The Converse of the Four Vertex Theorem

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Abstract

We establish the converse to the four vertex theorem without the positivity condition.

1 Introduction

Let $\boldsymbol{T}$ denote the unit circle of the complex plane $\boldsymbol{C}$ and let $\gamma : \boldsymbol{T} \to \boldsymbol{C}$ be the parametrisation of a smooth, simple and closed curve. Here the smoothness condition means that γ is infinitely many times continuously differentiable. If $\kappa : \boldsymbol{T} \to \boldsymbol{R}$ denotes the curvature function of γ then the four vertex theorem asserts that if κ is not a constant then κ has at least four critical points $p_i \in \boldsymbol{T}$, $i = 1, \dots, 4$ ordered counter clockwise such that p_1, p_3 are local maxima and p_2, p_4 are local minima and that furthermore $\kappa(p_1) > 0, \kappa(p_3) > 0$ and

$$\max(\kappa(p_2), \kappa(p_4)) < \min(\kappa(p_1), \kappa(p_3)).$$

This result was apparently first proved for the case of closed convex curves by Mukhopadhyaya [4]. For proofs in the case of simple closed curves see Fog [1], Jackson [3] and Vietoris [5].

The converse of this was studied by Gluck [2], who proved that if κ is a smooth and strictly positive function satisfying the above four vertex property then κ is the curvature function of a smooth, simple and closed curve. The purpose of this note is to establish the converse to the four vertex theorem without the positivity condition.

Theorem 1.1. *Suppose $\kappa : \boldsymbol{T} \to \boldsymbol{R}$ is not a constant and assume that κ has at least four critical points ordered counter clockwise $p_i \in T$, $i = 1, \dots, 4$ such that p_1, p_3 are local maxima and p_2, p_4 are local minima with $\kappa(p_1) > 0, \kappa(p_3) > 0$ and $\max(\kappa(p_2), \kappa(p_4)) < \min(\kappa(p_1), \kappa(p_3))$. If in addition κ is smooth then κ is the curvature function of a smooth, simple and closed curve.*

We remark that a smooth function $K : \boldsymbol{T} \to \boldsymbol{R}$ represents the curvature of a smooth, simple and closed curve parametrised by the arc length if and

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only if the following three conditions are satisfied:

(1) $$\int_0^{2\pi} K\, ds = 2\pi$$

(2) $$\int_0^{2\pi} e^{i\alpha(s)}\, ds = 0$$

(3) $$\int_t^{\tau} e^{i\alpha(s)}\, ds \neq 0 \text{ if } 0 \leq t < \tau < 2\pi$$

Here $\alpha(s) = \int_0^s K(t)\, dt$ and the parametrisation of the associated curve γ is given by

$$\gamma_K(t) = \int_0^t e^{i\alpha(s)}\, ds.$$

The above three conditions all have a geometric interpretation. The first condition expresses that the curve γ_K has a well determined tangent at $s = 0$, the second condition expresses that γ_K is a closed curve. Finally, the third condition expresses that γ_K is simple, that is, without self intersections.

We will say that κ is a *non–normalised curvature function* if

(4) $$\begin{cases} I = & \int_0^{2\pi} \kappa\, dt \neq 0 \text{ and} \\ K = & \frac{2\pi}{I}\kappa \text{ satisfies (2) and (3).} \end{cases}$$

In order to prove the theorem it is enough to show the existence of a smooth homeomorphism $\varphi : \boldsymbol{T} \to \boldsymbol{T}$ such that $\kappa \circ \varphi$ is a non–normalised curvature function. For letting ψ denote the inverse of φ and setting $K = \frac{2\pi}{I}\kappa \circ \varphi$, $I = \int_0^{2\pi} \kappa \circ \varphi\, dt$, then the curve Γ parametrised by $\Gamma(t) = \frac{2\pi}{I}\gamma_K(\psi(t))$ has κ as its curvature function.

2 Preliminary Results

The construction of the homeomorphisms required for the proof of theorem (1.1) will be based on the following observations.

Proposition 2.1. *Let $E_j \subset \boldsymbol{T}$, $j = 1, \ldots, 4$ be non–empty, pairwise disjoint open intervals that are ordered counter clockwise on the unit circle with $\bigcup \overline{E_j} = \boldsymbol{T}$. Let a, b be two positive numbers with $a \neq b$ and define the function k by*

$$k = \begin{cases} a \text{ on } E_1 \cup E_3 \\ b \text{ on } E_2 \cup E_4 \end{cases}$$

Suppose furthermore that $\int_0^{2\pi} k\, ds = 2\pi$. Then k is the curvature function of a closed convex curve parametrised by the arc length if and only if $E_3 = \{-w : w \in E_1\}$ and $E_4 = \{-w : w \in E_2\}$.

