On Stationary and Time-Dependent Solutions to the Linear Boltzmann Equation

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Abstract

This paper considers the stationary linear, space-dependent Boltzmann equation in the case of an interior source term together with an absorption term and general boundary reflections. First, mild L^1-solutions are constructed as limits of iterate functions. Then an H-theorem on a relative entropy function of two different solutions is studied. Finally a generalized inequality (of Dorrozes-Guiraud type) for the boundary terms in the H-theorem is proved.

1 Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics, (e.g. for discribing the neutron distribution in reactor physics, cf. [1]-[4]).

One fundamental question concerns the large time behavior of the function $f(\mathbf{x}, \mathbf{v}, t)$, representing the distribution of particles; in particular, the problem of convergence to a stationary equilibrium solution, when time goes to infinity. In our earlier papers [5]-[8] we have studied such convergence to equilibrium for the space-dependent linear Boltzmann equation with general boundary conditions and general initial data, under the assumption of existence of a corresponding stationary solution. For the proofs we use iterate functions, defined by an exponential form of the equation together with the boundary conditions, and we also use a general relative entropy functional for the quotient of the time dependent and the stationary solutions.

Then a fundamental question in kinetics concerns the existence and uniqueness of stationary solutions to the space-dependent transport equation, with general collision mechanism (including the case of inverse power forces), together with general boundary conditions, (including the periodic, specular and diffuse cases). We will study this problem in the angular cut-off case, using our earlier methods with iterate functions $F^n(\mathbf{x},\mathbf{v})$, (representing the distribution of particles having undergone at most n collisions, inside the body or at the boundary).

This will be done in the case of an interior source term $\alpha_0 G(\mathbf{x},\mathbf{v})$, where $\alpha_0 > 0$ is a constant and G is a given function, together with an absorption term $\alpha F(\mathbf{x},\mathbf{v}), \alpha > 0$, and general boundary reflections, (see Section 3). We will also (in Section 4) prove an H-theorem for a relative entropy function of two solutions with different (absorption) coefficients; in order to study the problem of convergence, when the coefficients go to zero. Finally (in Section 5) we give a generalization of an inequality of Dorrozes-Guiraud type for the boundary terms in the H-theorem.

For bounded gain operators the problem of existence and uniqueness for solutions to the linear Boltzmann equation has been studied earlier by a different technique, cf. ref. [3]. But in our approach unbounded operators also are included, e.g. the case of hard inverse power collision forces.

2 Preliminaries

We consider the stationary transport equation for a distribution function $F(\mathbf{x},\mathbf{v})$, depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$. The stationary linear Boltzmann equation in the case of given interior source $\alpha_0 G(\mathbf{x},\mathbf{v})$, where $\alpha_0 > 0$ is a constant and $G \geq 0$ is a given (measurable) function, together with an absorption term $\alpha F(\mathbf{x},\mathbf{v})$, $\alpha > 0$, is in the strong form

$$\alpha F(\mathbf{x},\mathbf{v}) + \mathbf{v}\nabla_{\mathbf{x}} F(\mathbf{x},\mathbf{v}) = \alpha_0 G(\mathbf{x},\mathbf{v}) + (QF)(\mathbf{x},\mathbf{v}). \tag{2.1}$$

The collision term can be written

$$(QF)(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} [Y(\mathbf{x},\mathbf{v}'_*)F(\mathbf{x},\mathbf{v}') - Y(\mathbf{x},\mathbf{v}_*)F(\mathbf{x},\mathbf{v})] \cdot B(\theta, |\mathbf{v}-\mathbf{v}_*|) d\theta d\zeta d\mathbf{v}_*, \tag{2.2}$$

where $Y \geq 0$ is a known distribution function, and $B \geq 0$ is given by the collision process. Here $\mathbf{v}, \mathbf{v}_*$ are the velocities before and $\mathbf{v}', \mathbf{v}'_*$ the velocities after a binary collision, and $\Omega = \{(\theta,\zeta) : 0 \leq \theta \leq \hat{\theta}, 0 \leq \zeta < 2\pi\}$ is the impact plane. In the angular cut-off case with $\hat{\theta} < \frac{\pi}{2}$ the gain and the loss term can be separated

$$(QF)(\mathbf{x},\mathbf{v}) = \int_V K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) F(\mathbf{x},\mathbf{v}') d\mathbf{v}' - L(\mathbf{x},\mathbf{v}) F(\mathbf{x},\mathbf{v}), \tag{2.3}$$

where L is the collision frequency

$$L(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} B(\theta, w) Y(\mathbf{x},\mathbf{v}_*) d\theta d\zeta d\mathbf{v}_*, w = |\mathbf{v}-\mathbf{v}_*|. \tag{2.4}$$

In the case of nonabsorbing body we have

$$L(\mathbf{x},\mathbf{v}) = \int_V K(\mathbf{x},\mathbf{v}\to\mathbf{v}'')d\mathbf{v}''. \tag{2.5}$$

One physically interesting case is that with inverse k-th power collision forces

$$B(\theta, w) = b(\theta) w^\gamma,\ \gamma = \frac{k-5}{k-1}, \tag{2.6}$$

with hard forces for $k>5$, Maxwellian for $k=5$, and soft forces for $3<k<5$.

