

INEQUALITIES FOR WEIERSTRASS PRODUCTS

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ABSTRACT

The aim of this note is to find new inequalities for the Weierstrass products $\prod_{i=1}^n (1+x_i)$ and $\prod_{i=1}^n (1-x_i)$, which are an analogues of Alzer's inequalities [A].

1. INTRODUCTION

Several authors interested in Weierstrass products $\prod_{i=1}^n (1+x_i)$ and $\prod_{i=1}^n (1-x_i)$. In [B, Pages 104–105] proved

$$1 - \sum_{i=1}^n x_i \leq \prod_{i=1}^n (1-x_i) \leq \left(1 + \sum_{i=1}^n x_i\right)^{-1},$$

and

$$1 + \sum_{i=1}^n x_i \leq \prod_{i=1}^n (1+x_i) \leq \left(1 - \sum_{i=1}^n x_i\right)^{-1},$$

where $x_i \in [0, 1]$, $i = 1, 2, \dots, n$, and in the second inequality assumed that $\sum_{i=1}^n x_i < 1$. In the last years these products attacked by many other inequalities which have been published. For example, Klamkin and Newman [KN] discovered several extensions and new inequalities:

$$(n+1)^n \prod_{i=1}^n x_i \leq \prod_{i=1}^n (1+x_i), \quad (n-1)^n \prod_{i=1}^n x_i \leq \prod_{i=1}^n (1-x_i),$$

and in [K] presented another inequality:

$$(n+1)^n \prod_{i=1}^n (1-x_i) \leq (n-1)^n \prod_{i=1}^n (1+x_i),$$

where $x_i \in [0, 1]$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^n x_i = 1$. An another example, the Ky Fan's (see [B, Page 5]) inequality:

$$(1.1) \quad \left(\prod_{i=1}^n \frac{x_i}{1-x_i}\right)^{1/n} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)},$$

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where $x_i \in (0, \frac{1}{2}]$, $i = 1, 2, \dots, n$. In this paper, we are interesting in the Alzer's inequalities [A] of Weierstrass Products. Here, we present a generalization for the Alzer's inequalities by using the same arguments proof as in Alzer's paper [A]. In Section 2 we establish new inequalities for Weierstrass products which are an analogues for these inequalities in [A]. Finally, we apply the applications of [A] to get new inequalities for trigonometric and integral sums and products.

2. INEQUALITIES FOR WEIERSTRASS PRODUCTS

Let us start by proving two lemmas which are an analogues of the two lemmas in [A].

Lemma 2.1. *For all real $x_i \in [0, 1]$, $i = 1, 2, \dots, n$, and $a \in [0, 1]$, we have*

$$(2.1) \quad \prod_{i=1}^n x_i \left[1 + a \sum_{i=1}^n (1 - x_i) \right] + \prod_{i=1}^n (1 - x_i) \left[1 + a \sum_{i=1}^n x_i \right] \leq \frac{1}{2}a + 1.$$

Proof. Using the same arguments as in the proof of [A, Lemma 1] we get as follows. Let

$$S(x_1, \dots, x_n) = \prod_{i=1}^n x_i \left[1 + a \sum_{i=1}^n (1 - x_i) \right] + \prod_{i=1}^n (1 - x_i) \left[1 + a \sum_{i=1}^n x_i \right],$$

since $a, x_i \in [0, 1]$ for all i , we arrive at

$$\begin{aligned} S(x_1, \dots, x_n) - S(x_1, \dots, x_{n-1}) &= -a(1 - x_n) \prod_{i=1}^{n-1} x_i \sum_{i=1}^n (1 - x_i) - (1 - a)(1 - x_n) \prod_{i=1}^{n-1} x_i \\ &\quad - ax_n \prod_{i=1}^{n-1} (1 - x_i) \sum_{i=1}^n x_i - (1 - a)x_n \prod_{i=1}^{n-1} (1 - x_i) \leq 0. \end{aligned}$$

Hence, inductively on n we have

$$S(x_1, \dots, x_n) \leq S(x_1) = -2a \left(x_1 - \frac{1}{2} \right)^2 + 1 + \frac{1}{2}a \leq \frac{1}{2}a + 1,$$

as requested. □

The next lemma is a counterpart of Ky Fan's Inequality 1.1 and analogues of [A, Lemma 2].

