COUNTING OCCURRENCES OF 132 IN AN EVEN PERMUTATION

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Abstract

We study the generating function for the number of even (or odd) permutations on n letters containing exactly $r \geq 0$ occurrences of 132. It is shown that finding this function for a given r amounts to a routine check of all permutations in S_{2r}.

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1. Introduction

Let $[n] = \{1, 2, \dots, n\}$ and $\mathfrak{S}_n$ denote the set of all permutations of $[n]$. We shall view permutations in $\mathfrak{S}_n$ as words with n distinct letters in $[n]$. A pattern is a permutation $\sigma \in \mathfrak{S}_k$, and an occurrence of σ in a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is a subsequence of π that is order equivalent to σ. For example, an occurrence of 132 is a subsequence $\pi_i\pi_j\pi_k$ $(1 \leq i < j < k \leq n)$ of π such that $\pi_i < \pi_k < \pi_j$. We denote by $\tau(\pi)$ the number of occurrences of τ in π, and we denote by $s_\sigma^r(n)$ the number of permutations $\pi \in \mathfrak{S}_n$ such that $\sigma(\pi) = r$.

In the last decade much attention has been paid to the problem of finding the numbers $s_\sigma^r(n)$ for a fixed $r \geq 0$ and a given pattern τ (see [1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 16, 17, 18, 19]). Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to patterns of length 3. Using two simple involutions (*reverse* and *complement*) on $\mathfrak{S}_n$ it is immediate that with respect to being equidistributed, the six patterns of length three fall into the two classes $\{123, 321\}$ and $\{132, 213, 231, 312\}$. Noonan [13] proved that

$$s_{123}^1(n) = \frac{3}{n}\binom{2n}{n-3}.$$

A general approach to the problem was suggested by Noonan and Zeilberger [14]; they gave another proof of Noonan's result, and conjectured that

$$s_{123}^2(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$$

and

$$s_{132}^1(n) = \binom{2n-3}{n-3}.$$

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The first conjecture was proved by Fulmek [8] and the latter conjecture was proved by Bóna in [6]. A conjecture of Noonan and Zeilberger states that $s_\sigma^r(n)$ is P-recursive in n for any r and τ. It was proved by Bóna [4] for $\sigma = 132$. Mansour and Vainshtein [11] suggested a new approach to this problem in the case $\sigma = 132$, which allows one to get an explicit expression for $s_{132}^r(n)$ for any given r. More precisely, they presented an algorithm that computes the generating function $\sum_{n\geq 0} s_{132}^r(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group $\mathfrak{S}_{2r}$. The algorithm has been implemented in C, and yields explicit results for $1 \leq r \leq 6$.

Let π be any permutation. The number of *inversions* of π is given by $i_\pi = |\{(i,j) : \pi_i > \pi_j,\ i < j\}|$. The *signature* of π is given by $\mathrm{sign}(\pi) = (-1)^{i_\pi}$. We say π is an *even permutation* [respectively; *odd permutation*] if $\mathrm{sign}(\pi) = 1$ [respectively; $\mathrm{sign}(\pi) = -1$]. We denote by E_n [respectively; O_n] the set of all even [respectively; odd] permutations in $\mathfrak{S}_n$. Clearly, $|E_n| = |O_n| = \frac{1}{2}n!$ for all $n \geq 2$. The following lemma holds immediately by definitions.

Lemma 1.1. *For any permutation* π, $\mathrm{sign}(\pi) = (-1)^{21(\pi)}$.

We denote by $e_\sigma^r(n)$ [respectively; $o_\sigma^r(n)$] the number of even [respectively; odd] permutations $\pi \in E_n$ [respectively; $\pi \in O_n$] such that $\sigma(\pi) = r$.

Apparently, for the first time the relation between even (odd) permutations and pattern avoidance problem was suggested by Simion and Schmidet in [16] for $\sigma \in \mathfrak{S}_3$. In particularly, Simion and Schmidt [16] proved that

$$e_{132}^0(n) = \frac{1}{2(n+1)}\binom{2n}{n} + \frac{1}{n+1}\binom{n-1}{(n-1)/2} \text{ and } o_{132}^0(n) = \frac{1}{2(n+1)}\binom{2n}{n} - \frac{1}{n+1}\binom{n-1}{(n-1)/2}.$$

In this paper, as a consequence of [12], we suggest a new approach to this problem in the case of even (or odd) permutations where $\sigma = 132$, which allows one to get an explicit expression for $e_{132}^r(n)$ for any given r. More precisely, we present an algorithm that computes the generating functions $E_r(x) = \sum_{n\geq 0} e_{132}^r(n)x^n$ and $O_r(x) = \sum_{n\geq 0} o_{132}^r(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group $\mathfrak{S}_{2r}$. The algorithm has been implemented in C, and yields explicit results for $0 \leq r \leq 6$.

2. Recall definitions and preliminary results

To any $\pi \in \mathfrak{S}_n$ we assign a bipartite graph G_π in the following way. The vertices in one part of G_π, denoted V_1, are the entries of π, and the vertices of the second part, denoted V_3, are the occurrences of 132 in π. Entry $i \in V_1$ is connected by an edge to occurrence $j \in V_3$ if i enters j. For example, let $\pi = 57614283$, then π contains 5 occurrences of 132, and the graph G_π is presented on Figure 1.