The equation (2.1) is supplemented with (general) boundary conditions

$$F_-(\mathbf{x},\mathbf{v}) = (1-\beta)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) F_+(\mathbf{x},\tilde{\mathbf{v}}) d\tilde{\mathbf{v}}, \tag{2.7}$$

where β is a constant, $0\le\beta\le 1$. The function $R\ge 0$ satisfies

$$\int_V R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})d\mathbf{v} \equiv 1, \tag{2.8}$$

and $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x}\in\Gamma$. The functions F_- and F_+ represent the ingoing and outgoing trace functions corresponding to F. Furthermore, in the specular reflection case, the function R is represented by Dirac measure $R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) = \delta(\mathbf{v}-\tilde{\mathbf{v}}+2\mathbf{n}(\mathbf{n}\mathbf{v}))$, and in the diffuse reflection case $R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) = |\mathbf{n}\mathbf{v}|W(\mathbf{x},\mathbf{v})$ with some given function $W\ge 0$ (e.g. Maxwellian function).

Now using differentiation along the characteristics, the equation (2.1) can formally be written

$$\begin{aligned}&\frac{d}{dt}(F(\mathbf{x}+t\mathbf{v},\mathbf{v})) = \alpha_0 G(\mathbf{x}+t\mathbf{v},\mathbf{v}) + \\ &+\int_V K(\mathbf{x}+t\mathbf{v},\mathbf{v}'\to\mathbf{v})F(\mathbf{x}+t\mathbf{v},\mathbf{v}')d\mathbf{v}' - [\alpha + L(\mathbf{x}+t\mathbf{v},\mathbf{v})]F(\mathbf{x}+t\mathbf{v},\mathbf{v}).\end{aligned} \tag{2.9}$$

Let

$$t_b \equiv t_b(\mathbf{x},\mathbf{v}) = \inf_{\tau\in\mathbb{R}_+}\{\tau : \mathbf{x}-\tau\mathbf{v}\notin D\}$$

and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x},\mathbf{v}) = \mathbf{x} - t_b\mathbf{v}$. Here t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b = \mathbf{x}-t_b\mathbf{v}$ to the point $\mathbf{x}$.

Then we have the following *mild form* of the stationary linear Boltzmann equation

$$F(\mathbf{x},\mathbf{v}) = F_-(\mathbf{x}_b,\mathbf{v}) + \int_0^{t_b} [(QF)(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \alpha_0 G(\mathbf{x}-\tau\mathbf{v},\mathbf{v})]d\tau \tag{2.10}$$

and the *exponential form*

$$\begin{aligned} F(\mathbf{x},\mathbf{v}) &= F_-(\mathbf{x}_b,\mathbf{v})e^{-\int_0^{t_b}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds} + \\ &+ \int_0^{t_b} e^{-\int_0^{\tau}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds}[\alpha_0 G(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \\ &+ \int_V K(\mathbf{x}-\tau\mathbf{v},\mathbf{v}'\to\mathbf{v})F(\mathbf{x}-\tau\mathbf{v},\mathbf{v}')d\mathbf{v}']d\tau. \end{aligned} \tag{2.11}$$

3 Construction of stationary solutions

We construct mild L^1-solutions to our problem as limits of iterate functions F^n, when $n\to\infty$. Let first $F^{-1}(\mathbf{x},\mathbf{v})\equiv 0$ for all $\mathbf{x},\mathbf{v}\in\mathbb{R}^3$. Then define, for given function F^{n-1} the next iterate F^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation);

$$F_-^n(\mathbf{x},\mathbf{v}) = (1-\beta)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})F_+^{n-1}(\mathbf{x},\tilde{\mathbf{v}})d\tilde{\mathbf{v}}, \tag{3.1}$$

$$\mathbf{n}\mathbf{v}<0, \mathbf{x}\in\Gamma=\partial D,\ \mathbf{v}\in V=\mathbb{R}^3;$$

$$\begin{aligned} F^n(\mathbf{x},\mathbf{v}) &= F_-^n(\mathbf{x}-t_b\mathbf{v},\mathbf{v})e^{-\int_0^{t_b}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds} + \\ &+ \int_0^{t_b} e^{-\int_0^{\tau}(\alpha+L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds}[\alpha_0 G(\mathbf{x}-\tau\mathbf{v},\mathbf{v}) + \\ &+ \int_V K(\mathbf{x}-\tau\mathbf{v},\mathbf{v}'\to\mathbf{v})F^{n-1}(\mathbf{x}-\tau\mathbf{v},\mathbf{v}')d\mathbf{v}']d\tau, \end{aligned} \tag{3.2}$$

$$\mathbf{x}\in D\setminus\Gamma_-(\mathbf{v}),\ \mathbf{v}\in V=\mathbb{R}^3.$$

Let also $F^n(\mathbf{x},\mathbf{v})\equiv 0$ for $\mathbf{x}\in\mathbb{R}^3\setminus D$.

Now we get a monotonicity lemma, which is essential in the following, and which can be proved by induction.

