q-NARAYANA NUMBERS AND THE FLAG h-VECTOR OF $J(\mathbf{2} \times \mathbf{n})$

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Abstract. The Narayana numbers are $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$. There are several natural statistics on Dyck paths with a distribution given by $N(n,k)$. We show the equidistribution of Narayana statistics by computing the flag h-vector of $J(\mathbf{2} \times \mathbf{n})$ in different ways. In the process we discover new Narayana statistics and provide co-statistics for which the Narayana statistics in question have a distribution given by Fürlinger and Hofbauers q-Narayana numbers. We also interpret the h-vector in terms of semi-standard Young tableaux, which enables us to express the q-Narayana numbers in terms of Schur functions.

1. Introduction

The *Narayana numbers*,

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1},$$

appear in many combinatorial problems. Some examples are the number of noncrossing partitions of $\{1, 2, \dots, n\}$ of rank k [3], the number of 132-avoiding permutations with k descents [8], and also several problems involving Dyck paths.

A *Dyck path* of length $2n$ is a path in $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to (n,n) using steps $v = (0,1)$ and $h = (1,0)$, which never goes below the line $x = y$. The set of all Dyck paths of length $2n$ is denoted $\mathcal{D}_n$. A statistic on $\mathcal{D}_n$ having a distribution given by the Narayana numbers will in the sequel be referred to as a *Narayana statistic*. The first Narayana statistics to be discovered were

- $\operatorname{des}(w)$: the number of *descents (valleys)* (sequences hv) in w, [7],
- $\operatorname{ea}(w)$: the number of *even ascents*, i.e., the number of letters v in an even position in w, [4],
- $\operatorname{lnfs}(w)$: the number of *long non-final sequences*, more precisely the number of sequences vvh and hhv in w, [5].

Recently, [1], a new Narayana statistic, hp, was discovered and it counts the number of *high peaks*, i.e., peaks not on the diagonal $x = y$. Also, in [11, 12] Sulanke found numerous new Narayana statistics with the help of a computer. For terminology on posets in what follows, we refer the reader to [9].

We will show that des, hp and lnfs arise when computing the flag h-vector of the lattice $J(\mathbf{2} \times \mathbf{n})$ of order ideals in the poset $\mathbf{2} \times \mathbf{n}$ in different ways.

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In Section 2 we will show how the statistics descents and high peaks arise when considering different linear extensions of $\mathbf{2} \times \mathbf{n}$. This will give the equidistribution of the descent-set and the set of high-peaks. In Section 3 we consider a shelling of the order complex $\Delta(J(\mathbf{2} \times \mathbf{n}))$ to show that the set of long non-final sequences has the same distribution as the descent set over Dyck paths.

There is a q-analog of the Narayana numbers,

$$N_q(n,k) = \frac{1}{[n]}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+k},$$

introduced by Fürlinger and Hofbauer in [2]. To each statistic we treat we will associate a co-statistic together with which the Narayana statistic has a joint distribution given by the q-Narayana numbers.

2. Descents and High peaks

Let P be any finite graded poset with a smallest element $\hat{0}$ and a greatest element $\hat{1}$ and let ρ be the rank function of P with $\rho(P) := \rho(\hat{1}) = n$. For $S \subseteq [n-1]$ let

$$\alpha_P(S) := |\{c \text{ is a chain of } P : \rho(c) = S\}|,$$

and

$$\beta_P(S) := \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

The functions $\alpha_P, \beta_P : 2^{[n-1]} \to \mathbb{Z}$ are the *flag f-vector* and the *flag h-vector* of P respectively.

