Discrete Approximation of the Curvature of Arcs

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Abstract

We study a (global) definition of curvature applicable to non-smooth curves, based on the relationship between the curvature vector and the variation of arc length.

1 Introduction

The purpose of this note is to first study the concept of curvature for piecewise linear arcs. Thereafter we consider the situation when a sequence $\{\gamma_\nu\}$ of such arcs approximates a smooth arc γ. We discuss the approximation properties of the curvatures of the arcs γ_ν to the curvature of the arc γ.

Let $I \subset \boldsymbol{R}$ be a closed interval $[a, b]$ and let $\gamma : I \to \boldsymbol{R}^d$ be a smooth arc. Letting $\frac{d}{ds}$ denote the differentiation with respect to arc length, we denote by $T(s) = \frac{d\gamma}{ds}$ and $\kappa(s) = \frac{dT}{ds}$, the unit tangent and the curvature vector of γ, respectively. If the dimension $d = 2$ we recall that if $T(s) = (\xi, \eta)$ then the principal unit normal to γ is given by $(-\eta, \xi)$. Since in this case $N(s)$ and $\kappa(s)$ are parallel, we can write $\kappa(s) = k(s)N(s)$, and the quantity $k(s)$ is called the curvature of γ. We also recall, that if the dimension $d \geq 3$ then the Euclidean norm of the curvature vector is the curvature of γ.

In order to motivate our definition of curvature for non–smooth curves, we recall the relationship between the curvature vector and the variation of arc length. Let $Q : I \to \boldsymbol{R}^d$ be a smooth vector valued function. Setting $\gamma_\epsilon = \gamma + \epsilon Q$ we see that γ_ϵ is a smooth arc for ϵ in a neighbourhood of 0. The arc length $L(\epsilon)$ of γ_ϵ is given by

$$L(\epsilon) = \int |\frac{d\gamma}{ds} + \epsilon \frac{dQ}{ds}|\, ds,$$

where $|w|$ denotes the Euclidean norm of a vector in $\boldsymbol{R}^d$. Again, L is a smooth function of ϵ in a neighbourhood of 0. Letting $\dot{L}$ denote the derivative of L with respect to ϵ evaluated for $\epsilon = 0$, we find that

$$\dot{L} = \int < \frac{d\gamma}{ds}, \frac{dQ}{ds} > ds$$

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so that if $Q(a) = Q(b) = 0$ we have that

$$\dot{L} = -\int < \kappa(s), Q(s) > \, ds. \tag{1}$$

Here $< v, w >$ denotes the scalar product of two vectors in $\boldsymbol{R}^d$.

Hence the curvature vector $\kappa(s)$ measures the variation of the arc length.

Let $\mathcal{P} = \{P_0, \ldots, P_M\} \subset \boldsymbol{R}^d$ be an ordered finite set of distinct points and let $\Gamma = \Gamma_{\mathcal{P}}$ be the corresponding piecewise linear arc connecting the nodes $P_0, \ldots, P_M$, in this order. Since Γ may be interpreted as closed or open, we shall mention explicitly what interpretation we are using. In any case, our convention is that $P_0 \neq P_M$. We will also use the convention that indices are counted modulo M+1, so that for instance P_{-1} corresponds to P_M.

We take the relation (1) as our starting point for the definition of a curvature vector for Γ. To this end, let $Q_0, \ldots, Q_M \in \boldsymbol{R}^d$ and $\epsilon \in \boldsymbol{R}$. We will use the convention that $Q_0 = Q_M = 0$ if Γ is open. Set $P_i^\epsilon = P_i + \epsilon Q_i$ and let Γ^ϵ be the corresponding curve. Letting $T_i = (P_i - P_{i-1})/|P_i - P_{i-1}|$ and

$$N_i = T_i - T_{i+1} \tag{2}$$

we see that if $L(\epsilon)$ denotes the arc length of Γ^ϵ then

$$\dot{L} = \Sigma < Q_i, N_i > . \tag{3}$$

This formula is valid when Γ is open, because $Q_0 = Q_M = 0$ in that case.