Lemma 3.1 $F^n(\mathbf{x},\mathbf{v}) \geq F^{n-1}(\mathbf{x},\mathbf{v}), \mathbf{x}\in D, \mathbf{v}\in V, \mathbf{n}\in\mathbb{N}.$

Using differentiation along the characteristics, we get by (3.2) that

$$\begin{aligned} &\frac{d}{dt}(F^n(\mathbf{x}+t\mathbf{v},\mathbf{v})) = \alpha_0 G(\mathbf{x}+t\mathbf{v},\mathbf{v}) - \alpha F^n(\mathbf{x}+t\mathbf{v},\mathbf{v}) + \\ &+ \int_V K(\mathbf{x}+t\mathbf{v},\mathbf{v}'\to\mathbf{v})F^{n-1}(\mathbf{x}+t\mathbf{v},\mathbf{v}')d\mathbf{v}' - L(\mathbf{x}+t\mathbf{v},\mathbf{v})F^n(\mathbf{x}+t\mathbf{v},\mathbf{v}). \end{aligned} \tag{3.3}$$

Then integrating (3.3), it follows by Green's formula that

$$\begin{aligned}
&\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \iint_{\Gamma V} F^n_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma = \\
&= \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \iint_{\Gamma V} F^n_-(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma + \\
&+ \iint_{DV} L(\mathbf{x},\mathbf{v})[F^{n-1}(\mathbf{x},\mathbf{v}) - F^n(\mathbf{x},\mathbf{v})]d\mathbf{x}d\mathbf{v},
\end{aligned} \tag{3.4}$$

where by (2.8) and (3.1)

$$\iint_{\Gamma V} F^n_-(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma = (1-\beta)\iint_{\Gamma V} F^{n-1}_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma \tag{3.5}$$

Now by Lemma 3.1 it follows that

$$\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \beta \iint_{\Gamma V} F^n_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma \le \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} \tag{3.6}$$

So, if $G \in L^1(D \times V)$, then we have for all $\alpha > 0$ that

$$\iint_{DV} F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} \le \frac{\alpha_0}{\alpha}\iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} < \infty. \tag{3.7}$$

Then Levi's theorem (on monotone convergence) gives existence of a mild (defined by (2.10)) L^1-solution $F(\mathbf{x},\mathbf{v}) = \lim_{n\to\infty} F^n(\mathbf{x},\mathbf{v})$ to the stationary linear Boltzmann equation (2.1) with (2.3), (2.7), and $F \equiv F_{\alpha,\beta,\alpha_0}$ satisfies for all $\alpha, \alpha_0 > 0, 0 \le \beta \le 1$, the inequality

$$\begin{aligned}
&\alpha \iint_{DV} F(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \beta \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma \le \\
&\le \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}.
\end{aligned} \tag{3.8}$$

Furthermore, if $L(\mathbf{x},\mathbf{v})F(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$, then we get equality in (3.8) together with uniqueness in the relevant function space, cf. [6] and also [3].

So, for instance, if $\beta = \rho \cdot \alpha, \alpha_0 = \alpha > 0, \rho \ge 0$, then

$$\iint_{DV} F(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \rho \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma = \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}. \tag{3.9}$$

In summary, we have the following existence theorem for solutions to our stationary linear Boltzmann equation with general boundary reflections.

Theorem 3.2 *Assume that K, L and R are nonnegative, measurable functions, such that (2.5) and (2.8) hold, and $L(\mathbf{x},\mathbf{v}) \in L^1_{\text{loc}}(D \times V)$. Let $\alpha, \alpha_0 > 0$ and $0 \le \beta \le 1$ be constants, and $G(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$ with $\iint G d\mathbf{x}d\mathbf{v} > 0$.*

a) Then there exists a mild L^1-solution $F(\mathbf{x},\mathbf{v})$ to the problem (2.1)-(2.4) with (2.7). This solution, depending on α, α_0 and β, satisfies the inequality (3.8).

b) Moreover, if $L(\mathbf{x},\mathbf{v})F(\mathbf{x},\mathbf{v}) \in L^1(D\times V)$, then the trace of the solution F satisfies the boundary condition (2.7) for a.e. $(\mathbf{x},\mathbf{v}) \in \Gamma \times V$. Furthermore, mass conservation, giving equality in (3.8), holds, together with uniqueness in the relevant L^1-space.

Remarks:

1) For the case $\alpha = \alpha_0 = 0, \beta = 0$, we have in an earlier paper obtained uniqueness of mild L^1-solutions to the stationary linear Boltzmann equation, using a general entropy functional cf. [8]; cf. also Section 4 in this paper.

2) The statement in Theorem 3.2 (b) on existence of traces follows e.g. from Proposition 3.3, Chapter XI, in [3].

3) The assumption $LF \in L^1(D\times V)$ is for instance, satisfied for the solution F in the case of inverse power collision forces, cf. (2.6), together with specular or diffuse boundary reflections. This follows from a statement on global boundedness of higher velocity moments, cf. Theorem 4.1 in [9],

$$\begin{aligned}&\iint_{DV} (1+v^2)^{\sigma/2} F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \leq C_\sigma < \infty,\\ &\sigma \geq \max(0,\gamma), -1 < \gamma = (k-5)/(k-1) < 1,\end{aligned} \tag{3.10}$$

if $(1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) \in L^1(D\times V)$.