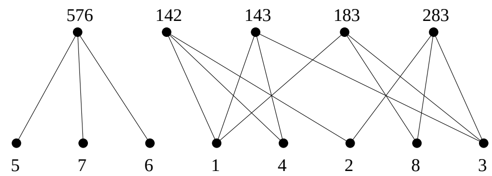

Figure 1. Graph G_π for $\pi = 57614283$

We denote by G_π^n the connected component of G_π containing entry n. Let $\pi(i_1), \dots, \pi(i_s)$ be the entries of π belonging to G_π^n, and let $\sigma = \sigma_\pi \in \mathfrak{S}_s$ be the corresponding permutation. We say that $\pi(i_1), \dots, \pi(i_s)$ is the *kernel* of π and denote it $\ker \pi$, σ is called the *shape* of the kernel, or the *kernel shape*, s is called the *size* of the kernel, and the number of occurrences of 132 in $\ker \pi$ is called the *capacity* of the kernel. For example, for $\pi = 57614283$ as above, the kernel equals 14283, its shape is 14253, the size equals 5, and the capacity equals 4.

Theorem 2.1. (Mansour and Vainshtein [12, Theorem 1]) *Let $\pi \in \mathfrak{S}_n$ such that $132(\pi) = r$, then the size of the kernel of π is at most $2r + 1$.*

We say that ρ is a *kernel permutation* if it is the kernel shape for some permutation π. Evidently ρ is a kernel permutation if and only if $\sigma_\rho = \rho$. Let $\rho \in \mathfrak{S}_s$ be an arbitrary kernel permutation. We denote by $\mathfrak{S}(\rho)$ the set of all the permutations of all possible sizes whose kernel shape equals ρ. For any $\pi \in \mathfrak{S}(\rho)$ we define the *kernel cell decomposition* as follows. The number of cells in the decomposition equals $s(s+1)$. Let $\ker \pi = \pi(i_1), \dots, \pi(i_s)$; the *cell* $C_{ml} = C_{ml}(\pi)$ for $1 \le l \le s+1$ and $1 \le m \le s$ is defined by

$$C_{ml}(\pi) = \{\pi(j)\ i_{l-1} < j < i_l,\ \pi(i_{\rho^{-1}(m-1)}) < \pi(j) < \pi(i_{\rho^{-1}(m)})\},$$

where $i_0 = 0$, $i_{s+1} = n + 1$, and $\pi(0) = 0$ for any π. If π coincides with ρ itself, then all the cells in the decomposition are empty. An arbitrary permutation in $\mathfrak{S}(\rho)$ is obtained by filling in some of the cells in the cell decomposition. A cell C is called *infeasible* if the existence of an entry $a \in C$ would imply an occurrence of 132 that contains a and two other entries $x, y \in \ker \pi$. Clearly, all infeasible cells are empty for any $\pi \in \mathfrak{S}(\rho)$. All the remaining cells are called *feasible*; a feasible cell may, or may not, be empty. Consider the permutation $\pi = 67382451$. The kernel of π equals 3845, its shape is 1423. The cell decomposition of π contains four feasible cells: $C_{13} = \{2\}$, $C_{14} = \varnothing$, $C_{15} = \{1\}$, and $C_{41} = \{6, 7\}$ (see Figure 2). All the other cells are infeasible; for example, C_{32} is infeasible, since if $a \in C_{32}$, then $a\pi'(i_2)\pi'(i_4)$ is an occurrence of 132 for any π' whose kernel is of shape 1423.

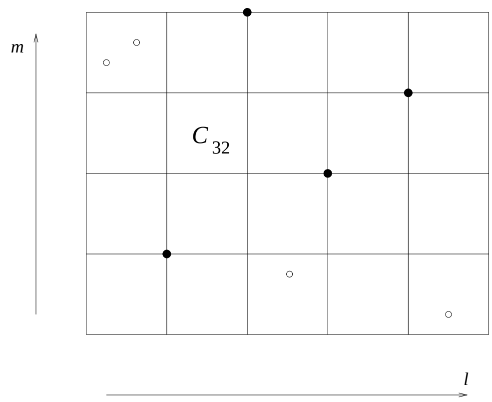

FIGURE 2. Kernel cell decomposition for $\pi \in \mathfrak{S}(1423)$

As another example, permutation $\widetilde{\pi} = 11\,10\,7\,12\,4\,6\,5\,8\,3\,9\,1\,2$ belongs to the same class $\mathfrak{S}(1423)$. Its kernel is $7\,12\,8\,9$, and the feasible cells are $C_{13} = \{4, 6, 5\}$, $C_{14} = \{3\}$, $C_{15} = \{1, 2\}$, $C_{41} = \{11, 10\}$.

Given a cell C_{ij} in the kernel cell decomposition, all the kernel entries can be positioned with respect to C_{ij}. We say that $x = \pi(i_k) \in \ker\pi$ lies *below* C_{ij} if $\rho(k) < i$, and *above* C_{ij} if $\rho(k) \geq i$. Similarly, x lies to the *left* of C_{ij} if $k < j$, and to the *right* of C_{ij} if $k \geq j$. As usual, we say that x lies to the *southwest* of C_{ij} if it lies below C_{ij} and to the left of it; the other three directions, northwest, southeast, and northeast, are defined similarly. Let us define a partial order $\prec$ on the set of all feasible cells by saying that $C_{ml} \prec C_{m'l'} \neq C_{ml}$ if $m \geq m'$ and $l \leq l'$.

Theorem 2.2. (Mansour and Vainshtein [12])

(i) $\prec$ *is a linear order.*

(ii) *Let* C_{ml} *and* $C_{ml'}$ *be two nonempty feasibly cells such that* $l < l'$. *Then for any pair of entries* $a \in C_{ml}$, $b \in C_{ml'}$, *one has* $a > b$.

(iii) *Let* C_{ml} *and* $C_{m'l}$ *be two nonempty feasibly cells such that* $m < m'$. *Then any entry* $a \in C_{ml}$ *lies to the right of any entry* $b \in C_{m'l}$.