Let PL(Γ) denote the class of continuous, real valued and piecewise linear functions on γ with nodes at $P_0, \ldots, P_M$. We notice that we have two natural scalar products on PL(Γ). For $\alpha, \beta \in$ PL(Γ) they are defined by

$$(\alpha, \beta) = \int \alpha\beta \, ds \tag{4}$$

and

$$< \alpha, \beta > = \Sigma \alpha(P_i)\beta(P_i). \tag{5}$$

In order to find the analogue of (1) we next study the relationship between these two scalar products. To this end we introduce two operators $\mathcal{B}$ and $\mathcal{B}_0$. We let $\mathcal{B}$ be the symmetric and positive operator on PL(Γ) for which

$$(\alpha, \beta) = < \alpha, \mathcal{B}\beta > \tag{6}$$

for all $\alpha, \beta \in$ PL(Γ). Letting PL$_0$(Γ) be the class of functions $f \in$ PL(Γ) that vanish at the end points of Γ, we define $\mathcal{B}_0$ as the positive and symmetric operator on PL$_0$(Γ) for which (6) holds for all α, β in PL$_0$(Γ).

For $0 \leq \nu \leq M$ we now define $f_\nu \in$ PL(Γ) as the functions satisfying $f_\nu(P_\mu) = 1$ if $\nu = \mu$ and 0 otherwise. The operator $\mathcal{B}$ is represented in the basis $\{f_\nu\}_0^M$ by the $(M+1) \times (M+1)$–matrix $(B_{\mu\nu})$ where

$$B_{\mu\nu} = \int f_\mu f_\nu \, ds, \quad 0 \leq \mu, \nu \leq M \tag{7}$$

It is easily seen that $\int f_\mu f_\nu \, ds = 0$ if $|\mu - \nu| > 1$, $\int f_\mu f_\nu \, ds = |P_\mu - P_\nu|/6$ if $|\mu - \nu| = 1$ and $\int f_\nu^2 \, ds = (|P_\nu - P_{\nu-1}| + |P_\nu - P_{\nu+1}|)/3$. Setting $a_\nu = |P_\nu - P_{\nu-1}|/6$ we therefore have that the matrix $(B_{\mu\nu}),\ 0 \le \mu, \nu \le M$ is given by

$$\begin{pmatrix} 2(a_0 + a_M) & a_1 & . & . & . & . & a_0 \\ a_1 & 2(a_1 + a_2) & a_2 & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & a_{M-1} & 2(a_{M-1} + a_M) & a_M \\ a_0 & . & . & . & . & a_M & 2(a_0 + a_M) \end{pmatrix}$$

Since $\{f_\nu\}_1^{M-1}$ is a basis for $\mathrm{PL}_0(\Gamma)$ we have that the operator $\mathcal{B}_0$ is represented by the $(M-1) \times (M-1)$–matrix $(B^0_{\mu\nu}),\ 1 \le \mu, \nu \le M-1$, where again

$$B^0_{\mu\nu} = \int f_\mu f_\nu \, ds, \quad 1 \le \mu, \nu \le M-1. \tag{8}$$

This matrix has the form

$$\begin{pmatrix} 2(a_1 + a_2) & a_2 & . & . & . & . & . \\ a_2 & 2(a_2 + a_3) & a_3 & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & a_{M-2} & 2(a_{M-2} + a_{M-1}) & a_{M-1} \\ . & . & . & . & . & a_{M-1} & 2(a_{M-1} + a_M) \end{pmatrix}$$

We now denote the inverses of $\mathcal{B}$ and $\mathcal{B}_0$ by $\mathcal{E}$ and $\mathcal{E}_0$, respectively. We extend $\mathcal{E}$ and $\mathcal{E}_0$ to act on vector valued functions simply by letting them act on each component separately. We define $\mathcal{N}$ as the $\boldsymbol{R}^d$–valued piecewise linear function on Γ with knots at $P_0, \ldots, P_M$ for which

$$\mathcal{N}(P_i) = N_i, \quad 0 \le i \le M, \tag{9}$$

where N_i is defined by (2). We can now define the curvature vector for a closed polygonal curve.

Definition 1.1. *Let Γ be a closed polygon in $\boldsymbol{R}^d$ with nodes at $P_0, \ldots, P_M$ and let $\mathcal{N}$ be defined by (9). The curvature vector $\mathcal{K}$ of Γ is then defined by $\mathcal{K} = \mathcal{E}(\mathcal{N})$.*

For Γ a non-closed curve we define $\mathcal{N}_0$ as the $\boldsymbol{R}^d$–valued piecewise linear function on Γ with knots at $P_0, \ldots, P_M$ for which

$$\mathcal{N}_0(P_i) = \begin{cases} N_i & \text{if } 1 \le i \le M-1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

We can now define the curvature vector $\mathcal{K}_0$ for a non-closed polygon Γ.