For a proof of (3.10) we can multiply equation (3.3) by $(1+v^2)^{\sigma/2}$, and integrate, (using Green's formula) getting an equation analogous to (3.4). Then the gain and loss terms can be estimated, using an inequality for the velocities in a binary collision, cf. [5],

$$\begin{aligned}&(1+(v')^2)^{\sigma/2} - (1+v^2)^{\sigma/2} \leq \\ &\leq C_1 w \cos\theta (1+v_*)^{\max(1,\sigma-1)} (1+v^2)^{\frac{\sigma-2}{2}} - \\ &- C_2 w \cos^2\theta (1+v^2)^{\frac{\sigma-1}{2}},\end{aligned}$$

with constants $C_1, C_2 > 0$ and $\sigma > 0$, together with some elementary estimate,

$$-w^{\gamma+1} \leq (1+v_*)^{\gamma+1} - 2^{-1}(1+v^2)^{\frac{\gamma+1}{2}}.$$

The function Y in (2.2) is here assumed to satisfy the following conditions

$$\int_V (1+v_*)^{\gamma+\max(2,\sigma)} \sup_{x\in D}(Y(\mathbf{x},\mathbf{v}_*)) d\mathbf{v}_* < \infty,$$

$$\int_V \inf_{x\in D}(Y(\mathbf{x},\mathbf{v}_*)) d\mathbf{v}_* > 0.$$

For further details on boundedness of higher velocity moments, see [9], and also our earlier papers [5]-[8].

4 An H-theorem (for a Relative Entropy function)

In this section we will prove an entropy theorem for the quotient of two solutions from Section 3, $\bar{F}$ and F, with coefficients $\bar{\alpha}$, $\bar{\alpha}_0, \bar{\beta}$ and α, α_0, β respectively. So we start from the equations

$$\begin{aligned}
&\frac{d}{dt}[\bar{F}(\mathbf{x}+t\mathbf{v},\mathbf{v})] = \bar{\alpha}_0 G(\mathbf{x}+t\mathbf{v},\mathbf{v}) - \bar{\alpha}\bar{F}(\mathbf{x}+t\mathbf{v},\mathbf{v}) + \\
&+\int K(\mathbf{x}+t\mathbf{v},\mathbf{v}'\to\mathbf{v})\bar{F}(\mathbf{x}+t\mathbf{v},\mathbf{v}')d\mathbf{v}' - L(\mathbf{x}+t\mathbf{v},\mathbf{v})\bar{F}(\mathbf{x}+t\mathbf{v},\mathbf{v}), \\
&\bar{F}_-(\mathbf{x},\mathbf{v}) = (1-\bar{\beta})\int \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})\bar{F}_+(\mathbf{x},\tilde{\mathbf{v}})d\tilde{\mathbf{v}},
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\frac{d}{dt}[F(\mathbf{x}+t\mathbf{v},\mathbf{v})] = \alpha_0 G(\mathbf{x}+t\mathbf{v},\mathbf{v}) - \alpha F(\mathbf{v}+t\mathbf{v},\mathbf{v}) + \\
&+\int K(\mathbf{x}+t\mathbf{v},\mathbf{v}'\to\mathbf{v})F(\mathbf{x}+t\mathbf{v},\mathbf{v}')d\mathbf{v}' - L(\mathbf{x}+t\mathbf{v},\mathbf{v})F(\mathbf{x}+t\mathbf{v},\mathbf{v}), \\
&F_-(\mathbf{x},\mathbf{v}) = (1-\beta)\int \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})F_+(\mathbf{x},\tilde{\mathbf{v}})d\tilde{\mathbf{v}}
\end{aligned} \tag{4.2}$$

To prove an H-theorem for the convex function $\varphi(z) = (z-1)^2, z = \bar{F}/F$, we begin with the following calculations

$$\begin{aligned}
&\frac{d}{dt}\Big[\Big(\frac{\bar{F}(\mathbf{x}+t\mathbf{v},\mathbf{v})}{F(\mathbf{x}+t\mathbf{v},\mathbf{v})}-1\Big)^2\cdot F(\mathbf{x}+t\mathbf{v},\mathbf{v})\Big] \equiv \\
&\equiv \frac{d}{dt}\Big[\Big(\frac{\bar{F}}{F}\Big)^2 F - 2\bar{F} + F\Big](\mathbf{x}+t\mathbf{v},\mathbf{v}) \equiv \\
&\equiv \Big[2\frac{\bar{F}}{F}\frac{d\bar{F}}{dt} - \Big(\frac{\bar{F}}{F}\Big)^2\frac{dF}{dt} - 2\frac{d\bar{F}}{dt} + \frac{dF}{dt}\Big](\mathbf{x}+t\mathbf{v},\mathbf{v}) = \\
&= 2\Big[\frac{\bar{F}}{F}-1\Big]\cdot[\bar{\alpha}_0 G - \bar{\alpha}\bar{F} + \int K\bar{F}(\mathbf{v}')d\mathbf{v}' - L\bar{F}] + \\
&+ \Big[1-\Big(\frac{\bar{F}}{F}\Big)^2\Big]\cdot[\alpha_0 G - \alpha F + \int KF(\mathbf{v}')d\mathbf{v}' - LF],
\end{aligned}$$

where we have shortened the notations to the essential variables.