Let π be any permutation with a kernel permutation ρ, and assume that the feasible cells of the kernel cell decomposition associated with ρ are ordered linearly according to $\prec$, $C^1, C^2, \dots, C^{f(\rho)}$. Let d_j be the size of C^j. For example, let $\pi = 67382451$ with kernel permutation $\rho = 1423$, as on Figure 2, then $d_1 = 2$, $d_2 = 1$, $d_3 = 0$, and $d_4 = 1$.
We denote by $l_j(\rho)$ the number of the entries of ρ that lie to the north-west from C^j or lie to the south-east from C^j. For example, let $\rho = 1423$, as on Figure 2, then $l_1(\rho) = 3$, $l_2(\rho) = 2$, $l_3(\rho) = 3$, and $l_4(\rho) = 4$. Clearly, $l_1(\rho) = s(\rho) - 1$ and $l_{f(\rho)} = s(\rho)$ for any nonempty kernel permutation ρ.

Lemma 2.3. *For any permutation* π *with a kernel permutation* ρ,

$$\mathrm{sign}(\pi) = (-1)^{\left(\sum_{1\leq i\leq j\leq f(\rho)} d_i d_j + \sum_{j=1}^{f(\rho)} d_j l_j(\rho)\right)} \cdot \mathrm{sign}(\rho) \cdot \prod_{j=1}^{f(\rho)} \mathrm{sign}(C^j).$$

Proof. To verify this formula, let us count the number of occurrences of the pattern 21 in π. There four possibilities for an occurrence of 21 in π. The first possibility is an occurrence occurs in one of the cells C^j, so in this case there are $\sum_{j=1}^{f(\rho)} 21(C^j)$ occurrences. The second possibility is an occurrence occurs in ρ, so there are $21(\rho)$ occurrences. The third possibility is an occurrence of two elements which the first belongs to ρ and the second belongs to C^i, so there are $\sum_{j=1}^{f(\rho)} d_j l_j(\rho)$ (see Theorem 2.2) occurrences. The fourth possibility is an occurrence of two elements which the first belongs to C^i and the second belongs to C^j where $i < j$ (Theorem 2.2 yields every entry of C^i is greater than every entry of C^j for all $i < j$), so there are $\sum_{1\leq i<j\leq f(\rho)} d_i d_j$ occurrences. Therefore, by Lemma 1.1 we have

$$\mathrm{sign}(\pi) = (-1)^{\sum_{j=1}^{f(\rho)} 21(C^j)}(-1)^{21(\rho)}(-1)^{\sum_{j=1}^{f(\rho)} d_j l_j(\rho)}(-1)^{\sum_{1\leq i<j\leq f(\rho)} d_i d_j},$$

equivalently, $\mathrm{sign}(\pi) = (-1)^{\left(\sum_{1\leq i\leq j\leq f(\rho)} d_i d_j + \sum_{j=1}^{f(\rho)} d_j l_j(\rho)\right)} \cdot \mathrm{sign}(\rho) \cdot \prod_{j=1}^{f(\rho)} \mathrm{sign}(C^j)$. □

We say the vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a *binary vector* if $v_i \in \{0, 1\}$ for all i, $1 \leq i \leq n$. We denote the set of all binary vectors of length n by $\mathcal{B}^n$. For any $\mathbf{v} \in \mathcal{B}^n$, we define $|\mathbf{v}| = v_1 + v_2 + \cdots + v_n$. For example, $\mathcal{B}^2 = \{(0,0), (0,1), (1,0), (1,1)\}$ and $|(1,1,0,0,1)| = 3$.

Let ρ be any kernel permutations, we denote by X_a^ρ [respectively; Y_a^ρ] the set of all the binary vectors $\mathbf{v} \in \mathcal{B}^{f(\rho)}$ such that $(-1)^{|\mathbf{v}|+s(\rho)} = a$ [respectively; $(-1)^{|\mathbf{v}|} = a$]. For any $\mathbf{v} \in \mathcal{B}^{f(\rho)}$, we define

$$z_\rho(\mathbf{v}) = (-1)^{\sum_{1\leq i<j\leq f(\rho)} v_i v_j + \sum_{j=1}^{f(\rho)} l_j(\rho) v_j} \mathrm{sign}(\rho).$$

Let ρ be any kernel permutations and $\mathbf{v} = (v_1, v_2, \dots, v_{f(\rho)}) \in \mathcal{B}^{f(\rho)}$, we denote by $\mathfrak{S}(\rho; \mathbf{v})$ the set of all permutations of all sizes with kernel permutation ρ such that the corresponding cells C^j satisfy $(-1)^{d_j} = (-1)^{v_j}$, in such a context $\mathbf{v}$ is called a *length argument vector* of ρ. By definitions, the following result holds immediately.

Lemma 2.4. *For any kernel permutation ρ,*

$$\mathfrak{S}(\rho) = \bigsqcup_{\mathbf{v} \in \mathcal{B}^{f(\rho)}} \mathfrak{S}(\rho; \mathbf{v}).$$

Let ρ be any kernel permutations and let $\mathbf{v} = (v_1, v_2, \dots, v_{f(\rho)})$, $\mathbf{u} = (u_1, u_2, \dots, u_{f(\rho)}) \in \mathcal{B}^{f(\rho)}$, we denote by $\mathfrak{S}(\rho; \mathbf{v}, \mathbf{u})$ the set of all permutations in $\mathfrak{S}(\rho; \mathbf{v})$ such that the corresponding cells C^j satisfy $\mathrm{sign}(C^j) = 1$ if and only if $u_j = 0$, in such a context $\mathbf{u}$ is called a *signature argument vector* of ρ. By Lemma 2.4, the following result holds immediately.

