimate for generators of positively dominated semigroups.

**Theorem 5.1** (Rozenblum-Solomyak'97). *Let  $G$ ,  $g$  and  $B \in \mathcal{P}$  be as above and suppose that  $\int_a^\infty M_B(t) dt < \infty$  for some  $a > 0$ . If  $A \in \mathcal{PD}(B)$  then*

$$N_-(A - V) \leq \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_\Omega M_B(t)G(tV(x)) dx, \quad (5.4)$$

as long as the expression on the right-hand side is finite.

The assumption that  $V$  is form-bounded with respect to  $B$  with a bound smaller than one in conjunction with  $A \in \mathcal{PD}(B)$  implies that  $V$  is form-bounded with respect to  $A$  with a bound less than one.

In Section 6 we shall apply Theorem 5.1 to prove the Lieb-Thirring inequality for  $H_{AB} - V$ .

Rozenblum has also developed an abstract machinery which, in our situation, allow us to carry over *any*, sufficiently regular, bound for  $N_-(L_0 - V)$  to  $N_-(H_{AB} - V)$  because the diamagnetic inequality is valid for  $H_{AB}$ . For our purpose Rozenblum's result for Schrödinger-like operators suffices, i.e. [21, Theorem 4]. We customize it to our situation.

**Theorem 5.2** (Rozenblum'00). *Assume  $B \in \mathcal{P}$ ,  $A \in \mathcal{PD}(B)$  and that  $V$  is a non-negative, measurable function which is infinitesimally form-bounded with respect to  $B$ . Suppose that, for some  $p > 0$ ,*

$$N_-(B - qV) \leq Kq^p \quad (5.5)$$

for all  $q > 0$  and some positive constant  $K$ . Then

$$N_-(A - qV) \leq eC_p Kq^p. \quad (5.6)$$

In Section 7 we carry over some recently obtained bounds for the two-dimensional Schrödinger operator, with a regularizing positive (Hardy) term added, to the operator  $H_{AB} - V$ .

## 6. LIEB-THIRRING INEQUALITY FOR $H_{AB} - V$

Having the diamagnetic inequality in Theorem 1.1 as well as the abstract CLR-estimate in Theorem 5.1 at our disposal we are ready to prove Theorem 1.2.

Before proceeding to the proof, observe that the assumption  $V \in L^p(\mathbb{R}^2)$ ,  $p > 1$ , in Theorem 1.2 implies that  $V$  is  $L_0$  form-bounded with relative bound zero and, whence, Theorem 1.1 implies that  $V$  is  $H_{AB}$  form-bounded with relative bound zero. Thus, according to the KLMN Theorem [17, Theorem X.17], the form sum  $H_{AB} - qV$  generates a semi-bounded (from below), self-adjoint operator in  $L^2(\mathbb{R}^2)$ ; here we introduce a coupling constant  $q$  for later purpose. Furthermore, Weyl's essential spectrum theorem asserts that  $\sigma_{ess}(H_{AB} - V) = \sigma_{ess}(H_{AB}) = [0, \infty)$ , where the last equality follows from, e.g., [1]. In particular, the spectrum of  $H_{AB} - qV$  is discrete below zero.

*Proof of Theorem 1.2.* Now  $L_0 \in \mathcal{P}$  and the kernel of its semigroup  $e^{-tL_0}$  is given by  $Q(t; x, x) = (1/2\pi)t^{-1}$ . From Theorem 1.1 we have that  $H_{AB} \in \mathcal{PD}(L_0)$  and the kernel  $P(t; x, x)$  of its semigroup obeys  $|P(t; x, x)| \leq (1/2\pi)t^{-1}$ .