Now some (elementary) calculations give

$$\frac{d}{dt}\Big[\Big(\frac{\bar F}{F}-1\Big)^2 F\Big](\mathbf{x}+t\mathbf{v},\mathbf{v}) =$$
$$= \Big(\frac{\bar\alpha_0}{\alpha_0}-1\Big)^2\alpha_0 G - \Big(\frac{\bar F}{F}-\frac{\bar\alpha_0}{\alpha_0}\Big)^2\alpha_0 G +$$
$$+\frac{(\bar\alpha-\alpha)^2}{2\bar\alpha-\alpha}F - (2\bar\alpha-\alpha)\Big(\frac{\bar F}{F}-\frac{\bar\alpha}{2\bar\alpha-\alpha}\Big)^2 F -$$
$$-\int K(\mathbf{v}'\to\mathbf{v})\Big|\frac{\bar F(\mathbf{v})}{F(\mathbf{v})}-\frac{\bar F(\mathbf{v}')}{F(\mathbf{v}')}\Big|^2 F(\mathbf{v}')d\mathbf{v}' +$$
$$+\int K(\mathbf{v}'\to\mathbf{v})\Big(\frac{\bar F(\mathbf{v}')}{F(\mathbf{v}')}-1\Big)^2 F(\mathbf{v}')d\mathbf{v}' -$$
$$-L(\mathbf{v})\Big(\frac{\bar F(\mathbf{v})}{F(\mathbf{v})}-1\Big)^2 F(\mathbf{v}) \tag{4.3}$$

where $L(\mathbf{v}) = \int K(\mathbf{v}\to\mathbf{v}')d\mathbf{v}'$.

Then by integration of (4.3) using Green's formula, and also by the relation

$$\begin{aligned}&\iint_{VV} K(\mathbf{v}\to\mathbf{v}')d\mathbf{v}'\Big(\frac{\bar F(\mathbf{v})}{F(\mathbf{v})}-1\Big)^2 F(\mathbf{v})d\mathbf{v} =\\ &\iint_{VV} K(\mathbf{v}'\to\mathbf{v})\Big(\frac{\bar F(\mathbf{v}')}{F(\mathbf{v}')}-1\Big)^2 F(\mathbf{v}')d\mathbf{v}d\mathbf{v}',\end{aligned} \tag{4.4}$$

it follows that

$$\iint_{\Gamma V}\Big(\frac{\bar F_+(\mathbf{x},\mathbf{v})}{F_+(\mathbf{x},\mathbf{v})}-1\Big)^2 F_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma -$$
$$-\iint_{\Gamma V}\Big(\frac{\bar F_-(\mathbf{x},\mathbf{v})}{F_-(\mathbf{x},\mathbf{v})}-1\Big)^2 F_-(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma +$$
$$+\alpha_0\iint_{DV}\Big(\frac{\bar F(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})}-\frac{\bar\alpha_0}{\alpha_0}\Big)^2 G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} +$$
$$+(2\bar\alpha-\alpha)\iint_{DV}\Big(\frac{\bar F(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})}-\frac{\bar\alpha}{2\bar\alpha-\alpha}\Big)^2 F(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} +$$
$$+\iiint_{DVV} K(\mathbf{x},\mathbf{v}'\to\mathbf{v})\Big|\frac{\bar F(\mathbf{x},\mathbf{v}')}{F(\mathbf{x},\mathbf{v}')}-\frac{\bar F(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})}\Big|^2 F(\mathbf{x},\mathbf{v}')d\mathbf{x}d\mathbf{v}d\mathbf{v}' =$$
$$= \frac{(\bar\alpha_0-\alpha_0)^2}{\alpha_0}\iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \frac{(\bar\alpha-\alpha)^2}{2\bar\alpha-\alpha}\iint_{DV} F(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}, \tag{4.5}$$

where (right-hand side)

$$RHS \le \Big[\frac{(\bar\alpha_0-\alpha_0)^2}{\alpha_0}+\frac{(\bar\alpha-\alpha)^2\alpha_0}{\alpha(2\bar\alpha-\alpha)}\Big]\iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v},$$

because of the massrelation

$$\alpha \iint_{DV} F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \beta \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma \leq \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}. \quad (4.6)$$

For the boundary terms we use, if $\beta = \bar{\beta} = 0$, a Darrozes-Guiraud inequality,

$$\int_V \Big(\frac{\bar{F}_-}{F_-} - 1\Big)^2 F_- |\mathbf{nv}| d\mathbf{v} \leq \int_V \Big(\frac{\bar{F}_+}{F_+} - 1\Big)^2 F_+ |\mathbf{nv}| d\mathbf{v}. \quad (4.7)$$