If P is a finite poset of cardinality p and $\omega : P \to [p]$ is a linear extension of P then the *Jordan-Hölder set*, $\mathcal{L}(P,\omega)$, of (P,ω) is the set of permutations $a_1 a_2 \cdots a_p$ such that $\omega^{-1}(a_1), \omega^{-1}(a_2), \ldots, \omega^{-1}(a_p)$ is a linear extension of P, in other words

$$\mathcal{L}(P,\omega) = \{\omega \circ \sigma^{-1} : \sigma \text{ is a linear extension of } P\}.$$

We will need the following theorem (Theorem 3.12.1 of [9]):

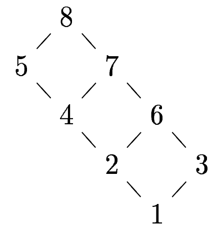
Theorem 1. *Let $L = J(P)$ be a distributive lattice of rank $p = |P|$, and let ω be a linear extension of P. Then for all $S \subseteq [p-1]$ we have that $\beta_L(S)$ is equal to the number of permutations $\pi \in \mathcal{L}(P,\omega)$ with descent set S.*

It will be convenient to code a Dyck path w in the letters $\{v_i\}_{i=1}^{\infty} \cup \{h_i\}_{i=1}^{\infty}$ by letting v_i and h_i stand for the ith vertical step and the ith horizontal step in w, respectively. Thus $vvhvhh$ is coded as $v_1 v_2 h_1 v_3 h_2 h_3$. We may write the set of elements of $\mathbf{2} \times \mathbf{n}$ as the disjoint union $C_1 \uplus C_2$ where $C_i = \{(i,k) : k \in [n]\}$ for $i = 1, 2$. For any linear extension σ of $\mathbf{2} \times \mathbf{n}$ let $W(\sigma)$ be the Dyck path $w_1 w_2 \cdots w_{2n}$ where

$$w_i = \begin{cases} v_j & \text{if } \sigma^{-1}(i) = (1,j) \text{ and} \\ h_j & \text{if } \sigma^{-1}(i) = (2,j). \end{cases}$$

It is clear that W is a bijection between the set of linear extensions of $\mathbf{2} \times \mathbf{n}$ and the set of Dyck paths of length $2n$.

FIGURE 1. The linear extension of $\mathbf{2} \times \mathbf{4}$ corresponding to the Dyck path $v_1v_2h_1v_3v_4h_2h_3h_4$.



Fix a Dyck path $W_0 \in \mathcal{D}_n$ and let $\omega_0 = W^{-1}(W_0)$. Now, if $\pi = \omega_0 \circ \sigma^{-1} \in \mathcal{L}(\mathbf{2} \times \mathbf{n}, \omega_0)$ let $W(\sigma) = w_1w_2\cdots w_{2n}$. Then $\pi(i) > \pi(i+1)$ if and only if w_{i+1} comes before w_i in W_0. In light of this we define, given Dyck paths W_0 and $w = w_1w_2\cdots w_{2n}$, the *descent set of w with respect to W_0* as

$$D_{W_0}(w) = \{i \in [2n-1] : w_{i+1} \text{ comes before } w_i \text{ in } W_0\}.$$

The descent set of $v_1h_1v_2v_3h_2h_3$ with respect to $v_1v_2h_1v_3h_2h_3$ is thus $\{2\}$. By Theorem 1 we now have:

Theorem 2. *Let W be any Dyck path of length $2n$ and let $S \subseteq [2n-1]$ and let $\beta_n = \beta_{J(\mathbf{2}\times\mathbf{n})}$. Then*

$$\beta_n(S) = |\{w \in \mathcal{D}_n : D_W(w) = S\}|.$$

For a given Dyck path W we define the statistics des_W, and MAJ_W by

$$\begin{aligned} \mathrm{des}_W(w) &= |D_W(w)|, \\ \mathrm{MAJ}_W(w) &= \sum_{i \in D_W(w)} i. \end{aligned}$$

Example 3. Two known Narayana statistics arise when fixing W in certain ways:

a) If $W = v_1v_2\cdots v_nh_1h_2\cdots h_n$ then $\mathrm{des}_W = \mathrm{des}$.
b) If $W = v_1h_1v_2h_2\cdots v_nh_n$ then $\mathrm{des}_W = \mathrm{hp}$. Thus as a consequence of Theorem 2 we have that the number of valleys and the number of high peaks have the same distribution over $\mathcal{D}_n$. This was first proved by Deutsch in [1].
c) If $W = v_1h_1v_2v_3\cdots v_nh_2h_3\cdots h_n$ then des_W counts valleys h_iv_j where $i > 1$ and high peaks of the form v_ih_1.

