FUNCTIONAL CALCULUS FOR NON-COMMUTING OPERATORS WITH REAL SPECTRA VIA AN ITERATED CAUCHY FORMULA

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ABSTRACT. We define a smooth functional calculus for a non-commuting tuple of (unbounded) operators A_j on a Banach space with real spectra and resolvents with temperate growth, by means of an iterated Cauchy formula. The construction is also extended to tuples of more general operators allowing smooth functional calculii. We also discuss the relation to the case with commuting operators.

1. INTRODUCTION

There are many different approaches to functional calculus for one or several operators acting on a Banach space, a common idea being that in order to define $f(P)$ where P is some operator and f a function of some suitable class, we represent $f(x)$ as a superposition of simpler functions $\omega_\alpha(x)$, for which $\omega_\alpha(P)$ can be defined and then define $f(P)$ as the corresponding superposition of the operators $\omega_\alpha(P)$. For instance, if P is a self-adjoint operator on a Hilbert space, we have

$$f(P) = \frac{1}{2\pi} \int \widehat{f}(t) e^{itP} dt, \tag{1.1}$$

corresponding to the representation of f as a superposition of exponential functions via Fourier's inversion formula. (Here $\widehat{f}$ denotes the standard Fourier transform of f. This approach has been developed by M. Taylor [23] and others.) Another example is when P is a bounded operator and f is holomorphic in a neighborhood of the spectrum, $\sigma(P)$, of P. Then

$$f(P) = \frac{1}{2\pi i} \int_\gamma f(z)(z-P)^{-1} dz \tag{1.2}$$

where γ is closed contour around $\sigma(P)$.

For problems of spectral asymptotics and scattering for partial differential operators, the representation (1.1) often has led to the sharpest known results (see Hörmander [14], Ivrii [15]), but the price to pay

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is that one has to get a good understanding of the associated unitary group for instance via the theory of Fourier integral operators or via propagation estimates. Often a formula like (1.2) is easier and more practical to use. (See for instance Agmon–Kannai [1], Seeley [19].) The advantage is that the resolvent $(z-P)^{-1}$ can be treated with simple means (like the theory of pseudodifferential operators).

If P is bounded, $f(z)$ is defined with its derivatives on the spectrum of P and has an extension $\widetilde{f}$ to a neighborhood of the spectrum such that $\overline{\partial}\widetilde{f}$ vanishes to infinite order on $\sigma(P)$, and if the resolvent only blows up polynomially when z tends to the spectrum, then Dynkin [11] used the Cauchy-Green formula

$$\widetilde{f}(w) = -\frac{1}{\pi}\int (z-w)^{-1}\partial_{\overline{z}}\widetilde{f}(z)L(dz),\ \ L(dz) = d(\operatorname{Re} z)d(\operatorname{Im} z),$$

to define

$$f(P) = -\frac{1}{\pi}\int (z-P)^{-1}\partial_{\overline{z}}\widetilde{f}(z)L(dz), \tag{1.3}$$

and he studied the corresponding functional calculus (also with other classes of functions f allowing for wilder resolvent behaviour). This work has been very influencial (see below).

Unknowingly of [11], Helffer and the second author [13] used (1.3) as a practical device in the study of magnetic Schrödinger operators in the framework of unbounded non-selfadjoint operators P; $\widetilde{f}$ is then the standard almost holomorphic extension of $f \in C_0^\infty(\mathbb{R})$. (We refrain from reviewing here the history of almost holomorphic extensions with roots in the work of Hörmander, Nirenberg, Dynkin and others.) It was soon realized that (1.3) is of great practical usefulness for many problems in spectral and scattering theory and in mathematical physics, because it is simple to manipulate without requiring holomorphy of the test-functions f. For instance, if P is an elliptic differential operator and f belongs to a suitable class of functions, it is very easy to show that $f(P)$ is a pseudodifferential operator ([13], [9]), and other applications were obtained in cases where f does not necessarily have compact support (E.B. Davies [8], A. Jensen, S. Nakamura [16]). Another application of (1.3) is in the area of trace formulae and effective Hamiltonians: For a given operator $P : \mathcal{H} \to \mathcal{H}$, one sometimes introduces an auxiliary (so called Grushin-, or in more special situations Feschbach-) problem:

$$(P-z)u + R_- u_- = v,\ \ R_+ v = v_+. \tag{1.4}$$

Here the auxiliary operators $R_+ : \mathcal{H} \to \mathcal{C}_+$, $R_- : \mathcal{C}_- \to \mathcal{H}$ should be chosen in such a way that the problem (1.4) has a unique solution

$$\mathcal{H} \ni u = Ev + E_+ v_+,\ \ \mathcal{C}_- \ni u_- = E_- v + E_{-+} v_+.$$

for all $v \in \mathcal{H}$, $v_+ \in \mathcal{C}_+$. Then it is well-known that the operator E_{-+} inherits many of the properties of P, and typically one looks for spaces $\mathcal{C}_\pm$ which are "smaller" in some sense, so that the study of E_{-+} may be easier than that of P. For trace formulae one can show under quite general assumptions that

$$\operatorname{tr} f(P) = \operatorname{tr} \frac{1}{\pi} \int \frac{\partial \widetilde{f}}{\partial z}(z)(E_{-+})^{-1} \frac{dE_{-+}}{dz}(z) L(dz). \tag{1.5}$$

which is very useful for instance when the spaces $\mathcal{C}_\pm$ are of finite (and here equal) dimensions.

The approach of Dynkin [11] has had a great influence on many later works devoted to general problems of functional calculus. In [20] J. Taylor introduced a notion of joint spectrum $\sigma(P) \subset \mathbb{C}^n$ for several commuting bounded operators $P_1, \ldots, P_m$ on a Banach space, defined in terms of the mapping properties of the operators. This spectrum is in general strictly smaller than the joint spectrum one obtains by regarding P_j as elements in some Banach subalgebra of $\mathcal{L}(B)$. In [21] he then constructed a general holomorphic functional calculus $\mathcal{O}(\sigma(P)) \to \mathcal{L}(B)$ and proved basic functorial properties. In simple cases, for instance if the function f is entire, one can use a simple multiple Cauchy formula to represent $f(P)$, but the general case is intricate, and Taylor's first construction was based on quite abstract Cauchy–Weil formulas; later on in [22] he made the whole construction with cohomological methods. In [2] was given a construction based on a multivariable notion of resolvent ω_{z-P} which permits a representation of the calculus analogous to formula (1.2). In special cases, for instance when the spectrum is real, such a representation was known earlier, and was used by Droste [10], following Dynkin's approach (1.3), to obtain a smooth functional calculus in the multivariable case for operators with real spectra. This approach is extended to more general spectra in [18].

Various versions of functional calculus have been used in the study of the joint spectrum of several commuting selfadjoint operators ([7], [5, 6]), and for nonselfadjoint operators with real spectra in [4].

The case of non-commuting operators is more difficult and more challenging. The monograph of Nazaikinski, Shatalov and Sternin [17] gives a nice treatment of such a theory and contains references to many earlier works of V.P. Maslov and others. The authors build the theory on the approximation of functions of several variables by linear combinations of tensor products. If $f(x_1, x_2, ..., x_m) = \prod_1^m f_j(x_j)$ is such a tensor product and P_j are operators on the same Banach space, that do not necessarily commute, it is natural to define $f(P_1, ..., P_m)$ as $f_1(P_1) \circ ... \circ f_m(P_m)$, and then approximate a general $f(x_1, ..., x_m)$ by linear combinations of tensor products, and define $f(P_1, ..., P_m)$ as the corresponding limit in the space of operators. A prototype for non-commutative functional calculus is given by the theory of pseudo-differential operators, with $x_1, x_2, ..., x_n, D_{x_1}, ..., D_{x_n}$ as the basic set of non-commuting operators.

Most approaches to the theory of pseudodifferential operators use direct methods rather than approximation by tensor products. In this paper we shall suggest a direct approach to smooth non-commutative functional calculus, based on a multivariable version (3.3) of (1.3). (Another possibility, that will not be explored here is to extend (1.1) to the multivariable case. Then, under suitable extra assumptions, one could also consider the Weyl quantization

$$f^w(P_1, \dots, P_m) = (2\pi)^{-1} \int \widehat{f}(t) e^{it \cdot P} dt,$$

with $t \cdot P = \sum t_j P_j$.) When $P_1, \dots, P_m$ are pseudodifferential operators with real principal symbols and f belongs to a suitable symbol class, it will be quite obvious from our formula that $f(P_1, \dots, P_m)$ is also a pseudodifferential operator, by extending the arguments from [13], [9]. We hope that the multivariable formula (3.3) will be a useful complement to existing multivariable functional calculii. It might provide a more direct alternative to some parts of the theory of in [17]. The purpose of the present paper is merely to establish some basis for this approach and to connect it to the one of J. Taylor and others ([20, 21, 4, 2, 3]) in the commutative case.

The plan of the paper is the following:

In Section 2, we introduce some special almost holomorphic extensions of smooth functions on the real domain.

In Section 3 we introduce the calculus using the formula (3.3). and in Section 4 we establish some additional properties. Thus we get a C_0^∞-calculus of several unbounded and non-commuting operators whose spectra are real and which have locally temperate growth of the resolvent near the real axis.

In Section 5, we relate our approach to a naive iterative approach, which amounts to treat the calculus as an operator valued distribution equal to a tensor product of 1-dimensional operator-valued distributions.

In Section 6, we review the Cayley transform and more general Möbius transforms of operators, as a tool to reduce many questions about unbounded operators to the bounded case.

In Section 7 we consider the commutative case and relate the theory to the Taylor approach. In particular we show that the (joint) Taylor spectrum and the support of our operator-valued distribution agree.

In Section 8, we discuss what happens when the operators have non-real spectra. In some cases there is a direct extension using formulae like (1.3) and (3.3), but there are also cases where such a functional calculus can be given differently already in the case of one operator (like for instance if we have a normal operator on a Hilbert space). The conclusion is that in all cases, one can get a multi-operator calculus by iterating suitable one-dimensional formulae, in a way that is well adapted to the spectrum of each of the individual operators.

In Section 9, we give some simple examples, and show in particular that the support (unlike the joint spectrum in the commutative case) is highly unstable under small perturbations.

In Section 10 we extend the calculus to the case of test-functions f that do not necessarily have compact support. This is of importance in applications to differential operators and spectral theory (see [16, 8]). For simplicity, in this and the two remaining sections, only the case of a single operator is considered, with the hope that the extension to the multi-operator case should be straight forward along the lines of the previous sections.

In Section 11 we show how to recover a generating operator from a given homomorphism from test-functions into the bounded operators on some Banach space. In the case of real spectrum it is important to have test-functions with a non-trivial behaviour near infinity, and we give an example of a homomorphism defined on the Schwartz-space $\mathcal{S}(\mathbb{R})$ which is not generated by any operator.

In Section 12 we establish the basic composition result $f(g(P)) = (f\circ g)(P)$ within the framework of the extended calculus of Section 10

2. Special almost holomorphic extensions

Lemma 2.1. *Let $f \in C_0^\infty(\mathbb{R}^m)$. Then there is a $\tilde{f} \in C_0^\infty(\mathbb{C}^m)$ with support in an arbitrarily small neighborhood of* $\operatorname{supp} f$ *such that*

$$\partial_{\bar{z}_j}\tilde{f} = \mathcal{O}(|\operatorname{Im} z_j|^\infty),\ 1 \le j \le n. \tag{2.1}$$

Proof. As a first attempt we take

$$\check{f}(z) = \frac{1}{(2\pi)^n}\int e^{iz\cdot\xi}\Big(\prod_{k=1}^m \chi\big(\langle\xi_k\rangle \operatorname{Im} z_k\big)\Big)\hat{f}(\xi)d\xi, \tag{2.2}$$

where $\hat{f} \in \mathcal{S}(\mathbb{R}^m)$ is the Fourier transform of f,

$$\langle\xi_k\rangle = \sqrt{1+|\xi_k|^2},\ z\cdot\xi = \sum_{k=1}^m z_k\xi_k,$$

and $\chi \in C_0^\infty(]-1,1[)$ is equal to 1 in a neighborhood of 0. Notice that the exponential factor is bounded on the support of the integrand so $\check{f} \in C^\infty(\mathbb{C}^m)$, and by modifying the choice of χ we may assume that $\check{f}$ has its support in an arbitrarily small tubular neighborhood of $\mathbb{R}^m$.