Lemma 2.5. *For any kernel permutation ρ,*

$$\mathfrak{S}(\rho) = \bigsqcup_{\mathbf{v} \in \mathcal{B}^{f(\rho)}} \mathfrak{S}(\rho; \mathbf{v}) = \bigsqcup_{\mathbf{v} \in \mathcal{B}^{f(\rho)}} \bigsqcup_{\mathbf{u} \in \mathcal{B}^{f(\rho)}} \mathfrak{S}(\rho; \mathbf{v}, \mathbf{u}).$$

For any $a, b \in \{0, 1\}$ we define

$$H_r(a, b) = \begin{cases} \frac{1}{2}(E_r(x) + (-1)^a E_r(-x)), & b = 0 \\ \frac{1}{2}(O_r(x) + (-1)^a O_r(-x)), & b = 1 \end{cases}.$$

By definitions, the following result holds immediately.

Lemma 2.6. *Let $a, b \in \{0, 1\}$. The generating function for all permutations π such that $132(\pi) = r$, $(-1)^{|\pi|} = (-1)^a$, and $\mathrm{sign}(\pi) = (-1)^b$ is given by $H_r(a, b)$.*

3. Main Theorem

The main result of this note can be formulated as follows. Denote by K the set of all kernel permutations, and by K_t the set of all kernel shapes for permutations in $\mathfrak{S}_t$. Let ρ be any kernel permutation, for any $a, b \in \{0, 1\}$ and any $r_1, \dots, r_{f(\rho)}$ we define

$$L^{\rho}_{r_1, \dots, r_{f(\rho)}}(a, b) = \sum_{\mathbf{v} \in X^{\rho}_{(-1)^a}} \sum_{\mathbf{u} \in Y^{\rho}_{(-1)^b z_\rho(\mathbf{v})}} \prod_{j=1}^{f(\rho)} H_{r_j}(v_j, u_j).$$

Theorem 3.1. *Let $r \geq 1$. For any $a, b \in \{0, 1\}$,*

$$H_r(a, b) = \sum_{\rho \in K_{2r+1}} \sum_{r_1 + \cdots + r_{f(\rho)} = r - c(\rho),\, r_j \geq 0} L^{\rho}_{r_1, \dots, r_{f(\rho)}}(a, b). \tag{3.1}$$

Proof. Let us fix a kernel permutation $\rho \in K_{2r+1}$, a length argument vector $\mathbf{v} = (v_1, \dots, v_{f(\rho)}) \in X_{(-1)^a}(\rho)$, and a signature argument vector $\mathbf{u} = (u_1, \dots, u_{f(\rho)}) \in Y^{f(\rho)}_{(-1)^b z_\rho(\mathbf{v})}$. Recall that the kernel ρ of any π contains exactly $c(\rho)$ occurrences of 132. The remaining $r - c(\rho)$ occurrences of 132 are distributed between the feasible cells of the kernel cell decomposition of π. By Theorem 2.2, each occurrence of 132 belongs entirely to one feasible cell, and the occurrences of 132 in different cells do not influence one another.

Let π be any permutation such that $132(\pi) = r$, $\mathrm{sign}(\pi) = (-1)^b$ and $(-1)^{|\pi|} = (-1)^a$ together with a kernel permutations ρ, length argument vector $\mathbf{v}$, and signature argument vector $\mathbf{u}$. Then by Lemma 2.5, the cells C^j satisfy the following conditions:

(1) $v_j = 0$ if and only if d_j is an even number,

(2) $u_j = 0$ if and only if $\mathrm{sign}(C^j) = 1$,

(3) $(-1)^{v_1+\dots+v_{f(\rho)}+s(\rho)} = (-1)^a$, and

(4) $(-1)^{u_1+\dots+u_{f(\rho)}} z_\rho(\mathbf{v}) = (-1)^b$.

Therefore, by Lemma 2.6 this contribution gives

$$x^{s(\rho)} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho),\, r_j\geq 0} \prod_{j=1}^{f(\rho)} H_{r_j}(v_j, u_j).$$

Hence by Lemma 2.5 and [12, Theorem 1], if summing over all the kernel permutations $\rho \in K_{2r+1}$, length argument vectors $\mathbf{v} \in X_{(-1)^a}(\rho)$, and signature argument vectors $\mathbf{u} \in Y^{f(\rho)}_{(-1)^b z_\rho(\mathbf{v})}$ then we get the desired result. □

Theorem 3.1 provides a finite algorithm for finding $E_r(x)$ and $O_r(x)$ for any given $r \geq 0$, since we have to consider all permutations in $\mathfrak{S}_{2r+1}$, and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition.

Proposition 3.2. *The only kernel permutation of capacity $r \geq 1$ and size $2r+1$ is*

$$\rho = 2r-1\,2r+1\,2r-3\,2r\,\dots2r-2j-3\,2r-2j\,\dots1\,4\,2.$$

Its parameters are given by $s(\rho) = 2r+1$, $c(\rho) = r$, $f(\rho) = r+2$, $\mathrm{sign}(\rho) = -1$, and $z_\rho(v_1,\dots,v_{r+2}) = (-1)^{\left(1+v_{r+2}+\sum_{1\leq i<j\leq r+2} v_iv_j\right)}$.

Proof. The first part of the proposition holds by [12, Prorposition]. Besides, by using the form of ρ we get $s(\rho) = 2r+1$, $c(\rho) = r$, $f(\rho) = r+2$; $\mathrm{sign}(\rho) = -1$, and $l_j(\rho) = 2r$ for all $j = 1,2,\dots,r+1$ and $l_{r+2}(\rho) = 2r+1$. Therefore, $z_\rho(v_1,\dots,v_{r+2}) = (-1)^{\left(1+v_{r+2}+\sum_{1\leq i<j\leq r+2} v_iv_j\right)}$. □

By this proposition, it suffices to search only permutations in $\mathfrak{S}_{2r}$. Below we present several explicit calculations.