Let  $\alpha > 0$  and define the auxiliary operators  $A_\alpha = H_{AB} + \alpha$  and  $B_\alpha = L_0 + \alpha$ . The assumptions  $L_0 \in \mathcal{P}$ ,  $H_{AB} \in \mathcal{PD}(L_0)$  imply that  $B_\alpha \in \mathcal{P}$  and  $A_\alpha \in \mathcal{PD}(B_\alpha)$ . For the kernel  $P_\alpha(t; x, x) = e^{-\alpha t}P(t; x, x)$  of the semigroup generated by  $A_\alpha$  we have therefore that  $|P_\alpha(t; x, x)| \leq Q_\alpha(t; x, x) = e^{-\alpha t}Q(t; x, x) = (1/2\pi)t^{-1}e^{-\alpha t}$ . Thus we may apply Theorem 5.1 which yields

$$N_-(A_\alpha - V) \leq \frac{1}{2\pi} \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_{\mathbb{R}^2} t^{-1} e^{-\alpha t} G(tV(x)) dx, \quad (6.1)$$

since  $M_B(t) = (1/2\pi)t^{-1}e^{-\alpha t}$ . We will not evaluate the integral in (6.1) as one may be inclined to do. Instead, for  $\gamma > 0$ , we recall that (see e.g. [13])

$$\begin{aligned} LT_\gamma &:= \sum_j |\lambda_j(H_{AB} - V)|^\gamma = - \int \alpha^\gamma dN_\alpha \\ &= \gamma \int_0^\infty \alpha^{\gamma-1} N_-(A_\alpha - V) d\alpha. \end{aligned} \quad (6.2)$$

We substitute (6.1) into (6.2) and get that

$$LT_\gamma \leq \frac{1}{2\pi} \frac{\gamma}{g(1)} \int_{\mathbb{R}^2} dx \int_0^\infty \alpha^{\gamma-1} d\alpha \int_0^\infty t^{-1} e^{-\alpha t} G(tV(x)) \frac{dt}{t}. \quad (6.3)$$

Making first the change of variables  $s = V(x)t$  and then the change of variables  $u = \alpha/V(x)$  we obtain that

$$LT_\gamma \leq C(\gamma) \int_{\mathbb{R}^2} V(x)^{\gamma+1} dx, \quad (6.4)$$

where

$$C(\gamma) = \frac{1}{2\pi} \frac{\gamma}{g(1)} \int_0^\infty \int_0^\infty s^{-2} e^{-us} G(s) u^{\gamma-1} ds du. \quad (6.5)$$

Now,  $\int_0^\infty u^{\gamma-1} e^{-us} du = s^{-\gamma} \Gamma(\gamma)$ , where  $\Gamma(\gamma)$  is the Gamma-function evaluated at  $\gamma$ . Choose  $G(s) = (s-k)_+$  for some  $k > 0$ ; this is Lieb's original choice. Then

$$\int_0^\infty s^{-\gamma} s^{-2} (s-k)_+ ds = \frac{1}{\gamma(\gamma+1)} k^{-\gamma}. \quad (6.6)$$

Moreover,

$$g(1) = \int_1^\infty e^{-ks} s^{-2} ds \geq \frac{e^{-k}}{k} - \frac{2}{k} g(1),$$

i.e.,  $1/g(1) \leq e^k(k+2)$ . Thus

$$C(\gamma) \leq C_1(\gamma) := \frac{1}{2\pi} \frac{e^k(k+2)}{(\gamma+1)k^\gamma}. \quad (6.7)$$

The optimization problem for the expression in (6.7) does not admit an exact solution. The following values were obtained numerically and give values for the three interesting values of  $\gamma$ , namely 1, 1/2 and 3/2:

$$C_1(\gamma) = \begin{cases} 0.5981 & \text{for } \gamma = 1/2, \\ 0.6175 & \text{for } \gamma = 1, \\ 0.7527 & \text{for } \gamma = 3/2. \end{cases}$$

□

## 7. EIGENVALUE ESTIMATES AND LARGE COUPLING CONSTANT ASYMPTOTICS FOR $H_{AB} - qV$

The (closed) quadratic form  $h$  of the unperturbed A-B Schrödinger operator  $H_{AB}$  can be written as

$$h[u] = \frac{h[u]}{2} + \frac{h[u]}{2}. \quad (7.1)$$

Let  $A = \min(\alpha, 1 - \alpha)$ . To one of the two terms in (7.1), we apply the Hardy type inequality by Laptev-Weidl [11]. This yields

$$h[u] \geq \frac{h[u]}{2} + A^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} dx. \quad (7.2)$$