We have

$$\partial_{\bar z_j}\check f(z) =$$
$$\frac{1}{(2\pi)^m}\int e^{iz\cdot\xi}\Big(\prod_{k=1,k\neq j}^m \chi\big(\langle\xi_k\rangle\operatorname{Im} z_k\big)\Big)\langle\xi_j\rangle\frac{i}{2}\chi'\big(\langle\xi_j\rangle\operatorname{Im} z_j\big)\hat f(\xi)d\xi.$$

On the support of the integrand we have $\langle\xi_j\rangle\sim 1/|\operatorname{Im} z_j|$ and using the rapid decay of $\hat f$ we get (2.1). Clearly $\check f|_{\mathbb{R}^m}=f$. Notice that the map $f\mapsto\check f$ is linear, and at least formally it is the tensor product of the 1-dimensional extension maps

$$C_0^\infty(\mathbb{R})\ni g\mapsto\check g(z)=\frac{1}{2\pi}\int e^{iz\xi}\chi\big(\langle\xi\rangle\operatorname{Im} z\big)\hat g(\xi)d\xi, \tag{2.3}$$

cf., Section 5 below. It is easy to see that (for any almost holomorphic extension $\check g$)

$$\check g(z)=\mathcal{O}(|\operatorname{Im} z|^\infty) \tag{2.4}$$

locally uniformly when $\operatorname{Re} z\notin\operatorname{supp} g$. In fact, if $g(x)$ has the Taylor expansion $\sum_\nu a_\nu(x-x_0)^\nu$ at some point x_0, then any almost holomorphic extension must have the expansion $\sum_\nu a_\nu(z-x_0)^\nu$ at this point.

Let f have support in $I_1\times\cdots\times I_n$, where I_j are bounded intervals. If $J_j\subset\subset\mathbb{R}$ are open intervals with $I_j\subset\subset J_j$, let $\psi_j\in C_0^\infty(J_j)$ be equal to 1 near $\overline{I_j}$ and consider

$$\tilde f(x)=\prod_1^m\psi_j(\operatorname{Re} z_j)\check f(z). \tag{2.5}$$

For $\operatorname{Re} z_j\in\operatorname{supp}\psi_j'$ we have $\check f(x)=\mathcal{O}(|\operatorname{Im} z_j|^\infty)$, so $\partial_{\bar z_j}\tilde f=\mathcal{O}(|\operatorname{Im} z_j|^\infty)$.

In the general case we first decompose f by a partition of unity into a finite sum of new functions f^ν, where each f^ν has support in a small box $I_1^\nu\times\cdots\times I_m^\nu$. Then we get $\tilde f^\nu$ with support arbitrarily close to $I_1^\nu\times\cdots\times I_m^\nu$, and if we sum the extensions $\tilde f^\nu$ we get an extension of f with support in an arbitrarily small neighborhood of $\operatorname{supp} f$. □

Notice that (2.1) is stronger than the usual requirement for almost holomorphic extensions:

$$\bar\partial\tilde f=\mathcal{O}(|\operatorname{Im} z|^\infty). \tag{2.6}$$

Also recall that if $\tilde f,\check f\in C_0^\infty(\mathbb{C}^m)$ are almost holomorphic extensions of the same $f\in C_0^\infty(\mathbb{R}^m)$, then

$$\tilde f-\check f=\mathcal{O}(|\operatorname{Im} z|^\infty); \tag{2.7}$$

this is just a special case of (2.4) above.

3. The calculus

Let $P_1, \dots, P_m \colon \mathcal{B} \to \mathcal{B}$ be densely defined closed operators on the complex Banach space $\mathcal{B}$. We assume that each P_j has real spectrum,

$$\sigma(P_j) \subset \mathbb{R}, \tag{3.1}$$

and that the resolvents have temperate growth locally near $\mathbb{R}$:

(3.2) For every $K \subset\subset \mathbb{C}$ there are $C_{K,j}, N_{K,j} \geq 0$ such that
$$\|(z-P_j)^{-1}\| \leq C_{K,j} |\operatorname{Im} z|^{-N_{K,j}}, \ z \in K \setminus \mathbb{R}.$$

Definition 1. For $f \in C_0^\infty(\mathbb{R}^m)$ we put

$$(3.3) \quad f(P_1, \dots, P_m) =$$
$$(-\frac{1}{\pi})^m \int \cdots \int (\partial_{\bar z_1} \cdots \partial_{\bar z_m} \tilde f)(z - P_1)^{-1} \cdots (z_m - P_m)^{-1} L(dz_1) \cdots L(dz_m),$$

where $\tilde f$ is a special almost holomorphic extension as in Lemma 2.1, and $L(dz_j)$ is the Lebesgue measure on $\mathbb{C} \sim \mathbb{R}^2$.

We first check that the right hand side of (3.3) is a bounded operator on $\mathcal{B}$ which depends on f but not on the choice of special extension $\tilde f$. The estimates (2.1) remain valid after differentiation so we have for every j that

$$\partial_{\bar z_1} \cdots \partial_{\bar z_m} \tilde f = \mathcal{O}(|\operatorname{Im} z_j|^\infty),$$

and taking geometrical means we get

$$\partial_{\bar z_1} \cdots \partial_{\bar z_m} \tilde f = \mathcal{O}(|\operatorname{Im} z_1|^\infty \cdots |\operatorname{Im} z_m|^\infty). \tag{3.4}$$

Using this in (3.3) we see that the integral converges in the space of bounded operators, and for every $K \subset\subset \mathbb{R}^m$ there exist constants $C_K, N_K \geq 0$ such that

$$\|I(\tilde f)\| \leq C_K \sum_{|\alpha| \leq N_K} \sup_K |\partial^\alpha f| \tag{3.5}$$

for every $f \in C_0^\infty(\mathbb{R}^m)$ with $\operatorname{supp} f \subset K$, where $I(\tilde f)$ is the right hand side of (3.3).

Let $\check f$ be another special extension of $f \in C_0^\infty(\mathbb{R}^m)$. Then

$$I(\tilde f) - I(\check f) =$$
$$\lim_{\epsilon \searrow 0} (-\frac{1}{\pi})^m \int \cdots \int \left(\partial_{\bar z_1} \cdots \partial_{\bar z_m} (\tilde f - \check f)(z_1, \dots, z_m)\right) \times$$
$$\left(\prod_1^m (1 - \chi(\operatorname{Im} z_j / \epsilon))\right) (z_1 - P_1)^{-1} \cdots (z_m - P_m)^{-1} \prod_1^m L(dz_j),$$

where $\chi \in C_0^\infty(\mathbb{R})$ is equal to 1 near the origin. Integration by parts gives

$$I(\tilde{f}) - I(\check{f}) =$$

$$\lim_{\epsilon \searrow 0} \left(\frac{1}{2\pi i}\right)^m \int \cdots \int (\tilde{f} - \check{f})(z_1, \ldots, z_m) \Big(\prod_1^m (\chi'(\operatorname{Im} z_j/\epsilon)) \Big)$$

$$(z_1 - P_1)^{-1} \cdots (z_m - P_m)^{-1} \prod_1^m \frac{L(dz_j)}{\epsilon}.$$

In view of (3.2) and (2.7), this limit is 0 and hence the definition (3.3) is independent of the choice of $\tilde{f}$.

It follows from (3.5) that

$$\|f(P_1, \ldots, P_m))\| \leq C_K \sum_{|\alpha| \leq N_K} \sup_K |\partial^\alpha f| \tag{3.6}$$

for every $f \in C_0^\infty(\mathbb{R}^m)$ with $\operatorname{supp} f \subset K$, which means that

$$C_0^\infty \ni f \mapsto f(P_1, \ldots, P_m) \in \mathcal{L}(\mathcal{B})$$

is an operator-valued distribution on $\mathbb{R}^m$. Let $\operatorname{supp}(P_1, \ldots, P_m)$ denote its support; clearly $f(P_1, \ldots, P_m)$ is welldefined for any smooth f defined in some neighborhood of $\operatorname{supp}(P_1, \ldots, P_m)$ and vanishing in a neighborhood of infinity.

Next we review Feynman notation:

Notation If f, P_j are as above and $\pi \colon \{1, \ldots, m\} \to \{1, \ldots, m\}$ is a permutation, we put

$$\begin{aligned}(3.7)\quad & f(\overset{\pi(1)}{P_1}, \ldots, \overset{\pi(m)}{P_m}) = \\ &= \Big(-\frac{1}{\pi}\Big)^m \int \cdots \int (\partial_{\bar{z}_1} \cdots \partial_{\bar{z}_m} \widetilde{f})(z_1, \ldots, z_m)(z_{\pi^{-1}(m)} - P_{\pi^{-1}(m)})^{-1} \\ &\qquad (z_{\pi^{-1}(m-1)} - P_{\pi^{-1}(m-1)})^{-1} \cdots (z_{\pi^{-1}(1)} - P_{\pi^{-1}(1)})^{-1} \prod_1^m L(dz_i).\end{aligned}$$

In simpler words, this is the same as (3.3) except that we rearrange the order of the resolvents, so that we have

$$(z_{j_m} - P_{j_m})^{-1}(z_{j_{m-1}} - P_{j_{m-1}})^{-1} \ldots (z_{j_1} - P_{j_1})^{-1},$$

with $\pi(j_1) = 1, \phi(j_2) = 2, \ldots$.

Example 1 (Some examples).

$$f(\overset{3}{P_1}, \overset{1}{P_2}, \overset{2}{P_3}) = \Big(-\frac{1}{\pi}\Big)^3 \int \int \int (\partial_{\bar{z}_1} \partial_{\bar{z}_2} \partial_{\bar{z}_3}) \tilde{f}(z_1, z_2, z_3)$$

$$(z_1 - P_1)^{-1}(z_3 - P_3)^{-1}(z_2 - P_2)^{-1} L(dz_1) L(dz_2) L(dz_3).$$

When no indices are suspended we use the usual ordering of operators as in compositions, so for the operator (3.3) we have

$$f(P_1,\dots,P_m) = f(\overset{m}{P_1},\dots,\overset{1}{P_m}).$$

This notation can also be extended to more complicated expressions. If $A\in\mathcal{L}(\mathcal{B})$, we can define

$$f(\overset{3}{P_1},\overset{1}{P_2})\overset{2}{A}=$$
$$\left(-\frac{1}{\pi}\right)^2\int\int(\partial_{\bar z_1}\partial_{\bar z_2})\tilde f(z_1,z_2)(z_1-P_1)^{-1}A(z_2-P_2)^{-1}L(dz_1)L(dz_2).$$

Notice that this is *not* an ordinary composition of $f(\overset{3}{P_1},\overset{1}{P_2})$ and A, while for instance

$$f(\overset{2}{P_1},\overset{1}{P_2})\overset{3}{A}= A\circ f(\overset{2}{P_1},\overset{1}{P_2})$$

and

$$f(\overset{2}{P_1},\overset{3}{P_2})\overset{1}{A}= f(\overset{1}{P_1},\overset{2}{P_2})\circ A.$$

□

4. Some further properties

Proposition 4.1. *Let $f\in C_0^\infty(\mathbb{R}^k)$, $g\in C_0^\infty(\mathbb{R}^\ell)$, $m=k+\ell$, and $P_1,\dots,P_m$ as above. Then*

$$f(P_1,\dots,P_k)\circ g(P_{k+1},\dots,P_m) = (f\otimes g)(P_1,P_2,\dots,P_m), \tag{4.1}$$

where $(f\otimes g)(x_1,\dots,x_m) = f(x_1,\dots,x_k)g(x_{k+1},\dots,x_m)$.

Proof. It follows directly from the definition since we can take $(f\otimes g)\tilde{} = \tilde f\otimes\tilde g$ as the special almost holomorphic extension of $f\otimes g$. □

Proposition 4.2. *Let $f\in C_0^\infty(\mathbb{R}^m)$ and $P_1,\dots,P_m$ as above. If $P_{k+1}=P_k$ for some $k\in\{1,\dots,k-1\}$, then*

$$f(P_1,\dots,P_k,P_{k+1},\dots,P_m) = f^{(k)}(P_1,\dots,P_k,P_{k+2},\dots,P_m), \tag{4.2}$$

where $f^{(k)}\in C_0^\infty(\mathbb{R}^{m-1})$ is given by $f^{(k)}(x_1,\dots,x_k,x_{k+2},\dots,x_m) = f(x_1,\dots,x_k,x_k,x_{k+2},\dots,x_m)$, (i.e., by restricting f to the subspace $x_{k+1}=x_k$).