More generally, for $\beta = \alpha\rho > 0, \bar{\beta} = \bar{\alpha}\rho > 0$, it follows from Theorem 5.1 (Section 5) with $\varphi(z) = (z-1)^2$ and $K = 1$, that

$$\begin{aligned} &\int_V \Big(\frac{\bar{F}_-(\mathbf{v})}{F_-(\mathbf{v})} - 1\Big)^2 F_-(\mathbf{v}) |\mathbf{nv}| d\mathbf{v} \leq \\ &\leq (1-\beta) \int_V \Big[\frac{(1-\bar{\beta})\bar{F}_+(\mathbf{v})}{(1-\beta)F_+(\mathbf{v})} - 1\Big]^2 F_+(\mathbf{v}) |\mathbf{nv}| d\mathbf{v}. \end{aligned} \quad (4.8)$$

Now, in the case $\beta = \bar{\beta} = 0$, the following H-theorem (on relative entropy) holds, cf. (4.5), (4.7)

$$\begin{aligned} &\alpha_0 \iint_{DV} \Big(\frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} - \frac{\bar{\alpha}_0}{\alpha_0}\Big)^2 G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\ &+(2\bar{\alpha} - \alpha) \iint_{DV} \Big(\frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} - \frac{\bar{\alpha}}{2\bar{\alpha}-\alpha}\Big)^2 F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\ &+ \iiint_{DVV} K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) \Big|\frac{\bar{F}(\mathbf{x},\mathbf{v}')}{F(\mathbf{x},\mathbf{v}')} - \frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})}\Big|^2 F(\mathbf{x},\mathbf{v}') d\mathbf{x} d\mathbf{v} d\mathbf{v}' \\ &\leq \Big[\frac{(\bar{\alpha}_0 - \alpha_0)^2}{\alpha_0} + \frac{\alpha_0(\bar{\alpha} - \alpha)^2}{\alpha(2\bar{\alpha} - \alpha)}\Big] \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}. \end{aligned} \quad (4.9)$$

Furthermore, let now $\alpha_0 = \alpha, \bar{\alpha}_0 = \bar{\alpha}$, and define δ by $\bar{\alpha} = \alpha(1-\delta)$, i.e. $\delta = 1 - \frac{\bar{\alpha}}{\alpha}$, and suppose that $0 < \delta < 1/2$.

Then, after division by $\alpha = \alpha_0$ in the H-theorem (4.9), it follows that

$$\begin{aligned} &\iint_{DV} \Big(\frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} - 1 + \delta\Big)^2 G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\ &+(1-2\delta) \iint_{DV} \Big(\frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} - \frac{1-\delta}{1-2\delta}\Big)^2 F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\ &+\frac{1}{\alpha} \iiint_{DVV} K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) \Big|\frac{\bar{F}(\mathbf{x},\mathbf{v}')}{F(\mathbf{x},\mathbf{v}')} - \frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})}\Big|^2 F(\mathbf{x},\mathbf{v}') d\mathbf{x} d\mathbf{v} d\mathbf{v}' \leq \\ &\leq \delta^2 \Big[1 + \frac{1}{1-2\delta}\Big] \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}. \end{aligned} \quad (4.10)$$

Further calculations (among others using an intermediate step through $\alpha, \bar{\alpha}_0$ between α, α_0 and $\bar{\alpha}, \bar{\alpha}_0$) give the following version of the H-theorem (when $\rho = 0$)

$$\begin{aligned}
&\iint_{DV} \Big(\frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} - 1\Big)^2 G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\
&+ \iint_{DV} \Big(\frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} - 1\Big)^2 F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\
&+ \frac{1}{\alpha} \iiint_{DVV} K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) \Big| \frac{\bar{F}(\mathbf{x},\mathbf{v}')}{F(\mathbf{x},\mathbf{v}')} - \frac{\bar{F}(\mathbf{x},\mathbf{v})}{F(\mathbf{x},\mathbf{v})} \Big|^2 F(\mathbf{x},\mathbf{v}') d\mathbf{x} d\mathbf{v} d\mathbf{v}' \leq \\
&\leq C \cdot \delta^2 \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v},
\end{aligned} \tag{4.11}$$

with constant $C > 0$.

In the rest of this section we will study a sequence of stationary solutions $\{F_N(\mathbf{x},\mathbf{v})\}_{N=2}^{\infty}$ by choosing the coefficients α and $\bar{\alpha}$ as follows. Let

$$\alpha = \alpha_N = \frac{1}{N}, \bar{\alpha} = \alpha_{N+1} = \frac{1}{N+1}, N = 2, 3, 4, \dots, \tag{4.12}$$

so $\delta = \delta_N \equiv 1 - \frac{\bar{\alpha}}{\alpha} = \frac{1}{N+1}$.