Proof. We begin by establishing the necessity. Let γ be a curve of length 2π whose curvature is given by $k = k(s)$. Since k is strictly positive it is well known that γ is convex. Let $\kappa = \kappa(\vartheta)$ represent the curvature of γ

as a function of the angle ϑ the tangent forms with the positive x–axis. Then it is easily seen that

$$\int_0^{2\pi} e^{i\vartheta} \frac{1}{\kappa(\vartheta)} \, d\vartheta = 0. \tag{5}$$

Let the function $F = F(\vartheta)$ parametrise γ with respect to the angle ϑ the tangent forms with the x–axis. Set now $e_j = \{\vartheta : F(\vartheta) \in \gamma(E_j)\}$, $j = 1, \ldots, 4$. Then the e_j's are non–empty, pairwise disjoint open intervals in $\boldsymbol{T}$ that are ordered counter clockwise and for which $\bigcup \overline{e_j} = \boldsymbol{T}$. Then $\kappa = a$ on $U = e_1 \cup e_3$ and $\kappa = b$ on $V = e_2 \cup e_4$. Since $a \neq b$ and $\int_0^{2\pi} e^{i\vartheta} \, d\vartheta = 0$ it follows from (5) that

$$\int_U e^{i\vartheta} \, d\vartheta = \int_V e^{i\vartheta} \, d\vartheta = 0. \tag{6}$$

Denote by $2l_j$ the length of the interval e_j and let c_j denote its centre. From (6) follows that $e^{ic_1} \sin l_1 + e^{ic_3} \sin l_3 = 0$ and $e^{ic_2} \sin l_2 + e^{ic_4} \sin l_4 = 0$. Since $l_j > 0$ for $j = 1, \ldots, 4$ it is easily seen that $c_3 = c_1 + \pi$, $c_2 = c_4 + \pi$, $l_3 = l_1$ and $l_4 = l_2$ which yields the necessity part of the proposition. The sufficiency part is an easy consequence of (1), (2) and (3). □

Proposition 2.2. *For $\alpha \in \boldsymbol{C}$ with $|\alpha| < 1$ let g_α denote the restriction to $\boldsymbol{T}$ of the Möbius transformation*

$$g_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha} w}.$$

Let $E_j \subset \boldsymbol{T}$, $j = 1, \ldots, 4$ be non–empty, pairwise disjoint open intervals that are ordered counter clockwise on the unit circle with $\bigcup \overline{E_j} = \boldsymbol{T}$. Suppose $E_3 = \{-w : w \in E_1\}$ and $E_4 = \{-w : w \in E_2\}$. If $g_\alpha(E_1) = -g_\alpha(E_3)$ and $g_\alpha(E_2) = -g_\alpha(E_4)$ then $\alpha = 0$.

Proof. The proof will be carried out by a contradiction argument. We assume therefore that $\alpha \neq 0$. Let z, ζ denote the end points of the interval E_1. The assumptions on g_α imply that $g_\alpha(-z) = -g_\alpha(z)$ and $g_\alpha(-\zeta) = -g_\alpha(\zeta)$. A straightforward computation shows that therefore $\bar{\alpha} z^2 = \alpha$ and $\bar{\alpha} \zeta^2 = \alpha$. Under the assumption that $\alpha \neq 0$ follows therefore that $z^2 = \zeta^2$ which is impossible since E_1 is nonempty with length strictly less than π. This contradiction establishes the proposition. □

We will need an infinitesimal version of the above propositions.