When $W = v_1v_2\cdots v_nh_1h_2\cdots h_n$ we drop the subscript and let $\mathrm{des} = \mathrm{des}_W$ and $\mathrm{MAJ} = \mathrm{MAJ}_W$. In [2] Fürlinger and Hofbauer defined the *q-Narayana numbers*, $N_q(n,k)$, by

$$N_q(n,k) := \sum_{w \in \mathcal{D}_n, \mathrm{des}(w)=k} q^{\mathrm{MAJ}(w)}.$$

We say that the bi-statistic $(\mathrm{des}, \mathrm{MAJ})$ has the q-Narayana distribution. We will later see that $N_q(n,k)$ can be written in an explicit form. By Theorem 2 we now have:

FIGURE 2. An example of a SSYT of shape $(6, 5, 4, 4, 2)$.

$$\begin{matrix} 1 & 2 & 2 & 3 & 5 & 5 \\ 2 & 3 & 4 & 4 & 6 & \\ 4 & 5 & 5 & 6 & & \\ 5 & 6 & 6 & 8 & & \\ 7 & 8 & & & & . \end{matrix}$$

Corollary 4. *For all $W \in \mathcal{D}_n$ the bi-statistic* $(\text{des}_W, \text{MAJ}_W)$ *has the q-Narayana distribution.*

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of positive integers. The index ℓ is called the *length*, $\ell(\lambda)$, of λ. A *semistandard Young tableau* (SSYT) of *shape* λ is an array $T = (T_{ij})$ of positive integers, where $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$, that is weakly increasing in every row and strictly increasing in every column. For any SSYT of shape λ let

$$x^T := x_1^{\alpha_1(T)} x_2^{\alpha_2(T)} \cdots ,$$

where $\alpha_i(T)$ denotes the number of entries of T that are equal to i. The *Schur function* $s_\lambda(x)$ of *shape* λ is the formal power series

$$s_\lambda(x) = \sum_T x^T,$$

where the sum is over all SSYTs T of shape λ. If T is any SSYT we let $\text{row}(T) = (\gamma_1(T), \gamma_2(T), \ldots)$ where $\gamma_i(T) = \sum_j T_{ij}$. Let $\langle 2^k \rangle$ be the partition $(2, 2, \ldots, 2)$ with k 2's.

Theorem 5. *For any $n > 0$ and $S \subseteq [2n-1]$, $|S| = k$, we have that $\beta_n(S)$ counts the number of* SSYT*s T of shape $\langle 2^k \rangle$ with* $\text{row}(T) = S$ *and with parts less than n.*

Proof. Let T be a SSYT as in the statement of the theorem. We want to construct a Dyck path $w(T)$ with descent set S.

Let $w(T) = w_1 w_1' w_2 w_2' \cdots w_{k+1} w_{k+1}'$ where

- w_1 is the word consisting of T_{12} vertical steps and w_1' is the word consisting of T_{11} horizontal steps,
- w_i is the word consisting of $T_{i2} - T_{(i-1)2}$ vertical steps and w_i' is the word consisting of $T_{i1} - T_{(i-1)1}$ horizontal steps, when $2 \leq i \leq k$,
- w_{k+1} is the word consisting of $n - T_{k2}$ vertical steps and w_{k+1}' is the word consisting of $n - T_{k1}$ horizontal steps.