Lemma 2.2. *For all real $x_i \in (0, \frac{1}{2}]$, $i = 1, 2, \dots, n$, and $a \geq 0$, we have*

$$(2.2) \quad \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1 - x_i)} \leq \frac{1 + a \sum_{i=1}^n x_i}{1 + a \sum_{i=1}^n (1 - x_i)},$$

with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \dots, n$.

Proof. To this theorem we present two different proofs: by induction argument, and by Ky Fan's Inequality 1.1.

(1) It is easy to see that the equality holds in 2.2 for $x_i = \frac{1}{2}$ where $i = 1, 2, \dots, n$. Now, we prove the lemma by induction on n . For $n = 1$, Inequality 2.2 yields

$$\frac{x_1}{1 - x_1} \leq \frac{1 + ax_1}{1 + a(1 - x_1)},$$

equivalently, $x_1 \leq \frac{1}{2}$ and the sign of equality holds if and only if $x_1 = \frac{1}{2}$. Now, let us suppose that

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \frac{1}{2}, \quad x_1 < x_{n+1},$$

and further we assume that Inequality 2.2 is true for n . Therefore, by induction hypothesis we get

$$\frac{\prod_{i=1}^{n+1} x_i}{\prod_{i=1}^{n+1} (1-x_i)} \leq \frac{1 + a \sum_{i=1}^n x_i}{1 + a \sum_{i=1}^n (1-x_i)} \cdot \frac{x_{n+1}}{1-x_{n+1}}.$$

If putting $p = a \sum_{i=1}^n x_i$, $q = a \sum_{i=1}^n (1-x_i)$, and $x = x_{n+1}$, then we have to prove that

$$\frac{x(1+p)}{(1-x)(1+q)} < \frac{1+p+ax}{1+q+a(1-x)},$$

equivalently,

$$a(q-p)x^2 + x \left(2(1+p)(1+q) - a(q-p) \right) - (1+p)(1+q) < 0.$$

Since, $p = q$ if and only if $x_1 = x_2 = \dots = x_n = \frac{1}{2}$ we have $p < q$ and since $0 \leq x \leq \frac{1}{2}$ we obtain

$$a(q-p)x^2 + x \left(2(1+p)(1+q) - a(q-p) \right) - (1+p)(1+q) < -\frac{a(q-p)}{4} < 0.$$

Thus we have the desired result.

(2) Since $x_i \leq \frac{1}{2}$ for all i , we have that $\frac{x_i}{1-x_i} \leq 1$. Therefore, by Inequality 1.1 we have

$$\prod_{i=1}^n \frac{x_i}{1-x_i} \leq \left[\prod_{i=1}^n \frac{x_i}{1-x_i} \right]^{1/n} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)} = \frac{a \sum_{i=1}^n x_i}{a \sum_{i=1}^n (1-x_i)}.$$

Hence, because of $\frac{x}{y} \leq 1$ if and only if $\frac{x}{y} \leq \frac{1+x}{1+y}$ where $x, y > 0$, and since $a \sum_{i=1}^n x_i \leq a \sum_{i=1}^n (1-x_i)$, we arrive at

$$\prod_{i=1}^n \frac{x_i}{1-x_i} \leq \frac{1 + a \sum_{i=1}^n x_i}{1 + a \sum_{i=1}^n (1-x_i)}.$$

□

For example, Theorem 2.2, for $a = 2$, yields [A, Lemma 2]. An another example, Theorem 2.2, for $a \rightarrow \infty$, yields

$$\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1-x_i)} \leq \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)},$$

Theorem 2.3. For all real $x_i \in (0, \frac{1}{2}]$, $i = 1, 2, \dots, n$, ($n \geq 2$), and $1 \leq a \leq \frac{3}{2}$, we have

$$(2.3) \quad \frac{\prod_{i=1}^n (1+x_i)}{\prod_{i=1}^n (1+(1-x_i))} \leq \frac{1 + a \sum_{i=1}^n x_i}{1 + a \sum_{i=1}^n (1-x_i)},$$

with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \dots, n$.