3.1. **The case** $r = 0$**.** Let us start from the case $r = 0$. Observe that Theorem 3.1 remains valid for $r = 0$, provided the left hand side of Equation 3.1 for $a = b = 0$ is replaced by $H_r(0,0) - 1 = \frac{1}{2}(E_r(x) + E_r(-x)) - 1$; subtracting 1 here accounts for the empty permutation. So, we begin with finding kernel shapes for all permutations in $\mathfrak{S}_1$. The only shape obtained is $\rho_1 = 1$, and it is easy to see that $s(\rho_1) = 1$, $c(\rho_1) = 0$, $f(\rho_1) = 2$,

$$X_1(\rho_1) = Y_{-1} = \{(1,0),(0,1)\}, \quad X_{-1}(\rho_1) = Y_1 = \{(0,0),(1,1)\},$$

and

$$z_{\rho_1}(0,0) = z_{\rho_1}(1,0) = z_{\rho_1}(1,1) = -z_{\rho_1}(0,1) = 1.$$

Therefore, Equation 3.1 for $a = b = 0$ gives

$$\begin{aligned}\tfrac{1}{2}(E_0(x)+E_0(-x))-1=&\\ =xH_0(1,0)H_0(0,0)+xH_0(1,1)H_0(0,1)+xH_0(1,0)H_0(0,1)+xH_0(1,1)H_0(0,0),&\end{aligned}\tag{3.2}$$

(Equation 3.1 for $a = 1$ and $b = 0$ gives

$$\tfrac{1}{2}(E_0(x)-E_0(-x)) = xH_0^2(0,0)+xH_0^2(0,1)+xH_0^2(1,0)+xH_0^2(1,1),\tag{3.3}$$

Equation 3.1 for $a = 0$ and $b = 1$ gives

$$\begin{aligned}\tfrac{1}{2}(O_0(x)+O_0(-x))=&\\ =xH_0(1,1)H_0(0,0)+xH_0(1,0)H_0(0,1)+xH_0(0,0)H_0(1,0)+xH_0(0,1)H_0(1,1),&\end{aligned}\tag{3.4}$$

and Equation 3.1 for $a = b = 1$ gives

$$\tfrac{1}{2}(O_0(x)-O_0(-x)) = 2xH_0(0,1)H_0(0,0)+2xH_0(1,1)H_0(1,0).\tag{3.5}$$

Out present aim to find explicitly $E_0(x)$ and $O_0(x)$, thus we need the following notation. We define

$$M_r(x) = E_r(x) - O_r(x) \text{ and } F_r(x) = E_r(x) + O_r(x)$$

for all $r \geq 0$. Clearly,

$$\begin{array}{ll} H_r(0,0)-H_r(0,1)=\frac{1}{2}(M_r(x)+M_r(-x)), & H_r(0,0)+H_r(0,1)=\frac{1}{2}(F_r(x)+F_r(-x)),\\ H_r(1,0)-H_r(1,1)=\frac{1}{2}(M_r(x)-M_r(-x)), & H_r(1,0)+H_r(1,1)=\frac{1}{2}(F_r(x)-F_r(-x)),\end{array}$$

for all $r \geq 0$. Therefore, by subtracting (respectively; adding) Equation 3.4 and Equation 3.2, and by subtracting (respectively; adding) Equation 3.5 and Equation 3.3 we get

$$\begin{cases} M_0(x)+M_0(-x)=2\\ M_0(x)-M_0(-x)=x(M_0^2(x)+M_0^2(-x))\end{cases} \text{and} \begin{cases} F_0(x)+F_0(-x)=2+x(F_0^2(x)-F_0^2(-x))\\ F_0(x)-F_0(-x)=x(F_0^2(x)+F_0^2(-x)).\end{cases}$$

Hence,

$$M_0(x) = 1+\frac{1-\sqrt{1-4x^2}}{2x} \text{ and } F_0(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Theorem 3.3. (i) *The generating function for the number of even permutations avoiding* 132 *is given by* (*see* [16])

$$E_0(x) = \frac{1}{2}\left(\frac{1-\sqrt{1-4x}}{2x}+1+\frac{1-\sqrt{1-4x^2}}{2x}\right).$$

(ii) *The generating function for the number of odd permutations avoiding* 132 *is given by* (*see* [16])

$$O_0(x) = \frac{1}{2}\left(\frac{1-\sqrt{1-4x}}{2x}-1-\frac{1-\sqrt{1-4x^2}}{2x}\right).$$

(iii) *The generating function for the number of permutations avoiding* 132 *is given by* (*see* [9])

$$F_0(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

3.2. **The Case $r=1$.** Since permutations in $\mathfrak{S}_2$ do not exhibit kernel shapes distinct from ρ_1, the only possible new shape is the exceptional one, $\rho_2 = 132$. Calculation of the parameters of ρ_2 gives $s(\rho_2)=3$, $c(\rho_2)=1$, $f(\rho_2)=3$,

$$X_1(\rho_2) = Y_{-1} = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\},$$
$$X_{-1}(\rho_2) = Y_1 = \{(0,0,0),(1,1,0),(1,0,1),(1,1,0)\},$$

and

$$z_{\rho_2}(0,0,0) = z_{\rho_2}(1,0,0) = z_{\rho_2}(0,1,0) = -z_{\rho_2}(1,1,0) = 1,$$
$$-z_{\rho_2}(0,0,1) = z_{\rho_2}(1,0,1) = z_{\rho_2}(0,1,1) = z_{\rho_2}(1,1,1) = 1.$$