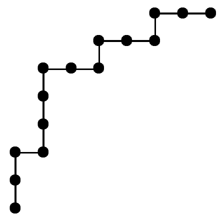
It is clear that $w(T)$ is indeed a Dyck path with descent set S, and each such Dyck path is given by $w(T)$ for a unique SSYT T. □

Theorem 6. *For all $n, k \geq 0$ we have*

$$N_q(n, k) = s_{\langle 2^k \rangle}(q, q^2, \ldots, q^{n-1}).$$

FIGURE 3. An illustration of Theorem 5 for $n = 7$.

$$\begin{matrix} 1 & 2 \\ 3 & 5 \\ 5 & 6 \end{matrix} \quad \rightarrow$$



Proof. By Theorem 5 we have that

$$\begin{aligned} \sum_{w \in \mathcal{D}_n, \mathrm{des}(w)=k} q^{\mathrm{MAJ}(w)} &= \sum_{|S|=k} \beta_n(S) q^{\sum_{s\in S} s} \\ &= \sum_T q^{\sum T_{ij}}, \end{aligned}$$

where the last sum is over all $SSYT$s T of shape $\langle 2^k \rangle$ with parts less than n. By the combinatorial definition of the Schur function this is equal to $s_{\langle 2^k \rangle}(q, q^2, \dots, q^{n-1})$, and the theorem follows. □

If we identify a partition λ with its diagram $\{(i,j) : 1 \le j \le \lambda_i\}$ then the *hook length*, $h(u)$, at $u = (x, y) \in \lambda$ is defined by

$$h(u) = |\{(x,j) \in \lambda : j \ge y\}| + |\{(i,y) \in \lambda : i \ge x\}| - 1,$$

and the *content*, $c(u)$, is defined by

$$c(u) = y - x.$$

We will use a result on Schur polynomials, commonly referred to as the *hook-content formula*, see [10, Theorem 7.21.2]. Let $[n] := 1 + q + \cdots + q^{n-1}$, $[n]! := [n][n-1]\cdots[1]$ and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[n-k]![k]!}.$$

Theorem 7 (Hook-content formula)**.** *For any partition λ and $n > 0$,*

$$s_\lambda(q, q^2, \dots, q^n) = q^{\sum i\lambda_i} \prod_{u \in \lambda} \frac{[n + c(u)]}{[h(u)]}.$$

We now have an alternative proof of the following result which was proved in [2], and is a special case of a result of MacMahon, stated without proof in [6, p. 1429].

Corollary 8 (Fürlinger, Hofbauer, MacMahon)**.** *The q-Narayana numbers are given by:*

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+k}$$

Proof. The Corollary follows from Theorem 6 after an elementary application of the hook-content formula, which is left to the reader. □

3. Long Non-final Sequences

In [5] Kreweras and Moszkowski defined a new Narayana statistic, lnfs. Recall that a *long non-final sequence* in a Dyck path is a subsequence of type vvh or hhv, and that the statistic lnfs is defined as the number of long non-final sequences in the Dyck path. We define the *long non-final sequence set*, $LS(w)$, of a Dyck path $w = a_1a_2\cdots a_{2n}$ to be

$$LS(w) = \{i \in [2n-1] : a_{i-1}a_ia_{i+1} = vvh \text{ or } a_{i-1}a_ia_{i+1} = hhv\}.$$

We will show that

$$\beta_n(S) = |\{w \in \mathcal{D}_n : LS(w) = S\}|.$$

To prove this we need some definitions.

An (abstract) *simplicial complex* Δ on a vertex set V is a collection of subsets F of V satisfying:

(i) if $x \in V$ then $\{x\} \in \Delta$,
(ii) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

The elements of Δ are called *faces* and a maximal face (with respect to inclusion) is called a *facet*. A simplicial complex is said to be *pure* if all its facets have the same cardinality. A linear partial order Ω on the set of facets of a pure simplicial complex Δ is a *shelling* if whenever $F <^{\Omega} G$ there is an $x \in G$ and $E <^{\Omega} G$ such that

$$F \cap G \subseteq E \cap G = G \setminus \{x\}.$$

A simplicial complex which allows a shelling is said to be *shellable*. Instead of finding a particular shelling we will find a partial order on the set of facets with the property that every linear extension is a shelling. In our attempts to prove that our partial order had this property we found ourselves proving Theorem 9 and Corollary 10. We therefore take the opportunity to take a general approach and define what we call a *pre-shelling*. Though we have found examples of pre-shellings implicit in the literature we have not found explicit references, so we will provide proofs.