Proof. It is easy to see that the equality holds in 2.3 for $x_i = \frac{1}{2}$ where $i = 1, 2, \dots, n$. Therefore, it remains to prove for numbers x_1, x_2, \dots, x_n are not all equal to $\frac{1}{2}$ then the Inequality 2.3 is valid with " $<$ " instead of " \leq ".

Let us prove Inequality 2.3 by induction on n . For $n = 2$ we have to prove, for $0 < x_1 < x_2 \leq \frac{1}{2}$,

$$\frac{(1+x_1)(1+x_2)}{(2-x_1)(2-x_2)} < \frac{1+ax_1+ax_2}{1+2a-ax_1-ax_2},$$

equivalently,

$$(3-2a+a(x_1+x_2-2x_1x_2))(x_1+x_2-1),$$

which is absolutely true for $0 < x_1 < x_2 \leq \frac{1}{2}$ and $1 < a \leq \frac{3}{2}$. Now, let us suppose that

$$0 < x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \frac{1}{2}, \quad x_1 < x_{n+1},$$

and further we suppose that Inequality 2.3 is hold for n . Therefore, by induction hypothesis we get

$$\prod_{i=1}^{n+1} \frac{1+x_i}{(1+(1-x_i))} \leq \frac{1+a\sum_{i=1}^n x_i}{1+a\sum_{i=1}^n (1-x_i)} \cdot \frac{1+x_{n+1}}{2-x_{n+1}}.$$

If putting $p = a\sum_{i=1}^n x_i$, $q = a\sum_{i=1}^n (1-x_i)$, and $x = x_{n+1}$, then we have to prove that

$$\frac{(1+x)(1+p)}{(2-x)(1+q)} < \frac{1+p+ax}{1+q+a(1-x)},$$

equivalently,

$$a(q-p)x^2 + 2x(1+q)(1+p-a) + (1+p)(a-1-q) < 0.$$

Since, $a = b$ if and only if $x_i = \frac{1}{2}$ for $i = 1, 2, \dots, n$, we have $p < q$. Therefore, since $x \leq \frac{1}{2}$ and $1 < a \leq \frac{3}{2}$ we obtain

$$a(q-p)x^2 + 2x(1+q)(1+p-a) + (1+p)(a-1-q) \leq -\frac{3a(q-p)}{4} < 0,$$

which this completes the proof. \square

For example, Theorem 2.3, for $a = 1$ yields the second inequality in [A, Theorem, Equation 2.2].

Corollary 2.4. For all real $x_i \in [\frac{1}{2}, 1)$, $i = 1, 2, \dots, n$, ($n \geq 2$), and $1 \leq a \leq \frac{3}{2}$, we have

$$(2.4) \quad \frac{\prod_{i=1}^n (1+x_i)}{\prod_{i=1}^n (1+(1-x_i))} \geq \frac{1+a\sum_{i=1}^n x_i}{1+a\sum_{i=1}^n (1-x_i)},$$

with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \dots, n$.

Proof. If we set $x_i = 1-y_i$, $y_i \in [\frac{1}{2}, 1)$, $i = 1, 2, \dots, n$, in Inequality 2.3 then we arrive to Inequality 2.4. \square

Theorem 2.5. For all real $x_i \in (0, \frac{1}{2}]$, $i = 1, 2, \dots, n$, ($n \geq 2$), and $\frac{2}{3} < a \leq 1$, we have

$$(2.5) \quad \frac{1+a\sum_{i=1}^n x_i}{1+a\sum_{i=1}^n (1-x_i)} \leq \frac{1+\prod_{i=1}^n x_i}{1+\prod_{i=1}^n (1-x_i)},$$

with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \dots, n$.

Proof. It is easy to see that the equality holds in 2.5 for $x_i = \frac{1}{2}$ where $i = 1, 2, \dots, n$. Therefore, it remains to prove for numbers x_1, x_2, \dots, x_n are not all equal to $\frac{1}{2}$ then the Inequality 2.5 is valid with " $<$ " instead of " \leq ".