Then the H-theorem (4.11) with $\bar{F} = F_{N+1}, F = F_N$ can be written (for $N = 2, 3, \dots$) as

$$\begin{aligned}
&\iint_{DV} \Big(\frac{F_{N+1}(\mathbf{x},\mathbf{v})}{F_N(\mathbf{x},\mathbf{v})} - 1\Big)^2 G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\
&+ \iint_{DV} \Big(\frac{F_{N+1}(\mathbf{x},\mathbf{v})}{F_N(\mathbf{x},\mathbf{v})} - 1\Big)^2 F_N(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\
&+ N \iiint_{DVV} K(\mathbf{x},\mathbf{v}' \to \mathbf{v}) \Big| \frac{F_{N+1}(\mathbf{x},\mathbf{v}')}{F_N(\mathbf{x},\mathbf{v}')} - \frac{F_{N+1}(\mathbf{x},\mathbf{v})}{F_N(\mathbf{x},\mathbf{v})} \Big|^2 F_N(\mathbf{x},\mathbf{v}') d\mathbf{x} d\mathbf{v} d\mathbf{v}' \leq \\
&\leq \frac{C}{N^2} \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}.
\end{aligned} \tag{4.13}$$

From (4.13) it follows (among others), if $0 < G(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$ that

$$\lim_{N \to \infty} \frac{F_{N+1}(\mathbf{x},\mathbf{v})}{F_N(\mathbf{x},\mathbf{v})} \equiv 1, \tag{4.14}$$

where $\alpha_N = \frac{1}{N} \to 0$ when $N \to \infty$.

Furthermore, we get (from (4.13) when $\rho = 0$) also the following theorem for the sequence of our solutions $\{F_N(\mathbf{x},\mathbf{v})\}_{N=2}^{\infty}$. The case $\rho > 0$ can be proved in the same manner, cf. (4.8).

Theorem 4.1 *Let $F_N(\mathbf{x}, \mathbf{v})$ be the solutions from Theorem 3.2 corresponding to (absorption) coefficients $\alpha = \alpha_k = 1/k, k = 2, 3, \ldots,$ and let $\alpha_0 = \alpha, \beta = \alpha\rho, \rho \geq 0,$ and assume that $G(\mathbf{x}, \mathbf{v}), L(\mathbf{x}, \mathbf{v})F_k(\mathbf{x}, \mathbf{v}) \in L^1(D \times V)$. Then the following estimates hold*

a)

$$\sum_{k=N}^{\infty} \iint_{DV} \Big(\frac{F_{k+1}(\mathbf{x}, \mathbf{v})}{F_k(\mathbf{x}, \mathbf{v})} - 1\Big)^2 G(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \leq \frac{C_1}{N}$$

b)

$$\sum_{k=N}^{\infty} \iint_{DV} \Big(\frac{F_{k+1}(\mathbf{x}, \mathbf{v})}{F_k(\mathbf{x}, \mathbf{v})} - 1\Big)^2 F_k(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \leq \frac{C_2}{N}$$

c)

$$\begin{aligned} &\sum_{k=N}^{\infty} k \iiint_{DVV} K(\mathbf{x}, \mathbf{v}' \to \mathbf{v}) \Big| \frac{F_{k+1}(\mathbf{x}, \mathbf{v}')}{F_k(\mathbf{x}, \mathbf{v}')} - \frac{F_{k+1}(\mathbf{x}, \mathbf{v})}{F_k(\mathbf{x}, \mathbf{v})} \Big|^2 \cdot \\ &\cdot F_k(\mathbf{x}, \mathbf{v}') d\mathbf{x} d\mathbf{v} d\mathbf{v}' \leq \frac{C_3}{N} \end{aligned} \tag{4.15}$$

with positive constants C_1, C_2, C_3.

Remark: We observe that

$\alpha_N = \alpha_{N-1}(1 - \delta_N) = \alpha_2 \prod_{k=3}^{N}(1 - \delta_k) = \alpha_2 \exp(\sum_{k=3}^{N} \ln(1 - \delta_k))$
$= \alpha_2 \exp(-\sum_{k=3}^{N} \delta_k + O(1) \sum_{k=3}^{N} \delta_k^2)$, using MacLaurin's formula, where $O(1)$ is a bounded function, when $N \to \infty$. So we find that $\alpha_N \to 0$, when $N \to \infty$, if and only if $\sum_3^\infty \delta_k = \infty$.

5 An inequality (of Dorrezes-Guiraud type) for the boundary terms (in an H-theorem)

We will here prove a generalization of an inequality (cf. [2], p.115, "A remarkable inequality"), concerning the relation between ingoing and outgoing boundary terms in an H-theorem for the relative entropy functional with general convex function.

Suppose that $\bar{F}$ and F are two functions satisfying the following boundary conditions, cf. (2.7)

$$\bar{F}_-(\mathbf{v}) = \bar{c} \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\tilde{\mathbf{v}} \to \mathbf{v}) \bar{F}_+(\tilde{\mathbf{v}}) d\tilde{\mathbf{v}}, \tag{5.1}$$

$$F_-(\mathbf{v}) = c \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\tilde{\mathbf{v}} \to \mathbf{v}) F_+(\tilde{\mathbf{v}}) d\tilde{\mathbf{v}}, \tag{5.2}$$

with (for simplicity) $\bar{c} = 1 - \bar{\beta}, c = 1 - \beta, 0 \leq \bar{\beta}, \beta < 1$.