Proposition 2.3. *For $\alpha \in \boldsymbol{C}$ with $|\alpha| < 1$ let g_α denote the restriction to $\boldsymbol{T}$ of the Möbius transformation*

$$g_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha} w}.$$

Let $f : \boldsymbol{T} \to \boldsymbol{R}$ denote the function defined by $f(e^{i\theta}) = 1$ whenever $|\theta - \frac{\pi}{2}| < \frac{\pi}{4}$ or $|\theta - \frac{3\pi}{2}| < \frac{\pi}{4}$ and zero elsewhere. Let a and b be two positive numbers such that $a \neq b$ and set $\kappa = a(1 - f) + bf$. Furthermore, let a and b be normalised so that $\int_0^{2\pi} \kappa \, dt = 2\pi$ and define $I_\alpha = \int_0^{2\pi} \kappa \circ g_\alpha$. Let A_α be defined by $A_\alpha(0) = 0$ and $A'_\alpha = \kappa \circ g_\alpha$. Suppose that α depends smoothly on a real parameter t such that $\alpha(0) = 0$. Letting $\dot{\alpha}$ denote the derivative with respect to the t–variable evaluated at $t = 0$ we have that $\dot{I} = 0$ and

$$\int_0^{2\pi} \dot{A}(t) e^{iA_0(t)} \, dt = z_0(\xi z_1 + \eta z_2).$$

Here ξ and η are defined by $\dot\alpha = \xi + i\eta$ and $z_0 = 2\sqrt{2}i(b-a)e^{i\pi w_0}$, $w_0 = (a+2b)/4$, $z_1 = (e^{-ib\frac{\pi}{2}} - 1)/b$ and $z_2 = (e^{ia\frac{\pi}{2}} - 1)/a$. Furthermore the vectors z_1 and z_2 are linearly independent over $\boldsymbol{R}$.

Proof. We begin by selecting a smooth branch of the argument for points in a neighbourhood of $\{e^{i\theta} : \theta_0 \le \theta \le \theta_1\}$, where $0 < \theta_0 < \theta_1 < 2\pi$. Let now G_α denote the argument of the inverse of g_α. It is easily seen that

$$\dot A(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{4} \\ (a-b)\dot G(\frac{\pi}{4}) & \text{if } \frac{\pi}{4} < t < \frac{3\pi}{4} \\ (a-b)(\dot G(\frac{\pi}{4}) - \dot G(\frac{3\pi}{4})) & \text{if } \frac{3\pi}{4} < t < \frac{5\pi}{4} \\ (a-b)(\dot G(\frac{\pi}{4}) - \dot G(\frac{3\pi}{4}) + \dot G(\frac{5\pi}{4})) & \text{if } \frac{5\pi}{4} < t < \frac{7\pi}{4} \\ (a-b)(\dot G(\frac{\pi}{4}) - \dot G(\frac{3\pi}{4}) + \dot G(\frac{5\pi}{4}) - \dot G(\frac{7\pi}{4})) & \text{if } \frac{7\pi}{4} < t < 2\pi \end{cases}.$$

Since the inverse of g_α is given by $g_{-\alpha}$ it is also easily seen that $\dot G(\theta) = 2Im(\dot\alpha e^{-i\theta})$, where $Im(w)$ denotes the imaginary part of w. Hence $\dot G(\theta + \pi) = -\dot G(t)$, so that

$$\dot A(t) = \begin{cases} 0 & \text{if } 0 < t < \frac{\pi}{4} \text{ or } \frac{7\pi}{4} < t < 2\pi \\ (a-b)\dot G(\frac{\pi}{4}) & \text{if } \frac{\pi}{4} < t < \frac{3\pi}{4} \\ (a-b)(\dot G(\frac{\pi}{4}) - \dot G(\frac{3\pi}{4})) & \text{if } \frac{3\pi}{4} < t < \frac{5\pi}{4} \\ (b-a)\dot G(\frac{3\pi}{4}) & \text{if } \frac{5\pi}{4} < t < \frac{7\pi}{4} \end{cases}.$$

In particular, we see that since $I_\alpha = A_\alpha(2\pi)$ we have that $\dot I = 0$. By using that $A_0(t+\pi) = \pi + A_0(t)$ and integrating it is easy to verify that the expression for $\dot A(t)$ holds. It remains to verify that the vectors z_1 and z_2 are linearly independent over $\boldsymbol{R}$. If not, then there would exist a real number $c \neq 0$ so that

$$c(1 - e^{-ib\frac{\pi}{2}}) = e^{ia\frac{\pi}{2}} - 1$$

The normalisation $\int_0^{2\pi} \kappa\, dt = 2\pi$ means that $a + b = 2$, so we find after some simplification that $c \sin b\frac{\pi}{4} = e^{(a+b)\frac{\pi}{4}} \sin a\frac{\pi}{4}$. Since $a, b \in (0,2)$ it follows that c must equal 0. This contradicts the assumption of linear dependence, which yields the proposition. □

3 Proof of the plane case

Let $f : \boldsymbol{T} \to \boldsymbol{R}$ denote the function defined by $f(e^{i\theta}) = 1$ whenever $|\theta - \frac{\pi}{2}| < \frac{\pi}{4}$ or $|\theta - \frac{3\pi}{2}| < \frac{\pi}{4}$ and zero elsewhere.