Let us prove Inequality 2.5 by induction on n . By simplify Inequality 2.5 we arrive to

$$(2.6) \quad an - 2a \sum_{i=1}^n x_i + (an + 1) \prod_{i=1}^n x_i - \prod_{i=1}^n (1 - x_i) - a \sum_{i=1}^n x_i \left(\prod_{i=1}^n x_i + \prod_{i=1}^n (1 - x_i) \right) \geq 0.$$

For $n = 2$ we have to prove, for $0 < x_1 < x_2 \leq \frac{1}{2}$,

$h(x_2) = 2a - 2a(x_1 + x_2) + (2a + 1)x_1x_2 - (1 - x_1)(1 - x_2) - a(x_1 + x_2)(x_1x_2 + (1 - x_1)(1 - x_2))$,
where $h : (x_1, \frac{1}{2}] \rightarrow \mathbb{R}$. So, $h''(x_2) = 2x(1 - 2x_1) > 0$, and $h'(x_2) = 0$ if and only if

$$x_2 = \frac{(x_1 - 1)^2}{1 - 2x_1} + \frac{a - 1}{2a(1 - 2x_1)} \geq \frac{(x_1 - 1)^2}{1 - 2x_1} - \frac{1}{4(1 - 2x_1)} = \frac{3}{4} - \frac{1}{2}x_1 \geq \frac{1}{2}.$$

Therefore, $h(x_2)$ is strictly decreasing on $(x_1, \frac{1}{2}]$ which gives

$$h(x_2) \geq h\left(\frac{1}{2}\right) = \left(\frac{1}{2} - x_1\right) \left(\frac{3a}{2} - 1\right) > 0.$$

Now, let us suppose that

$$0 < x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \leq \frac{1}{2}, \quad x_1 < x_{n+1},$$

and further we suppose that Inequality 2.3 is hold for n . Let us define a function $f : (0, \frac{1}{2}] \rightarrow \mathbb{R}$ by

$$f(x) = a(n + 1) - 2a \sum_{i=1}^n x_i - 2ax + (an + a + 1)x \prod_{i=1}^n x_i - (1 - x) \prod_{i=1}^n (1 - x_i) - a \left(x + \sum_{i=1}^n x_i \right) \left(x \prod_{i=1}^n x_i + (1 - x) \prod_{i=1}^n (1 - x_i) \right),$$

so we have to prove $f(x_{n+1}) > 0$. This by prove that f is a convex function, $f'(\frac{1}{2}) < 0$, and $f(\frac{1}{2}) > 0$. The function f is a convex function because $f''(x) = 2a \left(\prod_{i=1}^n (1 - x_i) - \prod_{i=1}^n x_i \right) \geq 0$. By direct calculations together with Lemma 2.1 we arrive at

$$f' \left(\frac{1}{2} \right) = \prod_{i=1}^n x_i \left(1 + a \sum_{i=1}^n x_i \right) + \prod_{i=1}^n (1 - x_i) \left(1 + a \sum_{i=1}^n x_i \right) - 2a \leq 1 + \frac{a}{2} - 2a \leq 1 - \frac{3a}{2} < 0.$$

To prove that $f(\frac{1}{2}) > 0$, let us write

$$f \left(\frac{1}{2} \right) = u(x_1, \dots, x_n) + v(x_1, \dots, x_n) + \frac{1}{2}(1 - a) \left(\prod_{i=1}^n (1 - x_i) - \prod_{i=1}^n x_i \right),$$

where

$$\begin{aligned} u(x_1, \dots, x_n) &= an - 2a \sum_{i=1}^n x_i + (an + 1) \prod_{i=1}^n x_i - \prod_{i=1}^n (1 - x_i) - a \sum_{i=1}^n x_i \left(\prod_{i=1}^n x_i + \prod_{i=1}^n (1 - x_i) \right), \\ v(x_1, \dots, x_n) &= \frac{a}{4} \prod_{i=1}^n (1 - x_i) \left(1 + 2 \sum_{i=1}^n x_i \right) - \frac{a}{4} \prod_{i=1}^n x_i \left(1 + 2 \sum_{i=1}^n (1 - x_i) \right). \end{aligned}$$

Since $\frac{2}{3} < a \leq 1$ we have $\frac{1}{2}(1-a)(\prod_{i=1}^n(1-x_i) - \prod_{i=1}^n x_i) \geq 0$, and by induction hypothesis we get $u(x_1, \dots, x_n) \geq 0$. Lemma 2.2, for $a = 2$, yields $v(x_1, \dots, x_n) > 0$. Hence, $f(\frac{1}{2}) > 0$ as requested. \square

For example, Theorem 2.5, for $a = 1$ yields the third inequality in [A, Theorem, Equation 2.2].

