Total Curvature and Rearrangements

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Abstract

We study to what extent rearrangements preserve the integrability properties of higher order derivatives. It is well-known that the second order derivatives of the rearrangement of a smooth function is not necessarily in L^1. We obtain a substitute for this fact. This is done by showing that the total curvature for the graph of the rearrangement of a function is bounded by the total curvature for the graph of the function itself.

1 Introduction

The purpose of this note is to study the regularity properties of the decreasing rearrangement of a function. Let f be a real-valued, bounded and measurable function on an interval $I = [a, b]$. Its decreasing rearrangement f^* is characterised by the following properties:

(a) f^* is bounded and decreasing on I;

(b) f^* is right continuous on $[a, b)$ and left continuous at b;

(c) f^* and f are equimeasurable, i.e.,

$$|\{x \in I : f^*(x) > \lambda\}| = |\{x \in I : f(x) > \lambda\}|$$

for all $\lambda \in \mathbb{R}$.

Here $|E|$ denotes the Lebesgue measure of the measurable set E. We refer to Hardy, Littlewood and Polya [1] for the classical theory. The monograph by Polya and Szegö [3] contains a wealth of applications of rearrangements to symmetrization and isoperimetric inequalities.

We recall that

$$\int \varphi(f^*)dx = \int \varphi(f)dx \tag{1}$$

for all continuous functions φ. The basic regularity result for rearrangements is that if $1 \leq p \leq \infty$ and if the derivative of f belongs to $L^p(I)$, then f^* has the same property. More precisely,

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$$\left\| \frac{df^*}{dx} \right\|_p \le \left\| \frac{df}{dx} \right\|_p, \tag{2}$$

where $\|f\|_p = (\int |f|^p dx)^{1/p}$.

We shall in this paper study how rearrangements preserve the integrability properties of higher order derivatives. We remark that it is easy to give examples of smooth functions f such that $\frac{d^2 f^*}{dx^2}$ does not belong to L^1. For example, letting

$$\begin{aligned} f(x) &= 2x^3 - 9x^2 + 12x, \quad 0 \le x \le 3 \\ g(x) &= (8x^3 - 36x^2 + 30x + 153)/32 \end{aligned}$$

then (see Talenti [4])

$$f^*(x) = \begin{cases} f(3-x) & x \in [0, 1/2] \cup [5/2, 3] \\ g(x) & x \in [1/2, 5/2]. \end{cases}$$

Notice however, that in this case $\frac{df^*}{dx}$ is of bounded variation.

For f a bounded function on $I = [a, b]$ let

$$\|f\|_C = \sup\{ | \int f \varphi'' dx | : \varphi \in C_0^\infty(a, b) \quad \text{and} \quad \|\varphi\|_\infty \le 1 \}. \tag{3}$$

Here $C_0^\infty(a, b)$ denotes the class of infinitely many times continuously differentiable functions supported in (a, b). We remark that if f is smooth, then

$$\|f\|_C = \int_I |f''| dx.$$

We shall establish the following analogue of (2).

Theorem 1.1. *Suppose f is real-valued, bounded and measurable on $[a, b]$. Then*

$$\|f^*\|_C \le \|f\|_C. \tag{4}$$

We shall derive (4) by analysing the total curvature of the graphs of f and f^*, respectively.

Let $\gamma(t)$, $a \le t \le b$ be a simple curve in the plane and let $X = \{\xi_0, \dots \xi_M\}$ be a partition of $[a, b]$, i.e., $a = \xi_0 < \xi_1 < \dots < \xi_M = b$ and let $e_i = \frac{\gamma(\xi_{i+1}) - \gamma(\xi_i)}{|\gamma(\xi_{i+1}) - \gamma(\xi_i)|}$, $0 \le i \le M - 1$.

Set

$$\mathcal{B}(\gamma, X) = \sum_{i=1}^{M-1} \delta_i$$

where δ_i is the length of the shortest arc on $S^1 = \{p \in \mathbb{R}^2 : |p| = 1\}$ joining e_{i-1} and e_i. Finally, the total curvature of γ is

$$\mathcal{B}(\gamma) = \sup_X \mathcal{B}(\gamma, X), \tag{5}$$

where the supremum is taken over all partitions X of $[a, b]$. We refer to Milnor [2] for the basic properties of the total curvature of arcs. We remark that if γ is a smooth curve with curvature k, then it can be shown (Milnor [2]) that

$$\mathcal{B}(\gamma) = \int |k| ds, \tag{6}$$

where the integration is taken with respect to the arc length of γ. For $f : [a, b] \to \mathbb{R}$ continuous let $T(f)$ denote the total curvature of the graph of f.

Theorem 1.2. *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then*

$$T(f^*) \leq T(f). \tag{7}$$

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2 Preliminary Results

We shall from now on let $I = [a, b]$ be an interval. Let $C(I)$ be the class of continuous and real-valued functions on I. If $f \in C(I)$, then f^* denotes the decreasing rearrangement. Notice that $f^* \in C(I)$ also. For $x \in I$ let $S(x) = a + b - x$. Notice that S maps I onto itself. If $g(x) = f(S(x))$, then

$$g^* = f^* \tag{8}$$

If $h(x) = -f(x)$, then

$$h^*(x) = -f^*(S(x)). \tag{9}$$

Let $X = \{\xi_0, \ldots, \xi_N\}$ be a partition of I and let $\gamma : I \to \mathbb{R}^2$ be a simple polygon with nodes at ξ_i, i.e., $\gamma : I \to \mathbb{R}^2$ is continuous, one-to-one and its restriction to the intervals $[\xi_i, \xi_{i+1}]$ is linear for $0 \leq i \leq N - 1$. Then it is well known (see Milnor [2]) that

$$\mathcal{B}(\gamma) = \mathcal{B}(\gamma, X). \tag{10}$$

In particular, if f is piecewise linear with nodes at ξ_i, $0 \leq i \leq N$, we have

$$T(f) = \sum_{1}^{N-1} |\varphi_{i+1} - \varphi_i| \tag{11}$$

where $\varphi_i \in (-\pi/2, \pi/2)$ is defined by

$$\tan \varphi_i = \frac{f(\xi_i) - f(\xi_{i-1})}{\xi_i - \xi_{i-1}}. \tag{12}$$

For $E \subset \mathbb{R}^d$ we let Int (E) and ∂E denote the interior and the boundary of the set E. Let

$$\mathcal{D} = \{x \in \mathbb{R} : 0 \leq x \leq \pi/2\}$$

and define $\gamma : \mathcal{D}^2 \to \mathcal{D}$ by

$$\cot \gamma(x, y) = \cot x + \cot y \quad \text{if} \quad (x, y) \in \text{Int } (\mathcal{D}^2)$$

and

$$\gamma(x, y) = \text{Min } (x, y) \quad \text{if} \quad (x, y) \in \partial \mathcal{D}^2.$$

Then γ is continuous on $\mathcal{D}^2$.

Proposition 2.1. *The function γ has the following properties:*

(i) $\gamma(x,y)=\gamma(y,x)$ *for* $(x,y)\in\mathcal{D}^2$*;*

(ii) $\gamma(x,\pi/2)=x$ *for* $x\in\mathcal{D}$*;*

(iii) $0\le\gamma(x,y)\le Min\ (x,y)\le x$ *for* $(x,y)\in\mathcal{D}^2$*;*

(iv) $0<\dfrac{\partial\gamma(x,y)}{\partial x}<1$ *for* $(x,y)\in Int\ (\mathcal{D}^2)$.

(v) $\dfrac{\partial\gamma(x,y)}{\partial x}<\dfrac{\partial\gamma(x,z)}{\partial x}$ *if* $x\in Int\ (\mathcal{D})$ *and* $0<y<z<\pi/2$.