Let Ω be a partial order on the set of facets of a pure simplical complex Δ. The *restriction*, $r_\Omega(F)$, of a facet F is the set

$$r_\Omega(F) = \{x \in F : \exists E \text{ s.t } E <^{\Omega} F \text{ and } E \cap F = F \setminus \{x\} \ \ \}.$$

We say that Ω is a *pre-shelling* if any of the equivalent conditions in Theorem 9 are satisfied.

Theorem 9. *Let Ω be a partial order on the set of facets of a pure simplicial complex Δ. Then the following conditions on Ω are equivalent:*

(i) *For all facets F, G we have*

$$r_\Omega(F) \subseteq G \text{ and } r_\Omega(G) \subseteq F \implies F = G.$$

(ii) *Δ is the disjoint union*

$$\Delta = \dot{\bigcup_F} [r_\Omega(F), F].$$

(iii) *For all facets F, G*

$$r_\Omega(F) \subseteq G \quad \Rightarrow \quad F \leq^\Omega G.$$

(iv) *For all facets F, G: if $F \not\geq^\Omega G$ then there is an $x \in G$ and $E <^\Omega G$ such that*

$$F \cap G \subseteq E \cap G = G \setminus \{x\}.$$

Proof. (i) $\Rightarrow$ (ii): Let F and G be facets of Δ. If there is an $H \in [r_\Omega(F), F] \cap [r_\Omega(G), G]$ then $r_\Omega(F) \subseteq G$ and $r_\Omega(G) \subseteq F$, so by (i) we have $F = G$. Hence the union is disjoint. Suppose that $H \in \Delta$, and let F_0 be a minimal element, with respect to Ω, of the set

$$\{F : F \text{ is a facet and } H \subseteq F\}.$$

If $r_\Omega(F_0) \not\subseteq H$ then let $x \in r_\Omega(F_0) \setminus H$ and let $E <^\Omega F_0$ be such that $F_0 \cap E = F_0 \setminus \{x\}$. Then $H \subseteq E$, contradicting the minimality of F_0. This means that $H \in [r(F_0), F_0]$.

(ii) $\Rightarrow$ (i): If $r_\Omega(F) \subseteq G$ and $r_\Omega(G) \subseteq F$ we have that $F \cap G \in [r_\Omega(F), F] \cap [r_\Omega(G), G]$, which by (ii) gives us $F = G$.

(i) $\Rightarrow$ (iii): If $r_\Omega(F) \subseteq G$ then by (i) we have either $F = G$ or $r_\Omega(G) \not\subseteq F$. If $F = G$ we have nothing to prove, so we may assume that there is an $x \in r_\Omega(G) \setminus F$. Then, by assumption, there is a facet $E_1 <^\Omega G$ such that

$$r_\Omega(F) \subseteq G \cap E_1 = G \setminus \{x\} \subset E_1.$$

If $E_1 = F$ we are done. Otherwise we continue until we get

$$F = E_k <^\Omega E_{k-1} <^\Omega \cdots <^\Omega E_1 <^\Omega G,$$

and we are done.

(iii) $\Leftrightarrow$ (iv): It is easy to see that (iv) is just the contrapositive of (iii)

(iii) $\Rightarrow$ (i): Immediate. □

The set of all partial orders on the same set is partially ordered by inclusion, i.e $\Omega \subseteq \Lambda$ if $x <^\Omega y$ implies $x <^\Lambda y$.

Corollary 10. *Let Δ be a pure simplicial complex. Then*

(i) *all shellings of Δ are pre-shellings,*

(ii) *if Ω is a pre-shelling of Δ and Λ is a partial order such that $\Omega \subseteq \Lambda$, then Λ is a pre-shelling of Δ with $r_\Lambda(F) = r_\Omega(F)$ for all facets F. In particular, the set of all pre-shellings of Δ is an upper ideal of the poset of all partial orders on the set of facets of Δ,*

(iii) *all linear extensions of a pre-shelling are shellings, with the same restriction function.*

Proof. (i): Follows immediately from Theorem 9(iv).