Definition 1.2. *Let Γ be a non closed polygon in $\boldsymbol{R}^d$ with nodes at $P_0, \ldots, P_M$ and let $\mathcal{N}_0$ be defined by (10). The curvature vector $\mathcal{K}_0$ of Γ is then defined by $\mathcal{K}_0 = \mathcal{E}_0(\mathcal{N}_0)$.*

2 Approximation

We will next study the approximation properties of the curvature vectors for polygons. Let $I \subset \boldsymbol{R}$ be a closed interval $[a,b]$ and suppose $\gamma : I \to \boldsymbol{R}^d$ is a smooth arc which is at least twice continuously differentiable. A collection $\mathcal{P}$ of points $P_0, \ldots, P_M$ is called a partition of γ if there are $t_0, \ldots, t_M$ such that $a \le t_0 < t_1 < \ldots < t_M \le b$ and $P_i = \gamma(t_i)$. We say that the partition is closed if $t_0 = a$ and $t_M < b$ and we say that it is non-closed if $t_0 = a$ and $t_M = b$. We will of course declare the polygon Γ associated to $\mathcal{P}$, closed or non closed simultaneously with $\mathcal{P}$. If $\mathcal{P}$ is closed we set $|\mathcal{P}| = \max |P_i - P_{i-1}|,\ 0 \le i \le M$ and in the case when $\mathcal{P}$ is non-closed we put $|\mathcal{P}| = \max |P_i - P_{i-1}|,\ 1 \le i \le M$. We recall that we count indices modulo $M+1$.

We need to compare functions defined on Γ and γ, respectively. To this end we define the map $\mathcal{R} : I \to \Gamma$ as follows. If $t_i \le t \le t_{i+1}$ we can write $t = (1-\theta)t_i + \theta t_{i+1}$ and we set $\mathcal{R}(t) = (1-\theta)P_i + \theta P_{i+1}$. This defines $\mathcal{R}$ on I if $\mathcal{P}$ is non closed. If $\mathcal{P}$ is closed and $t_M < t \le b$ we write $t = (1-\theta)t_M + \theta b$ and we set $\mathcal{R}(t) = (1-\theta)P_M + \theta P_0$. If $f \in \mathrm{PL}(\Gamma)$ we denote by $f(t)$ the value of f at $\mathcal{R}(t)$.

We can now formulate our approximation result.

Theorem 2.1. *Let $I \subset \boldsymbol{R}$ be a closed interval $[a,b]$ and suppose $\gamma : I \to \boldsymbol{R}^d$ be a smooth arc which is at least twice continuously differentiable. Also assume that γ is parametrised by the arc length, i.e. $|\gamma'(t)| = 1$ for all $t \in I$. Let κ denote the curvature vector of γ.*

Suppose that $\mathcal{P} = \mathcal{P}^{(\mu)}$ is a sequence of non closed partitions of γ with curvature vectors $\mathcal{K}_0 = \mathcal{K}_0^{(\mu)}$. If $\lim |\mathcal{P}^{(\mu)}| = 0$ then for every compact set $F \subset (a,b)$ we have that

$$\limsup\{|\kappa(t) + \mathcal{K}_0^{(\mu)}(t)| : t \in F\} = 0.$$

If γ is a smooth and closed curve and the partitions $\mathcal{P} = \mathcal{P}^{(\mu)}$ are closed with $\lim |\mathcal{P}^{(\mu)}| = 0$ then

$$\limsup\{|\kappa(t) + \mathcal{K}_0^{(\mu)}(t)| : t \in I\} = 0.$$

We will need some preliminary results before giving the proof of the theorem. We begin by considering two matrices related to $\mathcal{B}$ and $\mathcal{B}_0$. Let $(B_{\mu\nu})$ and $(B^0_{\mu\nu})$ be the matrices defined by (7) and (8). We define the matrices $\beta = (\beta_{\mu\nu})$ and $\beta^0 = (\beta^0_{\mu\nu})$ by

$$\beta_{\mu\nu} = B_{\mu\nu}/B_{\mu\mu}, \quad 0 \le \mu,\nu \le M \tag{11}$$

and

$$\beta^0_{\mu\nu} = B^0_{\mu\nu}/B^0_{\mu\mu}, \quad 1 \le \mu,\nu \le M-1. \tag{12}$$

If $\xi \in \boldsymbol{R}^n$ is an n–dimensional vector we put $||\xi||_\infty = \max\{|\xi_i| : 1 \le i \le n\}$.