Proof. The first three properties are obvious from the definition of γ. The last follow from the identity

$$\frac{\partial\gamma(x,y)}{\partial x}=\frac{\cot^2 x+1}{\cot^2\gamma+1},\qquad (x,y)\in\text{Int }(\mathcal{D}^2),$$

which completes the proof of the proposition. □

The function γ will be used for computing the rearrangements of piecewise linear functions. The following lemma gives its basic role.

Lemma 2.1. *Let $I_1=(a_1,b_1)$ and $I_2=(a_2,b_2)$ be two disjoint, open and bounded intervals of positive length. Let I be an interval of length $|I_1|+|I_2|$. Set $E=I_1\cup I_2$ and assume $f:E\to\mathbb{R}$ has a linear restriction to the subintervals I_1 and I_2 with $f(I_1)=f(I_2)$. Let $(\alpha,\beta)\in Int\ (D^2)$ and assume $|f'|=\tan\alpha$ in I_1 and $|f'|=\tan\beta$ in I_2 and set $\gamma=\gamma(\alpha,\beta)$. Then there is a decreasing linear function $g:I\to\mathbb{R}$ such that*

$$g'=-\tan\gamma$$

and

$$|\{x\in I:g(x)>\lambda\}|=|\{x\in E:f(x)>\lambda\}| \tag{13}$$

for all $\lambda\in\mathbb{R}$.

Proof. Let $J=(A,B)$, $A<B$, the range of f, i.e.,

$$J=f(E)=f(I_1)=f(I_2).$$

We may assume $f(b_1)=B$, otherwise we replace f by $f(a_1+b_1-x)$ on I_1. Similarly, we may assume $f(a_2)=B$ so $f(a_1)=f(b_2)=A$.

There is also no loss in generality in assuming $a_1<b_1=a_2<b_2$ so that f is continuous in $E=(a_1,b_2)$. Elementary geometry shows that if g is the linear function on E with $g(a_1)=B$ and $g(b_2)=A$, then g satisfies (13) and $g'=-\tan\gamma$. The lemma is proved. □

We shall next show some inequalities involving the function γ. We first define $a_n:\mathcal{D}^n\times\mathcal{D}^n\to\mathbb{R}$ and $b_n:\mathcal{D}^n\times\mathcal{D}^n\to\mathbb{R}$ by $a_1(x,y)=\gamma(x,y)$, $b_1(x,y)=x+y$ if $x,y\in\mathcal{D}$.

If $n\ge2$ and $x,y\in\mathcal{D}^n$, we set

$$a_n(x,y)=\gamma(x_1,y_1)+\sum_1^{n-1}|\gamma(x_i,y_i)-\gamma(x_{i+1},y_{i+1})|,$$

$$b_n(x,y)=x_1+y_1+\sum_1^{n-1}(|x_i-x_{i+1}|+|y_i-y_{i+1}|)$$

We next define $\alpha_n, \beta_n : \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D} \to \mathbb{R}$ by

$$\alpha_1(x,y,t) = \gamma(x,y) + |t - \gamma(x,y)|, \qquad \beta_1(x,y,t) = x + y + |t - x|$$

for $x, y, t \in \mathcal{D}$. If $n \geq 2$ and if $x, y \in \mathcal{D}^n$, $t \in \mathcal{D}$, we set

$$\begin{aligned} \alpha_n(x,y,t) &= a_n(x,y) + |t - \gamma(x_n, y_n)|, \\ \beta_n(x,y,t) &= b_n(x,y) + |t - x_n|. \end{aligned}$$

We can now give some basic inequalities.

Proposition 2.2. *Let $n \geq 1$ and let $x, y \in \mathcal{D}^n$, $t \in \mathcal{D}$. Then*

$$a_n(x,y) \leq b_n(x,y), \tag{14}$$

$$\alpha_n(x,y,t) \leq \beta_n(x,y,t). \tag{15}$$

We shall base the proof of Proposition 2.2 on the following lemma.

Lemma 2.2. *Suppose $f : \mathcal{D} \to \mathbb{R}$ satisfies $0 \leq f' \leq 1$. Let $\theta \in \mathcal{D}$ and $A \in \mathbb{R}$ and set*

$$g(x) = x + |x - \theta| - f(x) - |f(x) - A|.$$

Then $g(x) \geq g(\theta)$ for all $x \in \mathcal{D}$.

Proof. Let $h(x) = f(x) + |f(x) - A|$. Clearly

$$0 \leq h' \leq 2 \quad \text{in} \quad \mathcal{D}.$$

If $0 \leq \theta < \pi/2$, we have that $g' = 2 - h' \geq 0$ in the interval $(\theta, \pi/2)$. If $0 < \theta \leq \pi/2$, we see that $g' = -h' \leq 0$ in $(0, \theta)$ so in all cases $g(x) \geq g(\theta)$. □

Proof of Proposition 2.2. We begin by verifying the case $n = 1$. If $x, y, t \in \mathcal{D}$, we have that

$$a_1(x,y) \leq x \leq x + y = b_1(x,y)$$

which establishes (14) in this case. If $t \geq \gamma(x,y)$, we have $\alpha_1(x,y,t) = t \leq x + |t - x| \leq \beta_1(x,y,t)$. If $0 \leq t \leq \gamma(x,y)$, we have $\alpha_1(x,y,t) = 2\gamma(x,y) - t \leq 2\gamma(x,y) \leq x + y \leq \beta_1(x,y,t)$ which establishes (15) when $n = 1$. Let now $n \geq 2$ and assume that (14) and (15) hold in the range $1, 2, \ldots, n-1$. For $x \in \mathbb{R}^n$ let $\hat{x} \in \mathbb{R}^{n-1}$ be the vector $(x_2, x_3, \ldots, x_n)$ and set $x^* = (x_2, \hat{x})$. Let $e_n = b_n - a_n$, $\epsilon_n = \beta_n - \alpha_n$. If $x, y \in \mathcal{D}^n$, $t \in \mathbb{R}$, it follows from Lemma 2.2 that

$$e_n(x,y) \geq e_n(x^*, y^*) = e_{n-1}(\hat{x}, \hat{y}) \geq 0.$$

Similarly $\epsilon_n(x,y,t) \geq \epsilon_n(x^*, y^*, t) = \epsilon_{n-1}(\hat{x}, \hat{y}, t) \geq 0$. Hence the proposition follows by induction. □

3 The Main Inequality

We shall in this section develop the main step in the proof of Theorem 1.2. We begin by defining $\Gamma : \mathcal{D}^3 \to \mathcal{D}$ by setting

$$\Gamma(x,y,z) = \gamma(x, \gamma(y,z)) \quad \text{for} \quad x, y, z \in \mathcal{D}.$$

Notice that if $(x, y, t) \in \text{Int } (\mathcal{D}^3)$, then

$$\cot \Gamma(x, y, z) = \cot x + \cot y + \cot z, \tag{16}$$

so Γ is a symmetric function. We shall now define $A_n, B_n : \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D}^n \to \mathbb{R}$ by setting $A_1(x, y, z) = x + z + 2\Gamma(x, y, z)$, $B_1(x, y, z) = x + 2y + z$. It is easily seen that

$$A_1 \leq B_1. \tag{17}$$

For $n \geq 2$ and $x, y, z \in \mathcal{D}^n$ we now set

$$A_n(x, y, z) = x_1 + \Gamma(\omega_1) + \sum_{i=1}^{n-1} |\Gamma(\omega_{i+1}) - \Gamma(\omega_i)| + \Gamma(\omega_n) + z_n,$$

$$B_n(x, y, z) = \sum_{i=1}^{n-1} (|x_{i+1} - x_i| + |y_{i+1} - y_i| + |z_{i+1} - z_i|) + x_n + y_1 + y_n + z_1.$$

Here $\omega_j = (x_j, y_j, z_j)$, $1 \leq j \leq n$.