Therefore, by Theorem 3.1 we have

$$\begin{cases} 2(H_1(0,0)-H_1(0,1)) = \\ \quad = M_1(x)+M_1(-x) = \frac{x^3}{2}(M_0(-x)-M_0(x))(M_0^2(-x)+M_0^2(x)) \\ \\ 2(H_1(1,0)-H_1(1,1)) = \\ \quad = M_1(x)-M_1(-x) = 2x(M_0(x)M_1(x)+M_0(-x)M_1(-x)) \\ \qquad\qquad -\frac{x^3}{2}(M_0(-x)+M_0(x))(M_0^2(-x)+M_0^2(x))), \end{cases}$$

Using the expression for $M_0(x)$ (see the case $r=0$) we get

$$M_1(x) = \frac{1}{2}(-1+3x+2x^2) + \frac{1-3x-4x^2+4x^3}{2}(1-4x^2)^{-1/2}.$$

Similarly, if considering the expressions for $H_1(0,0)+H_1(0,1)$ and $H_1(1,0)+H_1(1,1)$ we get

$$F_1(x) = \frac{1}{2}(x-1) + \frac{1-3x}{2}(1-4x)^{-1/2}.$$

Theorem 3.4. (i) *The generating function for the number of even permutations containing* 132 *exactly once is given by*

$$E_1(x) = -\frac{1}{2}(1-2x-x^2) + \frac{1-3x}{4}(1-4x)^{-1/2} + \frac{1-3x-4x^2+4x^3}{4}(1-4x^2)^{-1/2}.$$

(ii) *The generating function for the number of odd permutations containing* 132 *exactly once is given by*

$$O_1(x) = -\frac{1}{2}(x+x^2) + \frac{1-3x}{4}(1-4x)^{-1/2} - \frac{1-3x-4x^2+4x^3}{4}(1-4x^2)^{-1/2}.$$

(iii) *The generating function for the number of permutations containing* 132 *exactly once is given by* (*see* [6])

$$F_1(x) = \frac{1}{2}(x-1) + \frac{1-3x}{2}(1-4x)^{-1/2}.$$

3.3. **The case $r=2$.** We have to check the kernel shapes of permutations in $\mathfrak{S}_4$. Exhaustive search adds four new shapes to the previous list; these are 1243, 1342, 1423, and 2143; besides, there is the exceptional $35142 \in \mathfrak{S}_5$. Calculation of the parameters s, c, f, z, X_a, Y_a is straightforward, and we get

Theorem 3.5. (i) *The generating function for the number of even permutations containing* 132 *exactly twice is given by*

$$E_2(x) = \frac{1}{2}x(x^3+3x^2-4x-1) + \frac{1}{4}(2x^4-4x^3+29x^2-15x+2)(1-4x)^{-3/2}$$
$$-\frac{1}{4}(16x^7-48x^6-76x^5+64x^4+36x^3-21x^2-5x+2)(1-4x^2)^{-3/2}.$$

(ii) *The generating function for the number of odd permutations containing* 132 *exactly once is given by*

$$O_2(x) = -\frac{1}{2}(x^4+3x^3-5x^2-4x+2) + \frac{1}{4}(2x^4-4x^3+29x^2-15x+2)(1-4x)^{-3/2}$$
$$+\frac{1}{4}(16x^7-48x^6-76x^5+64x^4+36x^3-21x^2-5x+2)(1-4x^2)^{-3/2}.$$

(iii) *The generating function for the number of permutations containing* 132 *exactly twice is given by* (*see* [12])

$$F_2(x) = \frac{1}{2}(x^2+3x-2) + \frac{1}{2}(2x^4-4x^3+29x^2-15x+2)(1-4x)^{-3/2}.$$

3.4. **The cases** $r = 3,4,5,6$**.** Let $r = 3,4,5,6$; exhaustive search in $\mathfrak{S}_6$, $\mathfrak{S}_8$, $\mathfrak{S}_{10}$, and $\mathfrak{S}_{12}$ reveals 20, 104, 503, and 2576 new nonexceptional kernel shapes, respectively, and we get

Theorem 3.6. *Let* $r = 3,4,5,6$*, then*

$$M_r(x) = \frac{1}{2}\left(A_r(x) + B_r(x)(1-4x^2)^{-r+1/2}\right) \text{ and } F_r(x) = \frac{1}{2}\left(C_r(x) + D_r(x)(1-4x)^{-r+1/2}\right)$$

where

$$A_3(x) = 2x^6+10x^5-24x^4-30x^3+23x^2+7x-2,$$

$$A_4(x) = 2x^8+14x^7-46x^6-90x^5+117x^4+85x^3-42x^2-8x+1,$$

$$A_5(x) = 2x^10+18x^9-76x^8-198x^7+360x^6+440x^5-355x^4-171x^3+62x^2+10x-2,$$

$$A_6(x) = 256x^{13}-446x^{12}-618x^{11}+194x^{10}-140x^9+798x^8+1404x^7-1702x^6$$
$$-1430x^5+815x^4+302x^3-88x^2-15x+4,$$

$$B_3(x) = 64x^{11}-320x^{10}-800x^9+1216x^8+1124x^7-972x^6-524x^5+312x^4+100x^3-43x^2$$
$$-7x+2,$$

$$B_4(x) = -256x^{15}+1792x^{14}+6112x^{13}-13120x^{12}-19840x^{11}+22224x^{10}+19054x^9$$
$$-14780x^8-8328x^7+4772x^6+1840x^5-775x^4-197x^3+56x^2+8x-1,$$

$$B_5(x) = 1024x^{19}-9216x^{18}-40064x^{17}+111744x^{16}+228896x^{15}-343264x^{14}-404056x^{13}$$
$$+398712x^{12}+321058x^{11}-234686x^{10}-137468x^9+78480x^8+33896x^7-15400x^6$$
$$-4780x^5+1723x^4+351x^3-98x^2-10x+2,$$