Proof. For simplicity we only consider the case $m = 2$, $k = 1$, so that $P_1 = P_2 =: P$. Then, using the resolvent identity,

$$f(\overset{2}{P},\overset{1}{P}) = \frac{1}{\pi^2}\int\int(\partial_{\bar{z}_1}\partial_{\bar{z}_2}\tilde{f})(z_1,z_2)(z_1-P)^{-1}(z_2-P)^{-1}L(dz_1)L(dz_2) =$$

$$\frac{1}{\pi^2}\int\int(\partial_{\bar{z}_1}\partial_{\bar{z}_2}\tilde{f})(z_1,z_2)(z_2-z_1)^{-1}(z_1-P)^{-1}L(dz_1)L(dz_2)+$$

$$\frac{1}{\pi^2}\int\int(\partial_{\bar{z}_1}\partial_{\bar{z}_2}\tilde{f})(z_1,z_2)(z_1-z_2)^{-1}(z_2-P)^{-1}L(dz_1)L(dz_2) =$$

$$-\frac{1}{\pi}\int(\partial_{\bar{z}_1}\tilde{f})(z_1,z_1)(z_1-P)^{-1}L(dz_1)-\frac{1}{\pi}\int(\partial_{\bar{z}_2}\tilde{f})(z_2,z_2)(z_2-P)^{-1}L(dz_2)$$

$$= -\frac{1}{\pi}\int\partial_{\bar{z}}\big(\tilde{f}(z,z)\big)(z-P)^{-1}L(dz),$$

which gives the result since $\tilde{f}(z,z)$ is an almost holomorphic extension of $f(x,x)$. □

5. Definition by iteration

It is possible to construct our functional calculus from the single operator case by iteration. To see this we first extend our previous construction to vector-valued functions. If $f \in C_0^\infty(\mathbb{R}^m, \mathcal{B})$ we can find a special almost holomorphic extension and define $f(P_1,\dots,P_m)$ in the same way as before, just being careful to put the factor $\partial_{\bar{z}_1}\cdots\partial_{\bar{z}_m}\tilde{f}$ on the right hand side of all the resolvents in formula (3.3). Again this definition is independent of the particular choice of extension, and the estimate (3.6) holds. Notice that $f(P_1,\dots,P_m) = 0$ if $\operatorname{supp}(f)\cap\operatorname{supp}(P_1,\dots,P_m) = \emptyset$, also when f is vectorvalued (where $\operatorname{supp}(P_1,\dots,P_m)$ is the support of our operator-valued distribution defined initially on scalar-valued testfunctions). For instance, if ϕ is scalarvalued, $u \in \mathcal{B}$, and $f(x_1,\dots,x_m) = \phi(x_1,\dots,x_m)u$, then $f(P_1,\dots,P_m) = \phi(P_1,\dots,P_m)u$. Moreover, if $f(x_1,\dots,x_k,x_{k+1},\dots,x_m)$ is $\mathcal{B}$-valued, and

$$g(x_1,\dots,x_k) = f(x_1,\dots,x_k,P_{k+1},\dots,P_m)$$

is defined as before, for each fixed $(x_1,\dots,x_k)$, then $g(x_1,\dots,x_k)$ is a function in $C_0^\infty(\mathbb{R}^k,\mathcal{B})$ and

$$f(P_1,\dots,P_m) = g(P_1,\dots,P_k).$$

Example 2. One can define, e.g., $f(\overset{3}{P_1},\overset{1}{P_2})\overset{2}{A}$, cf., Example 1, as $g(P_1)$, where

$$g(x_1) = A\circ f(x_1,P_2).$$

□

Remark 1. Since we use explicit integral formulas the necessary verifications for the statements above are easily made directly. However one can also obtain the multi-operator calculus in a more abstract way.

Spaces like $C_0^\infty(\mathbb{R}^k)$ are nuclear, and therefore they behave well under topological tensor products. Since

$$C_0^\infty(\mathbb{R}^m, \mathcal{B}) = C_0^\infty(\mathbb{R})\hat{\otimes}\cdots\hat{\otimes}C_0^\infty(\mathbb{R})\hat{\otimes}\mathcal{B}$$

it is therefore enough to define the functional calculus on decomposable elements $\phi_1(x_1)\otimes\cdots\otimes\phi_m(x_m)\otimes u$, for $u \in \mathcal{B}$, which is done by the single operator calculus. □

As an application we can prove

Proposition 5.1. *If $P_1, \dots, P_m$ are as above, then*

$$\operatorname{supp}(P_1 \dots, P_m) \subset \operatorname{supp}(P_1, \dots, P_k) \times \operatorname{supp}(P_{k+1}, \dots, P_m).$$

Proof. Let $P = P_1, \dots, P_k$ and $Q = P_{k+1}, \dots, P_m$, and similarily $(x_1, \dots, x_m) = (x, \xi)$. If $\phi(x, \xi)$ has support outside $\operatorname{supp}(P) \times \operatorname{supp}(Q)$, then $\xi \mapsto \phi(x, \xi)$ vanishes near $\operatorname{supp}(Q)$ if x belongs to (a neighborhood of) $\operatorname{supp}(P)$. Thus $x \mapsto \phi(x, Q)$ vanishes in a neighborhood of $\operatorname{supp}(P)$ and hence $\phi(P, Q) = 0$. □

Example 3. For one single operator P, the support coincides with the spectrum $\sigma(P)$, i.e., the complement of the resolvent set. In fact, suppose that $f \in C_0^\infty(\mathbb{R})$ has support in the resolvent set. Then we may assume that $\tilde{f}$ has support in the resolvent set as well. However, here the resolvent $(z-P)^{-1}$ is holomorphic, and thus

$$-\frac{1}{\pi}\int \partial_{\bar{z}}\tilde{f}(z)(z-P)^{-1}L(dz) = -\frac{1}{\pi}\int \partial_{\bar{z}}(\tilde{f}(z)(z-P)^{-1})L(dz) = 0$$

by Stokes' theorem. Thus $\operatorname{supp}(P) \subset \sigma(P)$. Conversely, if Ω is an open set in the complement of the support, then the operator-valued function $c(z) = (z-P)^{-1}$ has a holomorphic extension across $\mathbb{R}$ in Ω. In fact, if $F \in C_0^\infty(\Omega)$ is an almost holomorphic extension of a function f in $C_0^\infty(\Omega \cap \mathbb{R})$, then it is easy to see that

$$-\frac{1}{\pi}\int c(\zeta)\partial_{\bar{\zeta}}F(\zeta)\frac{L(d\zeta)}{\zeta - z} = c(z)F(z)$$

for each $z \in \Omega \setminus \mathbb{R}$. For any given point x^0 in $\Omega \cap \mathbb{R}$ we can choose F which is identically one in a neighborhood, and then the integral provides the holomorphic extension at x^0. One can conclude that Ω is contained in the resolvent set of P. Thus $\operatorname{supp}(P) = \sigma(P)$. □

6. The Cayley transform

In this section we shall consider closed operators on a complex Banach space $\mathcal{B}$ that are not necessarily densely defined. For such operators P one defines the spectrum as usual (namely as the complement in $\mathbb{C}$ of the set of z for which $z - P : \mathcal{D}(P) \to \mathcal{B}$ has a bounded inverse, where $\mathcal{D}(P)$ is the domain, equipped with the graph-norm $\|u\| + \|Pu\|$) and the spectrum $\sigma(P)$ becomes a closed subset of the complex plane. The point spectrum $\sigma_p(P) \subset \sigma(P)$ is the set of $z \in \mathbb{C}$ such that $z - P$

is not injective. In this section we only consider operators whose spectrum is not equal to the whole complex plane.

For any closed operator P on $\mathcal{B}$, we define its extended spectrum $\widehat{\sigma}(P)$ as $\sigma(P)$ if P is bounded and as $\sigma(P) \cup \{\infty\}$ if P is not bounded. Then $\widehat{\sigma}(P)$ is a compact subset of the extended plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If ψ is an automorphism of $\widehat{\mathbb{C}}$, a Möbius mapping, such that $\psi^{-1}(\infty)$ is outside the point spectrum of P, then $\psi(P)$ is a welldefined closed operator with extended spectrum $\psi(\widehat{\sigma}(P))$, and it is bounded, if and only if this set is bounded, i.e., if and only if $\psi^{-1}(\infty)$ is outside $\widehat{\sigma}(P)$. Moreover, $\psi(P)$ is densely defined if and only if the range of $P - \psi^{-1}(\infty)$ is dense (excluding the trivial case when ψ maps ∞ to itself, in which case P and $\psi(P)$ have identical domains). More precisely, $\mathcal{D}(\psi(P)) = \mathcal{R}(P - \psi^{-1}(\infty))$, where $\mathcal{D}$ and $\mathcal{R}$ indicate the domain and the range respectively. A simple way of checking these facts is to use that if $\psi(z) = (m_{1,1}z + m_{1,2})/(m_{2,1}z + m_{2,2})$, with $\det M \neq 0$, $M = \{m_{j,k}\}_{1\leq j,k\leq 2}$, then the graph of $\psi(P)$ is equal to $M(\mathrm{graph}\,(P))$, where M acts on $\mathcal{B} \times \mathcal{B}$ in the natural way and $\mathrm{graph}\,(P) = \{(Pu, u);\, u \in \mathcal{D}(P)\}$.

In this way, any closed operator P such that $\widehat{\sigma}(P) \subset_{\neq} \widehat{\mathbb{C}}$ can be transformed to a bounded operator. If $\widehat{\sigma}(P) \subset \widehat{\mathbb{R}}$, one can use the automorphism

$$C(z) = \frac{z+i}{z-i},$$

which maps $\widehat{\mathbb{R}}$ bijectively to the unit circle $\mathbb{T}$ and has the inverse

$$z = C^{-1}(w) = i\frac{w+1}{w-1}.$$

Thus C induces a 1-1 correspondence between closed operators A with real spectra and bounded operators B with $\sigma(B) \subset \mathbb{T}$, such that $B - 1$ is injective.

We also have the identity

$$|w|^2 - 1 = 4\frac{\operatorname{Im} z}{|z-i|^2},$$

which implies that $|\operatorname{Im} z| \sim d(w, \mathbb{T})$, for z close to $\mathbb{R}$ (i.e. w close to $\mathbb{T}$) with explicitly controled non-uniformity when $z \to \infty$ $(w \to 1)$. Furthermore, with A, B as above, we have

$$\frac{dw}{w-B} = \frac{A-i}{z-i}\frac{dz}{z-A}, \tag{6.1}$$

which implies that $(w-B)^{-1}$ has temperate growth locally near $\mathbb{T}_0 = \mathbb{T} \setminus \{1\}$ if and only if $(z-A)^{-1}$ has temperate growth locally near $\mathbb{R}$.

If this holds, we can define a functional calculus

$$C_0^\infty(\mathbb{T}_0) \to \mathcal{L}(\mathcal{B}),\ \phi \mapsto \phi(B),$$

as before, by the formula

$$\phi(B) = -\frac{1}{\pi}\int \partial_{\overline{w}}\widetilde{\phi}(w)\frac{L(dw)}{w-B},$$

where $\widetilde{\phi}$ is an almost holomorphic extension of ϕ with compact support.