We can now formulate the main result of this section.

Theorem 3.1. *Let $n \geq 1$ and suppose $\omega \in \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D}^n$. Then*

$$A_n(\omega) \leq B_n(\omega). \tag{18}$$

We will next introduce some notation. Let $U_n = \mathcal{D}^n \times \mathcal{D}^n \times \mathcal{D}^n$ and let

$$\Delta_n = B_n - A_n. \tag{19}$$

Put

$$\delta_n = \min_{U_n} \Delta_n \tag{20}$$

and let

$$D_n = \min\{\delta_1, \ldots, \delta_n\}.$$

From (17) follows

$$\delta_1 = D_1 = 0. \tag{21}$$

Also set

$$\Omega_n = \{\omega \in U_n : \Delta_n(\omega) = \delta_n\}$$

and notice $\Omega_n \neq \phi$ since Δ_n is continuous on U_n. For $\omega = (x, y, z) \in U_n$ and $1 \leq j \leq n$ let $\Gamma_j(\omega) = \Gamma(x_j, y_j, z_j)$.

Lemma 3.1. *Suppose $n \geq 2$, $\delta_n < 0$ and $D_{n-1} = 0$. Then n is odd and for all $\omega \in \Omega_n$*

$$\Gamma_{2j}(\omega) < Min\ (\Gamma_{2j-1}(\omega), \Gamma_{2j+1}(\omega)), \quad 2 \leq 2j < n. \tag{22}$$

$$\Gamma_1(\omega) > \Gamma_2(\omega), \quad \Gamma_n(\omega) > \Gamma_{n-1}(\omega) \tag{23}$$

$$\Gamma_{2j+1}(\omega) > Max\ (\Gamma_{2j}(\omega), \Gamma_{2j+2}(\omega)), \quad 2 \leq 2j < n - 2. \tag{24}$$

Proof. Let $\omega = (x, y, z) \in U_n,\ x, y, z \in \mathcal{D}^n$.

For $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ let $\hat{p} = (p_2, \ldots, p_n)$. Let $\hat{\omega} = (\hat{x}, \hat{y}, \hat{z}) \in U_{n-1}$. If $\Gamma_1(\omega) \le \Gamma_2(\omega)$, then using that $\Delta_{n-1}(\hat{\omega}) \ge D_{n-1} = 0$

$$\begin{aligned}\Delta_n(\omega) = \Delta_{n-1}(\hat{\omega}) + x_2 + |x_1 - x_2| - x_1 + y_1 + |y_1 - y_2| - y_2 \\ + z_1 - z_2 + |z_1 - z_2| \ge 0\end{aligned}$$

Similarly, if $\Gamma_n(\omega) \le \Gamma_{n-1}(\omega)$, then $\Delta_n(\omega) \ge 0$, which shows (23). Let now $1 < i < n$ and let $W = (X, Y, Z)$ where $X, Y, Z \in \mathcal{D}^{n-1}$

$$\begin{cases} X_j = x_j \\ Y_j = y_j \\ Z_j = z_j \end{cases}$$

for $1 \le j < i$ and

$$\begin{cases} X_j = x_{j+1} \\ Y_j = y_{j+1} \\ Z_j = z_{j+1} \end{cases}$$

for $i \le j \le n-1$. If $\Gamma_i(\omega)$ is between $\Gamma_{i-1}(\omega)$ and $\Gamma_{i+1}(\omega)$, then

$$\begin{aligned}\Delta_n(\omega) = \Delta_{n-1}(W) + |x_{i-1} - x_i| + |x_i - x_{i+1}| - |x_{i-1} - x_{i+1}| + \\ + |y_{i-1} - y_i| + |y_i - y_{i+1}| - |y_{i-1} - y_{i+1}| + \\ + |z_{i-1} - z_i| + |z_i - z_{i+1}| - |z_{i-1} - z_{i+1}| \ge 0.\end{aligned}$$

Using (23), we now see that (22) holds. Again using (23), we see that n must be odd. Finally (23) yields (24), which completes the proof of the lemma. □

For $f \in C(\mathcal{D})$ we let $m(f)$ denote the minimum of f on $\mathcal{D}$, i.e.,

$$m(f) = \min\{f(x) : x \in \mathcal{D}\}.$$

We shall now consider functions $g \in C(\mathcal{D})$ of the form

$$g(x) = |x - \alpha| + |x - \beta| - |f(x) - a| - |f(x) - b| + c, \tag{25}$$

where $\alpha, \beta \in \mathcal{D}$ and $a, b, c \in \mathbb{R}$. If (25) holds, we will say that g has the function f as its base. We say that $g \in \mathcal{M}_0$ if $g \in C(\mathcal{D})$ has the form (25) and

$$f(\xi) < \text{Min } (a, b)$$

whenever $g(\xi) = m(g)$. If

$$f(\xi) > \text{Max } (a, b)$$

whenever $g(\xi) = m(g)$ we will say that $g \in \mathcal{M}_1$.

For $\rho \in \mathbb{R}$ set $f_\rho(x) = \rho x$. Let Λ be the class of all $f \in C(\mathcal{D})$ such that f is continuously differentiable on Int $(\mathcal{D})$ with

$$0 < f' < 1 \quad \text{on} \quad \text{Int } (\mathcal{D}).$$

Lemma 3.2. *Suppose $g \in \mathcal{M}_1$ has f as its base function. Let $\xi \in \mathcal{D}$. If $f = f_0$, then $g(\xi) = m(g)$ if and only if $\xi \in [\alpha, \beta]$. If $f = f_1$, then $g(\xi) = m(g)$ if and only if Max $(\alpha, \beta) \le \xi \le \pi/2$. If $f \in \Lambda$, then $g(\xi) = m(g)$ if and only if $\xi =$ Max (α, β). Here the parameters α and β are defined by the relation (25).*

Proof. We may without loss of generality assume $\alpha \le \beta$ and set $h(x) = |x-\alpha| + |x-\beta|$.

If $f = f_0$, then $g = h + C$ for some constant C, which concludes the lemma in this case. Suppose now that $g(\xi) = m(g)$ and $f \in \Lambda \cup \{f_1\}$. Since f is increasing, we have for $x \ge \xi$ that

$$f(x) \ge f(\xi) > \text{Max } (a, b)$$

so from (25) follows that

$$g(x) = h(x) - 2f(x) + C, \qquad x \ge \xi.$$

Since f is strictly increasing and h is non-increasing on $(-\infty, \beta)$, we see that if ξ were less than β, then

$$g(\beta) < g(\xi),$$

which contradicts the definition of ξ. Hence $\xi \ge \beta$ if $f \in \Lambda \cup \{f_1\}$. If $x > \beta = \text{Max } (\alpha, \beta)$, then $h(x) = 2x - \alpha - \beta$. If now $f \in \Lambda$, then g is strictly increasing on $(\beta, \pi/2)$, so $g(\xi) = m(g)$ if and only if $\xi = \beta$ in this case. If $f = f_1$, then it is easily seen that $g(x) = g(\beta)$ for $x \ge \beta$ which completes the proof of the lemma. □

A straightforward modification of the proof of Lemma 3.2 yields the following result.