$$B_6(x) = 524288x^{24}+1175552x^{23}-1593344x^{22}-2324992x^{21}+1162752x^{20}+298112x^{19}$$
$$+2696448x^{18}+4856864x^{17}-7020288x^{16}-7464568x^{15}+6981056x^{14}$$
$$+5445696x^{13}-3868942x^{12}-2335450x^{11}+1324884x^{10}+627306x^9$$
$$-290536x^8-106510x^7+40772x^6+11046x^5-3543x^4-632x^3+176x^2+15x-4,$$

$$C_3(x) = 2x^3 - 5x^2 + 7x - 2,$$

$$C_4(x) = 5x^4 - 7x^3 + 2x^2 + 8x - 3,$$

$$C_5(x) = 14x^5 - 17x^4 + x^3 - 16x^2 + 14x - 2,$$

$$C_6(x) = 42x^6 - 44x^5 + 5x^4 + 4x^3 - 20x^2 + 19x - 4,$$

and

$$D_3(x) = -22x^6 - 106x^5 + 292x^4 - 302x^3 + 135x^2 - 27x + 2,$$

$$D_4(x) = 2x^9 + 218x^8 + 1074x^7 - 1754x^6 + 388x^5 + 1087x^4,$$

$$\begin{aligned} D_5(x) = & -50x^{11} - 2568x^{10} - 10826x^9 + 16252x^8 - 12466x^7 + 16184x^6 - 16480x^5 + 9191x^4 \\ & -2893x^3 + 520x^2 - 50x + 2, \end{aligned}$$

$$\begin{aligned} D_6(x) = & \, 4x^{14} + 820x^{13} + 32824x^{12} + 112328x^{11} - 205530x^{10} + 141294x^9 - 30562x^8 \\ & -67602x^7 + 104256x^6 - 74090x^5 + 30839x^4 - 7902x^3 + 1230x^2 - 107x + 4. \end{aligned}$$

Moreover, for $r = 3, 4, 5, 6$,

$$E_r(x) = \frac{1}{4}\left(A_r(x) + C_r(x) + D_r(x)(1-4x)^{-r+1/2} + B_r(x)(1-4x^2)^{-r+1/2}\right)$$

and

$$O_r(x) = \frac{1}{4}\left(A_r(x) - C_r(x) + D_r(x)(1-4x)^{-r+1/2} - B_r(x)(1-4x^2)^{-r+1/2}\right).$$

4. Further results and open questions

First of all, let us simplify the expression

$$L^{\rho}_{r_1,\dots,r_{f(\rho)}}(a,0) - L^{\rho}_{r_1,\dots,r_{f(\rho)}}(a,1),$$

where $a = 0, 1$, $r_j \geq 0$ for all j.

Lemma 4.1. *Let* $\mathbf{v} \in \{0,1\}^n$ *be any vector, and let* $a \in \{1, -1\}$. *Then*

$$\sum_{\mathbf{x}\in Y_a} \prod_{j=1}^{n} H_r(v_j, x_j) - \sum_{\mathbf{y}\in Y_{-a}} \prod_{j=1}^{n} H_r(v_j, y_j) = a \prod_{j=1}^{n} g_r(j),$$

where $g_r(j) = H_r(v_j, 0) - H_r(v_j, 1) = \frac{1}{2}(M_r(x) + (-1)^{v_j} M_r(-x))$ *for all* j.

Proof. Let us define an order on the set $\mathcal{B}^n$, we say the vector $\mathbf{v} < \mathbf{u}$ if there exists j such that $u_j + v_j = 1$, and $u_i = v_i$ for all $i \neq j$. We say the 2^n vectors $\mathbf{u}^1, \cdots, \mathbf{u}^{2^n}$ of $\mathcal{B}^n$ are satisfy the *ℓ-property* if

$$\mathbf{0} = (0, 0, \dots, 0) = \mathbf{u}^1 < \mathbf{u}^2 < \cdots < \mathbf{u}^{2^n},$$

and we say the 2^n vectors $\mathbf{u}^1, \cdots, \mathbf{u}^{2^n}$ are satisfy the *c-property* if the vectors

$$(\mathbf{u}^1_1, \dots, \mathbf{u}^1_m), \dots, (\mathbf{u}^{2^m}_1, \dots, \mathbf{u}^{2^m}_m)$$

are satisfy the ℓ-property for all $m = 1, 2, \ldots, n$. For example, the vectors of $\mathcal{B}^3$ are satisfy the c-property by

$$(0,0,0) < (1,0,0) < (1,1,0) < (0,1,0) < (0,1,1) < (1,1,1) < (1,0,1) < (0,0,1).$$

First of all, let us prove by induction on n that there exists an order of the vectors of $\mathcal{B}^n$ with c-property, for all $n \geq 1$. For $n = 1$ the c-property holds with $(0) < (1)$. Suppose that there exists an order of the vector of $\mathcal{B}^m$ with the c-property. Let $\mathbf{v}^j = (\mathbf{u}_1^j, \ldots, \mathbf{u}_m^j, 0)$ for all $j = 1, 2, \ldots, 2^m$ and $\mathbf{v}^{2^m+j} = (\mathbf{u}_1^{2^m+1-j}, \ldots, \mathbf{u}_m^{2^m+1-j}, 1)$ for all $j = 1, 2, \ldots, 2^m$. By definitions, $\mathbf{v}^1 = (0, 0, \ldots, 0)$ and $\mathbf{v}^1 < \cdots < \mathbf{v}^{2^{m+1}}$, so the ℓ-property holds for $m + 1$. Hence, by induction on m we get that there exists an order of the vector of $\mathcal{B}^n$ with the c-property.