Lemma 2.1. *The matrix $\beta = (\beta_{\mu\nu})$ is invertible on $\boldsymbol{R}^{M+1}$ and*

$$||\xi||_\infty/2 \le ||\beta\xi||_\infty \le 3||\xi||_\infty/2. \tag{13}$$

Furthermore, there are universal constants a and b such that if $a = (a_{\mu\nu})$ denotes the inverse of β then

$$|a_{\mu\nu}| \le a e^{-b||\mu-\nu||}.$$

Here $||x||$ denotes the distance from the integer x to 0 modulo $M+1$, i.e. $||x|| = \inf\{|x+s(M+1)| : s \in \boldsymbol{Z}\}$.

Proof. We first observe that $\beta_{\mu\nu} \geq 0$ and $\Sigma_\mu \beta_{\mu\nu} = 3/2$, which yields the right hand side of (13). Let the matrix $\alpha = (\alpha_{\mu\nu})$ be defined by $\alpha_{\mu\mu} = 0$ and $\alpha_{\mu\nu} = \beta_{\mu\nu}$ if $\mu \neq \nu$. Since $\Sigma_\mu \alpha_{\mu\nu} = 1/2$ it follows that $||\alpha\xi||_\infty \leq ||\xi||_\infty/2$. Since $\beta\xi = \xi + \alpha\xi$ it follows from the triangle inequality that $||\beta\xi||_\infty \geq ||\xi||_\infty - ||\alpha\xi||_\infty \geq ||\xi||_\infty/2$. Hence the invertability follows, together with the left hand side of (13).

Define the sequence $\{r_i\}$ by $r_0 = \beta_{0M}$ and $r_i = \beta_{ii-1}$ for $1 \leq i \leq M$. Extend next $\{r_i\}$ to be $(M+1)$–periodic on the integers $\boldsymbol{Z}$ and set $r_i^* = 1/2 - r_i$. Let $\Lambda = (\lambda_{\mu\nu})$ be the infinite matrix defined by $\lambda_{\mu\mu-1} = r_\mu$, $\lambda_{\mu\mu+1} = r_\mu^*$ and $\lambda_{\mu\nu} = 0$ otherwise. Let Λ^N be the N–th power of Λ and denote by $(\lambda_{\mu\nu}^{(N)})$ the coefficients of Λ^N. Since the coefficients of Λ are non negative and $\Sigma_\mu \lambda_{\mu\nu} = 1/2$ it follows that $e_{N+1} \leq e_N$ where $e_N = \sup\{\lambda_{\mu\nu} : \mu, \nu \in \boldsymbol{Z}\}$. Using that $e_1 \leq 1/2$ it therefore follows by induction that

$$0 \leq \lambda_{\mu\nu} \leq 2^{-N-1}. \tag{14}$$

We now claim that

$$\lambda_{\mu\nu} \leq 2^{-|\mu-\nu|}. \tag{15}$$

The inequality is obviously true for $N = 1$ and we proceed by induction. Clearly (14) verifies (15) when $\mu = \nu$. From the definition of Λ follows that

$$\lambda_{\mu\nu}^{(N+1)} \leq r_\mu \lambda_{\mu-1\nu}^{(N)} + r_\mu^* \lambda_{\mu+1\nu}^{(N)}.$$

Assuming that (15) holds for N we see that if $\mu > \nu$ then $\lambda_{\mu\nu}^{(N+1)} \leq 2^{-|\mu-\nu|}(2r_\mu + r_\mu^*/2) = 2^{-|\mu-\nu|}(3r_\mu/2 + 1/4)$ and since $0 \leq r_\mu \leq 1/2$ we see that (15) holds in this case. If $\mu < \nu$ we see that $\lambda_{\mu\nu}^{(N+1)} \leq 2^{-|\mu-\nu|}(r_\mu/2 + 2r_\mu^*) = 2^{-|\mu-\nu|}(1 - 3r_\mu/2)$ and using that $r_\mu \geq 0$ we see that (15) holds also in this case, which completes the induction argument. Multiplying (14) and (15) and taking the square root gives that $\lambda_{\mu\nu}^{(N)} \leq 2^{-(|\mu-\nu|+N+1)/2}$. Let $\mathcal{I}$ denote the identity operator on $\boldsymbol{Z}$ and let $\mathcal{A}$ denote the inverse of $\mathcal{I} + \Lambda$. Since we have that $\mathcal{A}$ is given by the Neumann series