Lemma 3.3. *Suppose $g \in \mathcal{M}_0$ has f as its base function. Let $\xi \in \mathcal{D}$. If $f = f_0$, then $g(\xi) = m(g)$ if and only if $\xi \in [\alpha, \beta]$. If $f = f_1$, then $g(\xi) = m(g)$ if and only if $0 \le \xi \le$ Min (α, β). If $f \in \Lambda$, then $g(\xi) = m(g)$ if and only if $\xi =$ Min (α, β). Here the parameters α, β are defined by the relation (25).*

Let $V \subset \{1, 2, \ldots, n\}$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We define $q_V(x,t)$ as the point $y \in \mathbb{R}^n$ with $y_i = x_i$ for $i \notin V$ and $y_i = t$ when $i \in V$. If $\omega = (x, y, z) \in U_n$, we put $Q_V(\omega, t) = (q_V(x,t), y, z)$ and

$$E_n^{\omega,V}(t) = \Delta_n(Q_V(\omega, t)). \tag{26}$$

In the special case when $V = \{k\}$, $1 \le k \le n$, we will write $E_n^{\omega,V}$ as $E_n^{\omega,k}$. For $\omega = (x, y, z) \in U_n$ we set

$$\theta_i(\omega) = \gamma(y_i, z_i), \qquad \lambda_{i,\omega}(t) = \gamma(t, \theta_i(\omega)).$$

We observe that $E_n^{\omega,k}$ has $\lambda_{k,\omega}$ as its base function. We remark that if $\omega \in \Omega_n$, then under the conditions of Lemma 3.1 we have

$$E_n^{\omega,k} \in \mathcal{M}_1 \tag{27}$$

for k odd and

$$E_n^{\omega,k} \in \mathcal{M}_0 \tag{28}$$

for k even.

The following result is an immediate consequence of the previous two lemmas. The verification is left to the reader.

Lemma 3.4. *Suppose $n \geq 3$, $\delta_n < 0$ and $D_{n-1} = 0$. Assume $\omega = (x, y, z) \in \Omega_n$ and $1 < k < n$. If k is odd, then*

$$\delta_n = E_n^{\omega,k}(Max\ (x_{k-1}, x_{k+1}))$$

and if k is even, then

$$\delta_n = E_n^{\omega,k}(Min\ (x_{k-1}, x_{k+1})).$$

If $\theta_k(\omega) = 0$ and $1 < k < n$, then

$$\delta_n = E_n^{\omega,k}(t) \quad \text{for all} \quad t \in [x_{k-1}, x_{k+1}].$$

If $\theta_k(\omega) > 0$, then

$$x_k \geq Max\ (x_{k-1}, x_{k+1}) \quad \text{for } k \text{ odd}$$

and

$$x_k \leq Min\ (x_{k-1}, x_{k+1}) \quad \text{for } k \text{ even.}$$

We shall next analyse the function $E_n^{\omega,V}$

Lemma 3.5. *Suppose $n \geq 3$, $\delta_n < 0$ and $D_{n-1} = 0$. Assume $j \geq 1$ satisfies $2j < n$ and set $V = \{1, 2, \ldots, 2j\}$. Let $\xi \in \mathcal{D}$ and assume $\omega = (x, y, z) \in \Omega_n$ satisfies*

$$x_1 = x_2 = \ldots = x_{2j} = \xi.$$

If $\xi \leq x_{2j+1}$, then

$$\delta_n = E_n^{\omega,V}(x_{2j+1})$$

so $Q_V(\omega, x_{2j+1}) \in \Omega_n$.

Proof. We need only treat the case when $\xi < x_{2j+1}$. Setting $\theta_i = \theta_i(\omega)$ we see from Lemma 3.1 that

$$\gamma(\xi, \theta_{2k-1}) > \gamma(\xi, \theta_{2k}),\ 1 \leq k \leq j.$$

From Proposition 2.1 follows that for all $t \in \mathcal{D}$

$$\gamma(t, \theta_{2k-1}) \geq \gamma(t, \theta_{2k}), \qquad \frac{\partial \gamma(t, \theta_{2k-1})}{\partial t} \geq \frac{\partial \gamma(t, \theta_{2k})}{\partial t},$$

whenever $1 \leq k \leq j$.

Also $\gamma(\xi, \theta_{2j}) < \Gamma_{2j+1}(\omega)$. Letting

$$a = \sup\{u \in [\xi, x_{2j+1}] : \gamma(t, \theta_{2j}) \leq \Gamma_{2j+1}(\omega) \text{ for } \xi \leq t \leq u\}$$

we have that $\xi < a \leq x_{2j+1}$. If $t \in [\xi, a]$, then

$$E_n^{\omega,V}(t) = -2t + 2\sum_{k=1}^{j}(\gamma(t, \theta_{2k}) - \gamma(t, \theta_{2k-1})) + \Phi,$$

where Φ is independent of t. Hence $E_n^{\omega,V}$ is decreasing on $[\xi, a]$ so $\delta_n = E_n^{\omega,V}(a)$ and $Q_V(\omega, a) \in \Omega_n$. In particular, $\gamma(a, \theta_{2j}) < \Gamma_{2j+1}(\omega)$ so we cannot have $a \in (\xi, x_{2j+1})$, i.e, $a = x_{2j+1}$, which yields the Lemma. $\square$

Lemma 3.6. *Suppose $n \geq 3$, $\delta_n < 0$ and $D_{n-1} = 0$. Assume that $j \geq 1$ satisfies $2j < n$ and put $V = \{2j, 2j+1\}$. Assume $\omega = (x, y, z) \in \Omega_n$ satisfies*

$$x_{2j} = x_{2j+1} \leq x_{2j-1}.$$

Then

$$\delta_n = E_n^{\omega,V}(x_{2j-1})$$

so $Q_V(\omega, x_{2j-1}) \in \Omega_n$.

Proof. Put $\xi = x_{2j} = x_{2j+1}$. We need only treat the case when $\xi < x_{2j-1}$. Setting $\theta_i = \theta_i(\omega)$ we find from Lemma 3.1 that

$$\gamma(\xi, \theta_{2j}) < \gamma(\xi, \theta_{2j+1}), \qquad \gamma(\xi, \theta_{2j}) < \Gamma_{2j-1}(\omega).$$

so from Proposition 2.1 it follows that for all $t \in \mathcal{D}$

$$\gamma(t, \theta_{2j}) \leq \gamma(t, \theta_{2j+1}), \qquad \frac{\partial \gamma(t, \theta_{2j})}{\partial t} \leq \frac{\partial \gamma(t, \theta_{2j+1})}{\partial t}.$$

Suppose now that $2j + 1 = n$. Let
$a = \sup\{u \in [\xi, x_{2j-1}] : \gamma(t, \theta_{2j}) \leq \Gamma_{2j-1}(\omega) \text{ for all } t \in [\xi, u]\}$. If $t \in [\xi, a]$, then

$$E_n^{\omega,V}(t) = 2(\gamma(t, \theta_{2j}) - \gamma(t, \theta_{2j+1})) + \Phi,$$

where Φ is independent of t. Hence $E_n^{\omega,v}$ is decreasing on $[\xi, a]$ so $\delta_n = E_n^{\omega,V}(a)$ and $Q_V(\omega, a) \in \Omega_n$.

In particular, $\gamma(a, \theta_{2j}) < \Gamma_{2j-1}(\omega)$, so we cannot have $a \in (\xi, x_{2j-1})$, i.e., $a = x_{2j-1}$ which establishes the Lemma in this case.