Let

$$W(\tilde{\mathbf{v}}) = cR(\tilde{\mathbf{v}} \to \mathbf{v})|\mathbf{n}\tilde{\mathbf{v}}|F_+(\tilde{\mathbf{v}})/(|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})). \tag{5.3}$$

Then $\int_V W(\tilde{\mathbf{v}})d\tilde{\mathbf{v}} \equiv 1$, so $W(\tilde{\mathbf{v}})$ is a weight-function, because of the boundary conditions (5.2) for F.

Suppose $\varphi(z)$ is a (general) strictly convex continuous function. Then we have, because of Jensens inequality, that

$$\varphi(\int_V g(\tilde{\mathbf{v}})W(\tilde{\mathbf{v}})d\mathbf{v}) \leq \int_V \varphi(g(\tilde{\mathbf{v}}))W(\tilde{\mathbf{v}})d\tilde{\mathbf{v}}. \tag{5.4}$$

Choose now the function $g = k\frac{\bar{F}}{F}$, with a constant $k > 0$, so

$$g(\tilde{\mathbf{v}}) = k\frac{\bar{F}_+(\tilde{\mathbf{v}})}{F_+(\tilde{\mathbf{v}})}, g(\mathbf{v}) = k\frac{\bar{F}_-(\mathbf{v})}{F_-(\mathbf{v})}, \tag{5.5}$$

for $\mathbf{n}\mathbf{v} < 0 < \mathbf{n}\tilde{\mathbf{v}}$.

Then we can prove the following generalization of the Dorrezes-Guiraud inequality for the boundary terms.

Theorem 5.1 *Suppose $\bar{F}$ and F satisfy the boundary conditions (5.1), (5.2), and let $\varphi(z)$ be a strictly convex continuous function. Then (for any constant $K > 0$)*

$$\begin{aligned} &\int_V \varphi\Big(K\frac{\bar{F}_-(\mathbf{v})}{F_-(\mathbf{v})}\Big)|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})d\mathbf{v} \leq \\ &\leq c\int_V \varphi\Big(K\frac{\bar{c}\bar{F}_+(\mathbf{v})}{cF_+(\mathbf{v})}\Big)|\mathbf{n}\mathbf{v}|F_+(\mathbf{v})d\mathbf{v}. \end{aligned} \tag{5.6}$$

Proof. By the boundary conditions (5.1) for $\bar{F}$ we get, cf. (5.3) and (5.5), that

$$
\begin{aligned}
& g(\mathbf{v}) = k\frac{\bar{F}_-(\mathbf{v})}{F_-(\mathbf{v})} = \\
& = k\bar{c}\int_V |\mathbf{n}\tilde{\mathbf{v}}|R(\tilde{\mathbf{v}} \to \mathbf{v})\bar{F}_+(\tilde{\mathbf{v}})d\tilde{\mathbf{v}}/(|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})) = \\
& = \frac{\bar{c}}{c}\int cR(\tilde{\mathbf{v}} \to \mathbf{v})F_+(\tilde{\mathbf{v}})k\frac{\bar{F}_+(\tilde{\mathbf{v}})}{F_+(\tilde{\mathbf{v}})}d\tilde{\mathbf{v}}/(|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})) = \\
& = \frac{\bar{c}}{c}\int W(\tilde{\mathbf{v}})g(\tilde{\mathbf{v}})d\tilde{\mathbf{v}}.
\end{aligned}
$$

Multiplying by $\frac{c}{\bar{c}}$, and taking $\varphi()$ of both sides, we find, by Jensens inequality (5.4), that

$$
\begin{aligned}
& \varphi\Big(\frac{c}{\bar{c}}g(\mathbf{v})\Big) = \varphi(\int W(\tilde{\mathbf{v}})g(\tilde{\mathbf{v}})d\tilde{\mathbf{v}}) \leq \\
& \leq \int \varphi(g(\tilde{\mathbf{v}}))W(\tilde{\mathbf{v}})d\tilde{\mathbf{v}} = \\
& = \int \varphi(g(\tilde{\mathbf{v}}))cR(\tilde{\mathbf{v}} \to \mathbf{v})|\mathbf{n}\tilde{\mathbf{v}}|F_+(\tilde{\mathbf{v}})d\tilde{\mathbf{v}}/(|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})).
\end{aligned}
$$

Then multiplying by $|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})$, and integrating $\int \ldots d\mathbf{v}$, we get

$$
\begin{aligned}
& \int_V \varphi\Big(\frac{c}{\bar{c}}g(\mathbf{v})\Big)|\mathbf{n}\mathbf{v}|F_-(\mathbf{v})d\mathbf{v} \leq \\
& \leq c\int \varphi(g(\tilde{\mathbf{v}}))|\mathbf{n}\tilde{\mathbf{v}}|F_+(\tilde{\mathbf{v}})d\tilde{\mathbf{v}},
\end{aligned}
$$

which gives the inequality (5.6), if we use (5.5) and set $k = \frac{\bar{c}}{c}K$.

Remark. If $k = c = \bar{c} = 1$, then we get the inequality in [2].

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