Clearly, $\phi \in C_0^\infty(\mathbb{T}_0)$ if and only if $\phi \circ C \in C_0^\infty(\mathbb{R})$, and as one would expect,

$$(\phi \circ C)(A) = \phi(B). \tag{6.2}$$

To see this, just notice that, by, (6.1)

$$\phi(B) = -\frac{1}{\pi}\int \partial_{\overline{w}}\widetilde{\phi}(w)\frac{L(dw)}{w-B} = \frac{1}{2\pi i}\int \overline{\partial}_w\widetilde{\phi}(w) \wedge \frac{dw}{w-B} =$$

$$\frac{1}{2\pi i}\int \overline{\partial}_z(\widetilde{\phi}\circ C)(z) \wedge \frac{A-i}{z-i}\frac{dz}{z-A} = -\frac{1}{\pi}\int \partial_{\overline{z}}(\widetilde{\phi}\circ C)(z)\frac{A-i}{z-i}\frac{L(dz)}{z-A},$$

and the last integral is equal to $\phi \circ C(A)$ by Stokes' theorem, since

$$(1 - \frac{A-i}{z-i})\frac{1}{z-A} = \frac{1}{z-i}$$

is holomorphic.

7. Commuting operators

In this section we shall see what happens if we impose the extra condition that $P_1, \ldots, P_m$ commute, but let us first recall the basic elements of Taylor's theory for commuting operators, [20] and [21]. If $A_1, \ldots, A_m$ is a tuple of commuting bounded operators on $\mathcal{B}$, then there is a compact set $\sigma(A) = \sigma(A_1, \ldots, A_m)$ in $\mathbb{C}^m$ called the joint (Taylor) spectrum. If A^j is a sequence of commuting tuples, all of which commute mutually, such that $A^j \to A$ in operator norm, then $\sigma(A^j) \to \sigma(A)$ in the Hausdorff sense (this is not true in general if they do not commute!). For each function f which is holomorphic in a neighborhood of $\sigma(A)$ one can define $f(A)$, depending continuously on f, such that it coincides with the obvious definition if f is a polynomial or entire function, and such that $(fg)(A) = f(A)g(A)$. Moreover, if $f = f_1, \ldots, f_n$, and $f(A) = f_1(A), \ldots, f_n(A)$, then the spectral mapping property holds, i.e., $\sigma(f(A)) = f(\sigma(A))$.

Let us now suppose that the spectrum of each A_k is real. By the spectral mapping property this holds if and only if the joint spectrum $\sigma(A)$ is contained in $\mathbb{R}^m$. Moreover, $w \in \mathbb{C}^n$ is outside the spectrum if and only if there are C_j in (A), the closed subalgebra of $\mathcal{L}(\mathcal{B})$ generated by $A_1, \ldots, A_m$, such that

$$\sum C_j(A_j - w_j) = 1.$$

The tuple A admits a continuous extension of the real-analytic functional calculus to a smooth one if and only this holds for each A_j, and this in turn is equivalent to the fact that the resolvent of each A_j has temperate growth in the Im-direction; it is also equivalent to that

$$\|e^{it\cdot A}\| \lesssim \langle t\rangle^M, \quad t \in \mathbb{R}^m,$$

for some $M > 0$, see, e.g., [4]. If A admits such a smooth functional calculus that extends the real-analytic functional calculus (induced in the natural way by the holomorphic functional calculus), then it is unique and the support of the corresponding operator-valued distribution is precisely $\sigma(A)$. Moreover, there is then an operator-valued form ω_{z-A} of bidegree $(m, m-1)$ in $\mathbb{C}^m \setminus \sigma(A)$, representing the resolvent of A, with

$$\|\omega_{z-A}\| \leq C|\mathrm{Im}\, z|^M,$$

and the smooth functional calculus can be represented by

$$f(A) = -\int \bar\partial_z \tilde f \wedge \omega_{z-A}, \tag{7.1}$$

if $\tilde f$ is a standard almost holomorphic extension of $f \in C^\infty(\mathbb{R}^n)$, i.e., such that $|\bar\partial_z \tilde f| = \mathcal{O}(|\mathrm{Im}\, z|^\infty)$, see [4].

As long as A_k are bounded, our functional calculus, constructed by means of (3.3), is defined for any $f \in C^\infty(\mathbb{R}^m)$, and we claim that it in fact coincides with (7.1). To see this, let us first assume that f is the restriction of an entire function F. Then we can take our special almost holomorphic extension to be equal to F in a neighborhood of $\mathbb{R}^m$, and it then follows from the iterated Cauchy formula that (3.3) gives the holomorphic functional calculus. Since the entire functions are dense in $C^\infty(\mathbb{R}^m)$, the claim follows. From the representation (7.1) it immediately follows that the support of the functional calculus, $\mathrm{supp}\,(A)$, is equal to $\sigma(A)$. The same statements hold if $\mathbb{R}^m$ is replaced by the real torus $\mathbb{T}^m$.

Let us now go back to our unbounded closed operators with real spectra. We say that two such operators P_1, P_2 commute if the resolvents $(z_1 - P_1)^{-1}$ and $(z_2 - P_2)^{-1}$ commute for all z_1 and z_2 in the resolvent sets. This holds if and only if the Cayley transforms $C(P_1)$ and $C(P_2)$ commute. If P_1 and P_2 are bounded this just means that they commute themselves. Now let $P_1, \ldots, P_m$ be as before, i.e., resolvents with temperate growth, but, in addition, commuting. It is convenient to extend our functional calculus to the algebra

$$\mathcal{A} = C_0^\infty(\mathbb{R}^m) \oplus (1),$$

of all smooth functions which are constant in some neighborhood of ∞.

Observe that if P_j are commuting, then

$$f(P_1, \ldots, P_m) = f(\overset{\sigma(1)}{P_1}, \ldots, \overset{\sigma(m)}{P_m})$$

for any permutation σ. From Propositions 4.1 and 4.2 we get

Proposition 7.1. *Suppose that $P_1, \dots, P_m$ are as above and commuting. Then*

$$f(P_1, \dots, P_m)g(P_1, \dots, P_m) = (fg)(P_1, \dots, P_m), \quad f, g \in \mathcal{A}. \tag{7.2}$$

Let $C(x_1, \dots, x_m) = (C(x_1), \dots, C(x_m))$ be the multiple Cayley transform, and suppose that P_j are commuting and have real spectra. Then each $C(P_j)$ has spectrum contained in $\mathbb{T}$ so the joint spectrum of $C(P)$ is contained in $\mathbb{T}^m$. If all P_j are bounded, then $C(z)$ is holomorphic in a neighborhood of $\sigma(P)$ and thus $\sigma(C(P))$ is contained in $\mathbb{T}_0^m = (\mathbb{T} \setminus \{1\})^m$ by the spectral mapping theorem. By another application of the same theorem it follows that

$$\sigma(P) = C^{-1}\big(\sigma(C(P)) \cap \mathbb{T}_0^m\big). \tag{7.3}$$

When P_j are unbounded and commuting let us take (7.3) as the definition of $\sigma(P)$.

Proposition 7.2. *If A_j are as above (real spectra and temperate resolvents) and in addition commuting, then*

$$\operatorname{supp}(A) = \sigma(A).$$

Proof. Let $B = C(A)$. We are to prove that $\sigma(B) \cap \mathbb{T}_0^m$ is equal to the support of

$$C_0^\infty(\mathbb{T}_0^m) \to \mathcal{L}(\mathcal{B}), \quad f \mapsto f(B). \tag{7.4}$$

By repeated use of (6.2) we have that $C(\operatorname{supp}(A))$ is equal to the support of (7.4), and so the proposition will follow.

To begin with, we shall extend (7.4) to a multiplicative mapping

$$\mathcal{G}(\mathbb{T}^m) \to \mathcal{L}(\mathcal{B}), \tag{7.5}$$

where $\mathcal{G}(\mathbb{T}^m)$ is the class of functions in $C^\infty(\mathbb{T}^m)$ that are real-analytic in a neighborhood of $\mathbb{T}^m \setminus \mathbb{T}_0^m$. Let $\chi_0(t)$ be a smooth function on $\mathbb{T}$ which is 1 in a neighborhood of a given compact set $K \subset \mathbb{T}_0$ and 0 in a neighborhood of 1. One can find an almost holomorphic extension $\widetilde{\chi}_0$ to a complex neighborhood of $\mathbb{T}$ such that $\widetilde{\chi}_0$ is 1 in a complex neighborhood of K, and 0 in a complex neighborhood of 1. Then

$$\widetilde{\chi}(w) = 1 - \prod_{j=1}^m \widetilde{\chi}_0(w_j)$$

is identically 0 in a complex neighborhood of K^m and identically 1 in a complex neighborhood of $\mathbb{T}^m \setminus \mathbb{T}_0^m$. After multiplication by a cutoff function (which is 1 in a neighborhood of $\mathbb{T}^m$), we may assume that $\widetilde{\chi}$ has compact support in $\mathbb{C}^m$. Now take $f \in \mathcal{G}(\mathbb{T}^m)$ and let F be the holomorphic extension at $\mathbb{T}^m \setminus \mathbb{T}_0^m$, and $\widetilde{f}_0$ a special almost holomorphic extension near $\mathbb{T}_0^m$. Then

$$\widetilde{f} = \widetilde{\chi} F + (1 - \widetilde{\chi})\widetilde{f}_0$$

is a special almost holomorphic extension of f which is even holomorphic in a complex neighborhood of $\mathbb{T}^m \setminus \mathbb{T}_0^m$.

Since we have temperate growth of the resolvents in $\mathbb{T}_0^m$, we can now define

$$f(B) = (-\frac{1}{\pi})^m \int ... \int \partial_{\overline{w}_1} ... \partial_{\overline{w}_m} \widetilde{f} \frac{L(dw_1)}{w_1 - B_1} \cdots \frac{L(dw_m)}{w_m - B_m}. \tag{7.6}$$

It is readily verified as in Section 3 that the integral is independent of the choice of $\widetilde{f}$. Also the multiplicativity follows by means of the resolvent identity as in Proposition 4.2 so we get the homomorphism (7.5).

Clearly (7.5) extends to a multiplicative mapping from functions which are C^∞ in a neighborhood of the support of (7.4) and real analytic in a neighborhood of $\mathbb{T}^m \setminus \mathbb{T}_0^m$. In particular; if $w \in \mathbb{T}_0^m$ is outside this support, then (7.5) applies to

$$\phi_j^w(x) = \frac{\bar{w}_j - \bar{x}_j}{|w - x|^2},$$

and since $\sum_j \phi_j^w(B)(w_j - B_j) = I$ it follows that $w \notin \sigma(B)$. Thus $\sigma(B) \cap \mathbb{T}_0^m$ is contained in the support of (7.4).

We claim that (7.5) coincides with the holomorphic functional calculus when f is real-analytic on the whole of $\mathbb{T}^m$. In fact; if $\widetilde{f}$ is an extension with compact support in $\mathbb{C}^m$ which is holomorphic in a complex neighborhood of $\mathbb{T}^m$, then it follows from Cauchy's formula that

$$\widetilde{f}(z) = (-\frac{1}{\pi})^m \int ... \int \partial_{\overline{w}_1} ... \partial_{\overline{w}_m} \widetilde{f}(w) \frac{L(dw_1)}{w_1 - z_1} ... \frac{L(dw_m)}{w_m - z_m}$$

there. Therefore, see e.g., [21], formula (7.6) defines $f(B)$ in the holomorphic functional calculus sense, and thus it coincides with our definition.

Lemma 7.3. *Suppose that $f \in C^\infty(\mathbb{T}^m)$ is real-analytic in $U \subset \mathbb{T}^m$. Then there are f_ϵ, $0 < \epsilon \leq 1$, holomorphic in some ϵ-independent neighborhood of $\mathbb{T}^m$, and a complex neighborhood $\widetilde{U}$ of U, such that $f_\epsilon \to f$ in $C^\infty(\mathbb{T}^m)$ and $f_\epsilon \to f$ in $O(\widetilde{U})$.*

To prove the lemma one defines f_ϵ by means of convolution with a Gaussian approximation of unity, and since we can make contour deformation in a complex neighborhood of U, we also get the convergence in $\mathcal{O}(\widetilde{U})$ for a suitable $\widetilde{U}$.