We shall now treat the remaining case, so we assume now that $2j+1 < n$. In this case $\gamma(\xi, \theta_{2j+1}) > \Gamma_{2j+2}(\omega)$ so we now set
$b = \sup\{u \in [\xi, x_{2j-1}] : \gamma(t, \theta_{2j}) \leq \Gamma_{2j-1}(\omega) \text{ and } \gamma(t, \theta_{2j+1}) \geq \Gamma_{2j+2}(\omega)$ for all $t \in [\xi, u]\}$. If $t \in [\xi, b]$, then

$$E_n^{\omega,V}(t) = -t + |t - x_{2j+2}| + 2(\gamma(t, \theta_{2j}) - \gamma(t, \theta_{2j+1})) + \psi,$$

where ψ is independent of t. Hence $E_n^{\omega,V}$ is decreasing on $[\xi, b]$ so $\delta_n = E_n^{\omega,V}(b)$ and $Q_V(\omega, b) \in \Omega_n$. In particular, $\gamma(b, \theta_{2j}) < \Gamma_{2j-1}(\omega)$ and $\gamma(b, \theta_{2j+1}) > \Gamma_{2j+2}(\omega)$, so we cannot have $b \in (\xi, x_{2j-1})$, i.e., $b = x_{2j-1}$. This concludes the proof of the Lemma. □

The next lemma will provide the crucial part of the proof of Theorem 3.1. For $\xi \in \mathbb{R}$ we let $Q_n(\xi)$ denote the point in $\mathbb{R}^n$ with all components equal to ξ.

Lemma 3.7. *Suppose $n \geq 2$, $\delta_n < 0$ and $D_{n-1} = 0$. Assume $W = (X, Y, Z) \in \Omega_n$. Then there exists a $\xi \in \mathcal{D}$ such that $(Q_n(\xi), Y, Z) \in \Omega_n$.*

Proof. Let $\Omega_n(W) = \{\omega = (x, y, z) \in \Omega_n : y = Y, z = Z\}$ and notice $W \in \Omega_n(W)$. For $\omega = (x, y, z) \in U_n$ let $N(\omega)$ be the largest integer $p \in \{1, \ldots, n\}$ such that $x_i = x_1$ for $1 \leq i \leq p$. Set

$$N = \max\{N(\omega) : \omega \in \Omega_n(W)\}$$

and pick $\omega = (x, y, z) \in \Omega_n(W)$ such that $N = N(\omega)$. Assume that $N < n$. We shall show that this assumption leads to a contradiction. Note that n is odd by Lemma 3.1, so that $n \geq 3$.

Suppose first that $N = n-1$. From Lemma 3.2 follows that $\delta_n = E_n^{\omega,n}(x_N)$ so $\zeta = Q_n(\omega, x_N) \in \Omega_n(\omega)$ with $N(\zeta) = n$. This contradicts the definition of N.

Suppose next that $N < n-1$. Put $\theta_i = \gamma(y_i, z_i)$. From Lemma 3.4 follows that $\delta_n = E_n^{\omega,N+1}(x_N)$ if $\theta_{N+1} = 0$. Hence, if $\theta_{N+1} = 0$ we have $\zeta = Q_{N+1}(\omega, x_N) \in \Omega_n(W)$ with $N(\zeta) \geq N+1$. Again this contradicts the definition of N, so we must have $\theta_{N+1} > 0$.

We can therefore from now on assume $\theta_{N+1} > 0$ and $1 \leq N \leq n-2$. Also recall that n must be an odd integer.

We first treat the case N is even, say $N = 2j$. Since $N+1$ must be odd with $\theta_{N+1} > 0$ it follows from Lemma 3.4 that $x_{N+1} \geq x_N$. Setting $V = \{1, \ldots, N\}$ it follows from Lemma 3.5 that $\zeta = Q_V(\omega, x_{N+1}) \in \Omega_n(W)$. But $N(\zeta) \geq N+1$, which again leads to a contradiction.

It remains only to treat the case when N is odd and $\theta_{N+1} > 0$. Setting $\rho_N = \text{Min}\ (x_N, x_{N+2})$ it follows from Lemma 3.4 that $x_{N+1} \leq \rho_N \leq x_N$. Putting $\eta = Q_{N+1}(\omega, \rho_N)$, we also see from Lemma 3.4 that $\eta \in \Omega_n(W)$. If $\rho_N = x_N$ then $N(\eta) \geq N+1$, which is a contradiction. If $\rho_N < x_N$, then $\rho_N = x_{N+2}$ so if $\eta = (\xi, Y, Z)$, then $\xi_{N+1} = \xi_{N+2} = \rho_N < x_N$. Hence η fulfils the assumptions of Lemma 3.6. Setting $S = \{N+1, N+2\}$, we therefore have $q = Q_S(p, x_N) \in \Omega_n(W)$. But $N(q) \geq N+2$ which again contradicts the definition of N.

So in all cases the assumption $N < n$ is impossible, which yields the Lemma. □

We can now prove the main result of this section.

Proof of Theorem 3.1. Since $\Delta_n(0) = 0$, we see that $\delta_n \leq 0$ for all $n \geq 1$. Hence it is enough to show $D_n = 0$ for all $n \geq 1$. From (17) follows $\delta_1 = D_1 = 0$. We shall now proceed by induction.

Suppose $n \geq 2$ and

$$D_{n-1} = 0. \tag{29}$$

We shall prove $D_n = 0$. It is enough to show $\delta_n = 0$. We shall argue by contradiction, so assume

$$\delta_n < 0. \tag{30}$$

Define the mapping $\rho : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\rho(x) = (x_n, \ldots, x_1) \quad \text{for} \quad x = (x_1, \ldots, x_n).$$

For $\omega = (x, y, z) \in U_n$ we set

$$R(\omega) = (\rho(z), \rho(y), \rho(x)) \in U_n.$$

Since $\Delta_n(R(\omega)) = \Delta_n(\omega)$, we have that

$$R : \Omega_n \to \Omega_n.$$

From Lemma 3.7 follows the existence of $\xi \in \mathcal{D}$, $y, z \in \mathbb{R}^n$ such that if $x = Q_n(\xi)$, then $\omega = (x, y, z) \in \Omega_n$.

Since $\rho(x) = x$ in this case, we have that $R(\omega) = (\rho(z), \rho(y), x) \in \Omega_n$. Using Lemma 3.7 one more time, we see that there is an $\eta \in \mathcal{D}$ such that if $p = Q_n(\eta)$, then $V = (p, \rho(y), x) \in \Omega_n$. Hence $W = R(V) \in \Omega_n$. Since $W = (x, y, p)$, we see by setting $\theta = \gamma(\xi, \eta)$ that

$$\begin{aligned}\delta_n &= \Delta_n(W) = y_1 + y_n - (\gamma(y_1, \theta) + \gamma(y_n, \theta)) + \\ &+ \sum_1^{n-1} (|y_i - y_{i+1}| - |\gamma(y_i, \theta) - \gamma(y_{i+1}, \theta)|) \\ &\geq 0\end{aligned}$$

by Proposition 2.1. This contradicts the assumption (30) which completes the proof by induction. □

4 Total Curvature of Piecewise Linear Functions

Let $I = [a, b]$ be an interval and let $f \in C(I)$. We will say that f is unimodular if there exists a $c \in [a, b]$ such that the restrictions $f|[a, c]$ and $f|[c, b]$ are both monotone. We shall begin by showing that if f is unimodular and piecewise linear, then $T(f^*) \leq T(f)$.