$$\mathcal{A} = \mathcal{I} + \Sigma_{N=1}^\infty (-1)^N \Lambda^N$$

we have that the coefficients $a_{\mu\nu}$ of $\mathcal{A}$ satisfy the bound $|a_{\mu\nu}| \leq c2^{-|\mu-\nu|/2}$ for some universal constant c. If now $0 \leq \mu, \nu \leq M+1$ we have that $\alpha_{\mu\nu} = \Sigma_s a_{(s(M+1)+\mu)\nu}$, where the summation is taken over all integers. With $b = \log\sqrt{2}$ we therefore have that $|\alpha_{\mu\nu}| \leq c(3e^{-b||\mu-\nu||} + 2\Sigma_{s\geq 2} e^{b(|\mu-\nu|-s(M+1)})$. Adding, we therefore see that there is a constant $c_1 > 0$ such that $|\alpha_{\mu\nu}| \leq c(3e^{-b||\mu-\nu||} + c_1 e^{-b(M+1)})$. Since $||\mu-\nu|| < M+1$ we see that (13) follows, which completes the proof of the lemma. ☐

Lemma 2.2. *The matrix $\beta^0 = (\beta_{\mu\nu}^0)$ is invertible on $\boldsymbol{R}^{M-1}$ and*

$$||\xi||_\infty/2 \leq ||\beta^0\xi||_\infty \leq 3||\xi||_\infty/2. \tag{16}$$

Furthermore, there are universal constants b and c such that if $a^0 = (a_{\mu\nu}^0)$ denotes the inverse of β^0 then

$$|a_{\mu\nu}^0| \leq ce^{-b|\mu-\nu|}.$$

Proof. Let the matrix $\alpha = (\alpha_{\mu\nu})$ be defined by $\alpha_{\mu\mu} = 0$ and $\alpha_{\mu\nu} = \beta^0_{\mu\nu}$ if $\mu \neq \nu$. Since $\Sigma_\mu \alpha_{\mu\nu} \leq 1/2$ it follows that $||\alpha\xi||_\infty \leq ||\xi||_\infty/2$. Since $\beta\xi = \xi + \alpha\xi$ it follows from the triangle inequality that $||\beta\xi||_\infty \geq ||\xi||_\infty - ||\alpha\xi||_\infty \geq ||\xi||_\infty/2$ and $||\beta\xi||_\infty \leq ||\xi||_\infty + ||\alpha\xi||_\infty \leq 3||\xi||_\infty/2$, which establishes (16). The arguments leading to (14) and (15) can be used for the matrix α to show that

$$0 \leq \alpha^{(N)}_{\mu\nu} \leq \min(2^{-(N+1)}, 2^{-|\mu-\nu|}) \leq 2^{-(|\mu-\nu|+N+1)/2}),$$

where $\alpha^{(N)}_{\mu\nu}$ denotes the coefficients of the N–th power α^N of α. We have the Neumann representation

$$a^0 = \mathcal{I} + \Sigma_1^\infty (-1)^N \alpha^N,$$

so $|a^0_{\mu\nu}| \leq 2^{-|\mu-\nu|/2}(1 + \Sigma_1^\infty 2^{-(N+1)/2})$, which completes the proof of the lemma. □

We will need the following elementary estimates for the matrices β and β^0.

Lemma 2.3. *Let β and β^0 be as above. If $\{\xi\}_0^M$ is an $(M+1)$–dimensional vector with $\omega = \max\{|\xi_i - \xi_{i-1}| : 0 \leq i \leq M\}$ then*

$$||\beta\xi - 3\xi/2||_\infty \leq \omega/2.$$

If $\{\xi\}_1^{M-1}$ is an $(M-1)$–dimensional vector with $\omega = \max\{|\xi_i - \xi_{i-1}| : 2 \leq i \leq M-1\}$ and $\eta = \beta^0\xi$ then

$$(17) \qquad |\eta_i - 3\xi_i/2| \leq \begin{cases} |\xi_1|/2 + \omega/2 & \text{if } i = 1, \\ \omega/2 & \text{if } 2 \leq i \leq M-2, \\ |\xi_{M-1}|/2 + \omega/2 & \text{if } i = M-1. \end{cases}$$

Proof. Letting $\eta = \beta\xi$ we have in the closed case that $|\eta_i - 3\xi_i/2| = |\beta_{ii-1}(\xi_{i-1} - \xi_i) + \beta_{ii+1}(\xi_{i+1} - \xi_i)| \leq \omega/2$ since the sum of the off-diagonal elements of β equals $1/2$ for each row. The same argument also gives the middle part of the inequality for β^0. Furthermore, $|\eta_1 - 3\xi_1/2| = |\beta^0_{12}(\xi_2 - \xi_1) + (1/2 - \beta^0_{12})\xi_1| \leq \omega/2 + |\xi_1|/2$ and $|\eta_{M-1} - 3\xi_{M-1}| = |\beta^0_{M-2M-12}(\xi_{M-2} - \xi_{M-1}) + (1/2 - \beta^0_{12})\xi_{M-1}| \leq \omega/2 + |\xi_{M-1}|/2$. □

We can now give the proof of the theorem.