To see that the support of (7.4) is contained in $\sigma(B)$, take any $\phi \in C_0^\infty(\mathbb{T}_0^m)$ with support outside $\sigma(B)$. If ϕ_ϵ are as in the lemma, then $\phi_\epsilon \to \phi$ in $\mathcal{G}(\mathbb{T}^m)$, so $\phi_\epsilon(B) \to \phi(B)$. On the other hand, since ϕ_ϵ are holomorphic in a complex neighborhood of $\sigma(B)$, and $\phi_\epsilon \to 0$ there, $\phi_\epsilon(B) \to 0$ by the continuity of the holomorphic functional calculus so $\phi(B) = 0$. Thus Proposition 7.2 is proved. $\square$

Remark 2. If B is a tuple of bounded operators with $\sigma(B) \subset \mathbb{T}^m$ there is an operator-valued $(m, m-1)$-form ω_{w-B} in $\mathbb{C}^m \setminus \sigma(B)$ such that

$$f(B) = -\int \overline{\partial}_w \widetilde{f} \wedge \omega_{w-B}, \tag{7.7}$$

if $\tilde{f}$ coincides with the holomorphic function f in a neighborhood of $\sigma(B)$ and has compact support. If B is as in the preceding proof, it is even possible to choose ω_{w-B} such that

$$\|\omega_{w-B}\| \lesssim d(w, \mathbb{T}^m)^{-M}$$

uniformly on compact sets in $\mathbb{T}^m_0$; this follows since one can define such a form ω_{w-B} as the functional calculus (7.5) acting on $s \wedge (\bar{\partial}_w s)^{m-1}$, where

$$s = \sum \phi_j^w(x) dw_j / 2\pi i.$$

By Lemma 7.3, or by a direct computation, one verifies that (7.7) can be used to define the functional calculus (7.5) (if $\tilde{f}$ is an almost holomorphic extension which is holomorphic in a neighborhood of $\mathbb{T}^m \setminus \mathbb{T}^m_0$) and from this formula it is obvious that the support of (7.4) is contained in $\sigma(B)$. □

Proposition 7.4. *Let A_j be as above (real spectra and temperate resolvents) and in addition commuting. If $\phi_1, \dots, \phi_n \in \mathcal{A}$, then $\phi_j(A)$ is a commuting tuple (of bounded operators) and $\sigma(\phi(A)) = \phi(\sigma(A))$.*

Proof. We first prove that if $f_j \in \mathcal{G}(\mathbb{T}^m)$, then $f(\sigma(B)) = \sigma(f(B))$. If $w \notin f(\sigma(B))$, then $\phi_j(x) = (\bar{w}_j - \bar{f}_j(x))/|f(x) - w|^2$ are analytic near $\sigma(B)$, and according to the previous proof, $\sum_j (w_j - f_j(B))\phi_j(B) = I$, and hence $w \notin \sigma(f(B))$. Thus $\sigma(f(B)) \subset f(\sigma(B))$.

We may assume that f is real. Assume that $f(x^0) = w$ and that $w \notin \sigma(f(B))$. Then (since $\sigma(f(B))$ is real) we can find C_j, by the holomorphic functional calculus, commuting with all B_k, such that $\sum_j (w_j - f_j(B))C_j = I$. However, for each j we can solve

$$f_j(x) - w_j = \sum (x_k - x_k^0) \psi_{jk}(x)$$

with $\psi_{jk}(x)$ in $\mathcal{G}(\mathbb{T}^m)$. It follows that $\sum_k (B_k - x_k^0) \sum_j C_j \psi_{jk}(B) = I$, and hence $x^0 \notin \sigma(B)$. Thus $w \notin f(\sigma(B))$.

We already know that $\phi_j(A)$ are bounded and commuting. By the definition of $\sigma(A)$, (6.2), and the first part of the proof, we have

$$\begin{aligned}\sigma(\phi(A)) = \sigma(\phi \circ C^{-1}(C(A))) = \phi \circ C^{-1}(\sigma(C(A))) = \\ = \phi(C^{-1}(\sigma(C(A)))) = \phi(\sigma(A)).\end{aligned}$$

□

We shall now see that $\phi(A)$ admits a smooth functional calculus if $\phi = \phi_1, \dots, \phi_n \in \mathcal{A}$ and A_k are as in Proposition 7.4. From the proposition we have that

$$\sigma(\phi(A)) = \{\xi + i\eta; (\xi, \eta) \in \sigma(\operatorname{Re}\phi(A), \operatorname{Im}\phi(A)\}.$$

Moreover, if g is smooth in a neighborhood of $\phi(\sigma(A))$, then $g \circ \phi \in \mathcal{A}$, in the sense that it coincides with an element in $\mathcal{A}$ in a neighborhood of $\sigma(A)$; thus $g \circ \phi(A)$ is defined.

Proposition 7.5. *Let A_k be as in Proposition 7.4 and let $\phi = \phi_1, \dots, \phi_n \in \mathcal{A}$. If ϕ is real then the resolvent of each $\phi_j(A)$ has temperate growth.*

If g is a smooth function in a neighborhood of $\sigma(\phi(A))$, then

$$g \circ \phi(A) = g(\phi(A)) \tag{7.8}$$

holds, if the right hand side is defined as $\tilde{g}(\operatorname{Re}\phi(A), \operatorname{Im}\phi(A))$, where $\tilde{g}(\xi, \eta) = g(\xi + i\eta)$.

Proof. If $g(w)$ is any polynomial in $\mathbb{C}^n$, then $g \circ \phi \in \mathcal{A}$ and (7.8) holds by Proposition 7.1. However, if g is entire, g_N are polynomials, and $g_N \to g$, then $g_N \circ \phi \to g \circ \phi$ in $\mathcal{A}$ and hence (7.8) holds for all entire g.

If ϕ is real, it follows that

$$\|e^{\phi(A)\cdot t}\| \leq C\langle t\rangle^M, \quad t \in \mathbb{R}^m,$$

and this implies (is actually equivalent to) that the resolvent of each $\phi_j(A)$ has temperate growth in the $\operatorname{Im} z_j$-direction. It also implies that $\phi(A)$ admits an extension of the holomorphic functional calculus to a smooth functional calculus, and moreover, that $g_N(\phi(A)) \to g(\phi(A))$ if g_N are entire functions (or polynomials) and $g_N \to g$ in C^∞ in a neighborhood of $\sigma(\phi(A))$ in $\mathbb{R}^n$. It follows that (7.8) holds for such g. The case with a complex ϕ follows by considering $\operatorname{Re}\phi, \operatorname{Im}\phi$. $\square$

8. Extension to operators with nonreal spectra

In this section we shall indicate an extension of the functional calculus to operators with not necessarily real spectrum.

Let $\mathcal{E}(\widehat{\mathbb{C}})$ be the space of smooth functions on $\widehat{\mathbb{C}}$, or equivalently the space of smooth functions $f(z)$, $z \in \mathbb{C}$ with $f(z) = g(1/z)$, for $|z| > 1$, where g is smooth on the unit disc. If $K \subset \widehat{\mathbb{C}}$ is closed, let $\mathcal{E}(K)$ be the space of germs of $\mathcal{E}(\widehat{\mathbb{C}})$-functions near K. We say that a closed operator A with $\widehat{\sigma}(A) \subset_{\neq} \widehat{\mathbb{C}}$ admits a smooth functional calculus

$$T : \mathcal{E}(\widehat{\sigma}(A)) \to \mathcal{L}(\mathcal{B}), \tag{8.1}$$

if T is a continuous algebra homomorphism that extends the holomorphic functional calculus $\mathcal{O}(\widehat{\sigma}(A)) \to \mathcal{L}(\mathcal{B})$. Such a T is an $\mathcal{L}(\mathcal{B})$-valued distribution with support $\operatorname{supp}(T)$ contained in $\widehat{\sigma}(A)$, and from applying T to $\phi(z) = 1/(z - w)$, $w \notin \operatorname{supp}(T)$, it follows that $\operatorname{supp}(T) = \sigma(A)$.

If A is bounded, then $\operatorname{Re} z$ and $\operatorname{Im} z$ are in $\mathcal{E}(\sigma(A))$, so $\operatorname{Re} A$ and $\operatorname{Im} A$ are bounded and continuous. It also follows that they both have real spectrum, and the continuity of T implies that their resolvents have temperate growth. We claim that

$$\sigma(\operatorname{Re} A, \operatorname{Im} A) = \{(x, y);\ x + iy \in \sigma(A)\}. \tag{8.2}$$

In fact; if we define $A^* = \operatorname{Re} A - i\operatorname{Im} A$, then $\sigma(A, A^*)$ is the image in $\mathbb{C}^2$ of $\sigma(\operatorname{Re} A, \operatorname{Im} A)$ under the biholomorphic mapping

$$(\xi, \eta) \mapsto (z, w) = (\xi + i\eta, \xi - i\eta),$$

by the spectral mapping property of the holomorphic functional calculus. Therefore,

$$\sigma(A, A^*) \subset \{(z, w) \in \mathbb{C}^2;\ w = \overline{z}\},$$

and since $\sigma(A)$ is the image of $\sigma(A, A^*)$ under $(z, w) \mapsto z$, (8.2) follows. It should be emphasized that such an extension T of the holomorphic functional calculus in general is not unique.

We now claim that the holomorphic functional calculus

$$\phi \mapsto \phi(\operatorname{Re} A, \operatorname{Im} A)$$

has an extension to all $\phi \in \mathcal{E}_{\mathbb{R}^2}(\sigma(\operatorname{Re} A, \operatorname{Im} A))$, i.e., functions ϕ that are smooth in some neighborhood of $\sigma(\operatorname{Re} A, \operatorname{Im} A)$ in $\mathbb{R}^2$. In fact, there is a closed $\mathcal{L}(\mathcal{B})$-valued $\overline{\partial}$-closed (2,1)-form $\omega_{(\xi,\eta)-(\operatorname{Re} A, \operatorname{Im} A)}$ in $\mathbb{C}^2 \setminus \sigma(\operatorname{Re} A, \operatorname{Im} A)$ such that $\|\omega_{(\xi,\eta)-(\operatorname{Re} A,\operatorname{Im} A)}\|$ has temperate growth when $\operatorname{Im}(\xi, \eta) \to 0$, in view of the discussion in the previous section. If $\Phi(\xi, \eta)$ is an almost holomorphic extension of ϕ to $\mathbb{C}^2$, with compact support, then

$$\phi(\operatorname{Re} A, \operatorname{Im} A) = -\int_{\mathbb{C}^2} \overline{\partial}_{\xi,\eta}\Phi \wedge \omega_{(\xi,\eta)-(\operatorname{Re} A,\operatorname{Im} A)} \tag{8.3}$$

is an absolutely convergent integral.

For $f \in \mathcal{E}(\sigma(A))$, let $\check{f}(x, y) = f(x + iy)$. This gives rise to an isomorphism

$$\mathcal{E}(\sigma(A)) \simeq \mathcal{E}_{\mathbb{R}^2}(\operatorname{Re} A, \operatorname{Im} A),$$

and we claim that

$$f(A) = \check{f}(\operatorname{Re} A, \operatorname{Im} A) \tag{8.4}$$

for all $f \in \mathcal{E}(\sigma(A))$, where the right hand side is defined by (8.3) and the left hand side is $T(f)$. To begin with, (8.4) clearly holds if f is a real-analytic polynomial, since the left hand side is multiplicative by assumption and the right hand side has the same property as part of the holomorphic functional calculus. The general case follows by approximation. Thus we have found a representation of $T(f) = f(A)$ as an explicit absolutely convergent integral over $\mathbb{C}^2$ for $f \in \mathcal{E}(\sigma(A))$.

If we have (8.1) but A is unbounded, then we just apply first an automorphism ψ of $\widehat{\mathbb{C}}$, that maps A to a bounded operator $\psi(A)$ and then express $T(f) = f(A) = f \circ \psi^{-1}(\psi(A))$ as an absolutely convergent integral

$$T(f) = -\int \bar{\partial}_{\xi',\eta'}(F \circ \psi^{-1}) \wedge \omega_{(\xi',\eta')-(\operatorname{Re}\psi(A),\operatorname{Im}\psi(A))},$$

where $F \circ \psi^{-1}$ is an almost holomorphic extension of $f \circ \psi^{-1}$, $\mathbb{C}^2_{\xi',\eta'} \supset \mathbb{R}^2_{x',y'}$ and $x' + iy' = \psi(x + iy)$.