Let  $H_{AB}(A^2 r^{-2})$ ,  $r = |x|^{-2}$ , denote the operator generated by the form on the right-hand side of (7.2). Since  $H_{AB}$  obeys the diamagnetic inequality, it follows from, e.g., the Trotter-Kato formula that  $H_{AB}(A^2 r^{-2})$  fulfills the diamagnetic inequality as well, in shorthand,

$H_{AB}(A^2r^{-2}) \in \mathcal{PD}(L_0 + A^2r^{-2})$ . The latter fact in conjunction with Theorem 5.2 allows us to carry over all well-known bounds for the two-dimensional Schrödinger operator  $L_0 + r^{-2} - V$  to the A-B Schrödinger operator  $H_{AB} - V$ . In order to take into account the influence of the value of  $A$ , we can write

$$L_0 + A^2r^{-2} - V \geq A^2L_0 + A^2r^{-2} - V = A^2(L_0 + r^{-2} - A^{-2}V).$$

Therefore, to estimate the number of eigenvalues for the operator with given  $A$ , we may use the existing estimates for the operator  $L_0 + r^{-2}$  with potential  $A^{-2}V$ .

Estimates of the number of negative eigenvalues for the Schrödinger operator  $H(r^{-2}, q) := L_0 + r^{-2} - qV$  in  $L^2(\mathbb{R}^2)$ , where we have introduced a coupling constant  $q > 0$ , have been studied in [25, 5, 10]. Following Solomyak [25], suppose that  $V$  belongs to the Orlicz space  $L \ln(1 + L)$  locally and let  $\{\zeta_j\}$ ,  $j \geq 0$ , be the sequence of averaged Orlicz norms over the annuli  $r \in (2^{j-1}, 2^j)$ ,  $j > 0$ , and over the unit disk for  $j = 0$ . Then, according to [25],  $N_-(H(r^{-2}, q)) \leq Cq \|\{\zeta_j\}\|_1$ . Since the function on the right-hand side of the latter is regular, this estimate is carried over to  $H_{AB} - qV$ . In a similar way, all estimates obtained in [5] and [10] hold for  $H_{AB} - qV$ . Of course, the factor  $A^{-2}$  must arise in the estimates, as it was explained above.

Another important case is the one of the radially symmetric potential considered in [10, Theorem 1.2]. We immediately get the following result.

**Theorem 7.1.** *If  $V$  is radially symmetric and  $V \in L^1(\mathbb{R}^2)$  then*

$$N_-(H_{AB} - V) \leq CA^{-2} \int_{\mathbb{R}^2} V(x) dx. \quad (7.3)$$

This, in particular, shows that the estimate obtained in [4, Theorem 2] holds with an absolute constant (i.e. independent of  $V$ ).

As soon as eigenvalue estimates having correct coupling constant behaviour are obtained, we can proceed in a standard way (see, e.g. [18], [6, 7] and, especially, [20] wherein the technology of proving asymptotic formulas for operators with magnetic field is explained in details), and obtain asymptotic formulas in the limit  $q \rightarrow \infty$ . We formulate just one of these results; similar asymptotic formulas hold also in all other situations (see [21], [20]), where the correct order eigenvalue estimates are obtained. We use, again, [10, Theorem 1.2] and get the following large coupling constant asymptotics for  $H_{AB} - qV$ .

**Theorem 7.2.** *If  $V$  is radially symmetric and  $V \in L^1(\mathbb{R}^2)$  then*

$$N_-(H_{AB} - qV) \sim Cq \int_{\mathbb{R}^2} V(x) dx \text{ as } q \rightarrow \infty. \quad (7.4)$$

**Acknowledgement.** M.M. thanks Professor J. Brasche for enlightening discussions and G.R. is grateful to Professor M. Z. Solomyak for useful consultations.

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