(ii): That Λ is a pre-shelling follows from Theorem 9(iv). If F is a facet then by definition $r_\Omega(F) \subseteq r_\Lambda(F)$, and if $r_\Omega(F) \subset r_\Lambda(F)$ for some facet F we would have a contradiction by Theorem 9(ii).

(iii): Is implied by (ii). □

Let P be a finite graded poset with a smallest element $\hat{0}$ and a greatest element $\hat{1}$. The *order complex*, $\Delta(P)$, of P is the simplicial complex of all chains of P. A simplicial complex Δ is *partitionable* if it can be written as

$$\Delta = [r(F_1), F_1] \dot\cup [r(F_2), F_2] \dot\cup \cdots \dot\cup [r(F_n), F_n], \tag{1}$$

where each F_i is a facet of Δ and r is any function on the set of facets such that $r(F) \subseteq F$ for all facets F. The right hand side of (1) is a *partitioning* of Δ. By Theorem 9(iii) we see that shellable complexes are partitionable. We need the following well known fact about partitionable order complexes. Let $\mathfrak{M}(P)$ be the set of maximal chains of P.

Lemma 11. *Let $\Delta(P)$ be partitionable and let*

$$\Delta(P) = \dot{\bigcup_c} [r(c), c] \tag{2}$$

be a partitioning of $\Delta(P)$. Then the flag h-vector is given by

$$\beta_P(S) = |\{c \in \mathfrak{M}(P) : \rho(r(c)) = S\}|.$$

Proof. Let $\gamma_P(S) = |\{c \in \mathfrak{M}(P) : \rho(r(c)) = S\}|$. Note that if c is a maximal chain then $\rho(c) = [0, \rho(\hat{1})]$. By (2) we have

$$\begin{aligned} \alpha_P(S) &= |\{c \in \Delta(P) : \rho(c) = S\}| \\ &= |\{c \in \mathfrak{M}(P) : \rho(r(c)) \subseteq S\}| \\ &= \sum_{T \subseteq S} \gamma_P(T), \end{aligned}$$

which, by inclusion-exclusion, gives $\gamma_P(S) = \beta_P(S)$. □

We will identify the set of facets of $\Delta(J(\mathbf{2} \times \mathbf{n}))$ with $\mathcal{D}_n$, the set of Dyck paths of length $2n$. We therefore seek a partial order on $\mathcal{D}_n$ which is a pre-shelling. Let $S = S(\mathcal{D}_n)$ be the set of mappings with elements

$$s_i(w) = \begin{cases} a_1 \cdots a_{i-1} vhv a_{i+3} \cdots a_{2n} & \text{if } a_i a_{i+1} a_{i+2} = vvh, \\ a_1 \cdots a_{i-1} hvh a_{i+3} \cdots a_{2n} & \text{if } a_i a_{i+1} a_{i+2} = hhv, \\ w & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq 2n-2$. Define a relation Ω_n, by $u <^{\Omega} w$ whenever $u \neq w$ and $u = \sigma_1 \sigma_2 \cdots \sigma_k(w)$ for some mappings $\sigma_i \in S$ (see Figure 4).

Lemma 12. *The relation Ω_n on $\mathcal{D}_n$ is a partial order.*

Proof. We need to prove that Ω_n is anti-symmetric. To do this we define a mapping $\sigma : \mathcal{D}_n \to \mathbb{N} \times \mathbb{N}$, where $\mathbb{N} \times \mathbb{N}$ is ordered lexicographically, with the property

$$u <^{\Omega} w \Rightarrow \sigma(u) < \sigma(w).$$

Define $\sigma(w) = (\mathrm{da}(w), \mathrm{MAJ}(w))$, where $\mathrm{da}(w)$ is the number of double ascents (sequences vv) in w. Now, suppose that $s_i \in S$ and $s_i(w) \neq w = a_1 a_2 \cdots a_{2n}$. Then $\mathrm{da}(s_i(w)) \leq \mathrm{da}(w)$, and if we have equality we must have $a_{i-1} a_i a_{i+1} a_{i+2} = vvhv$ or $a_{i-1} a_i a_{i+1} a_{i+2} = hhvh$ which implies $\mathrm{MAJ}(s_i(w)) < \mathrm{MAJ}(w)$, so σ has the desired properties. □