If we have several operators A_j that admit smooth functional calculii, $\mathcal{E}(\widehat{\sigma}(A_j)) \to \mathcal{L}(B)$, we can define

$$\mathcal{E}(\prod \widehat{\sigma}(A_j)) \to \mathcal{L}(\mathcal{B}) \tag{8.5}$$

as an iterated integral as in Section 3, just taking for $f(z_1, \ldots, z_m) \in \mathcal{E}(\prod \hat{\sigma}(A_j))$, a special almost holomorphic extension $\check{F}$ to $\mathbb{C}^{2m}$ of

$$\check{f}(x_1, y_1, ..., x_m, y_m) = f(x + iy_1, ..., x_m + iy_m)$$

such that

$$|\overline{\partial}_{\xi_1,\eta_1}\overline{\partial}_{\xi_2,\eta_2}...\overline{\partial}_{\xi_m,\eta_m}\check{F}(\xi,\eta)| = \mathcal{O}(|\mathrm{Im}\,(\xi_1,\eta_1)|^\infty \cdots |\mathrm{Im}\,(\xi_m,\eta_m)|^\infty)$$

in a neighborhood of $\sigma(\operatorname{Re} A_1, \operatorname{Im} A_1) \times ... \times \sigma(\operatorname{Re} A_m, \operatorname{Im} A_m)$. In case all A_j are bounded we then get the formula

$$\begin{aligned} f(A_1, \ldots, A_m) = \\ \pm \int_{\xi_1,\eta_1} \cdots \int_{\xi_m,\eta_m} \bar{\partial}_{\xi_1,\eta_1}\bar{\partial}_{\xi_2,\eta_2} \cdots \bar{\partial}_{\xi_m,\eta_m}\check{F}(\xi,\eta) \wedge \\ \omega_{(\xi_1,\eta_1)-(\operatorname{Re} A_1, \operatorname{Im} A_1)} \wedge \ldots \wedge \omega_{(\xi_m,\eta_m)-(\operatorname{Re} A_m, \operatorname{Im} A_m)}. \end{aligned}$$

For each unbounded A_j we first have to make an appropriate transformation with a Möbius mapping ψ as described above, but we omit the general resulting formula.

Remark 3. If T_j denotes the operator valued distrubution

$$T_j\phi = \phi(A_j),$$

then (8.5) is just the tensor product

$$T_1 \otimes ... \otimes T_m$$

and it could have been defined in a more abstract way; cf. Remark 1. □

9. Some further examples

The following example shows that small noncommutative perturbations of a pair of operators can blow up the support.

Example 4. Let $\mathcal{B} = \mathbb{C}^2$ and A the operator given by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $\sigma(A) = \{0,1\}$ and hence by the spectral mapping theorem for commuting operators

$$\sigma(A, A) = \{(0,0), (1,1)\}.$$

Now let $A_\epsilon = U_\epsilon^{-1} A U_\epsilon$, where

$$U_\epsilon = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix},$$

i.e., rotation with ϵ. Then clearly $A_\epsilon \to A$ in norm when $\epsilon \to 0$. We claim that $\operatorname{supp}(A, A_\epsilon)$ is the whole product set $\{0,1\} \times \{0,1\}$. Let us show that it contains the point $(0,1)$. To see this, take smooth functions $\phi_j(x_j)$ with small supports such that $\phi_1(x_1)$ is 1 in a neighborhood of 0 and $\phi_2(x_2)$ is 1 in a neighborhood of 1. Then

$$\phi_2(A_\epsilon) = U_\epsilon^{-1}\phi_2(A)U_\epsilon = A_\epsilon,$$

and

$$\phi_1(A) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A straight forward computation shows that $f(A, A_\epsilon) = \phi_1(A)\phi_2(A_\epsilon)$ is like

$$\begin{pmatrix} 0 & 0 \\ \epsilon & \epsilon^2 \end{pmatrix}$$

□

Let P_j and Q_j be tuples as before. Using that

$$(z - P_j)^{-1} - (z - Q_j)^{-1} = (z - P_j)^{-1}(Q_j - P_j)(z - Q_j)^{-1},$$

it is easy to check that

$$\|f(Q) - f(P)\| \lesssim \|Q - P\| = \sum \|Q_j - P_j\|.$$

Thus if f has support outside the spectrum of P, then $\|f(Q)\| \lesssim \|Q - P\|$, so even though not zero we can at least say that $f(Q)$ is small if Q is close to P.

Example 5. If P and Q are bounded (or at least if $[P, Q]$ is bounded), then

$$[(z-P)^{-1}, (w-Q)^{-1}] = (z-P)^{-1}(w-Q)^{-1}[P,Q](z-Q)^{-1}(w-P)^{-1},$$

and from this formula we get that

$$\|f(P,Q) - f(Q,P)\| \lesssim \|[P,Q]\|.$$

It also follows that $f(P,Q) - f(Q,P)$ is compact if $[P,Q]$ is compact.

□

10. Extended functional calculus.

Even though everything could be reduced by means of Cayley transform to the case of a bounded operator, we prefer a more direct treatment. We also restrict the attention from now on, to the case of one single operator, and hope that the extension to the case of several operators will turn out to be straight forward.

10.1. **The function space $\mathcal{E}$.** We define $\mathcal{E}(\widehat{\mathbb{R}}) = \mathcal{E} \subset C^\infty(\mathbb{R})$ to be the space of smooth functions on $\mathbb{R}$, which posess an asymptotic expansion,

$$f(x) \sim \sum_0^\infty a_k x^{-k},\ x \to \infty, \tag{10.1}$$

with $a_k \in \mathbb{C}$, in the sense that for every $N \in \mathbb{N}$:

$$f(x) = \sum_0^N a_k x^{-k} + x^{-N-1} r_{N+1}(x),\ |x| > 1, \tag{10.2}$$

where $r_{N+1}(x)$ is bounded with all its derivatives.

Proposition 10.1. *A continuous function on $\mathbb{R}$ belongs to $\mathcal{E}$ iff it has a bounded extension $\widetilde{f}$ to $\mathbb{C}$ with the property that $\frac{\partial \widetilde{f}}{\partial \overline{z}}$ is bounded and satisfies*

$$\frac{\partial \widetilde{f}}{\partial \overline{z}}(z) = \mathcal{O}_{N_0,N_1}(\langle z\rangle^{-N_1} |\mathrm{Im}\, z|^{N_0}),\ \forall N_0, N_1 \in \mathbb{N}. \tag{10.3}$$

Proof. Assume first that $f \in \mathcal{E}$. For $|x| > 1$, we introduce $y = -1/x$, $g(y) = f(x)$, and observe that the existence of an asymptotic expansion (10.1), (10.2) is equivalent to the fact that $g \in C^\infty(]-1, 1[)$ with $a_0 = g(0)$. Let $\widetilde{g}(y) \in C^\infty(D(0, 1))$ be an almost holomorphic extension of g with

$$\frac{\partial \widetilde{g}}{\partial \overline{y}}(y) = \mathcal{O}_N(|\mathrm{Im}\, y|^N),\ \forall N \in \mathbb{N}. \tag{10.4}$$

Consider $\widehat{f}(x) = \widetilde{g}(-1/x)$, $x \in \mathbb{C}$, $|x| > 1$. Using that

$$\frac{\partial}{\partial \overline{y}} = \overline{\frac{\partial x}{\partial y}} \frac{\partial}{\partial \overline{x}} = (\overline{\frac{\partial y}{\partial x}})^{-1} \frac{\partial}{\partial \overline{x}} = \overline{x}^2 \frac{\partial}{\partial \overline{x}},$$

and that $\mathrm{Im}\, y = |x|^{-2} \mathrm{Im}\, x$, we see that

$$\frac{\partial \widehat{f}}{\partial \overline{x}} = \mathcal{O}_N(\frac{|\mathrm{Im}\, x|^N}{|x|^{2N}}),\ \forall N.$$

In other words, $\widetilde{f} = \widehat{f}$ satisfies (10.3) in the region $|x| > 1$, and combining this with the standard construction in a bounded region, we get the desired extension $\widetilde{f}$.

Now let $f \in C(\mathbb{R})$ posess a bounded continuous extension $\widetilde{f}$ which satisfies (10.3). Put

$$\widetilde{g}(z) = -\frac{1}{\pi}\int \frac{\partial \widetilde{f}}{\partial \overline{w}}(w)(w-z)^{-1}L(dw), \tag{10.5}$$

and notice that the integral converges and that $\widetilde{g}$ is a bounded function which satisfies

$$\frac{\partial \widetilde{g}}{\partial \overline{z}} = \frac{\partial \widetilde{f}}{\partial \overline{z}}.$$

Consequently, $\widetilde{f} - \widetilde{g}$ is a bounded entire function on $\mathbb{C}$ and hence a constant, so

$$\widetilde{f}(z) = a_0 + \widetilde{g}(z), \;\; a_0 \in \mathbb{C}. \tag{10.6}$$

So far we only used that

$$\frac{\partial \widetilde{f}}{\partial \overline{z}}(z) = \mathcal{O}(\langle z\rangle^{-1-\epsilon}), \tag{10.7}$$

for some $\epsilon > 0$, and under this weaker assumption, we see that $\widetilde{g}$ is continuous and $\widetilde{g}(z) \to 0$, $|z| \to \infty$.

Now we use the full strength of (10.3), and write

$$\frac{1}{w-z} = -\sum_0^{N-1} \frac{w^k}{z^{k+1}} + \frac{w^N}{z^N(w-z)}. \tag{10.8}$$

Using this in (10.5), we get

$$\begin{aligned}\widetilde{g}(z) &= \sum_1^N \frac{1}{z^k}\frac{1}{\pi}\int \frac{\partial \widetilde{f}}{\partial \overline{w}}(w) w^{k-1} L(dw) \\ &+ \frac{1}{z^N}\left(-\frac{1}{\pi}\right)\int \frac{\partial \widetilde{f}(w)}{\partial \overline{w}} w^N \frac{1}{w-z} L(dw) = \sum_1^N z^{-k} a_k + \frac{1}{z^N} r_N(z)\end{aligned} \tag{10.9}$$

with the obvious definition of a_k, r_N. Using (10.3), we see that $r_{N|_\mathbb{R}}$is smooth and bounded together with all its derivatives. This and (10.6) imply that $f \in \mathcal{E}$. $\square$

Let $\mathcal{G}$ be the space of functions $f \in \mathcal{E}$ for which the series in (10.1) converges and is equal to $f(x)$ for $|x|$ sufficiently large. In other words, $\mathcal{G}$ is the space of smooth functions on $\mathbb{R}$ with a bounded holomorphic extension to a domain $\{z \in \mathbb{C}; |z| > R\}$ for some $R > 0$.

Proposition 10.2. *A continuous function f on $\mathbb{R}$ belongs to $\mathcal{G}$ iff it has a bounded extension $\widetilde{f}$ to $\mathbb{C}$, such that $\frac{\partial \widetilde{f}}{\partial \overline{z}}$ has compact support and satisfies*

$$\frac{\partial \widetilde{f}}{\partial \overline{z}} = \mathcal{O}(|\mathrm{Im}\, z|^N), \;\; \forall N \in \mathbb{N}. \tag{10.10}$$

The proof is just a slight variation of the one of Proposition 10.1 and will be omitted.

10.2. **The operator.** Let $\mathcal{B}$ be a complex Banach space and $P : \mathcal{B} \to \mathcal{B}$ a densely defined closed operator. We assume,

$$\sigma(P) \subset \mathbb{R}, \tag{10.11}$$

so that $(z-P)^{-1} \in \mathcal{L}(\mathcal{B})$ is well-defined and depends holomorphically on $z \in \mathbb{C} \setminus \mathbb{R}$. Assume,

$$\|(z-P)^{-1}\| \leq \mathcal{O}(|\operatorname{Im} z|^{-N_0^0}\langle z\rangle^{N_1^0}), \tag{10.12}$$

for some fixed $N_0^0, N_1^0 \in \mathbb{R}$.

For the $\mathcal{G}$-calculus, we will replace (10.12) by the weaker assumption (3.2) (with $P = P_1$).