Lemma 4.1. *Let $n \geq 1$ and assume*

$$x_n < x_{n-1} < \ldots < x_1 < x_0 \leq \xi_0 < \xi_1 < \ldots < \xi_{n-1} < \xi_n.$$

Put $a = x_n$, $b = \xi_n$. Suppose $y_0 > y_1 > \ldots > y_n$ and assume f is piecewise linear on $[a, b]$ with nodes $\{x_n, x_{n-1}, \ldots, x_0, \xi_0, \ldots, \xi_n\}$. Assume $f(x_i) = f(\xi_i) = y_i$, $0 \leq i \leq n$. Then

$$T(f^*) \leq T(f).$$

Proof. We define for $1 \leq i \leq n$ the angles $\alpha_i, \beta_i \in (0, \pi/2)$ by

$$\tan \alpha_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}, \ \tan \beta_i = \frac{y_{i-1} - y_i}{\xi_i - \xi_{i-1}}.$$

Notice that $f'(x) = \tan \alpha_i$ for $x \in (x_i, x_{i-1})$, and $f'(x) = -\tan \beta_i$ for $x \in (\xi_{i-1}, \xi_i)$. It is easily seen that

$$T(f) = \alpha_1 + \beta_1 + \sum_1^{n-1} (|\alpha_{i+1} - \alpha_i| + |\beta_{i+1} - \beta_i|).$$

Let $\epsilon = 1$ if $\xi_0 > x_0$ and zero otherwise. From Lemma 2.1 follows

$$T(f^*) = \epsilon \gamma(\alpha_1, \beta_1) + \sum_{i=1}^{n-1} |\gamma(\alpha_{i+1}, \beta_{i+1}) - \gamma(\alpha_i, \beta_i)|.$$

Hence the Lemma follows from Proposition 2.2. □

We will need the following variant of Lemma 4.1.

Lemma 4.2. *Let $m > n \geq 1$ and assume*

$$x_n < x_{n-1} < \ldots < x_1 < x_0 \leq \xi_0 < \xi_1 < \ldots < \xi_{m-1} < \xi_m.$$

Put $a = x_n$ and $b = \xi_m$. Suppose $y_0 > y_1 > \ldots > y_m$ and assume f is piecewise linear on $[a, b]$ with nodes $\{x_n, x_{n-1}, \ldots, x_0, \xi_0, \ldots, \xi_m\}$. Assume $f(x_i) = y_i$ for $0 \leq i \leq n$ and $f(\xi_i) = y_i$ for $0 \leq i \leq m$. Then

$$T(f^*) \leq T(f).$$

Proof. We define for $1 \le i \le n$ the angle $\alpha_i \in (0, \pi/2)$ by

$$\tan \alpha_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}.$$

For $1 \le i \le m$ we define $\beta_i \in (0, \pi/2)$ by

$$\tan \beta_i = \frac{y_{i-1} - y_i}{\xi_i - \xi_{i-1}}.$$

It is easily seen that

$$T(f) = \alpha_1 + \beta_1 + \sum_{i=1}^{n-1} (|\alpha_{i+1} - \alpha_i| + |\beta_{i+1} - \beta_i|) + |\beta_n - \beta_{n+1}| + T(g)$$

where $g = f|[\xi_n, b]$. Let $\epsilon = 1$ if $\xi_0 > x_0$ and zero otherwise. From Lemma 2.1 follows

$$T(f^*) = \epsilon\gamma(\alpha_1, \beta_1) + \sum_{1}^{n-1} (|\gamma(\alpha_{i+1}, \beta_{i+1}) - \gamma(\alpha_i, \beta_i)|) +$$
$$|\beta_{n+1} - \gamma(\alpha_n, \beta_n)| + T(g).$$

Hence the Lemma follows from Proposition 2.2. □

We can now analyse the total curvature of the rearrangement of a unimodular piecewise linear function.

Lemma 4.3. *Let $I = [a, b]$ be an interval. If $f \in C(I)$ is unimodular and piecewise linear, then*

$$T(f^*) \le T(f)$$

Proof. Let $c \in [a, b]$ be such that $f|[a, c]$ and $f|[c, b]$ are monotone. We may without loss of generality assume f is non-decreasing on $[a, c]$; otherwise we consider $-f$ instead. The result is trivial if f is also non-decreasing on $[c, b]$ so we may assume f is non-increasing on $[c, b]$. The result is also trivial if $f(c) \in \{f(a), f(b)\}$, so we will assume $f(c) > \text{Max }(f(a), f(b))$.

Put $x_0 = \inf\{x \in I : f(x) = f(c)\}$ and $\xi_0 = \sup\{x \in I : f(x) = f(c)\}$. Clearly $f(x) = f(c)$ for all $x \in [x_0, \xi_0]$. By approximation, it is enough to treat the case when f is strictly increasing on $[a, x_0]$ and f is strictly decreasing on $[\xi_0, b]$. Also, we may assume $f(b) \le f(a)$; otherwise we consider $g(x) = f(a + b - x)$. Set $M = \{x \in I : \ x \text{ is a node for } f\}$ and set $V = \{f(x) : x \in M\}$. Let $y_0 > \ldots > y_m$ be listing of the distinct numbers in V. For $1 \le i \le n$ let $x_i = \inf\{x \in I : f(x) = y_i\}$ and $\xi_i = \sup\{x \in I : f(x) = y_i\}$. For $n < i \le m$ let ξ_i be the unique solution of the equation $f(x) = y_i$, $x \in I$.

Clearly, f can be viewed as a piecewise linear function with nodes $\{x_n, \ldots, x_0,$
$\xi_0, \ldots, \xi_m\}$. If $m = n$ the lemma follows from Lemma 4.1. If $m > n$, then the lemma follows from Lemma 4.2. □

Let $I = [a, b]$ be an interval. We let $\mathcal{N}(I)$ denote the class of functions $f \in C(I)$ that satisfy the following two properties:

(i) There are two points $c_1, c_2 \in I$ such that $a < c_1 < c_2 < b$ and the restrictions $f|[a, c_1]$, $f|[c_1, c_2]$ and $f|[c_2, b]$ are all monotone.

(ii) Set $m = \text{Min}\,(f(a), f(b))$ and $M = \text{Max}\,(f(a), f(b))$. Then $m < f(x) < M$ for all $x \in (a, b)$.

We shall next establish the inequality $T(f^*) \le T(f)$ for the case when $f \in \mathcal{N}(I)$ and f is piecewise linear.

Lemma 4.4. *Let $n \ge 1$ and assume $x_0 < \ldots < x_n$, $\xi_n < \ldots < \xi_0$, $\eta_0 < \ldots < \eta_n$ and $y_0 > \ldots > y_n$. Assume $x_n \le \xi_n$, $\xi_0 \le \eta_0$, $a < x_0$ and $\eta_n < b$. Suppose $f \in C([a,b])$ is piecewise linear with nodes $\{a, x_0, \ldots, x_n, \xi_n, \ldots, \xi_0, \eta_0, \ldots, \eta_n, b\}$. Suppose furthermore that $f(a) > y_0$ and $f(b) < y_n$ and $y_i = f(x_i) = f(\xi_i) = f(\eta_i)$ for $0 \le i \le n$. Then*

$$T(f^*) \le T(f).$$

Proof. Let $y_{-1} = f(a)$, $x_{-1} = a$, $y_{n+1} = f(b)$, $\eta_{n+1} = b$ and define $a_i, b_i, c_i \in (0, \pi/2)$ by

$$\tan a_i = \frac{y_i - y_{i-1}}{x_{i-1} - x_i},\ \tan b_i = \frac{y_i - y_{i-1}}{\xi_i - \xi_{i-1}},\ \tan c_i = \frac{y_i - y_{i-1}}{\eta_{i-1} - \eta_i}.$$

It is easily seen that

$$\begin{aligned} T(f) = |a_1 - a_0| + \sum_1^{n-1} (|a_{i+1} - a_i| + |b_{i+1} - b_i| + |c_{i+1} - c_i|) + \\ + b_1 + c_1 + a_n + b_n + |c_{n+1} - c_n|. \end{aligned}$$

Let $\theta = |a_0 - \Gamma(a_1, b_1, c_1)|$ if $\xi_0 = \eta_0$ and $a_0 + \Gamma(a_1, b_1, c_1)$ otherwise. Let $\varphi = |c_{n+1} - \Gamma(a_n, b_n, c_n)|$ if $x_n = \xi_n$ and $c_{n+1} + \Gamma(a_n, b_n, c_n)$ otherwise. From the definition of Γ and Lemma 2.1 follows that

$$T(f^*) = \theta + \sum_1^{n-1} |\Gamma_{i+1} - \Gamma_i| + \varphi,$$

where $\Gamma_i = \Gamma(a_i, b_i, c_i)$, $1 \le i \le n$. We now set $\omega = (a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n) \in U_n$. We find by Theorem 3.1 that

$$\begin{aligned} T(f) - T(f^*) &\ge \Delta_n(\omega) + |a_0 - a_1| + |c_{n+1} - c_n| - a_0 + a_1 - c_{n+1} + c_n \\ &\ge \Delta_n(\omega) \ge 0 \end{aligned}$$

which establishes the Lemma. □

We can now study rearrangements of piecewise linear functions of the class $\mathcal{N}(I)$.