Corollary 2.6. *For all real $x_i \in [\frac{1}{2}, 1)$, $i = 1, 2, \dots, n$, ($n \geq 2$), and $1 \leq a \leq \frac{3}{2}$, we have*

$$(2.7) \quad \frac{1 + a \sum_{i=1}^n x_i}{1 + a \sum_{i=1}^n (1 - x_i)} \geq \frac{1 + \prod_{i=1}^n x_i}{1 + \prod_{i=1}^n (1 - x_i)},$$

with equality if and only if $x_i = \frac{1}{2}$ for all $i = 1, 2, \dots, n$.

Proof. If we set $x_i = 1 - y_i$, $y_i \in [\frac{1}{2}, 1)$, $i = 1, 2, \dots, n$, in Inequality 2.5 then we arrive to Inequality 2.7. \square

3. APPLICATIONS

In this section we present two applications of the pervious results. In [M] presented a great number of inequalities involving trigonometric functions. Here, we can be formulated another two inequalities which are the analogue of [A, Corollary 1].

Corollary 3.1.

(i) *For all $x_i \in (0, \frac{\pi}{4}]$, $i = 1, 2, \dots, n$, ($n \geq 2$), and $1 \leq a \leq \frac{3}{2}$, we have*

$$\frac{\prod_{i=1}^n (1 + \sin^2(x_i))}{\prod_{i=1}^n (1 + \cos^2(x_i))} \leq \frac{1 + a \sum_{i=1}^n \sin^2(x_i)}{1 + a \sum_{i=1}^n \cos^2(x_i)},$$

(ii) *For all $x_i \in [\frac{\pi}{4}, \frac{\pi}{2})$, $i = 1, 2, \dots, n$, ($n \geq 2$), and $1 \leq a \leq \frac{3}{2}$, we have*

$$\frac{\prod_{i=1}^n (1 + \sin^2(x_i))}{\prod_{i=1}^n (1 + \cos^2(x_i))} \geq \frac{1 + a \sum_{i=1}^n \sin^2(x_i)}{1 + a \sum_{i=1}^n \cos^2(x_i)},$$

Proof. If we replace x_i by $\sin^2(x_i)$ for all $i = 1, 2, \dots, n$, then the corollary follow immediately from Theorem 2.2 and Corollary 2.4. \square

For example, Corollary 3.1, for $a = 1$, yields [A, Corollary 1].

An another application for Theorem 2.3 and Corollary 2.4 we can present the following result.

Corollary 3.2.

(i) *For any integrable function $f : [0, n] \rightarrow (0, \frac{1}{2}]$ where $n \geq 2$, and $1 \leq a \leq \frac{3}{2}$, we have*

$$\frac{\prod_{i=1}^n \int_{i-1}^i (1 + f(t)) dt}{\prod_{i=1}^n \int_{i-1}^i (2 - f(t)) dt} \leq \frac{1 + a \int_0^n f(t) dt}{1 + a \int_0^n (1 - f(t)) dt},$$

(ii) *For any integrable function $f : [0, n] \rightarrow [\frac{1}{2}, 1)$ where $n \geq 2$, and $1 \leq a \leq \frac{3}{2}$, we have*

$$\frac{\prod_{i=1}^n \int_{i-1}^i (1 + f(t)) dt}{\prod_{i=1}^n \int_{i-1}^i (2 - f(t)) dt} \geq \frac{1 + a \int_0^n f(t) dt}{1 + a \int_0^n (1 - f(t)) dt},$$

Proof. If setting $x_i = \int_{i-1}^i f(t)dt$, then the corollary immediately follow from the Theorem 2.3 and Corollary 2.4. \square

For example, Corollary 3.2, for $a = 1$, yields [A, Corollary 2].

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