10.3. **The calculus.** For $f \in \mathcal{E}$ as in (10.1), we recall that we have (10.6) where $\widetilde{g}$ is given by (10.5). If $P : \mathcal{B} \to \mathcal{B}$ satisfies (10.11), (10.12), we define,

$$f(P) = a_0 1 - \frac{1}{\pi} \int \frac{\partial \widetilde{f}}{\partial \overline{z}}(z)(z-P)^{-1} L(dz). \tag{10.13}$$

In view of (10.3), (10.12), this clearly defines a bounded operator, but we need to check that the right hand side of (10.13) only depends on f and not on the choice of bounded extension $\widetilde{f}$ satisfying (10.3). Let $\check{f}$ be a second extension of f with the same properties. Then it is a standard fact that (2.7) holds for the difference of the two extensions, and this estimate can also be applied to the difference $\widetilde{g}(w) - \check{g}(w)$, where $\widetilde{g}(w) = \widetilde{f}(-1/w)$, $\check{g}(w) = \check{f}(-1/w)$, $|w| < 1$. We conclude that for all $N_0, N_1 \in \mathbb{N}$,

$$(\widetilde{f} - \check{f})(z) = \mathcal{O}_{N_0,N_1}(|\operatorname{Im} z|^{N_0}\langle z\rangle^{-N_1}), \tag{10.14}$$

for $z \in \mathbb{C}$. From this fact and (10.12), it is easy to see as in Section 3, that

$$-\frac{1}{\pi} \int \frac{\partial}{\partial \overline{z}} (\widetilde{f} - \check{f})(z)(z-P)^{-1} L(dz) = 0,$$

so the definition (10.13) is indeed independent of the choice of $\widetilde{f}$.

Notice that the map $\mathcal{E} \ni f \mapsto f(P) \in \mathcal{L}(\mathcal{B})$ is linear and continuous. ($\mathcal{E}$ is a Frechet space with C^∞-topology for the restriction of $f \in \mathcal{E}$ to any bounded interval and the $C^\infty(]-1,1[)$-topology for the function $f(-1/y)$.)

Example 6. if $\zeta \in C \setminus \mathbb{R}$, then $(\zeta - \cdot)^{-1} \in \mathcal{E}$ and $((\zeta - \cdot)^{-1})(P) = (\zeta - P)^{-1}$ is the resolvent. □

Let us establish a basic calculus result:

Proposition 10.3. *If $f_1, f_2 \in \mathcal{E}$, then $f_1 f_2 \in \mathcal{E}$, and*

$$(f_1 f_2)(P) = f_1(P) f_2(P). \tag{10.15}$$

Proof. write $f_j = a_{0,j} + g_j$, with $g_j(x) \sim a_{1,j}x^{-1} + a_{2,j}x^{-2} + \dots$, and recall that

$$\widetilde{g}_j(z) = -\frac{1}{\pi}\int \frac{\partial \widetilde{f}_j}{\partial \overline{w}}(w)(w-z)^{-1}L(dw),\ \frac{\partial \widetilde{f}_j}{\partial \overline{w}} = \frac{\partial \widetilde{g}_j}{\partial \overline{w}}. \tag{10.16}$$

Then,

$$\begin{aligned} f_1(P)f_2(P) &= (a_{0,1} + g_1(P))(a_{0,2} + g_2(P)) \\ &= (a_{0,1}a_{0,2} + a_{0,1}g_2 + g_1 a_{0,2})(P) + g_1(P)g_2(P), \end{aligned}$$

so it suffices to show (10.15) with f_j replaced by g_j. This verification can be done as in the proof of Proposition 4.2 and we omit the details. □

Application. If $f \in \mathcal{E}$, then $\sigma(f(P)) \subset \overline{f(\mathbb{R})}$, and if $\zeta \in \mathbb{C} \setminus \overline{f(\mathbb{R})}$, then

$$(\zeta - f(P))^{-1} = (\frac{1}{\zeta - f})(P).$$

Now consider the $\mathcal{G}$-calculus and let P satisfy (10.11), (3.2). If $f \in \mathcal{G}$, we still define $f(P)$ by (10.13) and show that it does not depend on the choice of $\widetilde{f}$ as in Proposition 10.2. Proposition 10.3 remains valid for the $\mathcal{G}$-calculus, and so does the application.

10.4. **Relation to the Cayley transform.** Consider the Cayley(-Möbius) transform C of Section 6.

If $f : \mathbb{R} \to \mathbb{C}$, $g : \mathbb{T} \to \mathbb{C}$, are related by

$$f = g \circ C, \tag{10.17}$$

then $f \in \mathcal{E} = \mathcal{E}(\widehat{\mathbb{R}})$ iff $g \in \mathcal{E}(\mathbb{T}) = C^\infty(\mathbb{T})$. Let $f \in \mathcal{E}$, $g \in C^\infty(\mathbb{T})$ be related by (10.17).

With P as before, define $Q \in \mathcal{L}(\mathcal{B})$, by

$$Q = C(P), \tag{10.18}$$

where the right hand side can either be defined by our calculus or more directly (but equivalently) as

$$C(P) = (P+i)(P-i)^{-1} = 1 + 2i(P-i)^{-1}.$$

We know that $\sigma(C(P)) \subset \mathbb{T}$, and as in Section 6 we get

$$g(Q) = f(P), \tag{10.19}$$

where $G(Q)$ is defined as prior to (6.2).

We have the same results for the $\mathcal{G}$-calculus. (If $f \in \mathcal{G} = \mathcal{G}(\mathbb{R})$, then g belongs to the space $\mathcal{G}(\mathbb{T})$ of C^∞-functions on $\mathbb{T}$ that are analytic near 1.)

11. Recovering P from the functional calculus

In this section we show that every functional calculus $\mathcal{E} \ni f \mapsto \mathrm{Op}\,(f) \in \mathcal{L}(B)$ with suitable properties, is of the form $\mathrm{Op}\,(f) = f(P)$ for some operator P as above. We will also get the corresponding result for the $\mathcal{G}$-calculus.

Assume we have a continuous linear map

$$\mathcal{E} \ni f \mapsto \mathrm{Op}\,(f) \in \mathcal{L}(B), \tag{11.1}$$

with the property

$$\mathrm{Op}\,(f_1)\mathrm{Op}\,(f_2) = \mathrm{Op}\,(f_1 f_2),\ \ f_j \in \mathcal{E}. \tag{11.2}$$

We further assume,

$$\sum_{g \in C_0^\infty(\mathbb{R})} \mathcal{R}(\mathrm{Op}\,(g)) \text{ is dense in } \mathcal{B}, \tag{11.3}$$

$$\bigcap_{g \in C_0^\infty} \mathcal{N}(\mathrm{Op}\,(g)) = 0, \tag{11.4}$$

where $\mathcal{N}$ =”nullspace of”, $\mathcal{R}$ =”range of”.

Lemma 11.1. *If $g_0 \in \mathcal{E}$ satisfies $g_0(x) \neq 0$ for all $x \in \mathbb{R}$, then* $\mathrm{Op}\,(g_0)$ *is injective with dense range.*

Proof. If $g \in C_0^\infty$, then $k = g/g_0 \in C_0^\infty$, $g = kg_0$, so

$$\mathrm{Op}\,(g) = \mathrm{Op}\,(g_0)\mathrm{Op}\,(k) = \mathrm{Op}\,(k)\mathrm{Op}\,(g_0).$$

Hence

$$\mathcal{R}(\mathrm{Op}\,(g)) \subset \mathcal{R}(\mathrm{Op}\,(g_0)),\ \ \mathcal{N}(\mathrm{Op}\,(g)) \supset \mathcal{N}(\mathrm{Op}\,(g_0)),$$

and the lemma follows. □

Put $\omega_z(x) = 1/(z-x)$, so that $\omega_z \in \mathcal{E}$ for $z \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 11.2. $\mathcal{D} := \mathcal{R}(\mathrm{Op}\,(\omega_z))$, $z \in \mathbb{C}\setminus\mathbb{R}$ *is independent of the choice of z.*

Proof. Let $z, w \in \mathbb{C}\setminus\mathbb{R}$, so that ω_w/ω_z, $\omega_z/\omega_w \in \mathcal{E}$. The lemma follows from applying Op to the relations

$$\omega_z = \frac{\omega_z}{\omega_w}\omega_w,\ \ \omega_w = \frac{\omega_w}{\omega_z}\omega_z.$$

□

Definition 2. . For $u = \mathrm{Op}\,(\omega_z)v \in \mathcal{D}$, $z \in \mathbb{C} \setminus \mathbb{R}$, $v \in \mathcal{B}$, we put $Pu = \mathrm{Op}\,(\omega_z(x)x)v = \mathrm{Op}\,(\frac{\cdot}{z-\cdot})v$.

We need to check that this definition does not depend on the choice of z, v, in the representation of u, so assume that we also have $u = \mathrm{Op}\,(\omega_{\widetilde{z}})(\widetilde{v})$, $\widetilde{z} \in \mathbb{C}\backslash\mathbb{R}$, $\widetilde{v} \in \mathcal{B}$. Using that $\mathrm{Op}\,(\omega_z)$, $\mathrm{Op}\,(\omega_{\widetilde{z}})$ are injective, we see that $\widetilde{v} = \mathrm{Op}\,(\omega_z/\omega_{\widetilde{z}})v$, and hence,

$$\mathrm{Op}\,(x\omega_{\widetilde{z}}(x))\widetilde{v} = \mathrm{Op}\,(x\omega_{\widetilde{z}})\mathrm{Op}\,(\frac{\omega_z}{\omega_{\widetilde{z}}})v = \mathrm{Op}\,(x\omega_{\widetilde{z}}\frac{\omega_z}{\omega_{\widetilde{z}}})v = \mathrm{Op}\,(x\omega_z)v.$$

Hence the definition of P does not depend on the choice of z, v.

We also see that $P : \mathcal{B} \to \mathcal{B}$ is a closed operator with domain $\mathcal{D}$, with $\sigma(P) \subset \mathbb{R}$, and with

$$(z-P)^{-1} = \mathrm{Op}\,(\omega_z). \tag{11.5}$$

On the other hand, if q is a seminorm on $\mathcal{E}$, then

$$q(\omega_z) \le C_0|\mathrm{Im}\,z|^{-N_0^q}\langle z\rangle^{N_1^q}, \tag{11.6}$$

for some $N_0^q, N_1^q \in \mathbb{N}$, and combining this with (11.5) and the fact that Op is continuous on $\mathcal{E}$ with values in $\mathcal{L}(\mathcal{B})$, we obtain

$$\|(z-P)^{-1}\| \le C_0|\mathrm{Im}\,z|^{-N_0^0}\langle z\rangle^{N_1^0}, \tag{11.7}$$

for some $N_0^0, N_1^0 \in \mathbb{N}$.

Proposition 11.3. $\mathrm{Op}\,(f) = f(P)$ *for all* $f \in \mathcal{E}$.

Proof. From (10.5), (10.6), we get by restriction to the real axis,

$$f = a_0 - \frac{1}{\pi}\int \frac{\partial \widetilde{f}}{\partial \overline{z}}(z)\omega_z L(dz), \tag{11.8}$$

where $\widetilde{f}$ is an almost holomorphic extension of f as in Proposition 10.1. Now (11.8) converges in $\mathcal{E}$, so

$$\begin{aligned}\mathrm{Op}\,(f) = a_0 1 - \frac{1}{\pi}\int \frac{\partial \widetilde{f}}{\partial \overline{z}}(z)\mathrm{Op}\,(\omega_z)L(dz) = \\ = a_0 1 - \frac{1}{\pi}\int \frac{\partial \widetilde{f}}{\partial \overline{z}}(z)(z-P)^{-1}L(dz) = f(P),\end{aligned}$$

where we used (11.5) for the second equality and (10.13) for the last one. $\square$

$\mathcal{G}$ is not a Frechet space but rather an inductive limit of such spaces: $\lim_{R\to\infty}\mathcal{G}_R$, where

$$\mathcal{G}_R = \{f \in \mathcal{G};\ f \text{ extends to a bounded holomorphic function in } |z| > R\}.$$

A sequence of functions converges in $\mathcal{G}$ iff there is some $R > 0$ such that it converges in $\mathcal{G}_R$. Assume that we have a (sequentially) continuous map

$$\mathcal{G} \ni f \mapsto \mathrm{Op}(f) \in \mathcal{L}(\mathcal{B}), \tag{11.9}$$

satisfying (11.2)–(11.4). Then we can still define a closed densely defined operator as above. Instead of (5.7), we get (3.2) and by the same proof as above, we have

Proposition 11.4. $\operatorname{Op}(f) = f(P)$ *for all* $f \in \mathcal{G}$.