Now we are ready to prove the lemma. Without loss of generality we can assume that $(0, 0, \ldots, 0) \in Y_a^n$ (which means $a = 1$); otherwise it is enough to replace a by $-a$. Let $\mathbf{x}^1, \ldots, \mathbf{x}^{2^n}$ all the vectors of $\mathcal{B}^n$ with the c-property. Using $(0, 0, \ldots, 0) \in Y_a^n$ together with the c-property we get that $\mathbf{x}^{2i-1} \in Y_a^n$ and $\mathbf{x}^{2i} \in Y_{-a}^n$ for all $i = 1, 2, \ldots, 2^{n-1}$. Therefore, for all $i = 1, 2, \ldots, 2^{n-1}$,

$$\begin{aligned}\sum_{i=1}^{2^{n-1}} \left(\prod_{j=1}^{n} H_r(v_j, \mathbf{x}_j^{2i-1}) - \prod_{j=1}^{n} H_r(v_j, \mathbf{x}_j^{2i}) \right) &= \sum_{i=1}^{2^{n-1}} \left((-1)^{i-1} g_r(1) \prod_{j=2}^{n} H_r(v_j, \mathbf{x}_j^{2i-1}) \right) = \\ &= g_r(1) \sum_{i=1}^{2^{n-2}} \left(\prod_{j=1}^{n-1} H_r(\widetilde{v}_j, \mathbf{y}_j^{2i-1}) - \prod_{j=1}^{n-1} H_r(\widetilde{v}_j, \mathbf{y}_j^{2i}) \right),\end{aligned}$$

where $\mathbf{y}^p = (\mathbf{x}_2^{2p}, \ldots, \mathbf{x}_n^{2p})$ for all $p = 1, 2, \ldots, 2^{n-1}$, and $\widetilde{v} = (v_2, v_3, \ldots, v_n)$. Using the c-property for $\mathbf{x}^1, \ldots, \mathbf{x}^{2^n}$ we get that the vectors $\mathbf{y}^1, \ldots, \mathbf{y}^{2^{n-1}}$ are satisfy the c-property in $\mathcal{B}^{n-1}$. Hence by induction on n (by definitions the lemma holds for $n = 1$), we get that the expression equals to $a \prod_{j=1}^n g_r(j)$. □

As a remark, the vector $(0, 0, \ldots, 0) \in Y_{z_\rho(\mathbf{v})}^\rho$ if and only if $z_\rho(\mathbf{v}) = 1$ for any kernel permutation ρ and vector $\mathbf{v}$. Therefore, by Theorem 3.1 and Lemma 4.1 we get the following result.

Theorem 4.2. *Let $a \in \{0, 1\}$ and $r \geq 0$. Then*

$$\begin{aligned}&\tfrac{1}{2}(M_r(x) + (-1)^a M_r(-x)) - \delta_{r+a,0} = \\ &= \sum_{\rho \in K_{2r+1}} x^{s(\rho)} \sum_{r_1, \ldots, r_{f(\rho)} = r - c(\rho),\, r_j \geq 0} \left(\sum_{\mathbf{v} \in X_{(-1)^a}(\rho)} 2^{-f(\rho)} z_\rho(\mathbf{v}) \prod_{j=1}^{f(\rho)} (M_{r_j}(x) + (-1)^{v_j} M_{r_j}(-x)) \right).\end{aligned}$$

As a remark, the above theorem yields two equations (for $a = 0$ and $a = 1$) that are linear on $M_r(x)$ and $M_r(-x)$. So, Theorem 4.2 provides a finite algorithm for finding $M_r(x)$ for any given $r \geq 0$, since we have to consider all permutations in $\mathfrak{S}_{2r+1}$, and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition which holds immediately by Proposition 3.2 and Theorem 4.2.

Proposition 4.3. *Let $r \geq 1$, $a \in \{0, 1\}$, and $\rho = 2r-1\, 2r+1\, 2r-3\, 2r \ldots 2r-2j-3\, 2r-2j \ldots 1\, 4\, 2$. Then the expression*

$$x^{s(\rho)} \sum_{r_1, \ldots, r_{f(\rho)} = r - c(\rho),\, r_j \geq 0} \left(\sum_{\mathbf{v} \in X_{(-1)^a}(\rho)} 2^{-f(\rho)} z_\rho(\mathbf{v}) \prod_{j=1}^{f(\rho)} (M_{r_j}(x) + (-1)^{v_j} M_{r_j}(-x)) \right)$$

is given by

$$\sum_{j=a}^{[(r+2)/2]} (-1)^{j-a+1} 2^{-r-2} \binom{r+2}{2j+1-a} x^{2r+1} (M_0(x) - M_0(-x))^j (M_0(x) + M_0(-x))^{r+2-j}.$$

By this proposition, it is sufficient to search only permutations in $\mathfrak{S}_{2r}$. Besides, using Theorem 4.2 and the case $r = 0$ together with induction on r we get the following result.

Theorem 4.4. *$M_r(x)$ is a rational function on x and $\sqrt{1-4x^2}$ for any $r \geq 0$.*

In view of our explicit results, we have even a stronger conjecture.

Conjecture 4.5. *For any $r \geq 1$, there exist polynomials $A_r(x)$, $B_r(x)$, $C_r(x)$, and $D_r(x)$ with integer coefficients such that*

$$E_r(x) = \tfrac{1}{4} A_r(x) + \tfrac{1}{4} B_r(x)(1-4x)^{-r+1/2} + \tfrac{1}{4} C_r(x)(1-4x^2)^{-r+1/2},$$

$$O_r(x) = \tfrac{1}{4} D_r(x) + \tfrac{1}{4} B_r(x)(1-4x)^{-r+1/2} - \tfrac{1}{4} C_r(x)(1-4x^2)^{-r+1/2},$$

where the coefficients of the polynomial $A_r(x) + D_r(x)$ are divided by 2.

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