Lemma 4.5. *Let $I = [a, b]$ and suppose $f \in \mathcal{N}(I)$ is piecewise linear. Then*

$$T(f^*) \le T(f).$$

Proof. Let $a < c_1 < c_2 < b$ be such that f has a monotone restriction to each of the intervals $[a, c_1], [c_1, c_2]$ and $[c_2, b]$. We may assume f is non-increasing on $[a, c_1]$ since otherwise we consider $-f$.

We may also assume that the restriction of f is not monotone on any of the intervals $[a, c_2]$ of $[c_1, b]$ since otherwise f is unimodular and the result follows from Lemma 4.3. Hence f must be non-increasing on the intervals $[a, c_1]$ and $[c_2, b]$ and non-decreasing on $[c_1, c_2]$. Consequently,

$$f(b) < f(c_1) < f(c_2) < f(a).$$

Let $I_1 = [a, c_2]$ and $I_2 = [c_1, b]$. Put $A_k = \inf\{x \in I_k : f(x) = f(c_k)\}$ and $B_k = \sup\{x \in I_k : f(x) = f(c_k)\}$. Then

$$a < A_1 \le B_1 < A_2 \le B_2 < b.$$

By approximation it is enough to treat the case when f is strictly monotone on the intervals $[a, A_1], [B_1, A_2]$ and $[B_2, b]$. Let A_0 solve the equation $f(x) = f(c_2)$, $x \in [a, A_1]$ and let B_3 solve the equation $f(x) = f(c_1)$, $x \in [B_2, b]$. Let $R = \{\xi_0, \dots, \xi_m\}$ be the set of nodes of f and let $\hat{a} = \sup\{\xi \in R : \xi < A_0\}$, $\hat{b} = \inf\{\xi \in R : \xi > B_3\}$.

It is easy to see that possibly after introducing additional nodes, we have that $g = f|[\hat{a}, \hat{b}]$ satisfies the assumptions of Lemma 4.4. Let $f_1 = f|[a, A_0]$, $f_2 = f|[B_3, b]$. Then

$$T(f) = T(g) + T(f_1) + T(f_2)$$

and

$$T(f^*) = T(g^*) + T(f_1) + T(f_2)$$

which yields the Lemma. □

5 Proof of the Main Results

We shall in this section finish the proofs of our main results. We begin with the following lemma.

Lemma 5.1. *Let $I = [a, b]$ be an interval. If $f \in C(I)$ is piecewise linear, then*

$$T(f^*) \le T(f). \tag{31}$$

Proof. Let $n \ge 2$ be the number of nodes of f. The result is trivial if $n = 2$. If $n = 3$, the result follows from Lemma 4.3. We shall prove (31) by induction over the number of nodes of f.

We shall therefore assume that $n \ge 4$ and that (31) holds for all piecewise linear functions with less than n nodes.

Let $V = \{1, \dots, n\}$, $V^* = \{2, \dots, n-1\}$, and let $\xi_1 = a < \xi_2 < \dots < \xi_n = b$ be the nodes of f. Set $\eta_i = f(\xi_i)$, $m = \min\{\eta_i : i \in V\}$, $M = \max\{\eta_i : i \in V\}$, $m^* = \min\{\eta_i : i \in V^*\}$ and $M^* = \max\{\eta_i : i \in V^*\}$. We will first treat the case when $M^* = M$. Pick $j \in V^*$ such that $\eta_j = M = M^*$. Set $g_1 = f|[a, \xi_j]$, $g_2 = f|[\xi_j, b]$. Let G_1 be the increasing rearrangement of g_1, and put $G_2 = g_2^*$. Define $\theta, \varphi \in [0, \frac{\pi}{2})$ by

$$\tan\theta = f'(\xi_j-), \qquad \tan\varphi = -f'(\xi_j+). \tag{32}$$

Then

$$T(f) = T(g_1) + T(g_2) + \theta + \varphi.$$

Define $\theta^*, \varphi^* \in [0, \pi/2)$ by

$$\tan\theta^* = G_1'(\xi_j-), \qquad \tan\varphi^* = -G_2'(\xi_j+). \tag{33}$$

Set $G(x) = G_1(x)$ if $a \le x \le \xi_j$, $G_2(x)$ if $\xi_j \le x \le b$. Now

$$T(G) = T(G_1) + T(G_2) + \theta^* + \varphi^*.$$

By the induction assumption $T(G_1) \leq T(g_1), T(G_2) \leq T(g_2)$. Since $0 \leq \theta^* \leq \theta, 0 \leq \varphi^* \leq \varphi$, we find $T(G) \leq T(f)$. Because G and f are equimeasurable, $f^* = G^*$. Since G is unimodular, we have $T(f^*) = T(G^*) \leq T(G) \leq T(f)$, which establishes the induction step in this case.

If $m^* = m$, the previous reasoning applied to $-f$ shows again that $T(f^*) \leq T(f)$. We are now left with the case $m < m^* \leq M^* < M$. We may assume $f(\xi_n) < M^*$, since otherwise we consider $-f$. Pick $j \in V^*$ such that $\eta_j = M^* < M$. Set $g_1 = f|[a, \xi_j]$, $g_2 = f|[\xi_j, b]$, and let $\theta, \varphi \in [0, \pi/2)$ be defined by (32). Then

$$T(f) = T(g_1) + T(g_2) + \theta + \varphi.$$

Let $g(x) = f(x)$ for $a \leq x \leq \xi_j$, $g(x) = g_2^*(x)$ for $\xi_j \leq x \leq b$. Then g and f are equimeasurable so $f^* = g^*$. Furthermore, $g \in C(I)$ is piecewise linear. Let $\varphi^* \in [0, \pi/2)$ be defined by $\tan \varphi^* = -g'(\xi_j+)$. Since $0 \leq \varphi^* \leq \varphi$, we have from the induction assumption that

$$T(g) = T(g_1) + T(g_2^*) + \theta + \varphi^* \leq T(f). \tag{34}$$

Set $\mu = \text{Min}\ \{f(x) : x \in [a, \xi_j]\}$. Then $\mu \leq M^*$ and if $\mu = M^*$, we must have $f(x) = M^*$ for $\xi_2 \leq x \leq \xi_j$ and consequently g is decreasing on $[a, b]$. Hence, if $\mu = M^*$, we have $f^* = g$ so (31) follows from (34) in this case.