The proof of the theorem will be carried out in two steps. First one chooses a homeomorphism η of the circle such that for some positive numbers a, b we have that $\kappa^* = \kappa \circ \eta \approx a(1-f) + bf$ on $\boldsymbol{T}$. This step is an easy consequence of the four vertex condition. The second step consists in showing the existence of a complex number β with $|\beta| < 1$ such that if g_β denotes the fractional transformation

$$g_\beta(w) = \frac{w - \beta}{1 - \bar\beta w},\ w \in \boldsymbol{T}$$

then $K = \kappa^* \circ g_\beta$ satisfies (4).

We need some definitions in order to construct the first preliminary homeomorphism. For $j = 1, \ldots, 4$ let $\theta_j = (j-1)\frac{\pi}{2}$ and set $q_j = e^{i\theta_j}$, $A_j = \{e^{i\theta} : |\theta - \theta_j| < \frac{\pi}{4} - \epsilon\}$ for a small positive number ϵ.

By continuity there are points $r_j \in \boldsymbol{T},\ j = 1, \dots, 4$ also ordered counter clockwise and positive numbers a, b such that $0 < a < b$ and $\kappa(r_1) = \kappa(r_3) = a,\ \kappa(r_2) = \kappa(r_4) = b$. Pick intervals $B_j \subset \boldsymbol{T}, j = 1, \dots, 4$ such that $r_j \in B_j$ and $|\kappa(w) - \kappa(r_j)| < \epsilon$ for all $w \in B_j$. Let η be any smooth orientation preserving C^∞–diffeomorphism of the circle such that $\eta(A_j) \subset B_j$ for $j = 1, \dots, 4$. It is now easily seen that $\kappa \circ \eta = a(1 - f) + bf + e$, where e is bounded with $\int_0^{2\pi} |e| dt < C\epsilon$ and C independent of ϵ.

We now claim that if ϵ has been chosen small enough then there is an $\beta \in \boldsymbol{C},\ |\beta| \leq \frac{1}{2}$ such that $\kappa^* \circ g_\beta$ is a non–normalised curvature function. To see this set $k^* = \kappa^* \circ g_\beta,\ k = a(1 - f \circ g_\beta) + bf \circ g_\beta$ and let

$$I = \int_0^{2\pi} k\, dt, \qquad I^* = \int_0^{2\pi} k^*\, dt$$

$K^* = \frac{2\pi}{I^*} k^*$ and $K = \frac{2\pi}{I} k$.

If ϵ has been chosen small enough then clearly $I^* \neq 0$ whenever $|\beta| \leq \frac{1}{2}$. Set $F^*(\beta, \epsilon) = \int_0^{2\pi} e^{i\alpha^*(s)}\, ds$ and $F(\beta) = \int_0^{2\pi} e^{i\alpha(s)}\, ds$, where α, α^* are defined by $\alpha(0) = \alpha^*(0) = 0$ and $\alpha' = K,\ \alpha^{*\prime} = K^*$. It follows from propositions (2.1) and (2.2) that $F(\beta) \neq 0$ whenever $\beta \neq 0, |\beta| < 1$. Noticing that $F(0) = 0$ we see from proposition (2.3) that the restriction of F to a sufficiently small circle centred at 0 has non vanishing winding number around 0. By a standard topology argument and the fact that $\lim_{\epsilon \downarrow 0} F^*(\beta, \epsilon) = F(\beta)$ it follows that if ϵ_0 has been chosen sufficiently small then for all $\epsilon \in (0, \epsilon_0)$ there is a $\beta,\ |\beta| < \frac{1}{2}$ such that $F^*(\beta, \epsilon) = 0$. Since the curve $\gamma(t) = \int_0^t e^{i\alpha(u)}\, du$ is simple and closed and $|\alpha - \alpha^*| \leq C\epsilon$ it follows that $\int_t^\tau e^{i\alpha^*(s)}\, ds \neq 0$ whenever $0 < t < \tau < 2\pi$, which completes the proof of the theorem.

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