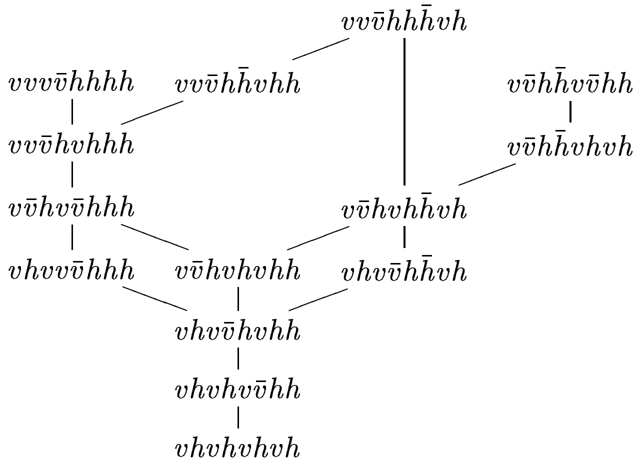
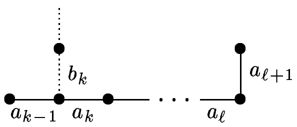


FIGURE 4. The partial order Ω_4 on $\mathcal{D}_4$, with long non-final sequences marked with bars.

FIGURE 5.



If v and w intersect maximally then it is plain to see that either $v = s(w)$ or $s(v) = w$ for some $s \in S$. It follows that if $w = a_1 a_2 \cdots a_{2n}$ then

$$r_{\Omega_n}(w) = \{a_1 + a_2 + \cdots + a_i : i \in LS(w)\},$$

so $\rho(r_{\Omega_n}(w)) = LS(w)$. It remains to prove that Ω_n is a pre-shelling.

Theorem 13. *For all $n \geq 1$ the partial order Ω_n is a pre-shelling of $\mathcal{D}_n$.*

Proof. We prove that Ω_n satisfies the contrapositive of condition (i) of Theorem 9. Suppose that $u = a_1 a_2 \cdots a_{2n} \neq w = b_1 b_2 \cdots b_{2n}$ and let k be the coordinate such that $a_i = b_i$ for $i < k$ and $a_k \neq b_k$. By symmetry we may assume that $a_k = h$. Now, if $a_{k-1} = h$ then the valley of u which is determined by the first v (at, say, coordinate $\ell + 1$) after k will correspond to an element

$$x = a_1 + \cdots + a_\ell \in r_{\Omega_n}(u) \setminus w$$

(see Figure 5).

If $a_{k-1} = v = b_{k-1}$, then if $\ell + 1$ is the coordinate for the first h after k we have that

$$x = b_1 + \cdots + b_\ell \in r_{\Omega_n}(w) \setminus u,$$

so Ω_n is a pre-shelling. □

If we define $\mathrm{MAJ}_\ell : \mathcal{D}_n \to \mathbb{N}$ by

$$\mathrm{MAJ}_\ell(w) = \sum_{i \in LS(w)} i,$$

we now have:

Corollary 14. *For all $n \geq 1$ we have*

$$\beta_n(S) = |\{w \in \mathcal{D}_n : LS(w) = S\}|,$$

In particular the bi-statistic $(\mathrm{lnfs}, \mathrm{MAJ}_\ell)$ *has the q-Narayana distribution.*

The Narayana statistic ea cannot in a natural way be associated to a shelling of $\Delta(J(\mathbf{2}\times\mathbf{n}))$. However, it would be interesting to find a co-statistic s for ea such that the bi-statistic (ea, s) has the q-Narayana distribution.

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