Proof of Theorem (2.1). We begin by considering N_i as defined by (2). Let $\Omega(r) = max|\gamma''(t) - \gamma''(u)|$, where the maximum is taken over all $t, u \in [a,b]$ for which $|u - t| \leq r$. Also, let $m = \max\{|\gamma''(t)| : t \in [a,b]\}$ and set $\omega(r) = \Omega(r) + (1+m)r$. We have from Taylor's formula that

$$(18) \qquad \gamma(t+h) - \gamma(t) = h\gamma'(t) + h^2\gamma''(t)/2 + h^2 q$$

where $|q| \leq \Omega(|h|)$. Since we have assumed that $|\gamma'(t)| = 1$ we have that $< \gamma'(t), \gamma''(t) >= 0$. Hence

$$(19) \qquad |\gamma(t+h) - \gamma(t)| = |h|(1 + he_1)$$

where $|e_1| \leq C\omega(|h|)$ for some constant C. Consequently, we have that

$$(20) \qquad \frac{\gamma(t+h) - \gamma(t)}{|\gamma(t+h) - \gamma(t)|} = \text{sign}(h)\gamma'(t) + |h|\gamma''(t)/2 + he_2$$

where $|e_2| \leq C\omega(|h|)$ for some constant C. Here $\text{sign}(h)$ equals 1 if $h > 0$ and -1 otherwise.

We will now consider the case when the polygon $\{P_i = \gamma(t_i)\}$ is closed. We set $\delta_i = t_{i+1} - t_{i-1}$ for $1 \leq i \leq M-1$, $\delta_0 = t_1 - t_0 + b - t_M$ and $\delta_M = b - t_{M-1}$. Using the above, we see that $N_i = -\delta_i \gamma''(t_i) + \delta_i e_3$, where $|e_3| \leq C\omega(\delta_i)$. Since we have by (7) that $B_{ii} = (|P_i - P_{i-1}| + |P_{i+1} - P_i|)/3$ it follows from (19) that

$$\delta_i / B_{ii} = 3 + E, \tag{21}$$

Letting $\mathcal{K}$ denote the the curvature vector we therefore find that $\beta\mathcal{K} = -3\xi/2 + e_3$, where $\xi_i = \gamma''(t_i)$ and $|e_3| \leq C\omega(|\mathcal{P}|$. From Lemma (2.3) follows that $||\beta\xi - 3\xi/2||_\infty \leq \Omega(|\mathcal{P}|)$. Using Lemma (2.1) we therefore have that $||\mathcal{K} + \xi||_\infty \leq C\omega(|\mathcal{P}|)$, which proves the theorem in the closed case.

We briefly outline the modifications of the above argument for the non-closed case. The only difference is the analysis at the endpoints. Proceeding as above one finds that $\beta^0(\mathcal{K}^0 + \xi) = e_4 + f$, where $||e_4||_\infty \leq C\omega(|\mathcal{P}|)$ and $||f||_\infty \leq Cm$, where $m = \max\{|\gamma''(t)| : t \in [a,b]\}$. Furthermore, $f_i = 0$ whenever $2 \leq i \leq M-2$. If a denotes the inverse of β^0 we have that if $\eta = af$ and $2 \leq i \leq M-2$ then by Lemma (2.2), $|\eta_i| \leq Cm(e^{-b|i-1|} + e^{-b|i+1-M|})$. Let now $a < a' < b' < b$ and set $\delta = \min(a'-a, b-b')$. It is easily seen that if $t_i \in [a', b']$ then $|i-1||\mathcal{P}| \geq \delta$ and $|i+1-M||\mathcal{P}| \geq \delta$, so that $|\eta_i| \leq Ce^{-b\delta/|\mathcal{P}|}$. Hence $\max\{|\mathcal{K}_i^0 + \xi_i| : t_i \in [a', b']\} \leq C(\omega(|\mathcal{P}|) + e^{-b\delta/|\mathcal{P}|}$, which yields the theorem in the non-closed case. □

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