Remark 4. In view of Proposition 11.3 it is natural to ask whether any continuous algebra homomorphism

$$\Phi : \mathcal{E}(\widehat{\mathbb{R}}) \to \mathcal{L}(\mathcal{B}) \tag{11.10}$$

corresponds to a closed operator A (with a resolvent with temperate growth as before) such that $\Phi(\phi) = \phi(A)$ for $\phi \in \mathcal{E}(\widehat{\mathbb{R}})$. Given such a Φ, there is a unique homomorphism

$$\widetilde{\Phi} : C^\infty(\mathbb{T}) \to \mathcal{L}(\mathcal{B}),$$

such that $\widetilde{\Phi}(f) = \Phi(f \circ C)$. If $B = \widetilde{\Phi}(\mathrm{id}\,) = \Phi(C)$ (where $\mathrm{id}\,(w) = w$, $w \in \mathbb{T}$), then $\widetilde{\Phi}(f) = f(B)$ for $f \in C^\infty(\mathbb{T})$, $\sigma(B) = \mathbb{T}$, and the resolvent has temperate growth near $\mathbb{T}$ (just apply to $f(z) = 1/(w - z)$). If the operator A exists, then $C(A) = \Phi(C) = B$, so therefore $B - 1$ must be injective. Conversely, if $B - 1$ is injective, it is easy to check that $A = C^{-1}(B)$ defines Φ. (Notice that the conditions (11.3), (11.4) ensure that $B - 1$ is injective and has dense range, respectively.)

The same conclusions hold if $\mathcal{E}(\widehat{\mathbb{R}})$ is replaced by $\mathcal{G}(\mathbb{R})$.

If we instead consider a similar homomorphism from $\mathcal{S}(\mathbb{R})$ or $C_0^\infty(\mathbb{R})$ things are different; then there is not necessarily always an operator like B. To see this, let

$$f(x) = x(2 + \sin x^m),$$

where $3 \le m \in \mathbb{N}$ and notice that f^*, i.e; the composition with f, induces a continuous homomorphism $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$. If $B = H^1(\mathbb{R})$, we can define a continous homomorphism $\mathcal{S} \to \mathcal{L}(H^1(\mathbb{R}))$, by letting $\Phi(\phi)$ be multiplication on $H^1(\mathbb{R})$ by $f^*\phi = \phi \circ f$. It is easy to see that this Φ cannot be extended to any function $\phi(x) = 1/(z - x)$, and therefore it does not correspond to any operator like A or B above. □

12. A $g(f(P)) = (g \circ f)(P)$ result.

As a preparation, we construct a suitable almost holomorphic extension of $\mathbb{R} \ni x \mapsto (\zeta - f(x))^{-1}$, when $f \in \mathcal{E}$, $\zeta \notin \overline{f(\mathbb{R})}$. Let $\widetilde{f}(z)$ be an almost holomorphic extension of f with

$$\frac{\partial \widetilde{f}}{\partial \overline{z}} = \mathcal{O}_N(1)\Big(\frac{|\operatorname{Im} z|}{\langle z\rangle^2}\Big)^N, \ \forall N \ge 0, \tag{12.1}$$

and

$$\nabla \widetilde{f}(z) = \mathcal{O}(\langle z\rangle^{-2}). \tag{12.2}$$

Then,

$$\widetilde{f}(z) = f(\operatorname{Re} z) + \mathcal{O}\Big(\frac{\operatorname{Im} z}{\langle z\rangle^2}\Big). \tag{12.3}$$

Let $\delta(\zeta) = \operatorname{dist}(\zeta, \overline{f(\mathbb{R})})$. From (12.3), it follows that

$$|\widetilde{f}(z) - \zeta| > \delta(\zeta)/2, \text{ if } \frac{|\operatorname{Im} z|}{\langle z\rangle^2} \ll \delta(\zeta). \tag{12.4}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be equal to 1 near 0, and put

$$\chi_\delta(z) = \chi\Big(\frac{C|\operatorname{Im} z|}{\delta\langle z\rangle^2}\Big), \tag{12.5}$$

where $C > 0$ is large enough, but independent of δ, z. Notice that when $\delta > 0$ is large enough, then $\chi_\delta(z) = 1$, for all $z \in \mathbb{C}$.

As an almost holomorphic extension of $x \mapsto (\zeta - f(x))^{-1}$, we take

$$F(\zeta, z) = \chi_{\delta(\zeta)}(z)\frac{1}{\zeta - \widetilde{f}(z)}. \tag{12.6}$$

By construction, we have

$$F(\zeta, z) = \frac{\mathcal{O}(1)}{\delta(\zeta)}. \tag{12.7}$$

Further,

$$\frac{\partial}{\partial\overline{z}}F(\zeta, z) = \frac{\partial}{\partial\overline{z}}(\chi_{\delta(\zeta)}(z))\frac{1}{\zeta - \widetilde{f}(z)} + \chi_{\delta(\zeta)}(z)\frac{1}{(\zeta - \widetilde{f}(z))^2}\frac{\partial\widetilde{f}}{\partial\overline{z}}(z). \tag{12.8}$$

Here,

$$\frac{\partial}{\partial\overline{z}}\chi_{\delta(\zeta)}(z) = \chi'\Big(\frac{C|\operatorname{Im} z|}{\delta\langle z\rangle^2}\Big)\frac{\partial}{\partial\overline{z}}\Big(\frac{C|\operatorname{Im} z|}{\delta\langle z\rangle^2}\Big)$$

has its support in a region

$$\frac{|\operatorname{Im} z|}{\delta\langle z\rangle^2} \sim 1, \tag{12.9}$$

and since

$$\frac{\partial}{\partial\overline{z}}\Big(\frac{C|\operatorname{Im} z|}{\delta\langle z\rangle^2}\Big) = \frac{\mathcal{O}(1)}{\delta\langle z\rangle^2},$$

we see that the first term in the right hand side of (12.8) is $\mathcal{O}(\delta^{-2}\langle z\rangle^{-2})$ and has its support in a region (12.9). The second term is $\mathcal{O}(1)\frac{1}{\delta^2}(\frac{|\operatorname{Im} z|}{\langle z\rangle})^N$ for all $N \geq 0$. We conclude that

$$\frac{\partial}{\partial\overline{z}}F(\zeta, z) = \mathcal{O}_N(1)\delta^{-2-N}\Big(\frac{|\operatorname{Im} z|}{\langle z\rangle^2}\Big)^N, \ \forall N \geq 0. \tag{12.10}$$

Essentially the same estimates show that

$$\nabla_z F(\zeta, z) = \mathcal{O}(1)\delta^{-2}\langle z\rangle^{-2}. \tag{12.11}$$

We also notice that

$$\frac{1}{\zeta-\widetilde{f}(z)}-F(\zeta,z)=(1-\chi(\frac{C|\mathrm{Im}\, z|}{\delta\langle z\rangle^2}))\frac{1}{\zeta-\widetilde{f}(z)}$$

is different from 0 only when

$$\frac{|\mathrm{Im}\, z|}{\delta\langle z\rangle^2}\geq\frac{1}{\widetilde{C}},$$

i.e. for

$$\delta(\zeta)\leq\frac{\widetilde{C}|\mathrm{Im}\, z|}{\langle z\rangle^2}. \tag{12.12}$$

Now let g be continuous on $\overline{f(\mathbb{R})}$ with a bounded uniformly Lipschitz extension $\widetilde{g}(\zeta)$, $\zeta\in\mathbb{C}$ satisfying

$$\frac{\partial\widetilde{g}}{\partial\overline{\zeta}}=\mathcal{O}(\mathrm{dist}\,(\zeta,\overline{f(\mathbb{R})})^\infty). \tag{12.13}$$

Consider

$$\widetilde{h}(z)=\widetilde{g}(\widetilde{f}(z)).$$

By the chain-rule,

$$\frac{\partial\widetilde{h}}{\partial\overline{z}}=\frac{\partial\widetilde{g}}{\partial\overline{\zeta}}(\widetilde{f}(z))\overline{\Big(\frac{\partial\widetilde{f}}{\partial z}\Big)}+\frac{\partial\widetilde{g}}{\partial\zeta}\frac{\partial\widetilde{f}}{\partial\overline{z}}.$$

Using that

$$\mathrm{dist}\,(\widetilde{f}(z),f(\mathbb{R}))=\mathcal{O}(\frac{|\mathrm{Im}\, z|}{\langle z\rangle^2}),$$

and the Lipschitz properties of $\widetilde{g},\widetilde{f}$, we get

$$\frac{\partial\widetilde{h}}{\partial\overline{z}}=\mathcal{O}_N(1)\Big(\frac{|\mathrm{Im}\, z|}{\langle z\rangle^2}\Big)^N,\ \forall N\geq 0. \tag{12.14}$$

It is also clear that $\widetilde{h}$ is a bounded continuous extension of $g\circ f$ with

$$\nabla\widetilde{h}=\mathcal{O}(\langle z\rangle^{-2}). \tag{12.15}$$

Consider

$$g(f(P)):=-\frac{1}{\pi}\int\frac{\partial\widetilde{g}}{\partial\overline{\zeta}}(\zeta)(\zeta-f(P))^{-1}L(d\zeta). \tag{12.16}$$

For $\zeta\in\mathbb{C}\setminus\overline{f(\mathbb{R})}$, we have

$$(\zeta-f(P))^{-1}=-\frac{1}{\pi}\int\frac{\partial}{\partial\overline{z}}(F(\zeta,z))(z-P)^{-1}L(dz), \tag{12.17}$$

and hence,

$$\begin{aligned} g(f(P)) &= (-\frac{1}{\pi})^2 \iint \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} \frac{\partial F(\zeta, z)}{\partial \overline{z}} (z-P)^{-1} L(dz) L(d\zeta) \\ &= -\frac{1}{\pi} \int \frac{\partial}{\partial \overline{z}} \Big(-\frac{1}{\pi} \int \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} (\zeta) F(\zeta, z) L(d\zeta) \Big) (z-P)^{-1} L(dz), \end{aligned} \tag{12.18}$$

where the first double integral converges in operator norm, so the same holds for the $\int (...) L(dz)$ integral in the last expression, which we can view as

$$\lim_{\epsilon \to 0} -\frac{1}{\pi} \int (1 - \chi_\epsilon(z)) \frac{\partial}{\partial \overline{z}} (...)(z-P)^{-1} L(dz). \tag{12.19}$$

Consider

$$\begin{aligned} &-\frac{1}{\pi} \int \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} (\zeta) F(\zeta, z) L(d\zeta) \\ &= -\frac{1}{\pi} \int \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} (\zeta) \frac{1}{\zeta - \widetilde{f}(z)} L(d\zeta) + \frac{1}{\pi} \int \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} (\zeta) \frac{1 - \chi_{\delta(\zeta)}(z)}{\zeta - \widetilde{f}(z)} L(d\zeta) \\ &\qquad = \widetilde{g}(\widetilde{f}(z)) + \frac{1}{\pi} \int \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} (\zeta) \frac{1 - \chi_{\delta(\zeta)}(z)}{\zeta - \widetilde{f}(z)} L(d\zeta). \end{aligned}$$

As already observed, the integrand in the last integral is $\neq 0$ only for $\delta(\zeta) \leq \mathcal{O}(1) \frac{|\mathrm{Im}\, z|}{\langle z \rangle^2}$, and using that $\frac{\partial \widetilde{g}}{\partial \overline{\zeta}}(\zeta) = \mathcal{O}(\delta(\zeta)^\infty)$, we see that

$$-\frac{1}{\pi} \int \frac{\partial \widetilde{g}}{\partial \overline{\zeta}} (\zeta) F(\zeta, z) L(d\zeta) = \widetilde{g}(\widetilde{f}(z)) + \mathcal{O}(1) \big(\frac{|\mathrm{Im}\, z|}{\langle z \rangle^2} \big)^\infty. \tag{12.20}$$

Using this in the last integral in (12.18), represented as a limit as in (12.19), together with the temperate growth of the resolvent, we get

$$g(f(P)) = -\frac{1}{\pi} \int \frac{\partial}{\partial \overline{z}} (\widetilde{g}(\widetilde{f}(z)))(z-P)^{-1} L(dz) = (g \circ f)(P). \tag{12.21}$$

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