We suppose now that $\mu < M^*$ and pick k, $1 \leq k < j$ such that $\eta_k = \mu$. Put $h_1 = f|[a, \xi_k]$, $h_2 = f|[\xi_k, \xi_j]$. Let H_1 be the decreasing rearrangement of h_1, H_2 the increasing rearrangement of h_2. Define H by

$$H(x) = \begin{cases} H_1(x) & \text{for} \quad a \leq x \leq \xi_k \\ H_2(x) & \text{for} \quad \xi_k \leq x \leq \xi_j \\ g(x) & \text{for} \quad \xi_j \leq x \leq b. \end{cases}$$

Then H and f are equimeasurable, $H \in C(I)$ is piecewise linear and arguing as in the derivation of (34) one finds

$$T(H) \leq T(g) \leq T(f).$$

By the construction the function $H \in \mathcal{N}(I)$ so $T(f^*) = T(H^*) \leq T(H) \leq T(f)$. The proof of the induction step is complete, which establishes the Lemma. □

Proof of Theorem 1.2. Let $X = \{\xi_0, \ldots, \xi_n\}$, $n \geq 1$, be a partition of I. For $f \in C(I)$ let

$$T(f, X) = \mathcal{B}(\gamma, X),$$

where γ is the graph of f. Let $\theta_i \in (-\pi/2, \pi/2)$,

$$\tan \theta_i = \frac{f(\xi_i) - f(\xi_{i-1})}{\xi_i - \xi_{i-1}}, \quad 1 \leq i \leq n.$$

Then

$$T(f, X) = \sum_1^{n-1} |\theta_{i+1} - \theta_i|.$$

Notice that if $f_n \in C(I)$, $f_n \to f$ uniformly, then $T(f_n, X) \to T(f, X)$. Also $f_n^* \to f^*$ uniformly.

Pick $f_n \in C(I)$ such that $f_n \to f$ uniformly and f_n is a piecewise linear function for all n. Then $T(f_n) \leq T(f)$ so

$$T(f^*, X) = \lim_{n\to\infty} T(f_n^*, X) \leq \limsup_{n\to\infty} T(f_n^*) \leq T(f)$$

by Lemma 5.1. Since

$$T(f^*) = \sup T(f^*, X),$$

where X ranges overall partitions of I, we have proved the Theorem. □

Lemma 5.2. *Suppose $f \in C(I)$, $I = [a, b]$ is piecewise linear. Then*

$$\|f\|_C = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} T(\epsilon f).$$

Also

$$\|f^*\|_C \le \|f\|_C.$$

Proof. Let $a = \xi_0 < \xi_1 < \ldots < \xi_n = b$ be the nodes of f. Set

$$Q_i = \frac{f(\xi_i) - f(\xi_{i-1})}{\xi_i - \xi_{i-1}}, \quad 1 \le i \le n.$$

Now

$$\begin{aligned} \frac{1}{\epsilon} T(\epsilon f) &= \frac{1}{\epsilon} \sum_1^{n-1} |\arctan(\epsilon Q_{i+1}) - \arctan(\epsilon Q_i)| \\ &\to \sum_1^{n-1} |Q_{i+1} - Q_i| = \|f\|_C \end{aligned}$$

as $\epsilon \downarrow 0$. Since f^* is piecewise linear, the lemma follows from Theorem 1.2. □

We shall next prove Theorem 1.1 in the case of smooth functions. We will use the Green function

$$G(x, \xi) = \begin{cases} (1-\xi)x & x \le \xi \\ (1-x)\xi & x \ge \xi. \end{cases} \tag{35}$$

For a measure μ on $(0, 1)$ set

$$G\mu(x) = \int_0^1 G(x, \xi) d\mu(\xi).$$

Lemma 5.3. *Let $I = [a, b]$. Suppose f is twice continuously differentiable on I. Then*

$$\|f^*\|_C \le \|f\|_C.$$

Proof. By rescaling there is no loss in generality in assuming that $I = [0, 1]$. Let $h = f''$. Then

$$f(x) = (1 - x)f(0) + xf(1) - Gh(x)$$

and $\|f\|_C = \int_0^1 |h(x)| dx$.

Let $X = \{\xi_0, \ldots, \xi_n\}$, $0 = \xi_0 < \ldots < \xi_n = 1$ be a partition of I. Let $F = A_X(f)$ denote the piecewise linear function in I whose set of nodes equals X and $F(\xi_i) = f(\xi_i)$. We claim that

$$\|A_X(f)\|_C \le \|f\|_C. \tag{36}$$

If g is twice continuously differentiable with $g'' \geq 0$, then $G = A_X(g)$ is convex. Hence

$$\|A_X(g)\|_C = G'(1-) - G'(0+) = g'(p) - g'(q)$$

for some $p, q \in (0,1)$. Since $g'' \geq 0$, we have that $g'(p) - g'(q) \leq g'(1) - g'(0) = \int g'' dx = \|g\|_C$. Hence $\|A_X(g)\|_C \leq \|g\|_C$. Notice that we can write $f = f_1 - f_2$ where f_1, f_2 are both twice continuously differentiable, convex and

$$\|f\|_C = \|f_1\|_C + \|f_2\|_C.$$

Hence (36) is proved. By selecting a suitable sequence $X^{(m)}$ of partitions we conclude the existence of a sequence $\{f_m\}_1^\infty$ of piecewise linear functions in $C(I)$ such that $\|f_m\|_C \leq \|f\|_C$ and $f_m \to f$ uniformly. If $\varphi \in C_0^\infty(0,1)$ with $|\varphi| \leq 1$, then the previous lemma gives that

$$\begin{aligned} |\int \varphi'' f^* dx| &= \lim_{m\to\infty} |\int \varphi'' f_m^* dx| \\ &\leq \limsup_{m\to\infty} \|f_m^*\|_C \leq \|f\|_C. \end{aligned}$$

Hence $\|f^*\|_C \leq \|f\|_C$ which shows the Lemma. □

Proof of Theorem 1.1. By rescaling we may without loss of generality assume $I = [0,1]$. Suppose $f \in C(I)$ with $\|f\|_C < \infty$. Then there is a measure μ on $(0,1)$ such that

$$f(x) = (1-x)f(0) + xf(1) - G\mu(x), \; x \in [0,1]. \tag{37}$$

In addition $\|f\|_C$ equals the total variation of μ. Notice that G is defined for all $x, \xi \in \mathbb{R}$ by (35). From (37) follows that f can be extended to a function F on $\mathbb{R}$ such that $|\int \varphi'' F dx| \leq \|\varphi\|_\infty \|f\|_C$ whenever $\varphi \in C_0^\infty(\mathbb{R})$. Let $\varphi \in C_0^\infty(-1,1)$ be nonnegative with $\int \varphi dx = 1$. For $\epsilon > 0$ set

$$\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon}).$$

Let $F_\epsilon = F * \varphi_\epsilon$ be the convolution of F with φ_ϵ. Putting $f_\epsilon = F_\epsilon | I$, we have that

$$\|f_\epsilon\|_C \leq \|f\|_C.$$

and $f_\epsilon \to f$ uniformly on I. If $\varphi \in C_0^\infty(0,1)$ with $|\varphi| \leq 1$, then the last lemma implies that

$$|\int \varphi'' f^* dx| = \lim_{\epsilon \downarrow 0} |\int \varphi'' f_\epsilon^* dx| \leq \|f\|_C.$$

The Theorem is proved. □

We conclude with the following corollary.

Corollary 5.1. *Let $n \geq 3$ and let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Let $a^* \in \mathbb{R}^n$ be the decreasing rearrangement of a. For $1 \leq k \leq n-1$ set*

$$\begin{aligned} \delta_k &= a_{k+1} + a_{k-1} - 2a_k, \\ \delta_k^* &= a_{k+1}^* + a_{k-1}^* - 2a_k^*. \end{aligned}$$

Then

$$\sum_1^{n-1} |\delta_k^*| \leq \sum_1^{n-1} |\delta_k|$$

Proof. Let $I = [1, n]$ and let $f \in C(I)$ be piecewise linear with nodes $\{1, \ldots, n\}$ and $f(i) = a_i$. Then $\|f\|_C = \sum_1^{n-1} |\delta_k|$ and $\|f^*\|_C = \sum |\delta_k^*|$ so the result follows from Theorem 1.1. □

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